MATH 420/507 Assignment 6

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Problem 1(a).

Proof. Since $f : \mathbb{R}^n \to \mathbb{C}$ is continuous, $\int_K |f| d\mu < \infty$ for any bounded set K, hence $f \in L^1_{loc}$. Fix x, then for any $\epsilon > 0$, there exist $\delta > 0$, such that $|y-x| < \delta \Rightarrow |f(y)-f(x)| \leq \epsilon$. Therefore $\left|\frac{1}{m(B(r,x))}\int_{B(r,x)} |f(y)-f(x)|dy\right| \leq \left|\frac{1}{m(B(r,x))}\int_{B(r,x)} \epsilon dy\right| = \left|A_r\epsilon\right| = \epsilon$ by *theorem 3.18*. Since x is arbitrary, $L_f = \mathbb{R}^n$. □

Problem 1(b).

Proof. Case 1: If $x \in E$, we have

$$\begin{aligned}
\lim_{r\to 0} \frac{1}{m(B(r,x))}\int_{B(r,x)} |f(y)-f(x)|dy &= \lim_{r\to 0} \frac{1}{m(B(r,x))}\int_{B(r,x)} |1_E(y)-1|dy \\
&= \lim_{r\to 0} \frac{1}{m(B(r,x))}\int_{B(r,x)} 1_{E^c}(y)dy \\
&= \lim_{r\to 0} \frac{1}{m(B(r,x))} m(B(r,x)\cap E^c) \\
&= \lim_{r\to 0} \frac{m(B(r,x)\cap E^c)}{m(B(r,x))} \\
&= \lim_{r\to 0} \frac{m(B(r,x)) - m(B(r,x)\cap E)}{m(B(r,x))} = 1
\end{aligned} \tag{1}$$

Therefore $x \notin L_f$.
Case 2: If $x \notin E$, $f(x) = 0$, we have

$$\begin{aligned}
\lim_{r\to 0} \frac{1}{m(B(r,x))}\int_{B(r,x)} |f(y)-f(x)|dy &= \lim_{r\to 0} \frac{1}{m(B(r,x))}\int_{B(r,x)} 1_E(y)dy \\
&= \lim_{r\to 0} \frac{1}{m(B(r,x))} m(B(r,x)\cap E) = 0
\end{aligned} \tag{2}$$

Therefore $x \in L_f$. In conclusion, we have $L_f = E^c$. □

Problem 1(c).

Proof. From part(a) we know that the continuous points of f is in L_f. Thus, it suffices to discuss the discontinuous points only ($x \in \mathbb{R}\backslash\mathbb{Z}$). Notice that for $x \in \mathbb{Z}$, we have

$$\begin{aligned}
\lim_{r\to 0} \frac{1}{m(B(r,x))}\int_{B(r,x)} |f(y)-f(x)|dy &= \lim_{r\to 0} \frac{1}{2r}\int_{x-r}^{x+r} |f(y)-f(x)|dy \\
&= \lim_{r\to 0} \frac{1}{2r}\int_{x-r}^{0} |x-1-x|dy + \int_{0}^{x+r} |x-x|dy \\
&= \lim_{r\to 0} \frac{1}{2r} r = \frac{1}{2} \neq 0
\end{aligned} \tag{3}$$

Therefore $x \notin L_f$, in conclusion we have $L_f = \mathbb{R}\backslash\mathbb{Z}$. □

Problem 2(a).

Proof. Since $F \in NBV$, we know that $F' \in L^1(m)$. Define $\tilde{F}(x) = \int_{-\infty}^{x} F'(t)dt$, then by *theorem 3.33* we have $\tilde{F} \in NBV$ and absolutely continuous, and $F' = \tilde{F}'$ a.e. Denote the set where $F' \neq \tilde{F}'$ as N, and hence we know N is a lebesgue null Borel set. By *theorem 3.35*, we have $\tilde{F}' \in L^1([a,b], m)$, $\tilde{F}(b) - \tilde{F}(a) = \int_a^b \tilde{F}'dt = \int_a^b F'dt$ if $[a,b] \cap N \neq \emptyset$. Since $F(b) - F(a) = \tilde{F}(b) - \tilde{F}(a)$, hence we can conclude that $F(b) - F(a) = \int_a^b F'dt$. $\square$

Problem 2(b).

