MATH 420/507 Assignment 5

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Problem 1(a).

Proof. Let $x, y \in \mathbb{R}, x \leq y$, then we have: **(1)** If $x, y \geq 0$ then $|y| \geq |x|$ and $\lfloor y \rfloor \geq \lfloor x \rfloor \Rightarrow F(y) \geq F(x)$. **(2)** If $x, y < 0$ then $0 < |y| \leq |x|$ and $\lfloor x \rfloor \leq \lfloor y \rfloor < 0 \Rightarrow F(y) \geq F(x)$. **(3)** If $x \leq 0, y \geq 0$ then $|x|\lfloor x \rfloor \leq 0 \leq |y|\lfloor y \rfloor \Rightarrow F(y) \geq F(x)$. Therefore F is non-decreasing. It is right continuous because: $\lim_{x\to y^+} f(x) = \lim_{x\to y^+} |x|\lfloor x \rfloor = \lim_{x\to y^+} |x| \lim_{x\to y^+} \lfloor x \rfloor = |y|\lfloor y \rfloor$ where $|x|$ and $\lfloor x \rfloor$ are right continuous functions. □

Problem 1(b).

Proof. Let m be the lebesgue measure and $\lambda : E \to \sum_{x \in E\cap\mathbb{Z}} |x|$ and $\rho : E \to \mu_F(E\backslash\mathbb{Z})$. Then if $k \in \mathbb{Z}$,

$$\mu_F(\{k\}) = |k|\lfloor k \rfloor - \lim_{n\to\infty} |k - \frac{1}{n}|\lfloor k - \frac{1}{n} \rfloor = |k|k - \lim_{n\to\infty} |k - \frac{1}{n}|(k-1) = |k| \tag{1}$$

By countable additivity, we have $\mu_F(E) = \mu_F(E \cap \mathbb{Z}) + \mu_F(E\backslash\mathbb{Z}) = \lambda(E) + \rho(E)$ for all set E, which means $\mu_F = \lambda + \rho$. To show this is a Lebesgue-Radon-Nikodym decomposition with respect to Lebesgue measure m, we first show:

(1)$\rho \ll m$: notice $E\backslash\mathbb{Z} = \bigcup_{k\in\mathbb{Z}} E \cap (k, k+1)$, suppose $m(E) = 0$ then we have $m(E \cap (k, k+1)) \leq m(E) = 0 \Rightarrow m(E \cap (k, k+1)) = 0$ for all $k \in \mathbb{Z}$ since $(E \cap (k, k+1)) \subset E$. Since $\lfloor x \rfloor = k$ for all $x \in (k, k+1)$, we have $\rho(E \cap (k, k+1)) = km(E \cap (k, k+1)) = 0$. Thus, $\rho(E\backslash\mathbb{Z}) = \sum_{k\in\mathbb{Z}} \rho(E \cap (k, k+1)) = 0 \Rightarrow \rho \ll m$. Next we will show:

(2)$\lambda \perp m$: Let $A = \mathbb{Z}, B = \mathbb{Z}^c$, then $A \cap B = \emptyset$ and $A \cup B = \mathbb{R}$. Since λ, m are non-negative measures by construction, it suffices to show $\lambda(B) = 0$ and $m(A) = 0$. Since $A = \mathbb{Z}$ is countable then $m(A) = 0$, and $\lambda(B) = 0$ because $B \cap \mathbb{Z} = \emptyset$, hence we've shown $\lambda \perp m$. In conclusion, we have proven that $\mu_F = \lambda + \rho$ is Lebesgue-Radon-Nikodym decomposition with respect to Lebesgue measure m. □

Problem 2(a).

Proof. $m \ll \mu$ is equivalent to show $m(E) = 0, \forall E \in \mathcal{M}$ such that $\mu(E) = 0$. Since $\mu(E) = 0$ only when $E = \emptyset$ and $m(\emptyset) = 0$, we've shown $m \ll \mu$. Then we show $dm \neq fd\mu$: Suppose $dm = fd\mu$ for some f, let $E = \{x\}$ where x is in $[0, 1]$, then we have $m(E) = 0$ and hence $\int_E fd\mu = 0$, which implies $f(x)\mu(\{x\}) = 0$. Since x is arbitrary and $\mu(\{x\}) = 1$, $f(x) = 0$ for all $x \in [0, 1]$. Pick $E' = [0, 1]$ then $m(E') = 1 \neq 0 = \int_{E'} fd\mu$ which leads to a contradiction. Therefore $m \ll \mu$ but $dm \neq fd\mu$ for any f. □

Problem 2(b).

