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Exact multivariate amplitude distributions for non-stationary Gaussian or algebraic fluctuations of covariances or correlations in correlated financial markets

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Abstract Correlated financial markets are perfect examples of highly non-stationary systems. In particular, sample averaged observables of time series as variances and correlation coefficients are continuously fluctuating, directly depending on the time window in which they are evaluated. Thus, models that describe the multivariate amplitude distributions of such systems are of considerable interest. Extending previous works, we apply a methodology, where a set of measured, non-stationary correlation matrices are viewed as an ensemble for which is set up a random matrix model. This ensemble is used to average the stationary multivariate amplitude distributions measured on short time scales and thus obtain for large time scales multivariate amplitude distributions which feature heavy tails. We explicitly use four cases, combining Gaussian and algebraic distributions to compare the distributions with empirical returns distributions using daily data from companies listed in the Standard & Poor's (S&P 500) stock market index in different periods of time. The comparison in the four cases with the empirical data reveals good agreement.

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1 Introduction

2 Data set

3 Exact multivariate amplitude distributions

3.1 Key concepts

Consider time series $S_k(t)$, $k = 1, 2, \ldots, K$ of stock prices for K companies. The values $S_k(t)$ are taken in fixed time steps Δt. In general, the data contain an exponential increase due to the drift. Thus, to measure the correlations independently of this trend, it is better to use logarithmic differences instead of returns

$$G_k(t) = \ln S_k(t+\Delta t) - \ln S_k(t) = \ln \frac{S_k(t+\Delta t)}{S_k(t)}. \quad (1)$$

Anyway, logarithmic differences and returns almost coincide if the time steps Δt are sufficiently short [1,2]

$$G_k(t) \approx r_k(t) = \frac{S_k(t+\Delta t) - S_k(t)}{S_k(t)}. \quad (2)$$

The returns are well known to have distributions with heavy tails, the smaller Δt, the heavier [3]. Furthermore, the sample standard deviations σ_k, referred to as volatilities, strongly fluctuate for different time windows of the same length T [3,4].

The mean of the logarithmic differences reads [5]

$$\langle G_k(t)\rangle_T = \frac{1}{T}\sum_{t=1}^{T} G_k(t). \quad (3)$$

To compare the different K companies, it is necessary to normalize the time series. The normalized time series are defined by [5,3]

$$M_k(t) = \frac{G_k(t) - \langle G_k(t)\rangle}{\sqrt{\langle G_k^2(t)\rangle_T - \langle G_k(t)\rangle_T^2}}, \quad (4)$$

where

$$\sigma_k = \sqrt{\langle G_k^2(t)\rangle_T - \langle G_k(t)\rangle_T^2} \quad (5)$$

is the volatility of the k company in the time window of length T. These values can be viewed as the elements of a $K \times T$ rectangular matrix M. With these normalizations and rescalings, it can be measured correlations in such a way that all companies and all stocks are treated on equal footing.

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The correlation coefficient for the stocks k and l is defined as [3]

$$C_{kl} = \left\langle M_k\left(t\right) M_l\left(t\right)\right\rangle_T = \frac{1}{T}\sum_{t=1}^{T} M_k\left(t\right) M_l\left(t\right), \tag{6}$$

which can be written as

$$C_{kl} = \frac{\left\langle G_k\left(t\right) G_l\left(t\right)\right\rangle_T - \left\langle G_k\left(t\right)\right\rangle_T \left\langle G_l\left(t\right)\right\rangle_T}{\sigma_k \sigma_l}. \tag{7}$$

The coefficients C_{kl} are the elements of a $K \times K$ square matrix C, the correlation matrix. The limiting values of these correlation coefficients

$$C_{kl}^{\mathrm{lim}} = \begin{cases} +1 & \text{completely correlated} \\ 0 & \text{completely uncorrelated} \\ -1 & \text{completely anticorrelated} \end{cases}. \tag{8}$$

The time average of C_{kl} can be viewed as the matrix product of the rectangular matrix M $(K \times T)$ with its transpose matrix $M^{\dagger}$ $(T \times K)$, divided by T. Thus, the correlation matrix can be written in the form

$$C = \frac{1}{T} M M^{\dagger}. \tag{9}$$

The correlation matrix C is real and symmetric. Using the correlation matrix C is possible to define the covariance matrix [6,7,8,9,5]

$$\Sigma = \sigma C \sigma, \tag{10}$$

where the diagonal matrix σ contains the volatilities σ_k, $k = 1, \ldots, K$.

3.2 General considerations

3.3 Gaussian-Gaussian distribution

3.4 Gaussian-Algebraic distribution

3.5 Algebraic-Gaussian distribution

3.6 Algebraic-Algebraic distribution

4 Comparison with empirical portfolio returns

4.1 Response functions on trade time scale

4.2 Response functions on physical time scale

5 Conclusion

6 Author contribution statement

TG proposed the research. JCHL carried out the analysis. All the authors contributed equally to analyzing the results and writing the paper.

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Appendix A Foreign exchange pairs used to analyze the spread impact

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