WEEK 2: LOG-LINEARIZATION & BLANCHARD-KAHN (1980)

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ECON 210C, 2025

TODAY'S AGENDA

1. Log-linearization
2. Blanchard-Kahn (1980) Method - *based on Eric Sims Graduate Macro notes*

WHY LOG-LINEARIZE?

- Solutions to most DSGE models involve non-linear difference equations
- Log-linearization is an approximation technique that allows us to
    1. express the solution as a system of **linear** differential equations
    2. express variables in percentage terms which is easy to interpret

LOG-LINEARIZATION - THREE STEPS

1. Log-transform all equilibrium conditions
2. Linearize (firt-order approx.) each equation around a point $\rightarrow$ steady-state
3. Re-arrange to express everything in terms of % deviation from steady-state

TAYLOR'S THEOREM

- An arbitrary function $f(x)$ can be expressed as a power series about a point $x^* \in X$:

$$f(x) = f(x^*) + \frac{f'(x^*)}{1!}(x - x^*) + \frac{f''(x^*)}{2!}(x - x^*)^2 + \frac{f'''(x^*)}{3!}(x - x^*)^3 + \ldots$$

- If the higher order derivatives are small, then the function can be well approximated linearly by:

$$f(x) \approx f(x^*) + \frac{f'(x^*)}{1!}(x - x^*)$$

- Straightforward to extend to multi-variate functions

$$f(x, y) \approx f(x^*, y^*) + \frac{f'_x(x^*, y^*)}{1!}(x - x^*) + \frac{f'_y(x^*, y^*)}{1!}(y - y^*)$$

GENERIC CASE

- Consider the non-linear function:

$$f(x, y) = \frac{g(x, y)}{h(x, y)}$$

- Taking logs on both sides we get:

$$\ln f(x, y) = \ln g(x, y) - \ln h(x, y)$$

- Next: compute the first order Taylor expansion for each term

FIRST ORDER TAYLOR EXPANSION

$$\ln f(x,y) \approx \ln f(x^*,y^*) + \frac{d\ln f(x^*,y^*)}{dx^*}(x-x^*) + \frac{d\ln f(x^*,y^*)}{dy^*}(y-y^*)$$

$$\ln g(x,y) \approx \ln g(x^*,y^*) + \frac{d\ln g(x^*,y^*)}{dx^*}(x-x^*) + \frac{d\ln g(x^*,y^*)}{dy^*}(y-y^*)$$

$$\ln h(x,y) \approx \ln h(x^*,y^*) + \frac{d\ln h(x^*,y^*)}{dx^*}(x-x^*) + \frac{d\ln h(x^*,y^*)}{dy^*}(y-y^*)$$

- Chain rule implies:

$$\frac{d\ln f(x)}{dx} = \frac{f'(x)}{f(x)}$$

- We will use this to simplify the above expressions

FIRST ORDER TAYLOR EXPANSION - *cont.*

- Combine the 3 expressions from the previous slide:

$$\ln f(x^*,y^*) + \frac{f_x^{'}(x^*,y^*)}{f(x^*,y^*)}(x-x^*) + \frac{f_y^{'}(x^*,y^*)}{f(x^*,y^*)}(y-y^*) =$$

$$\ln g(x^*,y^*) + \frac{g_x^{'}(x^*,y^*)}{g(x^*,y^*)}(x-x^*) + \frac{g_y^{'}(x^*,y^*)}{g(x^*,y^*)}(y-y^*) -$$

$$\left[\ln h(x^*,y^*) + \frac{h_x^{'}(x^*,y^*)}{h(x^*,y^*)}(x-x^*) + \frac{h_y^{'}(x^*,y^*)}{h(x^*,y^*)}(y-y^*)\right]$$

- Note that terms in pink cancel out

FIRST ORDER TAYLOR EXPANSION - *cont.*

- We are left with:

$$\frac{f_x^{'}(x^*,y^*)}{f(x^*,y^*)}(x-x^*)+\frac{f_y^{'}(x^*,y^*)}{f(x^*,y^*)}(y-y^*)=\frac{g_x^{'}(x^*,y^*)}{g(x^*,y^*)}(x-x^*)+\frac{g_y^{'}(x^*,y^*)}{g(x^*,y^*)}(y-y^*)-\left[\frac{h_x^{'}(x^*,y^*)}{h(x^*,y^*)}(x-x^*)+\frac{h_y^{'}(x^*,y^*)}{h(x^*,y^*)}(y-y^*)\right]$$

- Let $\tilde{x}=\frac{x-x^*}{x^*}$ be the percentage deviation of x from x^*
- Multiply and divide each term by either x^* or y^*

$$\frac{x^*f_x^{'}(x^*,y^*)}{f(x^*,y^*)}\tilde{x}+\frac{y^*f_y^{'}(x^*,y^*)}{f(x^*,y^*)}\tilde{y}=\frac{x^*g_x^{'}(x^*,y^*)}{g(x^*,y^*)}\tilde{x}+\frac{y^*g_y^{'}(x^*,y^*)}{g(x^*,y^*)}\tilde{y}-\left[\frac{x^*h_x^{'}(x^*,y^*)}{h(x^*,y^*)}\tilde{x}+\frac{y^*h_y^{'}(x^*,y^*)}{h(x^*,y^*)}\tilde{y}\right]$$

