MATH 420/507 Assignment 2

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Problem 1(a).

Proof. First we'll show monotonicity of J^*: Let $A, B \subset \mathbb{R}$ be two arbitrary sets where $A \subset B$, then for any open cover $\bigcup_{i=1}^{n}(a_i, b_i) \supset B$, we also have $\bigcup_{i=1}^{n}(a_i, b_i) \supset A$. Therefore by definition,

$$\{\sum_{j=1}^{N}(b_j - a_j)|A \subset \bigcup_{j=1}^{N}(a_j, b_j), a_j < b_j\} \subset \{\sum_{j=1}^{N}(b_j - a_j)|B \subset \bigcup_{j=1}^{N}(a_j, b_j), a_j < b_j\}$$

Hence $J^*(A) \leq J^*(B)$ by the definition of infimum. Next we'll show J^* is finitely subadditive: Let $\{A_i\}_{i=1}^{N}$ be finite sequences of sets in $\mathbb{R}$, where by definition:

$$J^*(A_i) := inf\{\sum_{j=1}^{N_i}(b_j^i - a_j^i)|A_i \subset \bigcup_{j=1}^{N_i}(a_j^i, b_j^i), n \in \mathbb{N}, a_j^i \leq b_j^i\}$$

$$J^*(\bigcup_{i=1}^{N} A_i) := inf\{\sum_{j=1}^{N}(b_j - a_j)|(\bigcup_{i=1}^{N} A_i) \subset \bigcup_{j=1}^{N}(a_j, b_j), n \in \mathbb{N}, a_j \leq b_j\}$$

By the property of infimum, we have:

$$J^*(A_i) + \frac{\epsilon}{N} > m(\bigcup_{j=1}^{N_i}(a_j^i, b_j^i)) = \sum_{j=1}^{N_i}(a_j^i, b_j^i)$$

and

$$\bigcup_{i=1}^{N} A_i \subset \bigcup_{i=1}^{N}(\bigcup_{j=1}^{N_i}(a_j^i, b_j^i))$$

since each $A_i \subset \bigcup_{j=1}^{N_i}(a_j^i, b_j^i)$

$$\begin{aligned} J^*(\bigcup_{i=1}^{N} A_i) &\leq J^*(\bigcup_{i=1}^{N}(\bigcup_{j=1}^{N_i}(a_j^i, b_j^i))) \leq m(\bigcup_{i=1}^{N}(\bigcup_{j=1}^{N_i}(a_j^i, b_j^i))) \leq \sum_{i=1}^{N} m(\bigcup_{j=1}^{N_i}(a_j^i, b_j^i)) \\ &\leq \sum_{i=1}^{N}(J^*(A_i) + \frac{\epsilon}{N}) = \sum_{i=1}^{N} J^*(A_i) + \epsilon \Rightarrow J^*(\bigcup_{i=1}^{N} A_i) \leq \sum_{i=1}^{N} J^*(A_i) \end{aligned} \tag{1}$$

Next we'll show J^* is not countably subadditive, by using the example from part(b): $J^*(\mathbb{Q} \cap [0, 1]) = 1$ (we'll prove this in part(b)). Let $E = \mathbb{Q} \cap [0, 1] = \bigcup_{i}^{\infty} E_j$ since there are countably many rational numbers in $[0,1]$, for each $E_j = \{e_j\}$, we can find an $a_j < b_j$ s.t. $e_j \in (a_j, b_j), b_j - a_j = \frac{\epsilon}{2^j}$. Then $J^*(E_j) \leq \frac{\epsilon}{2^j}$ by the definition of Jordan outer measure

$$\therefore \sum_{i=1}^{\infty} J^*(E_j) \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon \Rightarrow \sum_{i=1}^{\infty} J^*(E_j) = 0 \neq J^*(\mathbb{Q} \cap [0,1]) = 1$$

Therefore J^* is finitely subadditive but not countably subadditive $\square$

Problem 1(b).

