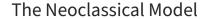
#### Lecture 16: Q-Theory of Investment

Juan Herreño UCSD

May 22, 2025



Time is continuous (same framework you saw in Econ 210B). Firms:

• Discounts time with rate *r* 

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- Perfect competition in input and output markets
- Production Function is CRS
- Hires labor, buys capital

Problem of a firm that takes input prices as given

Profits are given by revenues minus the wage bill minus capital expenditures

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- Perfect foresight

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You are experts in solving these problems

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(3)

(4)

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Question: What is the meaning of  $\lambda_t$ ?

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(6)



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Optimality conditions:

$$AF_n = w_t$$

$$e^{-rt} = \lambda_t$$

$$e^{-rt}A_tF_k = -\dot{\lambda}_t$$

(8)

(9)

$$\lim_{t\to\infty}\lambda_t k_t = 0.$$

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 $q_t = 1$ 

 $\lim e^{-rt}q_tk_t=0.$ 

 $AF_{k} = -\dot{q}_{t} + rq_{t}$ 

(13)

(14)

(15)

(16)

(17)

Optimality conditions:

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- The marginal value of investing is equal to  $q_t$
- The marginal cost of investing is losing 1 unit of profits
- Optimality requires to invest until marginal benefit equals marginal cost

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- There is no role for past variables

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- Capital should jump instantaneously
- therefore investment needs to be infinite for an instant

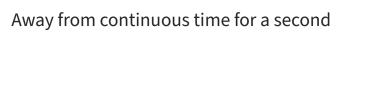
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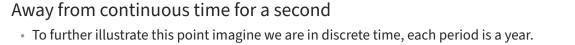
Imagine A increases to  $A_H$  and go down to A in the following instant

- Firms will infinitely invest in the present
- Firms will infinitely disinvest in the following instant

Against any possible intuition.  $r = AF_k$ 



You may think this are problems that arise only in the continuous time limit



- To further illustrate this point imagine we are in discrete time, each period is a year.
- The same equation applies

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$$1 + r_t = 1 + A_t \alpha K_t^{\alpha - 1} L_t^{1 - \alpha}$$

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Make a log-linear approximation. Hatted variables are log changes:

$$\hat{r}_t = \frac{r}{1+r} \left( \hat{a}_t - (1-\alpha)\hat{k}_t + (1-\alpha)\hat{l}_t \right)$$

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• where  $\hat{r_t} = \log \frac{1+r_t}{1+r}$ , solve for  $\hat{k}$ 

$$\hat{k}_t = -\hat{r}_t \left( \frac{1+r}{(1-\alpha)r} \right) + \frac{\hat{a}_t}{1-\alpha} + \hat{l}_t$$

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- Assume 100% of GDP could be transformed to capital.
- Capital-output ratios are between 2 and 4 (depending on land and housing)
- To increase capital by 31%, it would take 61%-124% of GDP
- With  $\delta$ : to increase capital by 14%, it would take 28-56% of GDP

Imagine firms notice that A will increase at time  $\tau$  ( $A_{\tau} > A$ )

- $\, \bullet \,$  Firms do nothing up until period  $\tau$
- invest  $+\infty$  at time  $\tau$

No role for news about the future

Idea: convex adjustment costs

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Convex costs

$$\phi(i,k) = \frac{\varphi}{2} \left(\frac{i}{k}\right)^2 k$$

(18)

Lucas (1967) is an early reference for convex costs on investment

• Note that defined in this way  $\phi(i, k)$  is homogeneous of degree 1

### New firm's problem

Objective

$$\max_{i_t, n_t} V = \int_0^\infty e^{-rt} \pi_t dt$$

 $\pi_t = A_t F(k_t, n_t) - w_t n_t - i_t - \frac{\varphi}{2} \left(\frac{i}{k}\right)^2 k$ 

Subject to

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(19)