- The above expression is linear in both $\tilde{x}$ and $\tilde{y}$ ✓

LOG-LINEARIZATION - RECAP

- Take logs
- Do a first order Taylor approximation
- Multiply/divide as necessary to get variables expressed as percentage deviations from x^*
- The final product is an expression linear in $\tilde{x}, \tilde{y}$:

$$\frac{x^* f_x'(x^*, y^*)}{f(x^*, y^*)}\tilde{x} + \frac{y^* f_y'(x^*, y^*)}{f(x^*, y^*)}\tilde{y} = \frac{x^* g_x'(x^*, y^*)}{g(x^*, y^*)}\tilde{x} + \frac{y^* g_y'(x^*, y^*)}{g(x^*, y^*)}\tilde{y} - \frac{x^* h_x'(x^*, y^*)}{f(x^*, y^*)}\tilde{x} - \frac{y^* h_y'(x^*, y^*)}{f(x^*, y^*)}\tilde{y}$$

LOG-LINEARIZATION - EXAMPLES

- Cobb-Douglas production function: $y_t = a_t k_t^{\alpha} n_t^{1-\alpha}$
- Taking logs

$$\ln y_t = \ln a_t + \alpha \ln k_t + (1-\alpha)\ln n_t$$

- First order Taylor approximation around y^*, a^*, k^*, n^*:

$$\ln y^* + \frac{1}{y^*}(y_t - y^*) = \ln a^* + \frac{1}{a^*}(a_t - a^*) + \alpha \ln k^* + \frac{\alpha}{k^*}(k_t - k^*)$$
$$+(1-\alpha)\ln n^* + \frac{(1-\alpha)}{n^*}(n_t - n^*)$$

- Pink terms cancel out. Re-arranging:

$$\frac{(y_t - y^*)}{y^*} = \frac{(a_t - a^*)}{a^*} + \alpha\frac{(k_t - k^*)}{k^*} + (1-\alpha)\frac{(n_t - n^*)}{n^*}$$

- As before, we define $\tilde{x}_t = \frac{(x_t - x^*)}{x^*}$:

$$\tilde{y}_t = \tilde{a}_t + \alpha\tilde{k}_t + (1-\alpha)\tilde{n}_t$$

LOG-LINEARIZING THE EULER EQUATION

BLANCHARD & KAHN (1980) METHOD

- After log-linearizing, we have a system of linear difference equations
- Next $\rightarrow$ how can we solve these systems?
- Four Steps
  1. Write model in state-space form
  2. Decouple the system using Jordan decomposition
  3. Intuition + algorithm to solve model
  4. Existence and uniqueness
- Dynare/Python/Julia implement this solution method
- More details (for example) here Eric Sims Notes

1. WRITE MODEL IN STATE-SPACE FORM

$$\mathbb{E}_t[x_{t+1}] = \Psi x_t$$

$$x_t \in \mathbb{R}^{p+m}, \quad x_t = \left[x_t^s, x_t^j\right]$$

$$x_t^s \in \mathbb{R}^m, \quad x_t^j \in \mathbb{R}^p$$

$$x_0^s \text{ is given}$$

- x_t is a vector of variables expressed in percentage deviations from *ss* (our $\tilde{x}_t$)
- We will separate this vector into two groups:
  1. p "jump" or forward-looking variables
  2. m state or predetermined variables
- Ψ governs the evolution of the system, given a starting point
- The initial value of state variables is given (eg. k_0 in the RBC model)
- **Q:** where do we start the jump variables?

1. WRITE MODEL IN STATE-SPACE FORM - EXAMPLE

- Log-linearized equilibrium conditions of (deterministic) neoclassical growth model:

$$-\sigma\tilde{c}_t = -\sigma\tilde{c}_{t+1} + \beta(\alpha - 1)R\tilde{k}_{t+1} \qquad \tilde{k}_{t+1} = \frac{1}{\beta}\tilde{k}_t - \frac{c}{k}\tilde{c}_t$$

- Re-arranging we get:

$$\underbrace{\begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \end{bmatrix}}_{\mathbb{E}_t[x_{t+1}]} = \underbrace{\begin{bmatrix} 1 - \frac{c}{k}\frac{\beta(\alpha-1)R}{\sigma} & \frac{(\alpha-1)R}{\sigma} \\ -\frac{c}{k} & \frac{1}{\beta} \end{bmatrix}}_{\Psi} \underbrace{\begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \end{bmatrix}}_{x_t}$$

- State: capital; Jump: consumption
- Note that I drop the expectation operator because the model is deterministic

