Problem Set 1. 210C
Due Friday 18th.

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1. Find the problem set in the online appendix of Nakamura and Steinsson (2018) Identification in Macroeconomics paper. You can find it in their websites. Solve it.

   See NS1.ipynb

2. A representative household wants to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t, N_t), \tag{1}$$

where C_t is a CES bundle between goods and services with elasticity of substitution η and weight parameters φ_g and φ_s

$$C_t = \left(\varphi_g^{1/\eta} C_{g,t}^{\frac{\eta-1}{\eta}} + \varphi_s^{1/\eta} C_{s,t}^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}. \tag{2}$$

C_{jt} is also a CES bundle that aggregates a continuum of varieties i in sector $j \in g, s$. Formally

$$C_{jt} = \left(\int_0^1 C_{ijt}^{\frac{\rho-1}{\rho}} di\right)^{\frac{\rho}{\rho-1}}, \tag{3}$$

for $j \in g, s$.

The household maximizes utility subject to a series of budget constraints

$$\int_0^1 P_{igt} C_{igt} di + \int_0^1 P_{ist} C_{ist} di + B_{t+1} \leq B_t(1 + i_{t-1}) + W_t N_t + T_t$$

taking output prices P_{ijt}, input prices W_t, nominal interest rates i_{t-1}, and lump-sum transfers T_t as given, as well as an initial condition for B in period zero.

   (a) Solve the cost minimization problem for each sector j. That is, taking the value of C_{jt} as given, find the allocation across varieties C_{ijt} $\forall i$, as a function of prices P_{ijt}, sectoral demand C_{jt}, the Lagrange multiplier with respect to the shape of the sectoral

CES aggregator, and parameters.
The household solves the following cost minimization problem for each sector j:

$$min_{C_{ijt}} \quad P_{ijt}C_{ijt} \quad \text{s.t.} \quad C_{jt} = \left(\int_0^1 C_{ijt}^{\frac{\rho-1}{\rho}} di \right)^{\frac{\rho}{\rho-1}}$$

First order condition for each variety i:

$$P_{ijt} = \lambda_{jt} \frac{C_{ijt}}{C_{jt}}^{-\frac{1}{\rho}} \qquad \forall\, i, \forall\, j = g, s$$

where λ_{jt} is the Lagrange multiplier with respect to the shape of the sectoral CES aggregator. Solving for C_{ijt} gives:

$$C_{ijt} = \left(\frac{P_{ijt}}{\lambda_{jt}} \right)^{-\rho} C_{jt}$$

(b) Find an expression for the ideal sectoral price index, that is, a price index P_{jt} such that $P_{jt}C_{jt} = \int_0^1 P_{ijt}C_{ijt}di$.
Replace C_{ijt} in the expression for the CES aggregator:

$$C_{jt} = \left(\int_0^1 \left[\left(\frac{P_{ijt}}{\lambda_{jt}} \right)^{-\rho} C_{jt} \right]^{\frac{\rho-1}{\rho}} di \right)^{\frac{\rho}{\rho-1}}$$

Re-arranging this gives:

$$\lambda_{jt} = \left(\int_0^1 P_{ijt}^{1-\rho} di \right)^{\frac{1}{1-\rho}}$$

We want to find the expression for P_{jt} that satisfies:

$$P_{jt}C_{jt} = \int_0^1 P_{ijt}C_{ijt}di$$

Plug in the expression for C_{ijt} on the right-hand side:

$$P_{jt}C_{jt} = \int_0^1 P_{ijt} \left(\frac{P_{ijt}}{\lambda_{jt}} \right)^{-\rho} C_{jt}di$$

$$P_{jt} = \lambda_{jt}^{\rho} \int_0^1 P_{ijt}^{1-\rho} di$$

Replace λ_{jt} with the expression we found before:

$$P_{jt} = \left(\int_0^1 P_{ijt}^{1-\rho} di \right)^{\frac{\rho}{1-\rho}} \left(\int_0^1 P_{ijt}^{1-\rho} di \right)^{\frac{1}{1-\rho}}$$

$$P_{jt} = \left(\int_0^1 P_{ijt}^{1-\rho} di \right)^{\frac{1}{1-\rho}}$$

(c) Solve the cost minimization problem across sectors. That is, taking the value of C_t as given, find the allocation across sectors C_{jt}, as a function of sectoral price indices P_{jt}, aggregate demand C_t, the Lagrange multiplier with respect to the shape of the CES aggregator across sectors, and parameters.