Proof. Denote $E = \mathbb{Q} \cap [0,1]$, we first show $J^*(E) \leq 1$: $\forall \epsilon > 0$, let $a_1 = -\epsilon, b_1 = 1 + \epsilon$, then

$$E \subset \bigcup_{j=1}^{N=1} [a_j, b_j], \sum_{j=1}^{N=1} (b_j - a_j) = 1 + 2\epsilon$$

$\therefore J^*(E) \leq 1 + 2\epsilon, \forall \epsilon > 0 \Rightarrow J^*(E) \leq 1$. We next show $J^*(E) \geq 1$:

$$J^*(E) = inf\{\sum_{j=1}^{N} (b_j - a_j) | E \subset \bigcup_{j=1}^{N} (a_j, b_j), n \in \mathbb{N}, a_j \leq b_j\}$$

$$\therefore E \subset \bigcup_{j=1}^{N} (a_j, b_j) \Rightarrow \overline{E} \subset \overline{\bigcup_{j=1}^{N} (a_j, b_j)} \Rightarrow [0,1] \subset \bigcup_{j=1}^{N} [a_j, b_j]$$

By rearranging $\bigcup_{j=1}^{N} [a_j, b_j]$, we could make it $\dot{\cup}_{j=1}^{N'} [a'_j, b'_j]$, where $\sum_{j=1}^{N} (a_j, b_j) \geq \sum_{j=1}^{N'} [a'_j, b'_j]$. Since all closed intervals here are disjoint, $[0,1] \subset [a'_j, b'_j], \exists a'_j, b'_j$. So $\sum_{j=1}^{N} (a_j, b_j) \geq \sum_{j=1}^{N'} [a'_j, b'_j] \geq 1$, $\therefore J^*(E) \geq 1$. Hence $J^*(\mathbb{Q} \cap [0,1]) = 1$ $\square$

Problem 1(c).

Proof. From (b) we know the Jordan outer measure: $J^*(\mathbb{Q} \cap [0,1]) = 1$
Suppose there is a countable union of intervals $\bigcup_{j=1}^{N} (a_j, b_j) \subset (\mathbb{Q} \cap [0,1])$
Since $\mathbb{Q}^c$ is dense in $[0,1]$, $\forall j, \exists q_j \in \mathbb{Q}^c$ such that $q_j \in (a_j, b_j)$, then $(a_j, b_j) \not\subset \mathbb{Q} \cap [0,1]$
Therefore no such $\bigcup_{j=1}^{N} (a_j, b_j)$ exists, which means $J_*(\mathbb{Q} \cap [0,1]) = \sup\{\emptyset\} = 0$
$J^*(\mathbb{Q} \cap [0,1]) \neq J_*(\mathbb{Q} \cap [0,1]), \mathbb{Q} \cap [0,1]$ is not Jordan measurable $\square$

Problem 2(a).

Proof. $\mathbb{Q}$ is dense in $[0,1] \Rightarrow \forall \epsilon > 0, \forall q_j \in \mathbb{Q} \cap [0,1]$, we can find a open cover U_j such that: $\exists a_j \in [0,1], q_j \in U_j = (a_j, a_j + \frac{\epsilon}{2^j}) \Rightarrow m_0(U_j) = \frac{\epsilon}{2^j}$. Then:

$$\mathbb{Q} \cap [0,1] \subset \bigcup_i^\infty U_j, m^*(\mathbb{Q} \cap [0,1]) \leq m^*(\bigcup_i^\infty U_j) = m_0(\bigcup_i^\infty U_j) = \sum_{j=1}^\infty \frac{\epsilon}{2^j} = \epsilon$$

since countable union of open sets is still open and lebesgue measurable

□

Problem 2(b).

Proof. Consider a modified cantor set C': Instead of taking out the middle $\frac{1}{3}$ part of the interval each time, $\forall \epsilon > 0$, we take out segment of length $\frac{\epsilon}{3^{2n-1}}$ from each interval each time (i.e. we take the middle $\frac{\epsilon}{3}$ off at the first round, then two segment of length $\frac{\epsilon}{3^3}$ off at the second round, and four segment of length $\frac{\epsilon}{3^5}$ off at the third round ... we take 2^{n-1} segment of length $\frac{\epsilon}{3^{2n-5}}$ at the nth round)
Then $C' = \bigcap\limits_{n=1}^{\infty} C_n$, $m(C') = m(\bigcap\limits_{n}^{\infty} C_n) = 1 - \sum\limits_{i=1}^{\infty} \frac{2^{n-1}}{3^{2n-1}}\epsilon = 1 - \frac{\frac{1}{3}}{1-\frac{2}{9}}\epsilon = 1 - \frac{3}{7}\epsilon < 1$
$\therefore C'$ is nowhere dense subset of $[0,1]$ (by 5(c)), measure arbitrary close to 1 but not equal □

Problem 2(c).

