MATH 420 Assignment 1

Jingyuan Hu (Juan) #41465155

Problem 1.

Proof. Let $\{\mathcal{A}_i\}$ be a family of σ-algebras, obviously $\emptyset \in \cap_i \mathcal{A}_i$ since $\emptyset$ is in every $\mathcal{A}_i$

Let $A \in \cap_i \mathcal{A}_i \Rightarrow A \in \mathcal{A}_i, \forall i$

Since all $\mathcal{A}_i$ are σ-algebras, $A^c \in \mathcal{A}_i, \forall i \Rightarrow A^c \in \cap_i \mathcal{A}_i$

Let $\{A_j\}_{j=1}^{\infty}$ be a countable subsets of $\cap_i \mathcal{A}_i$ where $A_j \in \cap_i \mathcal{A}_i, \forall j$, then we can get $A_j \in \mathcal{A}_i, \forall i, \forall j \Rightarrow \cup_j^{\infty} A_j \in \mathcal{A}_i, \forall i \Rightarrow \cup_j^{\infty} A_j \in \cap_i \mathcal{A}_i$

Therefore $\cap_i \mathcal{A}_i$ is also a σ-algebra $\square$

Problem 2.

Proof. $\mathcal{B}$ is generated by family of sets $\{(-\infty, a] | a \in \mathbb{R}\}$, hence $\{(-\infty, a] | a \in \mathbb{R}\} \subset \mathcal{B}$

Since $\mathcal{B}$ is σ-algebra, it is closed under complement $\Rightarrow$ family of sets of the form $\{(a, \infty) | a \in \mathbb{R}\}$ is also contained in $\mathcal{B}$

Family of sets $\{(a, b) | a, b \in \mathbb{R}\}$ is constructed by taking the intersections of sets in the form of $\{(a, \infty) | a \in \mathbb{R}\}$ with $a < b$

Similarly, family of sets $\{[a, b] | a, b \in \mathbb{R}\}$ is contained in $\mathcal{B}$, by taking the intersections of $\{(-\infty, a] | a \in \mathbb{R}\}$ with $a < b$, thus the compelement, family of sets $\{(-\infty, a) \cup (b, \infty] | a, b \in \mathbb{R}\}$ is also in $\mathcal{B}$

By taking the intersections of sets in the form of $\{(-\infty, a) \cup (b, \infty] | a, b \in \mathbb{R}\}$ and $\{(-\infty, a] | a \in \mathbb{R}\} \Rightarrow$ family of sets $\{(-\infty, a) | a \in \mathbb{R}\}$ is also in $\mathcal{B}$

Therefore σ-algebra $\mathcal{B}$ contains all open intervals

Next we'll show that $\mathcal{B} = \mathcal{B}_{\mathbb{R}}$:

where $\mathcal{B}$:= $\{\sigma$-algebra generated by set of all open intervals$\}$, $\mathcal{B}_{\mathbb{R}}$:= $\{\sigma$-algebra generated by set of all open sets$\}$

Let $A \in \mathcal{B}$, then A is an open intervals, thus also a open sets $\Rightarrow A \in \mathcal{B}_{\mathbb{R}}, \mathcal{B} \subseteq \mathcal{B}_{\mathbb{R}}$

Let $A \in \mathcal{B}_{\mathbb{R}}$, then A is an open set in $\mathbb{R}$, which is a countable (disjoint) union of open intervals (*Folland, Prop0.21*), $A = \cup_i^\infty E_i$ where $\{E_i\}_i^\infty$ are countable (disjoint) open intervals

Notice that $\{E_i\} \in \mathcal{B}$, so $\cup_i^\infty E_i \in \mathcal{B}$ by definition, hence $A \in \mathcal{B}$, so $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{B}$

In conclusion, $\mathcal{B} = \mathcal{B}_{\mathbb{R}}$ $\square$

Problem 3.

Proof. Denote the counting measure by μ on $(X, \mathcal{P}(X))$

If X is a countable set, each element e_i in X is correspond to a set $E_i = \{e_i\}, i \in \mathbb{N}$ by definition of countable, then the family of disjoint sets $\{E_i\}_i^\infty$ s.t. $X = \cup_i^\infty E_i$, where due to our construction only one element is in each of the sets, $\mu(E_i) = 1 < \infty, \forall i$, therefore μ is σ-finite

If the counting measure on $(X, \mathcal{P}(X))$ is σ-finite, there is a sequence of (assume all disjoint without loss of generality) sets $\{E_i\}_i^\infty$ s.t. $\cup_i^\infty E_i = X$, where $\mu(E_i) < \infty, \forall i$, hence each E_i is finite by the definition of counting measure, and countable union of finite sets is countable, therefore set X must be countable

$\square$

Problem 4(a).