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$$t - i_t - \frac{\varphi}{2} \left(\frac{i}{k}\right)^2 k \tag{20}$$

$$t - \frac{1}{2} \left( \frac{1}{k} \right)^{-k}$$

(19)

$$\mathcal{H} = e^{-rt} \left( AF(k_t, n_t) - w_t N_t - i_t - \frac{\varphi}{2} \left( \frac{i}{k} \right)^2 k \right) + \lambda_t i_t.$$

$$(n_t) - w_t N_t - i_t - \frac{\varphi}{2} \left(\frac{i}{k}\right)^2 k + \lambda_t i_t.$$
 (1)

$$\mathcal{H} = e^{-\kappa} \left( AF(\kappa_t, n_t) - w_t N_t - I_t - \frac{1}{2} \left( \frac{1}{\kappa} \right)^{-\kappa} \right) + \lambda_t I_t.$$

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$$\pi_t = A_t F(I_t)$$

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$$\left(\frac{i}{2}\right)^2 k$$

(22)

(19)

# **Optimality Conditions**

$$AF_{N} = w_{t}$$

$$1 + \varphi \frac{i_{t}}{k_{t}} = q_{t}$$

$$-\dot{q}_{t} + rq_{t} = AF_{k} + \frac{\varphi}{2} \left(\frac{i}{k}\right)^{2}$$

(23)

## Zoom in the important economics

$$1+\varphi\frac{i_t}{k_t}=q_t$$

Rewrite the equation as

$$\frac{i_t}{k_t} = h(q_t)$$

• The investment rate is **only** a function of  $q_t$ 

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- The investment rate is **only** a function of  $q_t$
- q is a sufficient statistic

# Two benefits for having more capital

You can solve the differential equation

$$-\dot{q}_t + rq_t = AF_k + \frac{\varphi}{2} \left(\frac{i}{k}\right)^2$$

And find that

$$q_0 = \int_0^\infty e^{-rt} \left[ A_t F_{kt} + \frac{\varphi}{2} \left( \frac{i_t}{k_t} \right)^2 \right] dt$$

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- Reductions in future costs of adjustment
- that is the marginal value of investment.

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$$q_0 = \int_0^\infty e^{-rt} \left[ A_t F_{kt} + \varphi \left( \frac{i_t}{k_t} \right)^2 \right] dt$$

a higher q calls for higher investment

$$\frac{\dot{q}_t}{\dot{q}_t} = \frac{q_t - q_t}{\varphi}$$

- Higher k brings down future MPK
- Lowering q

$$\frac{r_t}{k_t} = \frac{q_t}{\varphi}$$

$$\frac{i_t}{k_t} = \frac{q_t - 1}{\varphi}$$

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- Investment will be positive as long as q > 1
- How much to invest depends on  $\boldsymbol{\phi}$
- What happens in the limit  $\phi \rightarrow 0$ ?

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- It is a similar measurement problem to that of having measures of marginal cost!
- We do have measures of average q. Also called Tobin's Q
- To which we will refer to as Q.

#### Tobin's Q

• Tobin (1969) argued that firms should invest if

$$Q = \frac{\text{Value of the Firm Capital}}{\text{Replacement Cost of the Capital Stock}}$$

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- Our theory says that investment should be higher if q > 1
- Under what conditions Tobin's claim is sustained in our standard model?

• q is the marginal value of one unit of capital

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- Average *Q* is the average value of a unit of capital

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$$Q = \frac{\text{Value of the Firm Capital}}{\text{Replacement Cost of the Capital Stock}}$$

Can we learn about q with information about Q?

