Homework #7

Statistical Methods with R, Spring 2019

Due in class on Wednesday, March 20, 2019. *Instructions for "theoretical" questions: Answer all of the following questions. The theoretical problems should be neatly numbered, written out, and solved. Please do not turn in messy work. Working in small groups is allowed, but it is important that you make an effort to master the material and hand in your own work;Identical solutions will be considered as a violation of the Student Honor Code.* **Note that you are also required to turn in the computational portion of this assignment. It is on Canvas, under the name:**

`A4570SP19HW07-computational.ipynb`.

ON THE FRONT OF YOUR HOMEWORK CLEARLY PRINT THE FOLLOWING:

- Your full name.
- Your lecture number (either **APPM 4570** or **APPM 5570** or **STAT 4000** or **STAT 5000**).
- Homework number.
- **Points will be deducted if these instructions are not followed.**

Remember that writing style, clarity and completeness of explanations is always important. Justify your answers. (Be sure to place your homework in the correct pile, either "undergraduate" or "graduate".)

Theoretical Questions

1. Let $\vec{X} = (X_1, X_2, \ldots, X_n)$ be a random sample (*i.e.* i.i.d) from a Bernoulli distribution where the pmf of each r.v. is given by $f(x_i; p) = p^{x_i}(1-p)^{1-x_i}$ for $x_i = 0, 1$ and $i = 1, 2, \ldots, n$.

   (a) Find $\ln(L(p|\vec{X}))$, the *log-likelihood function* for the parameter p based on the given random sample above.

   (b) Now find the *maximum likelihood estimator*, $\widehat{p}$, of parameter p. Show all work

   (c) Find $E[\hat{p}]$. Is the estimator $\widehat{p}$ *unbiased*? Why or why not?

2. Let $X_1, X_2, \ldots, X_n$ be a random sample (*i.e.* i.i.d) from a Normal distribution with mean μ and variance σ^2. In class, we showed that $\widehat{\mu} = \overline{x}$ is the MLE point estimator of μ. We now consider σ.

   (a) Find $\ln(L(\sigma|\vec{X}))$, the *log-likelihood function* for the parameter σ based on the given random sample above.

   (b) Use the log-likelihood function from part (a) and the Invariance Principle to find the *maximum likelihood estimator* for σ and σ^2. Show all work.

   (c) Find $E\left[\widehat{\sigma^2}\right]$. Is the MLE $\widehat{\sigma^2}$ unbiased? Why or why not? Justify your answer. (*Hint:* Recall that we showed in class $E[s^2] = \sigma^2$.)

3. (a) In 1881 Michelson and Newcomb measured the time light took to travel a distance of 7400 meters. From a a study of their experimental setup and a descriptive study of their 64 measurements and experimental setup, we conclude that the data can be assumed to be i.i.d. These measurements yield the following sample quantities in microseconds (sec $\times 10^{-6}$):

   $$\bar{x} = 27.75, s = 5.08$$

   Construct an approximate 95% confidence interval for the time light takes to travel 7400 meters.

   (b) A journal article reports that a sample of size $n = 5$ was used as a basis for calculating a 95% CI for the true average natural frequency (Hz) of delaminated beams of a certain type. The resulting interval was $(229.764, 233.504)$. You decide that a confidence level of 99% is more appropriate than the 95% level used. What are the limits of the 99% interval?

♚PROBLEM #4 on the other side.♛

4. (**APPM 5570/STAT 5000 Students Only**) Read the following very carefully and then answer the given questions:

The Method of Moments (MoM) for finding Point Estimators

Definition #1: Let X be some r.v. with pmf or pdf given by $f(x)$, then the k^{th} *population* moment of X is defined to be $E[X^k]$ for $k = 1, 2, \ldots$.

Definition #2: If $X_1, X_2, \ldots, X_n$ is a sample of data then the k^{th} *sample* moment is defined to be $M_k = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i^k$ for $k = 1, 2, \ldots$.

Method of Moments: Now suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with pmf or pdf given by $f(x; \theta_1, \theta_2, \ldots, \theta_m)$ where $\theta_1, \theta_2, \ldots, \theta_m$ are parameters whose value is unknown. To find the estimators $\widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_m$ using the Method of Moments, we must equate the first m population moments to the first m sample moments (*i.e.* let $E[X^k] = M_k$ for $k = 1, 2, \ldots, m$); then solving for $\theta_1, \theta_2, \ldots, \theta_m$ in terms of $x_1, x_2, \ldots, x_n$ yields the estimators $\widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_m$.

Questions: Suppose $X_1, X_2, \ldots, X_n$ is a random sample (*i.e.* i.i.d) from a Gamma Distribution with "*shape*" parameter α and "*scale*" parameter β, moreover, note that $E[X] = \alpha\beta$ and $V(X) = \alpha\beta^2$.

(a) Find an expression for the second population moment $E[X^2]$ in terms of α and β.

(b) Now use the Method of Moments described above to find estimators $\widehat{\alpha}$ and $\widehat{\beta}$. (*Hint:* Note that $\beta = \alpha\beta^2/\alpha\beta$.)

(c) The following data was observed

3.27, 6.44, 2.25, 1.70, 5.89, 0.79, 6.54, 3.94, 7.77, 2.69

Use this information to approximate α and β using the estimators found in (b).