Proof. First show $\emptyset \in f^{-1}(\mathcal{B})$: $\exists B = \emptyset \in \mathcal{B}, f^{-1}(B) = f^{-1}(\emptyset) = \emptyset$

$\therefore \emptyset \in f^{-1}(\mathcal{B})$ by definition

Then we'll show it is closed under complement:

Let $A \in f^{-1}(\mathcal{B})$, want to show $A^c \in f^{-1}(\mathcal{B})$

$\exists B \in \mathcal{B}$ s.t. $f^{-1}(B) = A$, then $A^c = [f^{-1}(B)]^c = f^{-1}(B^c)$, where $B^c \in \mathcal{B}$, then $A^c \in f^{-1}(B)$

Show $f^{-1}(\mathcal{B})$ is also closed under countable unions:

Let $\{a_i\}_{i=1}^\infty \in f^{-1}(\mathcal{B})$ be a countable sequence of disjoint sets, then $\forall i, \exists B_i \in \mathcal{B}$ s.t. $f^{-1}(B_i) = A_i$

Then $\cup_{i=1}^\infty A_i = \cup_{i=1}^\infty f^{-1}(B_i) = f^{-1}(\cup_{i=1}^\infty B_i)$ where $\cup_{i=1}^\infty B_i \in \mathcal{B}$

$\therefore \cup_{i=1}^\infty A_i \in f^{-1}(\mathcal{B})$, $f^{-1}(\mathcal{B})$ is a σ-algebra on X $\square$

Problem 4(b).

Proof. First show $\emptyset \in f_*\mathcal{A}$: this can be easily shown since $\emptyset \subset Y, f^{-1}(\emptyset) = \emptyset \in \mathcal{A}$ since $\mathcal{A}$ is a σ-algebra

Then we'll show it is closed under complement:

Let $A \in f_*\mathcal{A}$, then by definition $A \subset Y, f^{-1}(A) \in \mathcal{A}$

$\therefore$ $A^c \subset Y$, and since $\mathcal{A}$ is a σ-algebra, $(f^{-1}(A))^c = f^{-1}(A^c) \in \mathcal{A} \Rightarrow A^c \in f_*\mathcal{A}$ by the properties of pre-image

Next show $f_*\mathcal{A}$ is also closed under countable unions:

Let $\{A_i\}_{i=1}^{\infty}$ by a sequences of disjoint sets in $f_*\mathcal{A}$, then $A_i \subset Y, f^{-1}(A_i) \in \mathcal{A}, \forall i$

Then $\cup_{i=1}^{\infty} A_i \subset Y$, and since $\mathcal{A}$ is a σ-algebra, $f^{-1}(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} f^{-1}(A_i) \in \mathcal{A}$ by the properties of pre-image $\square$

Problem 4(c).

Proof. First prove $\mathcal{M}(f^{-1}(\mathcal{E})) \subseteq f^{-1}(\mathcal{M}(\mathcal{E}))$:

It is sufficient to show: $(i) f^{-1}(\mathcal{M}(\mathcal{E}))$ is a σ-algebra, and $(ii) f^{-1}(\mathcal{E}) \subseteq f^{-1}(\mathcal{M}(\mathcal{E}))$ since by definition $\mathcal{M}(f^{-1}(\mathcal{E}))$ is the smallest σ-algebra generated by $f^{-1}(\mathcal{E})$

From part(a), we know that a pre-image of σ-algebra is still σ-algebra, which indicates (i)

Also by the properties of pre-image, $\mathcal{E} \subseteq \mathcal{M}(\mathcal{E})$ indicates (ii), therefore we've shown $\mathcal{M}(f^{-1}(\mathcal{E})) \subseteq f^{-1}(\mathcal{M}(\mathcal{E}))$

Next we'll prove $f^{-1}(\mathcal{M}(\mathcal{E})) \subseteq \mathcal{M}(f^{-1}(\mathcal{E}))$:

Define $\mathcal{C} := \{E \in \mathcal{M}(\mathcal{E}) | f^{-1}(E) \in \mathcal{M}(f^{-1}(\mathcal{E}))\}$

Then from part (b) we know that $\mathcal{C} = \mathcal{M}(\mathcal{E}) \cap f_*\mathcal{M}(f^{-1}(\mathcal{E}))$, where both $\mathcal{M}(\mathcal{E}), f_*\mathcal{M}(f^{-1}(\mathcal{E}))$ are σ-algebra and $f_*\mathcal{M}(f^{-1}(\mathcal{E})) = \{E \subset Y | f^{-1}(E) \in \mathcal{M}(f^{-1}(\mathcal{E}))\}$