Proof. Suppose there is a nowhere dense set $E \subset [0,1]$, then there must be an interval $(a,b) \subseteq E^c$ with $a < b$. Otherwise $\forall (a,b) \subset [0,1], (a,b) \subset E \Rightarrow E$ is dense in $[0,1]$, which leads to a contradiction. Hence $m(E) = m([0,1]) - m((a,b)) < 1$ □

Problem 2(d).

Proof. In part(b) we have a nowhere dense set, therefore

$$\nexists (a_j, b_j) \subset E \Rightarrow m_*(E) = \sup\{\emptyset\} = 0$$

□

Problem 2(e).

Proof. Since $K \subset E \Rightarrow m(E) \geq m(K) \Rightarrow m(E) \geq \sup\{m(K)|$ compact $K \subset E\}$
It suffices to show $m(E) \leq \sup\{m(K)|$ compact $K \subset E\}$:
If $m(E) < \infty$, then let $\epsilon_0 > 0, \exists$ closed set $F \subset E$ with $m(E \backslash F) \leq \epsilon_0$
$\therefore m(E\backslash F) = m(E) - m(F) \leq \epsilon_0 \Rightarrow m(E) \leq m(F) + \epsilon_0$
Let $K_n = F \cap [-n, n]$ hence K_n is compact since F is closed and $[-n, n]$ is compact
By continuity from below, $\lim_{n\to\infty} m(K_n) = m(F \cap \mathbb{R}) = m(F)$
By definition of limit, $\exists N \in \mathbb{N}, \forall n \geq N, m(F) - m(K_n) \leq \epsilon_0$
$m(E) \leq m(F) + \epsilon_0 \leq m(K_N) + 2\epsilon_0$ where K_N is a compact set $K_N \subset F \subset E$
Therefore $m(E) \leq \sup\{m(K)|$ compact $K \subset E\}$ when $m(E) < \infty$

If $m(E) = \infty$, to show $m(E) \leq \sup\{m(K)|$ compact $K \subset E\}$, it suffices to show:

$\forall M > 0, \exists K \subset E, m(K) \geq M$
Fixed $M + c > 0$, let $E_n = E \cap [-n, n] \subset E \Rightarrow \lim_{n\to\infty} m(E_n) = m(E) = \infty$ by continuity from below. Then $\exists N_E \in \mathbb{N}, \forall n \geq N_E, m(E_n) \geq M + c$
$\forall \epsilon_1, \exists$ closed set $F \subseteq E_{N_E} = E \cap [-N_E, N_E]$ s.t. $m(E_{N_E} \backslash F) \leq \epsilon_1$
$m(E_{N_E}) < \infty \Rightarrow m(F) < \infty$, so we have $m(F) \geq m(E_{N_E}) - \epsilon_1$
Let $K_n = F \cap [-n, n] \subset F$ hence K_n is compact since F is closed and $[-n, n]$ is compact
Then $\lim_{n\to\infty} m(K_n) = m(F) < \infty$ by continuity from below
$\forall \epsilon_2, \exists N_F \in \mathbb{N}, \forall n \geq N_F, m(F) - m(K_n) \geq M$, $m(K_{N_F}) \geq m(F) - \epsilon_2$
Therefore let $c = 3, \epsilon_1 = \epsilon_2 = 1$, for $n \geq \max\{N_E, N_F\}$, there exist a compact set K_n:
$m(K_n) \geq m(K_{N_F}) \geq m(F) - \epsilon_2 \geq m(E_{N_E}) - \epsilon_1 - \epsilon_2 \geq M + c - \epsilon_1 - \epsilon_2 > M$
$\therefore m(E) \leq \sup\{m(K) |$ compact $K \subset E\}$
In conclusion, $m(E) = \sup\{m(K) |$ compact $K \subset E\}$ □

Problem 3(a).

