Lecture 16: Q-Theory of Investment

Juan Herreño
UCSD

May 22, 2025

The Neoclassical Model

Time is continuous (same framework you saw in Econ 210B). Firms:

The Neoclassical Model

Time is continuous (same framework you saw in Econ 210B). Firms:

- Discounts time with rate r

The Neoclassical Model

Time is continuous (same framework you saw in Econ 210B). Firms:

- Discounts time with rate r
- Produces a homogeneous good

The Neoclassical Model

Time is continuous (same framework you saw in Econ 210B). Firms:

- Discounts time with rate r
- Produces a homogeneous good
- Perfect competition in input and output markets

The Neoclassical Model

Time is continuous (same framework you saw in Econ 210B). Firms:

- Discounts time with rate r
- Produces a homogeneous good
- Perfect competition in input and output markets
- Production Function is CRS

The Neoclassical Model

Time is continuous (same framework you saw in Econ 210B). Firms:

- Discounts time with rate r
- Produces a homogeneous good
- Perfect competition in input and output markets
- Production Function is CRS
- Hires labor, buys capital

Problem of a firm that takes input prices as given

Profits

Profits are given by revenues minus the wage bill minus capital expenditures

Profits

Profits are given by revenues minus the wage bill minus capital expenditures

$$\pi_t = A_t F(k_t, n_t) - w_t n_t - i_t \tag{1}$$

Profits

Profits are given by revenues minus the wage bill minus capital expenditures

$$\pi_t = A_t F(k_t, n_t) - w_t n_t - i_t \tag{1}$$

Capital adjusts given the law of motion

Profits

Profits are given by revenues minus the wage bill minus capital expenditures

$$\pi_t = A_t F(k_t, n_t) - w_t n_t - i_t \tag{1}$$

Capital adjusts given the law of motion

$$\dot{k}_t = i_t \tag{2}$$

Profits

Profits are given by revenues minus the wage bill minus capital expenditures

$$\pi_t = A_t F(k_t, n_t) - w_t n_t - i_t \tag{1}$$

Capital adjusts given the law of motion

$$\dot{k}_t = i_t \tag{2}$$

- We are assuming zero depreciation just for simplicity

Profits

Profits are given by revenues minus the wage bill minus capital expenditures

$$\pi_t = A_t F(k_t, n_t) - w_t n_t - i_t \tag{1}$$

Capital adjusts given the law of motion

$$\dot{k}_t = i_t \tag{2}$$

- We are assuming zero depreciation just for simplicity
- Abstracting from taxes, bonus depreciation, etc.

Profits

Profits are given by revenues minus the wage bill minus capital expenditures

$$\pi_t = A_t F(k_t, n_t) - w_t n_t - i_t \tag{1}$$

Capital adjusts given the law of motion

$$\dot{k}_t = i_t \tag{2}$$

- We are assuming zero depreciation just for simplicity
- Abstracting from taxes, bonus depreciation, etc.
- Perfect foresight

The firm’s problem

Objective

The firm's problem

Objective

$$\max_{i_t, n_t} V = \int_0^\infty e^{-rt} \pi_t dt \tag{3}$$

The firm's problem

Objective

$$\max_{i_t, n_t} V = \int_0^\infty e^{-rt} \pi_t dt \tag{3}$$

Subject to

The firm's problem

Objective

$$\max_{i_t, n_t} V = \int_0^\infty e^{-rt} \pi_t dt \tag{3}$$

Subject to

$$\pi_t = A_t F(k_t, n_t) - w_t n_t - i_t \tag{4}$$

$$\dot{k}_t = i_t \tag{5}$$

The firm's problem

Objective

$$\max_{i_t, n_t} V = \int_0^\infty e^{-rt} \pi_t dt \tag{3}$$

Subject to

$$\pi_t = A_t F(k_t, n_t) - w_t n_t - i_t \tag{4}$$

$$\dot{k}_t = i_t \tag{5}$$

You are experts in solving these problems

The firm's problem

Objective

$$\max_{i_t, n_t} V = \int_0^\infty e^{-rt} \pi_t dt \tag{3}$$

Subject to

$$\pi_t = A_t F(k_t, n_t) - w_t n_t - i_t \tag{4}$$

$$\dot{k}_t = i_t \tag{5}$$

You are experts in solving these problems

$$\mathcal{H} = e^{-rt} \left(AF(k_t, n_t) - w_t N_t - i_t \right) + \lambda_t i_t. \tag{6}$$