From the result in Q1, $\mathcal{C}$ is the intersection of two σ-algebras, so $\mathcal{C}$ is also a σ-algebra

$\forall E \in \mathcal{E}$, we know $E \in \mathcal{M}(\mathcal{E})$ and $f^{-1}(E) \in f^{-1}(\mathcal{E}) \subseteq \mathcal{M}(f^{-1}(\mathcal{E})) \Rightarrow \mathcal{E} \subseteq \mathcal{C}$ and we already proved that $\mathcal{C}$ is a σ-algebra

(A) Thus $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{C}$ by minimality of the generated σ-algebra

$\forall E \in \mathcal{M}(\mathcal{E}) \Rightarrow f^{-1}(E) \in f^{-1}(\mathcal{M}(\mathcal{E})) \Rightarrow \mathcal{C} = \mathcal{M}(\mathcal{E}) = f_*\mathcal{M}(f^{-1}(\mathcal{E}))$ by the definition of $\mathcal{C}$

$\therefore f^{-1}(\mathcal{M}(\mathcal{E})) = f^{-1}(f_*\mathcal{M}(f^{-1}(\mathcal{E}))) \subseteq \mathcal{M}(f^{-1}(\mathcal{E}))$

(B) Another way to interpret $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{C}$ is:

$\forall E \in \mathcal{M}(\mathcal{E}), f^{-1}(E) \in f^{-1}(\mathcal{M}(\mathcal{E})) \Rightarrow E \in \mathcal{C}, f^{-1}(E) \in \mathcal{M}(f^{-1}(\mathcal{E}))$ by the construction of $\mathcal{C}$

Then, $f^{-1}(\mathcal{M}(\mathcal{E})) \subseteq \mathcal{M}(f^{-1}(\mathcal{E}))$

In conclusion, $f^{-1}(\mathcal{M}(\mathcal{E})) = \mathcal{M}(f^{-1}(\mathcal{E}))$ $\square$

Problem 5(a).

Proof. From 4(b) we've shown that if $\mathcal{M}$ is a σ-algebra on X, $f_*\mathcal{M}$ is also a σ-algebra on Y

Show that $f_*\mu(E)$ is a measure:

$f_*\mu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$, $f_*\mu(E) = \mu(f^{-1}(E)) \geq 0, \forall E \in f_*\mathcal{M}$ since μ is a measure

Also, $f_*\mu(\cup_{i=1}^{\infty} A_i) = \mu(f^{-1}(\cup_{i=1}^{\infty} A_i)) = \mu(\cup_{i=1}^{\infty} f^{-1}(A_i)) = \sum_{i=1}^{\infty} \mu(f^{-1}(A_i)) = \sum_{i=1}^{\infty} f_*\mu(A_i)$

where A_i is a sequence of disjoint set on $f_*\mathcal{M}$

Therefore $(Y, f_*\mathcal{M}, f_*\mu)$ is a measure space $\square$

Problem 5(b).

Proof. $f_*\mu(y_0) = \mu(f^{-1}(y_0)) = \mu(X)$ since $f(x) = y_0, \forall x \in X$

$f_*\mu(y) = \mu(f^{-1}(y)) = \mu(\emptyset) = 0$ for all $y \neq y_0$ in Y

Then for an arbitrary set $E \in f_*\mathcal{M}$, $f_*\mu(E) = \mu(X)$ if $y_0 \in E$, otherwise $f_*\mu(E) = 0$, which works as an indicator function for sets in Y, that tells us whether y_0 is in the given set, this is known as Dirac Measure. $\square$

Problem 6(a).

Proof. We first verify that $\sum_{j=1}^{\infty} a_j\mu_j \geq 0$ since $\mu_j \geq 0$ by definition of measure and $\{a_j\}_{j=1}^{\infty} \subset (0, \infty)$ are all positive, also $\sum_{j=1}^{\infty} a_j\mu_j(\emptyset) = 0$ since $\mu_j(\emptyset) = 0, \forall j$

Then, let $\{A_i\}_{i=1}^{\infty} \in \mathcal{M}$ be a countable sequence of disjoint sets (without loss of generality), we can show countable additivity:

$\sum_{j=1}^{\infty} a_j\mu_j(\cup_{i=1}^{\infty} A_i) = \sum_{j=1}^{\infty} a_j[\sum_{i=1}^{\infty} \mu_j(A_i)] = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_j\mu_j(A_i) = \sum_{i=1}^{\infty}(\sum_{j=1}^{\infty} a_j\mu_j(A_i))$

(This can be done since all measure, μ_j must be nonnegative) $\square$

Problem 6(b).