#### Hayashi 1982

- Under a number of conditions, there is an equivalent result.
  - Firms have CRS technology
  - The adjustment cost technology is CRS
  - Perfect competition
  - Efficient Financial Markets
- Then  $q_t = Q_t$
- Hayashi (1982) presented this result in continuous time.
- I will show you the result in discrete time, which for me is more intuitive

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- Profits are given by

$$\pi_t = AF(K_t, L_t) - W_t L_t - I_t - \psi\left(\frac{I_t}{K_t}\right) K_t$$

Firms maximize their value

$$V_0 = \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \pi_t$$

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Subject to the law of motion of capital

$$K_{t+1} = K_t(1-\delta) + I_t$$

q<sub>t</sub> the lagrange multiplier

• The problem has very similar first order conditions

$$q_{t} = 1 + \psi'\left(\frac{I_{t}}{K_{t}}\right)$$

$$q_{t} = \frac{1 - \delta}{1 + r}q_{t+1} + \frac{1}{1 + r}\left(A_{t+1}F_{k}(K_{t+1}, L_{t+1}) - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right) - \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}}\right)$$

 $AF_{I}(K_{t},L_{t})=W_{t}$ 

 $q_{t} = \frac{1-\delta}{1+r}q_{t+1} + \frac{1}{1+r}\left(A_{t+1}F_{k}(K_{t+1}, L_{t+1}) - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right) - \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}}\right)$ 

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• And multiply by  $K_{t+1}$ 

$$1-\delta$$
 1 (...

 $q_{t}K_{t+1} = \frac{1-\delta}{1+r}q_{t+1}K_{t+1} + \frac{1}{1+r}\left(A_{t+1}F_{k}(K_{t+1}, L_{t+1})K_{t+1} - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right)K_{t+1} - \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)I_{t+1}\right)$ 

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• Now use  $q_{t+1} = 1 + \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)$ , to replace  $\psi'$ 

$$q_{t}K_{t+1} = \frac{1-\delta}{1+r}q_{t+1}K_{t+1} + \frac{1}{1+r}\left(A_{t+1}F_{k}(K_{t+1}, L_{t+1})K_{t+1} - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right)K_{t+1} - (q_{t+1}-1)I_{t+1}\right)$$

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$$1 - \delta \qquad 1 /$$

• Use the law of motion of capital in period t + 1 to replace away  $I_{t+1}$ 

 $q_{t}K_{t+1} = \frac{1-\delta}{1+r}q_{t+1}K_{t+1} + \frac{1}{1+r}\left(A_{t+1}F_{k}(K_{t+1}, L_{t+1})K_{t+1} - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right)K_{t+1} - \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)I_{t+1}\right)$ 

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Rearrange

$$q_t K_{t+1} = \frac{1}{1+r} q_{t+1} K_{t+2} + \frac{1}{1+r} \left( A_{t+1} F_k(K_{t+1}, L_{t+1}) K_{t+1} - \psi \left( \frac{I_{t+1}}{K_{t+1}} \right) K_{t+1} - I_{t+1} \right)$$

- Now use Euler's Theorem  $F(K,L) = F_k(K,L)K + F_L(K,L)L$
- And, perfect competition in the labor market  $W = AF_{I}(K, L)$

$$q_{t}K_{t+1} = \frac{1}{1+r}q_{t+1}K_{t+2} + \frac{1}{1+r}\left(A_{t+1}F(K_{t+1}, L_{t+1}) - W_{t+1}L_{t+1} - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right)K_{t+1} - I_{t+1}\right)$$

Notice that the term in parenthesis is just profits tomorrow

$$q_t K_{t+1} = \frac{1}{1+r} q_{t+1} K_{t+2} + \frac{1}{1+r} \pi_{t+1}$$

• Iterate forward and use the TVC to get rid of the terminal term

$$q_t K_{t+1} = \sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^t \pi_{t+1}$$

Use efficiency of financial markets:

$$q_t K_{t+1} = V_t^e$$

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- Divide by the stock of capital at the beginning of tomorrow's period

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Realize that the RHS is the average Q

$$q_t = Q_t$$

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- Why do we want to test this theory?
  - It is our benchmark model
  - Many educated guesses come from this framework
  - The investment effects of tax reforms
  - The investment effects of monetary policy
  - And so on...