The firm's problem

Objective

$$\max_{i_t, n_t} V = \int_0^{\infty} e^{-rt} \pi_t dt \tag{3}$$

Subject to

$$\pi_t = A_t F(k_t, n_t) - w_t n_t - i_t \tag{4}$$
$$\dot{k}_t = i_t \tag{5}$$

You are experts in solving these problems

$$\mathcal{H} = e^{-rt} \left(AF(k_t, n_t) - w_t N_t - i_t \right) + \lambda_t i_t . \tag{6}$$

Question: What is the meaning of λ_t?

λ vs. q

λ vs. q

λ_t is the marginal value of one unit of investment at time t to time-zero firm value

λ vs. q

λ_t is the marginal value of one unit of investment at time t to time-zero firm value

- It is the *present value* marginal value of investment

λ vs. q

λ_t is the marginal value of one unit of investment at time t to time-zero firm value

- It is the *present value* marginal value of investment

We will introduce a new concept, the current marginal value q_t

$$q_t = \lambda_t e^{rt} \tag{7}$$

λ vs. q

λ_t is the marginal value of one unit of investment at time t to time-zero firm value

- It is the *present value* marginal value of investment

We will introduce a new concept, the current marginal value q_t

$$q_t = \lambda_t e^{rt} \tag{7}$$

Optimality Conditions λ

$$\mathcal{H} = e^{-rt}\left(AF(k_t, n_t) - w_t n_t - I_t\right) + \lambda_t i_t. \tag{8}$$

Optimality Conditions λ

$$\mathcal{H} = e^{-rt}\left(AF(k_t, n_t) - w_t n_t - I_t\right) + \lambda_t i_t. \tag{8}$$

Optimality conditions:

Optimality Conditions λ

$$\mathcal{H} = e^{-rt}\left(AF(k_t, n_t) - w_t n_t - I_t\right) + \lambda_t i_t. \tag{8}$$

Optimality conditions:

$$AF_n = w_t \tag{9}$$
$$e^{-rt} = \lambda_t \tag{10}$$
$$e^{-rt}A_t F_k = -\dot{\lambda}_t \tag{11}$$
$$\lim_{t\to\infty} \lambda_t k_t = 0. \tag{12}$$

Optimality Conditions q

$$\mathcal{H} = e^{-rt}\left(AF(k_t, n_t) - w_t n_t - I_t\right) + \lambda_t i_t. \tag{13}$$

Optimality Conditions q

$$\mathcal{H} = e^{-rt}\left(AF(k_t, n_t) - w_t n_t - I_t\right) + \lambda_t i_t. \tag{13}$$

Optimality conditions:

$$\begin{aligned} AF_n &= w_t & (14)\\ q_t &= 1 & (15)\\ AF_k &= -\dot{q}_t + rq_t & (16)\\ \lim_{t\to\infty} e^{-rt} q_t k_t &= 0. & (17) \end{aligned}$$

Zoom in the key result

$$q_t = 1$$

Zoom in the key result

$$q_t = 1$$

What is the meaning of this equation?

Zoom in the key result

$$q_t = 1$$

What is the meaning of this equation?

- The marginal value of investing is equal to q_t

Zoom in the key result

$$q_t = 1$$

What is the meaning of this equation?

- The marginal value of investing is equal to q_t
- The marginal cost of investing is losing 1 unit of profits

Zoom in the key result

$$q_t = 1$$

What is the meaning of this equation?