Proof. $\forall E \in \mathcal{P}(X), \sum_{j=1}^{\infty} a_j\delta_{x_j}(E) \leq \sum_{j=1}^{\infty} a_j\delta_{x_j}(X) = \sum_{j=1}^{\infty} a_j$

since $E \subset X, \{x_j\}_{j=1}^{\infty}$ is in X

$\therefore$ The measure is finite if $(i) \sum_{j=1}^{\infty} a_j < \infty$ and $(ii) a_j \geq 0, \forall j$, due to the property of measure

We can write X as $X = \cup_i^\infty \{x_j\} \cup A$ where $A = X \backslash (\cup_i^\infty \{x_j\})$, then clearly none of $\{x_j\}_{j=1}^\infty$ is in $A \Rightarrow \sum_{j=1}^\infty a_j \delta_{x_j}(A) = 0 < \infty$

And for each $\{x_j\}$, $\sum_{j=1}^\infty a_j \delta_{x_j}(\{x_j\}) = a_j$, therefore the measure is σ-finite as long as $0 \leq a_j < \infty$ □

Problem 7.

Proof. Suppose μ is a measure, then consider $j < 0, E_i = \{\lceil i^{-\frac{2}{j}} \rceil\}$, then we should have:

$\mu(\cup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \mu(E_i) = \sum_{i=1}^\infty (\lceil i^{-\frac{2}{j}} \rceil)^j \leq \sum_{i=1}^\infty (i^{-\frac{2}{j}})^j = \sum_{i=1}^\infty i^{-2} < \infty$

since $j < 0, i^{-\frac{2}{j}} \leq \lceil i^{-\frac{2}{j}} \rceil$

However $\mu(\cup_{i=1}^\infty E_i) = \infty$ by the definition of $\mu(E)$ since $\# \cup_{i=1}^\infty E_i = \infty$

This leads to a contradiction, therefore $j \geq 0$

Next we'll show that if $j \geq 0 \Rightarrow \mu(E)$ is a measure:

First we can easily show that $\mu(\emptyset) = \sum_{n \in \emptyset} n^j = 0$ and $\mu(E) = \sum_{n \in E} n^j \geq 0$ since $n > 0 (E \in \mathcal{P}(\mathbb{Z}_+))$

Let $\{E_i\}_i^\infty$ be a family of disjoint sets in $\mathbb{Z}_+$:

(Without loss of generality, we assume $E \neq \emptyset, \#E \geq 1$)

Case 1: $\exists i$ s.t. $\#E_i = \infty$, then $\# \cup_i^\infty E_i = \infty, \mu(\cup_i^\infty E_i) = \infty$

Case 2: $\forall i$ s.t. $\#E_i < \infty$, $\cup_i^\infty E_i \in \mathcal{P}(\mathbb{Z}_+)$, and $\# \cup_i^\infty E_i = \infty$, therefore $\mu(\cup_i^\infty E_i) = \infty$

Hence, to show $\mu(\cup_i^\infty E_i) = \sum_i^\infty \mu(E_i)$, it suffices to show $\sum_i^\infty \mu(E_i) = \infty$

Where $\sum_i^\infty \mu(E_i) = \sum_i^\infty \sum_{n \in E_i} n^j \geq \sum_i^\infty \sum_{n \in E_i} n^0 = \sum_i^\infty (\#E_i) = \infty$

Therefore, $\mu(E)$ is a measure on $(\mathbb{Z}_+, \mathcal{P}(\mathbb{Z}_+))$ if and only if $j \geq 0$ □

Problem 8.

Proof. Since $E \subset \mathbb{R}$ is countable, we can let $E = \cup_{i=1}^\infty \{e_i\}$, where $\{e_i\}_{i=1}^\infty$ is a countable set of points on $\mathbb{R}$

Hence $\forall \epsilon > 0, \forall e_i$, we can find a_i, b_i with $a_i < b_i$ such that the open interval $(a_i, b_i) \subset \mathbb{R}$ and $e_i \in (a_i, b_i)$ where $b_i - a_i = \frac{\epsilon}{2^i}$, then $E \subset \cup_i^\infty (a_i, b_i)$

$\therefore m^*(E) \leq \sum_{i=1}^\infty m_0((a_i, b_i)) = \sum_{i=1}^\infty \frac{\epsilon}{2^i} = \epsilon$ by the definition of the outer measure

Since $m^*(E) \geq 0$, we know that $m^*(E) = 0$ □