Lecture 2: CES - Cobb-Douglas economies

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April 2, 2025

Goals for Today

- So far you have worked with economies where every agent is a price-taker
- Highly unsatisfactory to study inflation.
  - Inflation is an aggregated measure of individual price changes
  - Price changes obviously depend on the prices firms **choose**
  - Studying how firms choose prices in models where no one set prices is not promising
- Today it is a “framework” lecture
  - Firms will take input prices as given
  - Choose input combinations
  - And set prices subject to demand curves
- We will do this in the most tractable framework we have: the Dixit-Stiglitz framework
- The DS framework is widely used in macro/trade/IO/urban/labor/etc. You should know it.

Digression

- Pedagogically, economists teach perfect competition as a benchmark
- ... and then imperfect competition as an extension
- I personally think that is not the right approach
- Imperfect competition is the **general** case
- and perfect competition is a **limiting** scenario

Model

- Household problem

$$\max_{\{C_{it}\},N_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t, N_t)$$

$$\text{subject to: } \int_0^1 P_{it}C_{it} + A_{t+1} \leq W_t N_t + A_t(1+i)$$

$$C_t = \left(\int_0^1 C_{it}^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}}$$

- Note that $\theta \geq 1$ will give you the elasticity of substitution across varieties. Do not need a continuum
- As $\theta \to \infty$, function is linear, goods are perfect substitutes. As $\theta \to 1$, you get Leontief, no substitutability.
- C_t is oftentimes called a Dixit-Stiglitz aggregator.
- In this case, each variety is atomistic. Changes in C_{it} alone do not affect aggregates

Result 1: Two stage budgeting

- We could write a Lagrangean/Bellman to solve the problem directly
- Turns out there is an easier way
- If upper nest is separable (in this case N is separable from any C_i in u), and lower nest is homothetic (which DS aggregator is), then we can split the problem in two stages:
  - Take C as given, and solve the allocation of expenditure across varieties
  - Solve the outer nest using standard consumer theory

First Stage: Cost minimization

$$\min_{\{C_{it}\}} \int_0^1 P_{it}C_{it}di \text{ subject to: } \left(\int_0^1 C_{it}^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}} \geq \bar{C}$$

- Main advantage: Static problem with one constraint. Lagrange multiplier λ
- FOC:

$$P_{it} = \lambda_t \left(\frac{C_{it}}{C_t}\right)^{-1/\theta}$$

- Take ratio of the FOC for two varieties i and j

$$\frac{C_{it}}{C_{jt}} = \left(\frac{P_{it}}{P_{jt}}\right)^{-\theta}$$

- Making obvious that the elasticity of substitution between any two varieties is θ.

Price Indices

- Summarize prices in the economy with a single index. Infinite choices. Two appealing ones:
- The relative demand price index, which is the Lagrange multiplier λ_t.

$$\frac{C_{it}}{C_t} = \left(\frac{P_{it}}{\lambda_t}\right)^{-\theta}$$

- The ideal price index. Define P_t as satisfying

$$P_t C_t = \int_0^1 P_{it} C_{it} di$$

- How do these compare?

Result 2: The ideal price index = relative demand price index

- Under CES these two appealing options are the same index
- Evaluate the constraint $\left(\int_0^1 C_{it}^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}} = C_t$, and plug the demand curve $\frac{C_{it}}{C_t} = \left(\frac{P_{it}}{\lambda_t}\right)^{-\theta}$

$$\lambda_t = \left(\int_0^1 P_{it}^{1-\theta} di\right)^{1/(1-\theta)}$$

- Use the definition of the ideal price index $P_t C_t = \int_0^1 P_{it} C_{it} di$ and plug-in the demand curve

$$P_t = \lambda_t$$

- This property is very specific to CES. Not necessarily true in other demand systems.

Second stage: Utility maximization

$$\max_{\{C_{it}\},N_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t, N_t)$$

$$\text{subject to: } P_t C_t + A_{t+1} \leq W_t N_t + A_t(1+i)$$

- Same problem you know
- Used the definition $P_t C_t = \int_0^1 P_{it} C_{it} di$
- In the background we have

$$\frac{C_{it}}{C_t} = \left(\frac{P_{it}}{P}\right)^{-\theta}$$

Very tractable demand structure for an arbitrary number of goods.

Monopolistic Competition

- The assumption is that one, and only one, firm produces each variety *i*
- So I can index a firm and a variety with the same index.
- The firm is Neoclassical as in 210A

$$\max_{L_{it},K_{it},Y_{it},P_{it}} P_{it}Y_{it} - R_tK_{it} - w_tL_{it}$$

- But it has one additional constraint

$$Y_{it} \le K_{it}^{\alpha}L_{it}^{1-\alpha}$$
$$Y_{it} = C_t\left(P_{it}/P_t\right)^{-\theta}$$

- Notice that monopolistic competition in a closed economy implies $Y_{it} = C_{it}$.
- Can apply two-stage budgeting as well

First Stage. Cost minimization

$$\min_{K_{it},L_{it}} R_t K_{it} + w_t L_{it} \text{ subject to: } K_{it}^{\alpha} L_{it}^{1-\alpha} \geq \bar{Y} \tag{1}$$

- Lagrange multiplier ψ_t. FOC:

$$R_t K_{it} = \alpha Y_t \psi_t$$
$$w_t L_{it} = (1-\alpha) Y_t \psi_t$$

- Evaluate the constraint $K_{it}^{\alpha} L_{it}^{1-\alpha} = Y_{it}$ at the optimum

$$\psi_t = \left(\frac{R_t}{\alpha}\right)^{\alpha} \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}$$

- Total cost function $TC_{it} = R_t K_{it} + w_t L_{it}$ and evaluate at the optimum

$$TC_{it} = \psi_{it} Y_{it}$$

- Making evident that

$$mc_{it} = \psi_t$$

- Could have known that by simply remembering that the Lagrange multiplier is the effect on

Second Stage: Profit maximization

- Profit maximization

$$\max_{Y_{it}, P_{it}} (P_{it} - mc_{it}) Y_{it}$$

$$\text{subject to: } Y_{it} = C_t (P_{it}/P_t)^{-\theta}$$

- FOC and simplify:

$$P_{it} = \frac{\theta}{\theta - 1} mc_{it} \equiv \mu \times mc_{it}$$

- Price is a constant markup over marginal cost. Profits are:

$$(\mu - 1) mc_{it} C_{it} \geq 0$$

- In the limit where $\theta \to \infty$, $\mu \to 1$ and $\text{Profits}_{it} \to 0$.

Extensions

- Framework amenable to extensions. Examples:
  - Nested CES: Consumers choose among sectors. Sectors are composed of varieties. Two layer-CES.
  - Trade models: Consumers choose among origin of imports. Origin countries have different industries. Industries are composed by varieties. Three-layer CES.
  - Easy to extend to production functions with Decreasing Returns to scale.