Proof. Suppose μ has a Lebesgue decomposition with respect to lebesgue measure m, say $\mu = \lambda + \rho$, where $\lambda \perp m$ and $\rho \ll m$. Then $\exists E, F \in \mathcal{M}$ where $E \cup F = X, E \cap F = \emptyset$, E is m-null and F is λ-null. Thus, $F = \emptyset$ since if there are some $x \in F$, $\mu(\{x\}) = 1$ but $\lambda(\{x\}) = 0$ and $m(\{x\}) = 0 \Rightarrow \rho(\{x\}) = 0$ then the left hand side is 1 and the right hand side is 0, which leads to a contradiction. Therefore $E = X$, but we have $m(X) = 1 \neq 0$ since we know $E = X$ is $m-$null, which also leads to contradiction. In conclusion, μ does not have a Lebesgue decomposition with respect to lebesgue measure m □

Problem 2(c).

Proof. The counting measure μ is not σ-finite on $(X, \mathcal{M})$ so the decomposition and thoerem cannot apply. □

Problem 3(a).

Proof. Define set function $\lambda(E) = \int_E f d\mu$ for all $E \in \mathcal{N}$, then λ is well-defined since $f \in L^1(\mu)$. We next show λ is signed measure: $\lambda(\emptyset) = \int_\emptyset f d\mu = 0$, and $\lambda(E) = \int_E f d\mu \in (-\infty, \infty)$ since $\int_E f d\mu \le \int_X |f| d\mu < \infty$. Let $\{E_j\}_j^\infty$ be a collection of disjoint sets in $\mathcal{N}$, then by dominated convergence theorem (let $f_n = f \sum_j^n 1_{E_j}$ and $|f_n| < |f|$ which is integrable) we have:

$$\lambda(\bigcup_j E_j) = \int_{\bigcup_j E_j} f d\mu = \int_X f 1_{\bigcup_j E_j} d\mu = \int_X f \sum_j 1_{E_j} d\mu = \sum_j \int_X f 1_{E_j} d\mu = \sum_j \lambda(E_j) \tag{2}$$

Then we have shown λ is a signed measure, notice that λ is $\sigma-$finite since $\lambda(X) = \int_X f d\mu < \infty$. Also, $\forall E \in \mathcal{N}$ where $\nu(E) = 0$, we have $\nu(E) = 0 \Rightarrow \mu(E) = 0 \Rightarrow \lambda(E) = 0$, hence $\lambda \ll \nu$. By the Lebesgue-Radon-Nikodym thoerem, the Radon-Nikodym derivative $g = d\lambda/d\nu$ exists and $g \in L^1(\nu)$. So we have $d\lambda = g d\nu, d\lambda = f d\mu$, which means $\lambda(E) = \int_E g d\nu = \int_E f d\mu$ where g is unique (any other such $g' = g$ ν-a.e. from the theorem). $\square$

Problem 3(b).

Proof. For $\mathcal{N} = \{\emptyset, X\}$, g is measurable with respect to $\mathcal{N} \iff g^{-1}(U) \in \mathcal{N}, \forall$ open set U in $\mathbb{R}$. Therefore we have g is a constant function, say $g = c$, then $\int_X g d\nu = c\nu(X) = \int_X f d\mu \Rightarrow c = \frac{\int_X f d\mu}{\nu(X)} \Rightarrow g = \frac{\int_X f d\mu}{\mu(X)}$. Similarly for $\mathcal{N} = \{\emptyset, E, E^c, X\}$, we have $g = a1_E + b1_{E^c}$. If $\mu(E) \ne 0$, then $\int_E g d\nu = a\nu(E) = \int_E f d\mu, \int_{E^c} g d\nu = b\nu(E^c) = \int_{E^c} f d\mu \Rightarrow a = \frac{\int_E f d\mu}{\nu(E)} = \frac{\int_E f d\mu}{\mu(E)}$ and $b = \frac{\int_{E^c} f d\mu}{\nu(E^c)} = \frac{\int_{E^c} f d\mu}{\mu(E^c)}$. If $\mu(E) = 0$, then $b = \frac{\int_{E^c} f d\mu}{\mu(E^c)}$ and a can be arbitrary since $\int_E g d\nu = a\nu(E) = \int_E f d\mu = 0$ for all a. $\square$

Problem 4(a).