Proof. First show F is increasing, without loss of generality, assume $x > y$
Then there could be three cases:
$(i) x > y \geq 0$
$F(x) - F(y) = \mu((0, x]) - \mu((0, y]) \geq 0$ since $(0, y] \subset (0, x]$
$(ii) x \geq 0 > y$
$F(x) - F(y) = \mu((0, x]) + \mu((y, 0]) \geq 0$ since $\mu(E) \geq 0, \forall E \subset R$
$(iii) 0 \geq x > y$
$F(x) - F(y) = -\mu((x, 0]) + \mu((y, 0]) \geq 0$ since $(x, 0] \subset (y, 0], \mu((x, 0]) \leq \mu((y, 0])$
Next show F is right-continuous, i.e. $F(x+) = F(x)$:
Since μ is finite measure and by continuity from above
For $x \geq 0$, $\lim_{y\downarrow x} F(y) = \lim_{y\downarrow x} \mu((0, y])) = \mu((0, x])) = F(x)$
For $x < 0$, $\lim_{y\downarrow x} F(y) = \lim_{y\downarrow x} -\mu((y, 0])) = -\mu((x, 0])) = F(x)$ □

Problem 3(b).

Proof. (i) m_F defined on $\mathbb{R} \Rightarrow m_F(E) \leq m_F(R) < \infty$
(ii) If $m_F(\{x_0\}) = 0$, we must have $\lim_{\delta\to 0}(m_F(x_0 - \delta, x_0 + \delta)) = \lim_{\delta\to 0}(F(x_0 + \delta) - F(x_0 - \delta)) = 0$, which is the definition of continuity for F. If F is continous at x_0, we know that $\forall \epsilon > 0, \exists \delta > 0, F(x_0 + \delta) - F(x_0 - \delta) \leq \epsilon \Rightarrow m_F(\{x_0\}) \leq \epsilon$
(iii) $m_F((a, b]) = F(b) - F(a) = \lfloor b \rfloor - \lfloor a \rfloor$ is an counting measure that counts the number of integers in $(a, b]$ □

Problem 4(a).

Proof. For $d \geq 0, l_j \geq 0$, we have $\sum_{j=1}^{\infty} l_j^d \geq 0$, hence $H_d(E) := \sup_{\delta>0} H_{d,\delta}(E) \geq 0$. To show $H_d(\emptyset) := \sup_{\delta>0} H_{d,\delta}(\emptyset) = 0$, it is equivalent to show: $\forall \delta > 0, H_{d,\delta}(\emptyset) = 0$. Let $a = l = 0 < \delta$ then $\emptyset \subset (0, 0) = \emptyset \Rightarrow H_{d,\delta}(\emptyset) \leq 0 \Rightarrow H_{d,\delta}(\emptyset) = 0$

Next we'll show monotonicity: Let E, F be two sets in $\mathbb{R}$ such that $E \subset F$, then any countable union of open sets that cover F must cover E:

$$\{\sum_{j=1}^{\infty} l_j^d | F \subset \bigcup_{j=1}^{\infty}(a_j, a_j + l_j), 0 \leq l_j \leq \delta)\} \subset \{\sum_{j=1}^{\infty} l_j^d | E \subset \bigcup_{j=1}^{\infty}(a_j, a_j + l_j), 0 \leq l_j \leq \delta)\}$$

By property of infimum, $H_{d,\delta}(E) \leq H_{d,\delta}(F)$. Then $\sup\limits_{\delta>0} H_{d,\delta}(E) \leq \sup\limits_{\delta>0} H_{d,\delta}(F)$

Next we'll show subadditivity:

$H_d(A) \leq \sum\limits_{j=1}^{\infty} H_d(A_j) = \sup\limits_{\delta>0} H_{d,\delta}(A) \leq \sum\limits_{j=1}^{\infty} \sup\limits_{\delta>0} H_{d,\delta}(A_j)$ where $A \subset \bigcup\limits_{j=1}^{\infty} A_j$, $\{A_j\}$ is a sequence of disjoint sets in $\mathbb{R}$