- The marginal value of investing is equal to q_t
- The marginal cost of investing is losing 1 unit of profits
- Optimality requires to invest until marginal benefit equals marginal cost

Implications of the key result

If $q_t = 1$, then $\dot{q}_t = 0$

Implications of the key result

If $q_t = 1$, then $\dot{q}_t = 0$
So equation $AF_k = -\dot{q}_t + rq_t$

Implications of the key result

If $q_t = 1$, then $\dot{q}_t = 0$
So equation $AF_k = -\dot{q}_t + rq_t$
Transforms into

Implications of the key result

If $q_t = 1$, then $\dot{q}_t = 0$
So equation $AF_k = -\dot{q}_t + rq_t$
Transforms into

$$AF_k = r$$

It is as if firms are solving a static problem

Implications of the key result

If $q_t = 1$, then $\dot{q}_t = 0$
So equation $AF_k = -\dot{q}_t + rq_t$
Transforms into

$$AF_k = r$$

It is as if firms are solving a static problem

- Equate marginal product of capital to r at each point in time

Implications of the key result

If $q_t = 1$, then $\dot{q}_t = 0$
So equation $AF_k = -\dot{q}_t + rq_t$
Transforms into

$$AF_k = r$$

It is as if firms are solving a static problem

- Equate marginal product of capital to r at each point in time
- There is no role for future variables

Implications of the key result

If $q_t = 1$, then $\dot{q}_t = 0$
So equation $AF_k = -\dot{q}_t + rq_t$
Transforms into

$$AF_k = r$$

It is as if firms are solving a static problem

- Equate marginal product of capital to r at each point in time
- There is no role for future variables
- There is no role for past variables

The economics of the basic problem

Imagine a firm starts with $k_0 < k^*$, for k^* consistent with $AF_k^* = r$

The economics of the basic problem

Imagine a firm starts with $k_0 < k^*$, for k^* consistent with $AF_k^* = r$

- Capital will jump immediately to k^*

The economics of the basic problem

Imagine a firm starts with $k_0 < k^*$, for k^* consistent with $AF_k^* = r$

- Capital will jump immediately to k^*
- Investment is not well-defined at the time of adjustment

The economics of the basic problem

Imagine a firm starts with $k_0 < k^*$, for k^* consistent with $AF_k^* = r$

- Capital will jump immediately to k^*
- Investment is not well-defined at the time of adjustment
- it will be zero otherwise

The economics of the basic problem

Imagine a firm starts with $k_0 < k^*$, for k^* consistent with $AF_k^* = r$

- Capital will jump immediately to k^*
- Investment is not well-defined at the time of adjustment
- it will be zero otherwise

Not having a properly defined investment function is obviously problematic. Economics:

The economics of the basic problem

Imagine a firm starts with $k_0 < k^*$, for k^* consistent with $AF_k^* = r$

- Capital will jump immediately to k^*
- Investment is not well-defined at the time of adjustment
- it will be zero otherwise

Not having a properly defined investment function is obviously problematic. Economics:

- Capital should *jump* instantaneously

The economics of the basic problem

Imagine a firm starts with $k_0 < k^*$, for k^* consistent with $AF_k^* = r$

- Capital will jump immediately to k^*
- Investment is not well-defined at the time of adjustment
- it will be zero otherwise

Not having a properly defined investment function is obviously problematic. Economics:

- Capital should *jump* instantaneously
- therefore investment needs to be infinite for an instant

The economics of the basic problem

Imagine A increases to A_H and go down to A in the following instant

- Firms will infinitely invest in the present

The economics of the basic problem

Imagine A increases to A_H and go down to A in the following instant

- Firms will infinitely invest in the present
- Firms will infinitely disinvest in the following instant

Against any possible intuition. $r = AF_k$

Away from continuous time for a second

You may think this are problems that arise only in the continuous time limit

Away from continuous time for a second

- To further illustrate this point imagine we are in discrete time, each period is a year.

Away from continuous time for a second

- To further illustrate this point imagine we are in discrete time, each period is a year.
- The same equation applies

$$AF_k = r$$

Away from continuous time for a second

- To further illustrate this point imagine we are in discrete time, each period is a year.
- The same equation applies

$$AF_k = r$$

- Assume that $F(K, L) = K^{\alpha}L^{1-\alpha}$. Therefore:

$$1 + r_t = 1 + A_t \alpha K_t^{\alpha-1} L_t^{1-\alpha}$$

Away from continuous time for a second

- To further illustrate this point imagine we are in discrete time, each period is a year.
- The same equation applies

$$AF_k = r$$

- Assume that $F(K, L) = K^{\alpha}L^{1-\alpha}$. Therefore:

$$1 + r_t = 1 + A_t \alpha K_t^{\alpha-1} L_t^{1-\alpha}$$

- Make a log-linear approximation. Hatted variables are log changes:

$$\hat{r}_t = \frac{r}{1+r}\left(\hat{a}_t - (1-\alpha)\hat{k}_t + (1-\alpha)\hat{l}_t\right)$$