The household solves the following cost minimization problem:

$$min_{C_{jt}} \quad P_{gt}C_{gt} + P_{st}C_{st} \quad \text{s.t.} \quad C_t = \left(\varphi_g^{1/\eta} C_{g,t}^{\frac{\eta-1}{\eta}} + \varphi_s^{1/\eta} C_{s,t}^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}$$

First order condition for each sector j:

$$P_{jt} = \lambda_t (\frac{C_{jt}}{C_t})^{-\frac{1}{\eta}} \varphi_j^{\frac{1}{\eta}} \quad \forall\ j = g, s$$

Re-arranging this gives:

$$C_{jt} = \varphi_j \left(\frac{P_{jt}}{\lambda_t}\right)^{-\eta} C_t$$

(d) Find an expression for the ideal aggregate price index, that is, a price index P_t such that $P_t C_t = P_{st}C_{st} + P_{gt}C_{gt}$.

Replacing C_{jt} in the expression for the CES aggregator and solving for the Lagrange multiplier λ_t gives:

$$\lambda_t = \left[\varphi_g P_{gt}^{1-\eta} + \varphi_s P_{st}^{1-\eta}\right]^{\frac{1}{1-\eta}}$$

We want to find the expression for P_t that satisfies:

$$P_t C_t = P_{st}C_{st} + P_{gt}C_{gt}$$

Plugging in the expression for C_{jt} and re-arranging gives:

$$P_t = \left[\varphi_g P_{gt}^{1-\eta} + \varphi_s P_{st}^{1-\eta}\right] \lambda_t^{\eta}$$

Using the expression for λ_t we found before gives:

$$P_t = \left[\varphi_g P_{gt}^{1-\eta} + \varphi_s P_{st}^{1-\eta}\right]^{\frac{1}{1-\eta}}$$

(e) Find the optimality conditions of the household for bond holdings, labor supply, and consumption.

Households maximize the flow of expected utility subject to a sequence of budget constraints:

$$\max_{C_t,N_t,B_{t+1}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t, N_t) + \lambda_t \Big[B_t(1 + i_{t-1} + W_T N_t + T_t - P_t C_t - B_{t+1}) \Big]$$

The first order conditions for the household problem are:

$$\begin{aligned} U_c^{'}(C_t, N_t) &= \lambda_t P_t \\ U_n^{'}(C_t, N_t) &= -\lambda_t W_t \\ \lambda_t &= \beta \mathbb{E}_t \Big[\lambda_{t+1}(1 + i_t) \Big] \end{aligned}$$

Combining the first two equations gives us the intra-temporal trade-off between consumption and leisure. Combining the first with the third equation gives us the Euler equation/intertemporal trade-off.

(f) Firms in each sector j produce with a technology linear in labor that depends on sector-specific productivity. $Y_{ijt} = A_{jt} L_{ijt}$. Firms are monopolistic competitors, and take wages as given. Firms understand the structure of demand, and must satisfy demand $Y_{ijt} = C_{ijt}$. Solve the profit maximization problem of the firms and find an expression for the optimal price firms set. Profits are given by $(P_{ijt} Y_{ijt} - W_t L_{ijt})$, so there is a fully integrated labor market in the economy.

Firms in each sector j maximize profits subject to the technological constraint given by the production function and subject to the demand schedule that we found in point (a).

Profits can be expressed as:

$$\begin{aligned} \Pi_{ijt} &= P_{ijt} Y_{ijt} - W_t L_{ijt} \\ &= P_{ijt} \Big(\frac{P_{ijt}}{P_{jt}} \Big)^{-\rho} Y_{jt} - \frac{W_t}{A_{jt}} \Big(\frac{P_{ijt}}{P_{jt}} \Big)^{-\rho} Y_{jt} \end{aligned}$$

where I replaced L_{ijt} with Y_{ijt}/A_{jt} and Y_{ijt} with the demand schedule.

The first order condition for the profit maximization problem gives us the optimal price:

$$P_{ijt} = \frac{\rho}{\rho - 1} \frac{W_t}{A_{jt}}$$

(g) Are firms charging the same prices within sectors? Are firms charging the same price across sectors? Are firms charging the same markup within sectors? Are firms charging the same markup across sectors?

Yes, firms charge the same price within sector. Not necessarily across sectors given that productivities could differ across sectors, $A_{gt} \neq A_{st}$. Yes, they charge the same markup within and across sectors.

(h) Do markups depend on η, ρ, or both η, and ρ? What is the intuition?
Markups depend only on ρ. Markups are independent of η because firms do not internalize the effect that their own pricing decisions have on sector-level demand.

(i) Will firms across sectors sell different quantities? What economic objects determine the differences (if any)? What is the intuitive explanation?

Yes, they will sell different quantities. The objects that determine possible differences are sectoral productivity A_{jt} and the weight parameters φ_j.