By property of infimum, there exist an open cover of intervals $\{I_j^k\}_k$ that cover A_j, then $A \subset \bigcup\limits_{j=1}^{\infty} \bigcup\limits_{k=1}^{\infty} A_j$, and $H_{d,\delta}(A_j) + \frac{\epsilon}{2^j} \geq \sum\limits_{k=1}^{\infty} l_k^d$. Then:

$$\forall \epsilon > 0, \sum_{j=1}^{\infty} H_{d,\delta}(A_j) \geq \sum_{j=1}^{\infty}(\sum_{k=1}^{\infty} l_k^d - \frac{\epsilon}{2^j}) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} l_k^d - \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} l_k^d - \epsilon \geq H_{d,\delta}(A) - \epsilon$$

Therefore $H_{d,\delta}(A) \leq \sum\limits_{j=1}^{\infty} H_{d,\delta}(A_j)$, hence we've proven that H_d is an outer measure □

Problem 4(b).

Proof. $H_0(E) := \sup\limits_{\delta>0} H_{0,\delta}(E) = \lim\limits_{\delta\to 0} \inf\{\sum\limits_{j=1}^{\infty} 1 | E \subset \bigcup\limits_{j=1}^{\infty} (a_j, a_j + l_j), 0 \leq l_j \leq \delta)\}$

$H_0(E) \geq 0$ since $\sum\limits_{j=1}^{\infty} 1 \geq 0$. Let $a = l = 0$ then $\emptyset \subset (0, 0) = \emptyset \Rightarrow H_0(\emptyset) \leq 0 \Rightarrow H_0(\emptyset) = 0$

Show $H_0(E) = \#E$ when E is finite, we'll start by showing $H_0(E) \leq \#E$:

$$H_0(E) \leq \#E \Leftrightarrow \sup_{\delta>0} H_{0,\delta}(E) \leq \#E \Leftrightarrow \forall\delta > 0, \inf\{\sum_{j=1}^{\infty} 1 | E \subset \bigcup_{j=1}^{\infty}(a_j, a_j+l_j), 0 \leq l_j \leq \delta)\} \leq \#E$$

It suffices to show that: $\forall\delta > 0$, there is an open cover so that the summation is $\leq \#E$:

For a finite E we have finite elements, say $\{e_i\}_{i=1}^N$, for each e_i we can find a open cover such that $e_i \in (a_j, a_j + l)$ where $l = \min\big(\delta, \min\{|a - b| : a \neq b, a, b \in E\}\big)$ so that no two elements are in the same open interval, therefore we need exactly N intervals, by the property of infimum we have

$$\sup_{\delta>0} H_{0,\delta}(E) = \inf\{\sum_{j=1}^{\infty} 1 | E \subset \bigcup_{j=1}^{\infty}(a_j, a_j + l_j), 0 \leq l_j \leq \delta)\} \leq N = \#E$$

Then we'll show: $H_0(E) \geq \#E \Leftrightarrow \exists\delta > 0$, so that for all open cover that covers E the summation is larger than $\#E$. Pick $\delta = \min\{|a - b| : a \neq b, a, b \in E\}$, since set E is

finite and now no two elements are in the same open interval, then we need at least N open intervals to cover the set E, hence we know $H_0(E) \geq \#E$. Therefore, we conclude that $H_0(E) = \#E$ when E is finite.
We'll next show $H_0(E) = \infty$ if E is infinite:
$\forall M > 0, M \in \mathbb{N}, \exists P \subset E, H_0(P) \geq M$(otherwise E is not infinite) and from part(a) and the conclusion above for finite case, we know $H_0(E) \geq H_0(P) = M$, therefore $H_0(E) = \infty$.
Therefore $H_0(E)$ is a counting measure in $\mathbb{R}$ □

Problem 4(c).