Away from continuous time for a second

- To further illustrate this point imagine we are in discrete time, each period is a year.
- The same equation applies

$$AF_k = r$$

- Assume that $F(K,L) = K^{\alpha}L^{1-\alpha}$. Therefore:

$$1 + r_t = 1 + A_t \alpha K_t^{\alpha-1} L_t^{1-\alpha}$$

- Make a log-linear approximation. Hatted variables are log changes:

$$\hat{r}_t = \frac{r}{1+r}\left(\hat{a}_t - (1-\alpha)\hat{k}_t + (1-\alpha)\hat{l}_t\right)$$

- where $\hat{r}_t = \log \frac{1+r_t}{1+r}$, solve for $\hat{k}$

$$\hat{k}_t = -\hat{r}_t\left(\frac{1+r}{(1-\alpha)r}\right) + \frac{\hat{a}_t}{1-\alpha} + \hat{l}_t$$

Away from continuous time for a second

$$\hat{k}_t = -\hat{r}_t \left(\frac{1+r}{(1-\alpha)r} \right) + \frac{\hat{a}_t}{1-\alpha} + \hat{l}_t$$

Away from continuous time for a second

$$\hat{k}_t = -\hat{r}_t \left(\frac{1+r}{(1-\alpha)r} \right) + \frac{\hat{a}_t}{1-\alpha} + \hat{l}_t$$

- Assume an exogenous decrease of 1% in interest rates.

Away from continuous time for a second

$$\hat{k}_t = -\hat{r}_t \left(\frac{1+r}{(1-\alpha)r} \right) + \frac{\hat{a}_t}{1-\alpha} + \hat{l}_t$$

- Assume an exogenous decrease of 1% in interest rates.
- Capital would have to increase 31.5%

Away from continuous time for a second

$$\hat{k}_t = -\hat{r}_t \left(\frac{1+r}{(1-\alpha)r} \right) + \frac{\hat{a}_t}{1-\alpha} + \hat{l}_t$$

- Assume an exogenous decrease of 1% in interest rates.
- Capital would have to increase 31.5%
- Including reasonable depreciation would change this number to 14%.

Away from continuous time for a second

$$\hat{k}_t = -\hat{r}_t\left(\frac{1+r}{(1-\alpha)r}\right) + \frac{\hat{a}_t}{1-\alpha} + \hat{l}_t$$

- Assume an exogenous decrease of 1% in interest rates.
- Capital would have to increase 31.5%
- Including reasonable depreciation would change this number to 14%.
- Letting labor increase would further increase this number

Away from continuous time for a second

$$\hat{k}_t = -\hat{r}_t \left(\frac{1+r}{(1-\alpha)r} \right) + \frac{\hat{a}_t}{1-\alpha} + \hat{l}_t$$

- Assume an exogenous decrease of 1% in interest rates.
- Capital would have to increase 31.5%
- Including reasonable depreciation would change this number to 14%.
- Letting labor increase would further increase this number
- Assume 100% of GDP could be transformed to capital.

Away from continuous time for a second

$$\hat{k}_t = -\hat{r}_t \left(\frac{1+r}{(1-\alpha)r} \right) + \frac{\hat{a}_t}{1-\alpha} + \hat{l}_t$$

- Assume an exogenous decrease of 1% in interest rates.
- Capital would have to increase 31.5%
- Including reasonable depreciation would change this number to 14%.
- Letting labor increase would further increase this number
- Assume 100% of GDP could be transformed to capital.
- Capital-output ratios are between 2 and 4 (depending on land and housing)

Away from continuous time for a second

$$\hat{k}_t = -\hat{r}_t \left(\frac{1+r}{(1-\alpha)r} \right) + \frac{\hat{a}_t}{1-\alpha} + \hat{l}_t$$

- Assume an exogenous decrease of 1% in interest rates.
- Capital would have to increase 31.5%
- Including reasonable depreciation would change this number to 14%.
- Letting labor increase would further increase this number
- Assume 100% of GDP could be transformed to capital.
- Capital-output ratios are between 2 and 4 (depending on land and housing)
- To increase capital by 31%, it would take 61%-124% of GDP
- With δ: to increase capital by 14%, it would take 28-56% of GDP