Proof. Define an increasing and right continuous function $G : \mathbb{R} \to \mathbb{R}$ where:

$$G(x) = \begin{cases} 0 & \text{if } x < a \\ F(x+) - F(a+) & \text{if } a \leq x < b \\ F(b) & \text{if } x \geq b \end{cases}$$

Since F is increasing, by *theorem 3.23* F and G are differentiable a.e., and $F' = G'$ a.e. on $[a,b]$, and $G \in NBV$, then by *theorem 3.29* we have the corresponding measure μ_G where $G(x) = \mu_G\big((-\infty, x]\big)$ on $[a,b]$. Notice μ_G is finite since $\mu_G(X) = F(b) < \infty$ and we know m is σ-finite and positve. Then we have the Lebesgue-Radon-Nikodym representation of μ_G : $\mu_G = \lambda + fdm$ where $\lambda \perp m$ and $fdm \ll m$. By the prove of *theorem 3.8*, we know λ is a positive measure when μ_G, m are both finite postive measures, which they are on $[a,b]$. Observe that for $x \in [a,b]$

$$\lim_{h \downarrow 0} \frac{G(x+h) - G(x)}{h} = \lim_{h \downarrow 0} \frac{\mu_G((x, x+h])}{m((x, x+h])} = f(x) \tag{4}$$

and similarly,

$$\lim_{h \uparrow 0} \frac{G(x+h) - G(x)}{h} = \lim_{h \downarrow 0} \frac{G(x-h) - G(x)}{-h} = \lim_{h \downarrow 0} \frac{G(x) - G(x-h)}{h} = \lim_{h \downarrow 0} \frac{\mu_G((x-h, x])}{m((x-h, x])} = f(x) \tag{5}$$

so we have $G' = f$. Therefore, we have

$$F(b) - F(a) \geq G(b) - G(a) = \mu_G\big((a,b]\big) \geq \int_{(a,b]} fdm = \int_{(a,b]} G'dm = \int_{(a,b]} F'dm \tag{6}$$

and we can conclude that $F(b) - F(a) \geq \int_a^b F'dm$.

Another way is to define G equal to $F(x)$ if $x < b$ and equal to $F(b)$ if $x \geq b$, then $G' = F'$ a.e. on $[a,b]$. Consider $f_k(x) = \frac{G(x+h) - G(x)}{h}$ where $h = \frac{1}{k}$, then we have $f_k \to f$ a.e. and by Fatou's lemma,

$$\begin{aligned} \int_a^b G'dx &\leq \liminf_{k \to \infty} \int_a^b f_k(x)dx = \liminf_{h \to 0^+} \int_a^b \frac{G(x+h) - G(x)}{h} = \liminf_{h \to 0^+} \int_a^b \frac{G(x+h) - G(x)}{h} \\ &= \liminf_{h \to 0^+} \Big(\frac{1}{h} \int_b^{b+h} G(x)dx - \frac{1}{h} \int_a^{a+h} G(x)dx\Big) \leq G(b) - G(a) \leq F(b) - F(a) \end{aligned} \tag{7}$$

Therefore $F(b) - F(a) \geq \int_a^b F'(t)dt$. $\square$

Problem 3(a).

Proof. Since for any $\epsilon > 0$, there exist $\delta = \epsilon$, such that $\sum_{i=1}^{N} |b_j - a_j| \leq \delta$ (without loss of generality, assume $b_j \geq a_j$), we have

$$\sum_{j=1}^{N} |F(b_j) - F(a_j)| = \sum_{j=1}^{N} \big||b_j| - |a_j|\big| \leq \sum_{j=1}^{N} \big|b_j - a_j\big| = \sum_{j=1}^{N} b_j - a_j \leq \delta = \epsilon$$

Therefore we can conclude that F is absolutely continuous on $[-1,1]$. $\square$

Problem 3(b).

Proof. If $0 \leq x \leq 1$, left hand side: $F(x) - F(-1) = x - 1$, right hand side: $\int_{-1}^{0} 1dt + \int_{0}^{1} 1dt = -1 + x$ (Notice that although F' is not define at $x = 0$, the measure at that single point is 0). If $-1 \leq x < 0$, left hand side: $F(x) - F(-1) = -x - 1$, right hand side: $\int_{-1}^{x} -1dt = -1 - x$. In conclusion, we have $F(x) - F(-1) = \int_{-1}^{x} F'(t)dt$ on $[-1, 1]$. □

Problem 4(a).