Proof. Let $d_0 = \frac{\log 2}{\log 3}$ then by assumption we know $1 \geq d > d_0$. Want to show: $\sup_{\delta>0} H_{d,\delta}(C) = 0 \Leftrightarrow \forall \delta > 0, H_{d,\delta}(C) = 0 \Leftrightarrow \forall \epsilon > 0, \exists$ cover of C with $0 \leq l_j \leq \delta$ s.t. $\sum\limits_{j=1}^{\infty} l_j^d \leq \epsilon$

Since by the construction of cantor set, C_N consists of 2^N closed intervals $\{I_j\}_j^{2^N}$, each of which has length $\frac{1}{3^N}$. So if we consider $\{\hat{I}_j\}_j^{2^N}$, where each $\hat{I}_j$ is the open interval with the same center but twice the length of I_j (which is length $\frac{2}{3^N}$ now), then $\{\hat{I}_j\}_j^{2^N}$ forms an open cover of C_N (which is also an open cover of C). For $\epsilon > 0$, $\forall \delta > 0$, we can pick $N_{\delta,\epsilon}$ satisfying both $\frac{2}{3^N} < \delta$ and $3^{-N(d-d_0)} < \epsilon$, then by picking the open cover $\{\hat{I}_j\}_j^{2^N}$ as above where $l_j = \frac{2}{3^N}$ now, then we have:

$$\begin{aligned} H_{d,\delta}(C_N) &= \sum_{j=1}^{\infty} l_j^d = 2^N \times (\frac{2}{3^N})^d = 2^N \times 2^d \times 3^{-Nd} = 2^N \times 2^d \times 3^{-N(d-d_0)} \times 3^{-Nd_0} \\ &= 2^d \times 3^{-N(d-d_0)} \leq 2 \times 3^{-N(d-d_0)} < \epsilon \end{aligned} \tag{2}$$

Since $\{\hat{I}_j\}_j^{2^N}$ is also an open cover of C, then we have $H_d(C) \leq H_d(C_N) \leq \epsilon$, therefore $H_d(C) = 0$ if $d > \frac{\log 2}{\log 3}$ □

Problem 5(a).

Proof. $m(C_{\vec{\alpha}}) = m(K_0) \times (1-\alpha_1) \times (1-\alpha_2) \times ... = m(K_0) \times \prod\limits_{j=1}^{\infty}(1-\alpha_j)$, where $m(K_0) = m([0,1]) = 1$, therefore $m(C_{\vec{\alpha}}) = \prod\limits_{j=1}^{\infty}(1-\alpha_j)$ □

Problem 5(b).

Proof. Since $\mathbb{Q}$ is dense in $\mathbb{R} \cap [0,1)$, then for any $\beta \in [0,1)$, there exist a strictly decreasing sequence $\{q_n\} \in \mathbb{Q} \cap [0,1)$ s.t. the sequence converges to β. Since from part (a) of the question we already know that $m(C_{\vec{\alpha}}) = \prod\limits_{j=1}^{\infty}(1-\alpha_j)$, we can pick α_j so that $\prod\limits_{j=1}^{n}(1-\alpha_j) = q_n$. This can be done since $q_n \in [0,1)$ and $\frac{q_{n+1}}{q_n} = (1-\alpha_{n+1})$ where $\alpha_{n+1} = (1-\frac{q_{n+1}}{q_n}) \in [0,1)$ □

Problem 5(c).

Proof. The clousre of the generalized Cantor set $C_{\vec{\alpha}}$ is the same Cantor set since it is closed. Suppose $C_{\vec{\alpha}}$ contains an open interval of length δ inside, then by the construction of cantor set, we know that at after the first removal, the largest open interval cannot be larger than $\frac{1}{2}$, and after the Nth removal the largest open interval cannot be larger than $\frac{1}{2^N}$. Therefore we can pick N s.t. $\frac{1}{2^N} < \delta$, where such open interval cannot exist. Since δ is arbitrary, we've shown that it contains no open set and therefore is nowhere dense. □

Problem 6.

Proof. Without loss of generality, assume $f : \mathbb{R} \to \mathbb{R}$ is increasing (f is not necessary continuous). It suffices to show: $f^{-1}((-\infty, a]) \in \mathcal{B}_{\mathbb{R}}, \forall a \in \mathbb{R}$, and there could be two cases:
Case 1: $a \in \text{Range}(f)$, then $f^{-1}((-\infty, a]) = (-\infty, b]$ where $b = \sup\{x : f(x) \leq a\}$
Case 2: $a \notin \text{Range}(f)$, then if $f(x) < a, \forall x$ or $f(x) > a, \forall x$, we would have $f^{-1}((-\infty, a]) = \mathbb{R}$ or $\emptyset$. Otherwise $f^{-1}((-\infty, a]) = (-\infty, b)$ where $b = \sup\{x : f(x) \leq a\}$ but now a is not attainable. In conclusion, we know $f^{-1}((-\infty, a])$ are in the form of $\emptyset$, $\mathbb{R}$, $(-\infty, b)$, $(-\infty, b]$, which are all in $\mathcal{B}_{\mathbb{R}}$, hence we've proven f is Borel measurable □

Problem 7.