2. DECOUPLE THE SYSTEM USING JORDAN DECOMPOSITION

- An eigenvalue is a scalar, λ_i, and an eigenvector is a vector $\mathbb{v}_i$, that jointly satisfy:

$$\Psi \mathbb{v}_i = \lambda_i \mathbb{v}_i \qquad \forall \;\; i \in p+m$$

- Define the following two matrices

$$\Gamma = \begin{bmatrix} v_{1,1} & v_{2,1} & \cdot\cdot & v_{p+m,1} \\ \dots & \dots & \dots & \dots \\ v_{1,p+m} & v_{2,p+m} & \dots & v_{p+m,p+m} \end{bmatrix} \qquad \Lambda = diag(\lambda)$$

- We can decompose Ψ as: $\Psi\Gamma = \Gamma\Lambda \rightarrow \Psi = \Gamma\Lambda\Gamma^{-1}$

2. DECOUPLE THE SYSTEM USING JORDAN DECOMPOSITION - II

- Use Jordan decomposition to re-write model as:

$$\mathbb{E}_t[x_{t+1}] = \Gamma\Lambda\Gamma^{-1}x_t$$

- Pre-multiply each side by Γ^{-1}:

$$\mathbb{E}_t[\Gamma^{-1}x_{t+1}] = \Lambda\Gamma^{-1}x_t$$

- Change of variables: $w_t = \Gamma^{-1}x_t$. We get:

$$\mathbb{E}_t[w_{t+1}] = \Lambda w_t \rightarrow \mathbb{E}_t \begin{bmatrix} w_{1,t+1} \\ w_{2,t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix}$$

- Partition Λ in two:
  1. Λ_1 contains Q stable eigenvalues ($|\lambda_i| < 1$)
  2. Λ_2 contains B unstable eigenvalues ($|\lambda_j| > 1$)
- System is in VAR(1) form, Λ is diagonal $\rightarrow$ $w_{1,t}, w_{2,t}$ evolve independently of each other

2. DECOUPLE THE SYSTEM USING JORDAN DECOMPOSITION-III

- Iterating forward we can write:

$$\mathbb{E}_t[w_{1,t+T}] = \Lambda_1^T w_{1,t} \qquad\qquad \mathbb{E}_t[w_{2,t+T}] = \Lambda_2^T w_{2,t}$$

- Eigenvalues in Λ_1 are all stable $\rightarrow \quad \Lambda_1^T \rightarrow 0$ as $T \rightarrow \infty$
- Eigenvalues in Λ_2 are all unstable $\rightarrow \quad \Lambda_2^T \rightarrow \infty$ as $T \rightarrow \infty$
- This implies $\mathbb{E}_t[w_{2,t=T}] \rightarrow \infty$ which is inconsistent with transversality condition /bounded equilibrium unless $w_{2,t} = 0$
- Goal: find policy function/ updating rule for jump variables by setting $w_{2,t} = 0$

3. INTUITION + ALGORITHM TO SOLVE MODEL

- We can express w_t as:

$$\underbrace{w_{1,t}}_{Q\times 1} = \underbrace{G_{11}}_{Q\times p}\underbrace{x_t^j}_{p\times 1} + \underbrace{G_{12}}_{Q\times m}\underbrace{x_t^s}_{m\times 1} \qquad \text{where} \qquad \Gamma^{-1} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

$$\underbrace{w_{2,t}}_{B\times 1} = \underbrace{G_{21}}_{B\times p}\underbrace{x_t^j}_{p\times 1} + \underbrace{G_{22}}_{B\times m}\underbrace{x_t^s}_{m\times 1}$$

- Set $w_{2,t} = 0$:

$$\underbrace{\mathbf{0}}_{B\times 1} = G_{21}\, x_t^j + G_{22}\, x_t^s \quad \rightarrow x_t^j = -G_{21}^{-1} G_{22}\, x_t^s$$

- When will G_{21} $(B \times p)$ have an inverse? Necessary condition: $p = B$.
- In other words, we need as many jump variables as unstable eigenvalues

3. INTUITION + ALGORITHM TO SOLVE MODEL

- When $p = B$, we get:

$$x_t^j = \underbrace{-G_{21^{-1}G_{22}}}_{P} x_t^s$$

- Recall that the starting point of the state variables, x_0^s, is given
- We can use the policy function to find the period-zero value of the jump variables that yields $w_{2,0} = 0$
- Once we have x_0^s and x_0^j, we can use the expression in previous slide to get $w_{1,0}$
- Then, we can use $\mathbb{E}_t[w_{1,t+T}] = \Lambda_1^T w_{1,t}$ and $\mathbb{E}_t[w_{2,t+T}] = \Lambda_2^T w_{2,t}$ to compute the evolution of the system across time

4. EXISTENCE AND UNIQUENESS OF A SOLUTION

1. If $p = B$, there exists a unique bounded equilibrium
2. If $p < B$ (more explosive λ's than jump variables), there is no solution
3. If $p > B$, there is an infinite number of solutions.