The economics of the basic problem

Imagine firms notice that A will increase at time τ ($A_\tau > A$)

- Firms do nothing up until period τ
- invest $+\infty$ at time τ

No role for news about the future

Idea: convex adjustment costs

Investing takes time and resources

Idea: convex adjustment costs

Investing takes time and resources

- Installing capital is costly
- Planning the right level of capital is costly
- scrapping old capital is costly

Idea: convex adjustment costs

Investing takes time and resources

- Installing capital is costly
- Planning the right level of capital is costly
- scrapping old capital is costly

Convex costs

$$\phi(i,k) = \frac{\varphi}{2}\left(\frac{i}{k}\right)^2 k \tag{18}$$

Lucas (1967) is an early reference for convex costs on investment

- Note that defined in this way $\phi(i,k)$ is homogeneous of degree 1

New firm's problem

Objective

$$\max_{i_t, n_t} V = \int_0^\infty e^{-rt} \pi_t dt \tag{19}$$

Subject to

$$\pi_t = A_t F(k_t, n_t) - w_t n_t - i_t - \frac{\varphi}{2} \left(\frac{i}{k} \right)^2 k \tag{20}$$

$$\dot{k}_t = i_t \tag{21}$$

You are experts in solving these problems

New firm's problem

Objective

$$\max_{i_t, n_t} V = \int_0^\infty e^{-rt} \pi_t dt \tag{19}$$

Subject to

$$\pi_t = A_t F(k_t, n_t) - w_t n_t - i_t - \frac{\varphi}{2} \left(\frac{i}{k}\right)^2 k \tag{20}$$

$$\dot{k}_t = i_t \tag{21}$$

You are experts in solving these problems

$$\mathcal{H} = e^{-rt} \left(AF(k_t, n_t) - w_t N_t - i_t - \frac{\varphi}{2} \left(\frac{i}{k}\right)^2 k \right) + \lambda_t i_t. \tag{22}$$

New firm's problem

Objective

$$\max_{i_t,n_t} V = \int_0^\infty e^{-rt}\pi_t dt \tag{19}$$

Subject to

$$\pi_t = A_t F(k_t, n_t) - w_t n_t - i_t - \frac{\varphi}{2}\left(\frac{i}{k}\right)^2 k \tag{20}$$

$$\dot{k}_t = i_t \tag{21}$$

You are experts in solving these problems

$$\mathcal{H} = e^{-rt}\left(AF(k_t, n_t) - w_t N_t - i_t - \frac{\varphi}{2}\left(\frac{i}{k}\right)^2 k\right) + \lambda_t i_t. \tag{22}$$

Question: What is the meaning of λ_t?

Optimality Conditions

$$AF_N = w_t \tag{23}$$

$$1 + \varphi \frac{i_t}{k_t} = q_t \tag{24}$$

$$-\dot{q}_t + rq_t = AF_k + \frac{\varphi}{2}\left(\frac{i}{k}\right)^2 \tag{25}$$

Zoom in the important economics

$$1 + \varphi \frac{i_t}{k_t} = q_t$$

- Rewrite the equation as

$$\frac{i_t}{k_t} = h(q_t)$$

- The investment rate is **only** a function of q_t

Zoom in the important economics

$$1 + \varphi \frac{i_t}{k_t} = q_t$$

- Rewrite the equation as

$$\frac{i_t}{k_t} = h(q_t)$$

- The investment rate is **only** a function of q_t
- q is a *sufficient statistic*

Two benefits for having more capital

- You can solve the differential equation

$$-\dot{q}_t + rq_t = AF_k + \frac{\varphi}{2}\left(\frac{i}{k}\right)^2$$

- And find that

$$q_0 = \int_0^\infty e^{-rt}\left[A_t F_{kt} + \frac{\varphi}{2}\left(\frac{i_t}{k_t}\right)^2\right] dt$$

- Investment brings a sequence of marginal products

Two benefits for having more capital

- You can solve the differential equation

$$-\dot{q}_t + rq_t = AF_k + \frac{\varphi}{2}\left(\frac{i}{k}\right)^2$$

- And find that

$$q_0 = \int_0^\infty e^{-rt}\left[A_t F_{kt} + \frac{\varphi}{2}\left(\frac{i_t}{k_t}\right)^2\right] dt$$