Proof. The inequality can be shown by:

$$\frac{1}{\alpha}\int |f| dm \ge \frac{1}{\alpha}\int_{|f|>\alpha} |f| dm = \frac{1}{\alpha}\int_{|f|>\alpha} \alpha dm = \int_{|f|>\alpha} 1 dm = m(\{|f| > \alpha\}) \Rightarrow m(\{x : |f| > \alpha\}) \le \frac{1}{\alpha}\int |f| \tag{3}$$

$\square$

Problem 4(b).

Proof. Notice the ball $B(r, x) \subset [-r, r]^n, B(r, x) \supset [\frac{-r}{\sqrt{n}}, \frac{r}{\sqrt{n}}]^n$, hence we have $(\frac{2}{\sqrt{n}})^n r^n \le m(B(r, x)) \le 2^n r^n$. Denote this as $C_1 r^n \le m(B(r, x)) \le C_2 r^n$ where $C_1 = (\frac{2}{\sqrt{n}})^n, C_2 = 2^n$. There exist some $N \in \mathbb{N}$ such that f is not almost everywhere 0 on $B(N, 0)$, and we must have $\int_{B(N,0)} |f| dm = M > 0$. Fix x such that $|x| \ge 1$, then we have $(N + 1)|x| \ge |x| + N$ so $B((N + 1)|x|, x) \supset B(N, 0)$, and $\int_{B((N+1)|x|,x)} |f| dm \ge M$, then

$$Hf(x) \ge \frac{1}{m(B((N+1)|x|, x))}\int_{B((N+1)|x|,x)} |f| dm \ge \frac{M}{C_2(N+1)^n |x|^n} \tag{4}$$

Thus we have $Hf(x) \ge C/|x|^n$ for all x with $|x| \ge 1$, where $C = \frac{M}{C_2(N+1)^n}$. Now let $E_k = \{x : |x| \in (k-1, k]\}$ for each $k \in \mathbb{N}$, then $E_k = B(k, 0)\backslash B(k - 1, 0)$, so we have $m(E_k) = m(B(k, 0)) - m(B(k - 1, 0))) \ge C_1 k^n - C_1(k - 1)^n$. Let $K = C \times C_1$, then we have:

$$\begin{aligned}\int |Hf(x)| dm &\ge C\int \frac{1}{|x|^n} dm \ge C\sum_{k=2}^\infty \int_{E_k} \frac{1}{|x|^n} dm \ge C\sum_{k=2}^\infty \frac{C_1 k^n - C_1(k-1)^n}{k^n} \\ &\ge C\sum_{k=2}^\infty \frac{C_1 k^{n-1}}{k^n} = K\sum_{k=2}^\infty \frac{1}{k} \to \infty\end{aligned} \tag{5}$$

Therefore $\int |Hf(x)| dm = \infty$ and hence we can conclude that $Hf(x) \notin L^1$. $\square$

Problem 5(a).

Proof. Without loss of generality, assume $y > x$, then we have $|F(y) - F(x)| = |\int_x^y f(t)dt| \leq \int_x^y |f(t)|dt$. Fix $x \in \mathbb{R}$, consider an bounded interval $X = [x-1, x+1]$, since $f \in L^1_{loc}(m)$ we have $\int |f1_X| < \infty \Rightarrow f1_X \in L^1(m)$. Thus, $\forall \epsilon > 0, \exists |y - x| < \delta < 1$, let $E = (x, y)$, which is equivalent to say $m(E) < \delta \Rightarrow \int_E |f|dm = \int_x^y |f(t)|dt \leq \epsilon$ by *Corollary 3.6*. Therefore we can conclude that F is continuous. □

Problem 5(b).

Proof. Recall that F is differentiable at some point x_0 if and only if there exist a continuous function ϕ at x_0 such that $F(x) - F(x_0) = (x - x_0)\phi(x)$ where $\phi(x_0) = F'(x_0)$. Let $x_0 = 0$ here, since F is differentiable at 0 and $F(0) = 0$, we have function ϕ continuous at 0 such that $F(x) = x\phi(x)$ where $\phi(0) = F'(0)$. Thus, we have

$$\begin{aligned} A_r f(0) - F'(0) &= \frac{1}{2r}\int_{-r}^{r} f(t)dt - \phi(0) = \frac{1}{2r}[F(r) - F(-r)] - \phi(0) \\ &= \frac{1}{2r}[r\phi(r) + r\phi(-r)] - \phi(0) = \frac{1}{2}[\phi(r) + \phi(-r)] - \phi(0) \to 0 \text{ as } r \to 0^+ \end{aligned} \tag{6}$$

since ϕ is continous at $x_0 = 0$. Therefore $\lim_{r \to 0^+} A_r f(0) = F'(0)$, F is differentiable at 0. □

Problem 5(c).

