Solutions to Problem Set 2. 210C
Due Monday May 11th.

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1. The model economy is the New Keynesian model, given by

$$\hat{\pi}_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \tilde{y}_t + u_t \tag{1}$$
$$\tilde{y}_t = \mathbb{E}_t \tilde{y}_{t+1} - \sigma(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}) \tag{2}$$
$$\hat{i}_t = \phi_\pi \hat{\pi}_t + v_t \tag{3}$$

u_t and v_t are shifters to the Phillips curve and the Taylor rule, respectively. These disturbances follow independent AR(1) processes, given by

$$u_t = \rho_u u_{t-1} + \epsilon_t \tag{4}$$
$$v_t = \rho_u v_{t-1} + \xi_t \tag{5}$$

(a) Assume away the existence of u_t from the system. Make a guess that the solution is linear on the realization of v_t. Using the method of undetermined coefficients, find solutions for $\hat{\pi}_t$ and $\tilde{y}_t$.

Guess $\hat{\pi}_t = \phi_{\pi v} v_t$ and $\tilde{y}_t = \phi_{yv} v_t$.

Solutions:

$$\phi_{\pi v} = \frac{-\kappa\sigma}{(1-\beta\rho_v)(1-\rho_v) + \kappa\sigma(\phi_\pi - \rho_v)}$$
$$\phi_{yv} = \frac{-\sigma(1-\beta\rho_v)}{(1-\beta\rho_v)(1-\rho_v) + \kappa\sigma(\phi_\pi - \rho_v)}$$

The expressions for π and y can be found by pluggint these coefficients on the initial guess.

(b) Now assume away the existence of v_t from the system, but reintroduce u_t. Make a guess that the solution is linear on the realization of u_t. Using the method of undetermined coefficients, find solutions for $\hat{\pi}_t$ and $\tilde{y}_t$.

Guess $\hat{\pi}_t = \phi_{\pi v} u_t$ and $\tilde{y}_t = \phi_{yv} u_t$.

$$\phi_{\pi u} = \frac{1-\rho_u}{(1-rho_u)(1-\beta\rho_u)-\kappa\sigma(\rho_u-\phi_\pi)}$$
$$\phi_{yu} = \frac{\sigma(\rho_u-\phi_\pi)}{(1-rho_u)(1-\beta\rho_u)-\kappa\sigma(\rho_u-\phi_\pi)}$$

The expressions for π and y can be found by pluggint these coefficients on the initial guess.

(c) What did you learn about the covariance of inflation and the output gap as a function of the source of the shock? Discuss.

The shifter to the Taylor rule is a demand shock. The covariance of inflation and output in response to a demand shock is positive. On the other hand, the covariance between inflation and the output gap is negative in the case of a cost-push (supply) shock.

2. Under optimal monetary policy, in the simplest case we discussed in the slides (divine coincidence), the optimal interest rate was $\hat{i}_t = \hat{r}_t^n$. Imagine that a central bank then takes that literally to heart and you solve the new keynesian model given by

$$\hat{\pi}_t = \beta\mathbb{E}_t\pi_{t+1} + \kappa\tilde{y}_t \tag{6}$$
$$\tilde{y}_t = \mathbb{E}_t\tilde{y}_{t+1} - \sigma(\hat{i}_t - \mathbb{E}_t\hat{\pi}_{t+1} - \hat{r}_t^n) \tag{7}$$
$$\hat{i}_t = \hat{r}_t^n \tag{8}$$

(a) Check that $\hat{\pi}_t = \tilde{y}_t = 0$ is a possible solution to the model. Hint: A solution to the system does not violate any of the equations.

Yes. It is a possible solution to the model as it doesn't violate any of the equilibrium conditions. In particuar, it implies expectations anchored at $E_t\hat{\pi}_{t+1} = E_t\tilde{y}_{t+1} = 0$.

(b) Is this solution locally unique? In order to check uniqueness, use the state-space representation for the model and use the Blanchard-Khan method. Please state the conditions the eigenvalues should satisfy and check whether those conditions are indeed satisfied. Do you have a solution? Multiple solutions?

The system has two forward-looking variables. The Blanchard-conditions require that the number of eigenvalues that are larger than one in modulus be equal to the number of forward-looking or jump variables. In this case, uniqueness requires two eigenvalues larger than one in modulus.
However, only one eigenvalue will satisfy this condition under reasonable parametrization($\beta < 1$).
State-space representation:

$$\begin{bmatrix} E_t\hat{\pi}_{t+1} \\ E_t\tilde{y}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ \frac{-\sigma}{\beta} & \frac{\beta+\sigma\kappa}{\beta} \end{bmatrix}}_{\Lambda} \begin{bmatrix} \hat{\pi}_t \\ \tilde{y}_t \end{bmatrix}$$

We need to compute the eigenvalues of $Lambda$ to study uniqueness. The eigenvalues are the roots of the characteristic equation $det(\Lambda - \lambda I)$.
Alternatively, for a 2×2 matrix we can use the following condition.
Both eigenvalues lie outside the unit circle if and only if either Case I or Case II hold:

- **Case I:**

$$\begin{aligned} \det A &> 1 \\ \det A - \operatorname{tr} A &> -1 \\ \det A + \operatorname{tr} A &> -1 \end{aligned}$$

- **Case II:**

$$\begin{aligned} \det A - \operatorname{tr} A &< -1 \\ \det A + \operatorname{tr} A &< -1 \end{aligned}$$

Compute the Determinant and Trace of Λ

$$\begin{aligned} \det A &= \frac{1}{\beta} \cdot \frac{\beta + \sigma\kappa}{\beta} - \left(-\frac{\kappa}{\beta}\right)\left(-\frac{\sigma}{\beta}\right) \\ &= \frac{\beta + \sigma\kappa}{\beta^2} - \frac{\kappa\sigma}{\beta^2} \\ &= \frac{\beta}{\beta^2} = \frac{1}{\beta} \end{aligned}$$

Since $\beta < 1$, we have:

$$\boxed{\det A = \frac{1}{\beta} > 1}$$

Trace

$$\operatorname{tr} A = \frac{1}{\beta} + \frac{\beta + \sigma\kappa}{\beta} = \frac{1 + \beta + \sigma\kappa}{\beta}$$

Check the Conditions

Check Case I

$$\begin{aligned} \det A - \operatorname{tr} A &= \frac{1}{\beta} - \frac{1 + \beta + \sigma\kappa}{\beta} = \frac{-\beta - \sigma\kappa}{\beta} = -1 - \frac{\sigma\kappa}{\beta} < -1 \\ \det A + \operatorname{tr} A &= \frac{1}{\beta} + \frac{1 + \beta + \sigma\kappa}{\beta} = \frac{2 + \beta + \sigma\kappa}{\beta} > -1 \end{aligned}$$

Conclusion: Case I fails (middle condition fails)
Check Case II

$$\begin{aligned} \det A - \operatorname{tr} A &< -1 \quad \text{(holds)} \\ \det A + \operatorname{tr} A &> -1 \quad \text{(fails)} \end{aligned}$$

Conclusion: Case II also fails.
Since neither Case I nor Case II holds, we conclude that only one eigenvalue lies outside the unit circle. Thus, the solution is not unique.

(c) Suppose that instead of following the rule $\hat{i}_t = \hat{r}_t^n$, the central bank follows the rule $\hat{i}_t = \hat{r}_t^n + \phi_\pi \hat{\pi}_t$ for $\phi_\pi > 1$. Repeat questions (a) and (b).

Yes, $\hat{\pi}_t = \tilde{y}_t = 0$ is a possible solution to the model.
The system has two forward-looking variables. According to the Blanchard-Kahn conditions, uniqueness of equilibrium requires that the number of eigenvalues of the transition matrix that lie outside the unit circle equals the number of forward-looking (jump) variables.
State-space representation:

$$\begin{bmatrix} E_t\hat{\pi}_{t+1} \\ E_t\tilde{y}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ -\frac{\sigma(1-\beta\phi_\pi)}{\beta} & \frac{\beta+\sigma\kappa}{\beta} \end{bmatrix}}_{\Lambda} \begin{bmatrix} \hat{\pi}_t \\ \tilde{y}_t \end{bmatrix}$$

To assess uniqueness, we analyze the eigenvalues of Λ. Using the conditions stated above, we compute the determinant and trace of Λ.
Determinant:

$$\begin{aligned} \det\Lambda &= \frac{1}{\beta}\frac{\beta+\sigma\kappa}{\beta} - \left(-\frac{\kappa}{\beta} - \frac{\sigma(1-\beta\phi_\pi)}{\beta}\right) \\ &= \frac{\beta+\sigma\kappa}{\beta^2} - \frac{\kappa\sigma(1-\beta\phi_\pi)}{\beta^2} \\ &= \frac{\beta+\kappa\sigma\beta\phi_\pi}{\beta^2} = \frac{1+\kappa\sigma\phi_\pi}{\beta} \end{aligned}$$

Trace:

$$\operatorname{tr}\Lambda = \frac{1}{\beta} + \frac{\beta+\sigma\kappa}{\beta} = \frac{1+\beta+\sigma\kappa}{\beta}$$

Check Case I Conditions
(i) $\det\Lambda > 1$
Since $\beta < 1$ and all parameters are positive,

$$\det\Lambda = \frac{1+\kappa\sigma\phi_\pi}{\beta} > 1$$

(ii) $\det\Lambda - \operatorname{tr}\Lambda > -1$

$$\begin{aligned} \det\Lambda - \operatorname{tr}\Lambda &= \frac{1+\kappa\sigma\phi_\pi}{\beta} - \frac{1+\beta+\sigma\kappa}{\beta} \\ &= \frac{-\beta+\kappa\sigma(\phi_\pi-1)}{\beta} \end{aligned}$$

This is greater than -1 if:

$$-\beta+\kappa\sigma(\phi_\pi-1) > -\beta-\beta \quad \Leftrightarrow \quad \kappa\sigma(\phi_\pi-1) > -\beta$$

Which holds if $\phi_\pi > 1$
(iii) $\det\Lambda + \operatorname{tr}\Lambda > -1$

$$\det\Lambda + \operatorname{tr}\Lambda = \frac{1+\kappa\sigma\phi_\pi+1+\beta+\sigma\kappa}{\beta} = \frac{2+\beta+\kappa\sigma(1+\phi_\pi)}{\beta} > -1$$

If $\phi_\pi > 1$ (assumed in the question), then all three Case I conditions are satisfied. Therefore, both eigenvalues lie outside the unit circle. The solution is unique.