- Investment brings a sequence of marginal products
- Reductions in future costs of adjustment

Two benefits for having more capital

- You can solve the differential equation

$$-\dot{q}_t + rq_t = AF_k + \frac{\varphi}{2}\left(\frac{i}{k}\right)^2$$

- And find that

$$q_0 = \int_0^\infty e^{-rt}\left[A_t F_{kt} + \frac{\varphi}{2}\left(\frac{i_t}{k_t}\right)^2\right] dt$$

- Investment brings a sequence of marginal products
- Reductions in future costs of adjustment
- that is the marginal value of investment.

The mechanics of the q-Theory

- Imagine there are news about future profitability

The mechanics of the q-Theory

- Imagine there are news about future profitability
- This increases *A* for instance
- Increasing *q* by

$$q_0 = \int_0^{\infty} e^{-rt} \left[A_t F_{kt} + \varphi \left(\frac{i_t}{k_t} \right)^2 \right] dt$$

The mechanics of the q-Theory

- Imagine there are news about future profitability
- This increases *A* for instance
- Increasing q by

$$q_0 = \int_0^\infty e^{-rt}\left[A_t F_{kt} + \varphi\left(\frac{i_t}{k_t}\right)^2\right] dt$$

- a higher q calls for higher investment

$$\frac{i_t}{k_t} = \frac{q_t - 1}{\varphi}$$

- Higher k brings down future MPK
- Lowering q

When to invest

$$\frac{i_t}{k_t} = \frac{q_t - 1}{\varphi}$$

When to invest

$$\frac{i_t}{k_t} = \frac{q_t - 1}{\varphi}$$

- Investment will be positive as long as $q > 1$

When to invest

$$\frac{i_t}{k_t} = \frac{q_t - 1}{\varphi}$$

- Investment will be positive as long as $q > 1$
- How much to invest depends on φ

When to invest

$$\frac{i_t}{k_t} = \frac{q_t - 1}{\varphi}$$

- Investment will be positive as long as $q > 1$
- How much to invest depends on φ
- What happens in the limit $\varphi \to 0$?

How to implement the Q Theory

- So far we have a theory that ties investment to q

How to implement the Q Theory

- So far we have a theory that ties investment to q
- We do not have good measures for marginal q in the data

How to implement the Q Theory

- So far we have a theory that ties investment to q
- We do not have good measures for marginal q in the data
- It is a similar measurement problem to that of having measures of marginal cost!

How to implement the Q Theory

- So far we have a theory that ties investment to q
- We do not have good measures for marginal q in the data
- It is a similar measurement problem to that of having measures of marginal cost!
- We do have measures of *average* q. Also called *Tobin's* Q
- To which we will refer to as Q.

Tobin's Q

- Tobin (1969) argued that firms should invest if

$$Q = \frac{\text{Value of the Firm Capital}}{\text{Replacement Cost of the Capital Stock}}$$

- Capital is valued more within the firm, than outside of the firm.

Tobin's Q

- Tobin (1969) argued that firms should invest if

$$Q = \frac{\text{Value of the Firm Capital}}{\text{Replacement Cost of the Capital Stock}}$$

- Capital is valued more within the firm, than outside of the firm.
- Our theory says that investment should be higher if $q > 1$

Tobin's Q

- Tobin (1969) argued that firms should invest if

$$Q = \frac{\text{Value of the Firm Capital}}{\text{Replacement Cost of the Capital Stock}}$$

- Capital is valued more within the firm, than outside of the firm.
- Our theory says that investment should be higher if $q > 1$
- Under what conditions Tobin's claim is sustained in our standard model?

How to implement the Q Theory

- q is the marginal value of one unit of capital

How to implement the Q Theory

- q is the marginal value of one unit of capital
- Average Q is the average value of a unit of capital

$$Q = \frac{\text{Value of the Firm Capital}}{\text{Replacement Cost of the Capital Stock}}$$

How to implement the Q Theory

- q is the marginal value of one unit of capital
- Average Q is the average value of a unit of capital

$$Q = \frac{\text{Value of the Firm Capital}}{\text{Replacement Cost of the Capital Stock}}$$

- Can we learn about q with information about Q?