Proof. Let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$$

Observe that $\lim_{r \to 0^+} A_r f(0) = \lim_{r \to 0^+} \frac{1}{2r}\int_{-r}^{r} f(t)dt = \frac{1}{2} = f(0)$, and $F = \int_0^x f(t)dt$ is not differentiable at 0 since $\frac{F(x) - F(0)}{x} = \frac{1}{x}\int_0^x f(t)dt$ is 1 for $x > 0$ but 0 for $x < 0$. □

Problem 5(d).

Proof. Let $F(x) = \int_0^x f(t)dt$ where

$$f(t) := \begin{cases} \sin(\frac{1}{t}) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

We first show $F'(0) = 0$: by definition, need to show $\lim_{x \to 0} \frac{1}{x}\int_0^x \sin(\frac{1}{t})dt = 0$, assume $x > 0$ ($x < 0$ would be similar since f is an odd function). we have:

$$\lim_{x \to 0} \frac{1}{x}\int_{\frac{1}{x}}^{\infty} \frac{sin(u)}{u^2}du = 0$$

$$\lim_{y \to \infty} y\int_y^{\infty} \frac{sin(u)}{u^2}du = 0$$

by change of variable: $u = \frac{1}{t}, y = \frac{1}{x}$. Then integration by parts, this is equivalent to show $\lim_{y \to \infty} y\int_y^{\infty} \frac{dcos(u)}{u^2}du = 0$. This can be shown by:

$$\begin{aligned} y\int_y^{\infty} \frac{dcos(u)}{u^2}du &= y[\frac{cos(u)}{u^2}|_y^{\infty} - \int_y^{\infty} cos(u)(-\frac{2}{u^3})du] \\ &= y[\frac{cos(y)}{y^2} + 2\int_y^{\infty}(\frac{cos(u)}{u^3})du] \\ &= \frac{cos(y)}{y} + 2\int_y^{\infty} y(\frac{cos(u)}{u^3})du \end{aligned} \tag{7}$$

Denote $I_1 = \frac{cos(y)}{y}, I_2 = \int_y^\infty y(\frac{cos(u)}{u^3})du$, then we can show that $\lim_{y\to\infty} I_1 = 0, \lim_{y\to\infty} I_2 = 0$ since $|I_2| \leq \int_y^\infty \frac{y}{u^3}du = y\frac{c}{y^2} \to 0$. Therefore, we have proven that $F'(0) = 0$.
Then we have $A_r|f|(0) = \frac{1}{2r}\int_{-r}^r |\sin(\frac{1}{t})|dt = \frac{1}{r}\int_0^r |\sin(\frac{1}{t})|dt$, similarly let $u = \frac{1}{t}$, we have $A_r|f|(0) = \frac{1}{r}\int_{\frac{1}{r}}^\infty \frac{|\sin(u)|}{u^2}du$. Consider $n\pi < \frac{1}{r} < (n+1)\pi$, then we have

$$\begin{aligned}
\int_{\frac{1}{r}}^\infty \frac{|\sin(u)|}{u^2}du &\geq \int_{(n+1)\pi}^\infty \frac{|\sin(u)|}{u^2} \\
&= \sum_{j=n+1}^\infty \int_{j\pi}^{(j+1)\pi} \frac{|\sin(u)|}{u^2}du \text{ by Monotone Convergence Theorem} \\
&\geq \sum_{j=n+1}^\infty \int_{j\pi}^{(j+1)\pi} \frac{|\sin(u)|}{(j+1)^2\pi^2}du \\
&= \sum_{j=n+1}^\infty \frac{2}{\pi^2}\frac{1}{(j+1)^2} \\
&\geq C\int_{n+1}^\infty \frac{2}{\pi^2}\frac{1}{(u+1)^2}du = C\frac{1}{n+2}
\end{aligned} \tag{8}$$

But then $\frac{1}{r}\int_{\frac{1}{r}}^\infty \frac{|\sin(u)|}{u^2}du > n\pi \times \frac{C}{n+2} > \frac{C}{2} > 0$, therefore $\lim_{r\to 0^+} A_r|f - F'(0)|(0) \neq 0$. $\square$