Proof. By *proposition 2.1*, $f : (X, \mathcal{M}) \to (\bar{\mathbb{R}}, \mathcal{B}_{\bar{\mathbb{R}}})$ is measurable is equivalent to show:

$$(i) f^{-1}((r, \infty]) \in \mathcal{M}, \forall (r, \infty] \in \mathcal{E}$$

where $\mathcal{E}$ is the family of sets $\{(r, \infty] | r \in \mathbb{Q}\}$ and also to show $(ii) \mathcal{B}_{\bar{\mathbb{R}}}$ is generated by $\mathcal{E} \subset \mathcal{B}_{\bar{\mathbb{R}}}$ (the σ-algebra generated by the open sets in the extended real line $\bar{\mathbb{R}}$). Since we already know (i), therefore it suffices to show that $\sigma(\mathcal{E}) = \mathcal{B}_{\bar{\mathbb{R}}}$: Since $\mathcal{E} \subset \mathcal{B}_{\bar{\mathbb{R}}}$, we can easily get $\sigma(\mathcal{E}) \subset \mathcal{B}_{\bar{\mathbb{R}}}$. To show $\mathcal{B}_{\bar{\mathbb{R}}} \subset \sigma(\mathcal{E})$, we know $\mathcal{B}_{\bar{\mathbb{R}}} = \sigma(O(\bar{\mathbb{R}}))$ where $O(\bar{\mathbb{R}})$ is the collection of all open sets in $\bar{\mathbb{R}}$, then it suffices to show $O(\bar{\mathbb{R}}) \subset \sigma(\mathcal{E})$, since in assignment 1 question 2 we've proved that all open sets A in $\bar{\mathbb{R}}$ can be expressed as either $\bigcup_{i=1}^{\infty} (a, b)$ or $\bigcup_{i=1}^{\infty} (a, b) \cup [-\infty, c)$ (if $-\infty \in A$) or $\bigcup_{i=1}^{\infty} (a, b) \cup (c, \infty]$ (if $\infty \in A$). Hence it suffices to consider each type of the interval, in other words $(a, b), [-\infty, c), (c, \infty]$
(i)$(a, b) = \bigcup_{i=1}^{\infty} (r_i, r'_i)$
Where $\{r_i\}, \{r'_i\}$ are rational sequences approaching to a, b decreasingly/increasingly separately, which means (a, b) is also a countable union of open intervals with rational endpoints
(ii)$[\infty, c)$ or $(c, \infty]$, without loss of generality let $(c, \infty] = \bigcup_{i=1}^{\infty} (c_i, \infty]$ (The other side $[\infty, c)$ follows from the same prove). We can find a rational sequence $\{c_i\}$ that apporaches to b from above. Then $(c, \infty] = \bigcup_{i=1}^{\infty} (c_i, \infty]$ which means $(c, \infty]$ is also a countable union of open intervals with rational endpoints. Hence we've shown that $\mathcal{B}_{\bar{\mathbb{R}}} \subset \sigma(\mathcal{E})$, therefore $\mathcal{B}_{\bar{\mathbb{R}}} = \sigma(\mathcal{E})$. By *proposition 2.1*, $f : X \to \bar{\mathbb{R}}$ is $(\mathcal{M}, \mathcal{B}_{\bar{\mathbb{R}}})$-measurable □

Problem 8(a).

Proof. Suppose N is measurable: $N \in \mathcal{L}$, then by transition invariance we have: N_q is also measurable and $m(N_q) = m(N)$. Then we have :

$$m^*([0,1)) = m^*(\cup_{q\in\mathbb{Q}\cap[0,1)} N_q) = \sum_q^{\infty} m(N_q) = 1$$

Since $m(N_q) = m(N), \forall q \in \mathbb{Q} \cap [0,1)$, the summation $\sum_q^{\infty} m(N_q)$ is either ∞(if $m(N) > 0$) or 0 (if $m(N) = 0$), which leads to a contradiction. Therefore N is not measurable: $N \notin \mathcal{L}$. $\square$

Problem 8(b).