Hayashi 1982

- Under a number of conditions, there is an equivalent result.
  - Firms have CRS technology
  - The adjustment cost technology is CRS
  - Perfect competition
  - Efficient Financial Markets
- Then $q_t = Q_t$
- Hayashi (1982) presented this result in continuous time.
- I will show you the result in discrete time, which for me is more intuitive

Marginal q and Average Q in discrete time

- Time is discrete and the problem is very similar to what we covered before

Marginal q and Average Q in discrete time

- Time is discrete and the problem is very similar to what we covered before
- Profits are given by

$$\pi_t = AF(K_t, L_t) - W_t L_t - I_t - \psi\left(\frac{I_t}{K_t}\right)K_t$$

- Firms maximize their value

$$V_0 = \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \pi_t$$

Marginal q and Average Q in discrete time

- Time is discrete and the problem is very similar to what we covered before
- Profits are given by

$$\pi_t = AF(K_t, L_t) - W_t L_t - I_t - \psi\left(\frac{I_t}{K_t}\right) K_t$$

- Firms maximize their value

$$V_0 = \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \pi_t$$

- Subject to the law of motion of capital

$$K_{t+1} = K_t(1-\delta) + I_t$$

- q_t the lagrange multiplier

Marginal q and Average Q in discrete time

- The problem has very similar first order conditions

$$AF_L(K_t, L_t) = W_t$$

$$q_t = 1 + \psi'\left(\frac{I_t}{K_t}\right)$$

$$q_t = \frac{1-\delta}{1+r}q_{t+1} + \frac{1}{1+r}\left(A_{t+1}F_k(K_{t+!}, L_{t+1}) - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right) - \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}}\right)$$

Marginal q and Average Q in discrete time

$$q_t = \frac{1-\delta}{1+r} q_{t+1} + \frac{1}{1+r}\left(A_{t+1} F_k(K_{t+1}, L_{t+1}) - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right) - \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}}\right)$$

Marginal q and Average Q in discrete time

$$q_t = \frac{1-\delta}{1+r} q_{t+1} + \frac{1}{1+r}\left(A_{t+1}F_k(K_{t+1}, L_{t+1}) - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right) - \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}}\right)$$

- And multiply by K_{t+1}

$$q_t K_{t+1} = \frac{1-\delta}{1+r} q_{t+1} K_{t+1} + \frac{1}{1+r}\left(A_{t+1}F_k(K_{t+1}, L_{t+1})K_{t+1} - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right)K_{t+1} - \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)I_{t+1}\right)$$

Marginal q and Average Q in discrete time

$$q_t = \frac{1-\delta}{1+r} q_{t+1} + \frac{1}{1+r}\left(A_{t+1}F_k(K_{t+1}, L_{t+1}) - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right) - \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}}\right)$$

- And multiply by K_{t+1}

$$q_t K_{t+1} = \frac{1-\delta}{1+r} q_{t+1} K_{t+1} + \frac{1}{1+r}\left(A_{t+1}F_k(K_{t+1}, L_{t+1})K_{t+1} - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right)K_{t+1} - \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)I_{t+1}\right)$$

- Now use $q_{t+1} = 1 + \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)$, to replace ψ'

$$q_t K_{t+1} = \frac{1-\delta}{1+r} q_{t+1} K_{t+1} + \frac{1}{1+r}\left(A_{t+1}F_k(K_{t+1}, L_{t+1})K_{t+1} - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right)K_{t+1} - (q_{t+1}-1)I_{t+1}\right)$$

Marginal q and Average Q in discrete time

$$q_t = \frac{1-\delta}{1+r} q_{t+1} + \frac{1}{1+r}\left(A_{t+1}F_k(K_{t+1}, L_{t+1}) - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right) - \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}}\right)$$

- And multiply by K_{t+1}

$$q_t K_{t+1} = \frac{1-\delta}{1+r} q_{t+1} K_{t+1} + \frac{1}{1+r}\left(A_{t+1}F_k(K_{t+1}, L_{t+1})K_{t+1} - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right)K_{t+1} - \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)I_{t+1}\right)$$