(d) Even though the two monetary policy rules produce the same sequence of interest rates in at least one equilibrium, the two rules are not identical as you hopefully discovered. Please state an intuitive explanation of why that is the case, and how an off-equilibrium threat by the central bank is a key aspect of the difference. Your answer should be **maximum one short paragraph**. Extreme wordiness will be penalized.

The two rules produce the same sequence of interest rates. However, in the second case the monetary rule with $\phi_\pi > 1$ implies that the monetary authority is expected to move interest rates more than one to one in response to deviations of inflation from its steady-state value. The off-equilibrium threat of moving rates anchors inflation expectations and is crucial to guarantee the uniqueness of the model solution.

(e) Imagine an econometrician is interested in estimating the Taylor rule parameter ϕ_π. Please discuss the intuitive challenges in doing so. Can you think of a better alternative than running OLS regressions of interest rates on inflation rates? You are welcome to give your own ideas or gloss over papers you find online that discuss alternative approaches. Your answer should be **maximum one short paragraph**. Extreme wordiness will be penalized.

Open. The parameter ϕ_π measures the response of the nominal interest rate to movements in the inflation rate. Challenges: endogeneity/simultaneity. Identification requires some source of exogeneous variation in the inflation rate that is uncorrelated with other observed/unobserved macro variables that affect nominal rates.

3. The economy has an Euler equation $\tilde{y}_t = \mathbb{E}_t \tilde{y}_{t+1} - \sigma \hat{r}_t$. Assume that the central bank has perfect control over the real interest rate, so the central bank is choosing r_t directly, as opposed to choosing only i_t, and r_t being given by the Fisher equation. Assume further that the economy is stationary so the economy is in steady state in the infinite future $\lim_{k\to\infty} \mathbb{E}_t \tilde{y}_{t+k} = 0$

   Imagine that at time t the central bank chooses the whole sequence of interest rates, and in particular it chooses $\hat{r}_{t+k} = 0$ for all k except for a single period $t + k^*$ for k^* finite, where the interest rate takes the value $-r$. Intuitively, the central bank is announcing in period t a one-period monetary policy change that will happen in the present (if $k^* = 0$) or in the future (if $k^* > 0$). In the latter case, this is precisely what Forward Guidance is: a present announcement about the future path of the interest rate.

   (a) Iterate forward the Euler equation using the terminal condition.

$$\tilde{y}_t = -\sigma \sum_{j=0}^{\infty} E_t \hat{r}_{t+j}$$

   where the terminal condition is $lim_{k\to\infty} E_t \tilde{y}_{t+k} = 0$

   (b) What is the value of the output gap in period t if the central bank chooses $k^* = 0$?

$$\tilde{y}_t = \sigma r$$

   (c) What is the value of the output gap in period t if the central bank chooses $k^* = 10$? $k^* = 100$?

Same as in previous point. A change in the interest rate happening in the distance future that is announced today has the same power to move output as a surprise change in interest rates happening today.

(d) Intuitively, do you think that the relative magnitude of the response of the output gap today to an interest rate cut that takes place today vs. one that takes place in the distant future is an appealing or unappealing property of the New Keynesian model?

Open. Answer could relate to the strength of forward-looking forces implied by the model or how this is at odds with the empirical evidence on the effect of forward-guidance monetary policy shocks identified by the literature.

(e) Repeat your answers to questions (a) - (c) if the Euler equation had instead the following shape $\tilde{y}_t = \beta \mathbb{E}_t \tilde{y}_{t+1} - \sigma \hat{r}_t$, for a discount factor $\beta \in (0,1)$.

The outgap today for a change in interest rates in period k announced today is:

$$\tilde{y}_t = \sigma \beta^k r$$

(f) Relate your findings to the *Forward Guidance Puzzle*, and the *Discounted Euler Equation*. You will need to do a short google scholar search. You should not read full papers, but maybe read the abstract and parts of the introduction of one or two papers. Your answer should be **maximum one short paragraph** per concept. Extreme wordiness will be penalized.

Open. The forward guidance puzzle refers to the fact that the very strong effects of future policy announcements implies by the NK model seem to be at odds with the available empirical evidence. The Discounted Euler Equation is one way in which the NK model can be modified to dampen the strength of forward-looking forces. In particular, it implies that the effect of interest rate changes happening in some future period $t + s$ but announced today have an impact on output that is decreasing in s.