Proof. $N \subset [0,1) \Rightarrow m^*(N) \leq m^*([0,1)) = 1$. Note that we also have: $m^*([0,1)) = m^*(\cup_{q\in\mathbb{Q}\cap[0,1)} N_q) = \sum_q^{\infty} m^*(N_q)$, which means $m^*(N_q) > 0$, so $m^*(N) = m^*(N_q) > 0$. Therefore $0 < m^*(N) \leq 1$ $\square$

Problem 8(c).

Proof. $E \subset N, N \subset [0,1)$, let $E_r = \{x+r : x \in E\cap[0,1-r)\} \cup \{x+r-1 : x \in E\cap[1-r,1)\}$. Then $\forall r \in \mathbb{R} = \mathbb{Q} \cap [0,1), E_r \subset N_r$, and E_r are all disjoint sets. By transition invariance, $m^*(E) = m^*(E_r)$ and $m^*(\bigcup_r^{\infty} E_r) = \sum_r^{\infty} m^*(E_r) \leq m^*([0,1)) = 1$, hence $m(E) = m^*(E) = 0$ $\square$

Problem 8(d).

Proof. $E = \bigcup_{q\in\mathbb{Q}\cap[0,1)} (N_q \cap E)$ where $E \in \mathcal{L}, m(E) > 0$, then $m(E) = m^*(E) \leq \sum_q m^*(N_q \cap E)$, and $N_q \cap E$ is a subset of E. Suppose all $N_q \cap E$ is measurable, then by (c), we have $m^*(N_q \cap E) = m(N_q \cap E) = 0$. However, $\sum_q m^*(N_q \cap E) \geq m(E) > 0$, which leads to a contradiction. Therefore there must exist some q such that $N_q \cap E$ is non-measurable $\square$

Problem 8(e).

Proof. By subadditivity, we know $m^*([0,1)\backslash N) \leq m^*([0,1)) = 1$. Then it suffices to show $m^*([0,1)\backslash N) \geq 1$. Let $N^c = [0,1)\backslash N$, suppose $m^*(N^c) < 1$, write $m^*(N^c) = 1 - \epsilon$ for some $\epsilon \in (0,1]$, where by the definition we have $m^*(N^c) = \inf\{\sum_i^{\infty}(b_i - a_i) | N^c \subseteq \bigcup_i^{\infty}(a_i, b_i)\} = 1 - \epsilon$, then for $\delta = \frac{\epsilon}{2} > 0, \exists$ open cover $\{I_i\}_i^{\infty}, N^c \subseteq \bigcup_i^{\infty} I_i$ such that

$$m^*(N^c) \leq \sum_i^{\infty} m_0(I_i) = \sum_i^{\infty}(b_i - a_i) \leq 1 - \epsilon + \frac{\epsilon}{2} = 1 - \frac{\epsilon}{2}$$

let $U = \bigcup_i^{\infty} I_i$ then $N^c \subseteq U$, $m^*(U) = m(U) = \sum_i^{\infty} m_0(I_i) \leq 1 - \frac{\epsilon}{2} < 1$. Then:

$$1 = m([0,1)) = m(U) + m(U^c) \leq 1 - \frac{\epsilon}{2} + m(U^c)$$

so we have $m(U^c) \geq \frac{\epsilon}{2}$. Since $N^c \subseteq U$, then we know $U^c \subseteq N \Rightarrow m^*(N) \geq m^*(U^c) = \frac{\epsilon}{2}$ by subadditivity, and we already know $m^*(N^c) = 1 - \epsilon$ by assumption, so we have:

$$1 = m([0,1)) = m^*([0,1)) = m(U) + m(U^c) = m^*(U) + m^*(U^c) \leq m^*(N^c) + m^*(N) = 1 - \frac{\epsilon}{2}$$

which leads to a contradiction. Therefore $m^*(N^c) \geq 1 \iff m^*([0,1)\backslash N) \geq 1$. In conclusion, $m^*([0,1)\backslash N) = 1$ $\square$