- Now use $q_{t+1} = 1 + \psi'\left(\frac{I_{t+1}}{K_{t+1}}\right)$, to replace ψ'

$$q_t K_{t+1} = \frac{1-\delta}{1+r} q_{t+1} K_{t+1} + \frac{1}{1+r}\left(A_{t+1}F_k(K_{t+1}, L_{t+1})K_{t+1} - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right)K_{t+1} - (q_{t+1} - 1)I_{t+1}\right)$$

- Use the law of motion of capital in period $t+1$ to replace away I_{t+1}

$$q_t K_{t+1} = \frac{1-\delta}{1+r} q_{t+1} K_{t+1} + \frac{1}{1+r}\left(A_{t+1}F_k(K_{t+1}, L_{t+1})K_{t+1} - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right)K_{t+1} - (q_{t+1} - 1)(K_{t+2} - K_{t+1}\right.$$

Marginal q and Average Q in discrete time

- Rearrange

$$q_t K_{t+1} = \frac{1}{1+r} q_{t+1} K_{t+2} + \frac{1}{1+r} \left(A_{t+1} F_k(K_{t+1}, L_{t+1}) K_{t+1} - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right) K_{t+1} - I_{t+1} \right)$$

- Now use Euler's Theorem $F(K,L) = F_k(K,L)K + F_L(K,L)L$
- And, perfect competition in the labor market $W = AF_L(K,L)$

$$q_t K_{t+1} = \frac{1}{1+r} q_{t+1} K_{t+2} + \frac{1}{1+r} \left(A_{t+1} F(K_{t+1}, L_{t+1}) - W_{t+1} L_{t+1} - \psi\left(\frac{I_{t+1}}{K_{t+1}}\right) K_{t+1} - I_{t+1} \right)$$

- Notice that the term in parenthesis is just profits tomorrow

$$q_t K_{t+1} = \frac{1}{1+r} q_{t+1} K_{t+2} + \frac{1}{1+r} \pi_{t+1}$$

Marginal q and Average Q in discrete time

- Iterate forward and use the TVC to get rid of the terminal term

$$q_t K_{t+1} = \sum_{j=1}^{\infty} \left(\frac{1}{1+r} \right)^t \pi_{t+1}$$

- Use efficiency of financial markets:

$$q_t K_{t+1} = V_t^e$$

Marginal q and Average Q in discrete time

- Iterate forward and use the TVC to get rid of the terminal term

$$q_t K_{t+1} = \sum_{j=1}^{\infty} \left(\frac{1}{1+r} \right)^t \pi_{t+1}$$

- Use efficiency of financial markets:

$$q_t K_{t+1} = V_t^e$$

- V_t^e is the "ex-dividend" (after paying today's dividend) value of the firm equity
- Divide by the stock of capital at the beginning of tomorrow's period

$$q_t = \frac{V_t^e}{K_{t+1}}$$

Marginal q and Average Q in discrete time

- Iterate forward and use the TVC to get rid of the terminal term

$$q_t K_{t+1} = \sum_{j=1}^{\infty} \left(\frac{1}{1+r} \right)^t \pi_{t+1}$$

- Use efficiency of financial markets:

$$q_t K_{t+1} = V_t^e$$

- V_t^e is the "ex-dividend" (after paying today's dividend) value of the firm equity
- Divide by the stock of capital at the beginning of tomorrow's period

$$q_t = \frac{V_t^e}{K_{t+1}}$$

- Realize that the RHS is the average Q

$$q_t = Q_t$$

Relevance of this Result

- $q_t = Q_t$. So?

Relevance of this Result

- $q_t = Q_t$. So?
- This result makes the Q-theory operational

Relevance of this Result

- $q_t = Q_t$. So?
- This result makes the Q-theory operational
- Q is measurable

Relevance of this Result

- $q_t = Q_t$. So?
- This result makes the Q-theory operational
- Q is measurable
- We can now measure Q at different levels of aggregation
  - Firm level
  - Industry level
  - National level
- And test the theory

Relevance of this Result

- $q_t = Q_t$. So?
- This result makes the Q-theory operational
- Q is measurable
- We can now measure Q at different levels of aggregation
  - Firm level
  - Industry level
  - National level
- And test the theory
- Why do we want to test this theory?
  - It is our benchmark model
  - Many educated guesses come from this framework
  - The investment effects of tax reforms
  - The investment effects of monetary policy
  - And so on...