Proof. When $x \neq 0, F_1' = 2xsin(\frac{1}{x}) - cos(\frac{1}{x}), F_2' = 2xsin(\frac{1}{x^2}) - \frac{2cos(\frac{1}{x^2})}{x}, F_3' = 2xsin(\frac{1}{x^{4/3}}) - \frac{4}{3}\frac{cos(x^{-4/3})}{x^{1/3}}$ Thus, it suffices to show F_j is differentiable at 0. Observe that

$$\lim_{h\to 0} \frac{F_1(h) - F_1(0)}{h} = \lim_{h\to 0} \frac{h^2 sin(\frac{1}{h})}{h} = 0 \tag{8}$$

by squeeze theorem since $-h \leq h^2 \sin(\frac{1}{h}) \leq h$. Similarly we also have

$$\lim_{h\to 0} \frac{F_2(h) - F_2(0)}{h} = \lim_{h\to 0} \frac{h^2 sin(\frac{1}{h^2})}{h} = 0 \tag{9}$$

$$\lim_{h\to 0} \frac{F_3(h) - F_3(0)}{h} = \lim_{h\to 0} \frac{h^2 sin(\frac{1}{h^{\frac{4}{3}}})}{h} = 0 \tag{10}$$

Therefore we can conclude that each F_j is everywhere differentiable. □

Problem 4(b).

Proof. From (a) we have shown that F_1 is differntiable everywhere, hence F_1' is defined everywhere on $(-1, 1)$, then it suffices to show $F_1' \in L^1(m)$. Notice that Let

$$F_1'(x) = \begin{cases} 2xsin(\frac{1}{x}) - cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Therefore we have $\int_{(-1,1)} |F_1'(x)|dm < \int_{(-1,1)} |3|dm < \infty \Rightarrow F_1' \in L^1((-1, 1), m) \Rightarrow F_1$ is absolutely continuous on $(-1, 1)$ by *theorem 3.35*, and also $F_1 \in BV((-1, 1))$. □

Problem 4(c).

Proof. $F_2(x_n) = \frac{(-1)^n}{\frac{\pi}{2}+n\pi}$ where $x_n = \frac{1}{\sqrt{\frac{\pi}{2}+n\pi}}$, hence we have

$$\begin{aligned} T_F(1) - T_F(0) &= \sup\{\sum_{j=1}^{n} \left|F(x_j) - F(x_{j-1})\right| : n \in \mathbb{N}, 0 = x_0 < ... < x_n = 1\} \\ &\geq \sum_{j=1}^{n} \left|\frac{(-1)^j}{\frac{\pi}{2} + j\pi} - \frac{(-1)^{j-1}}{\frac{\pi}{2} + (j-1)\pi}\right| = \sum_{j=1}^{n} \left|\frac{1}{\frac{\pi}{2} + j\pi} + \frac{1}{\frac{\pi}{2} + (j-1)\pi}\right| \\ &\geq \sum_{j=1}^{n} \left|\frac{1}{\frac{\pi}{2} + j\pi}\right| \geq \frac{1}{2\pi} \sum_{j=1}^{n} \frac{1}{j} \to \infty \end{aligned} \tag{11}$$

Therefore we have $F_2 \notin BV((-1, 1))$. □

Problem 4(d).

Proof. $F_3' = 2xsin(\frac{1}{x^{4/3}}) - \frac{4}{3}\frac{cos(x^{-4/3})}{x^{1/3}}$, notice that $2xsin(\frac{1}{x^{4/3}}), \frac{4}{3}cos(x^{-4/3})$ are all bounded, therefore it suffices to show $x^{-1/3} \in L^1$: $\int_{-1}^{1} |x|^{-\frac{1}{3}}dx = 2\int_{0}^{1} r^{-\frac{1}{3}}dr = 2 \times \frac{3}{2}r^{2/3}|_0^1 = 3 < \infty$. Thus, we have shown that $F_3' \in L^1\big((-1, 1), m\big)$, and we already know F_3 is everywhere differentiable. Observe that $F(x) - F(-1) = \int_{-1}^{x} F'dt$ on $(-1, 1)$, by *theorem 3.35*, we can conclude that F_3 is absolutely continuous. □