MATH 420/507 Assignment 4

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Problem 1(a).

Proof. Let f be a characteristic function $\mathcal{X}_{[a,b]}$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}} e^{inx}f(x)dx = \lim_{n\to\infty}\int_{\mathbb{R}} e^{inx}\mathcal{X}_{[a,b]}dx = \lim_{n\to\infty}\int_a^b e^{inx}dx = \lim_{n\to\infty}\frac{i(e^{inb}-e^{ina})}{n} = 0 \tag{1}$$

Since $f \in L^1(dx)$, for all $\epsilon > 0$, we can find a simple function $g = \sum_{i=1}^{m} a_j\mathcal{X}_{E_j}$ such that $\int |f-g| \leq \epsilon$ and an N such that for all $n \geq N$, we have $\big|\int_{\mathbb{R}} e^{inx}\mathcal{X}_{E_j}dx\big| \leq \epsilon/[m\times\max\{|a_1|,...,|a_m|\}]$ for all j. Hence, $\big|\int_{\mathbb{R}} e^{inx}g(x)dx\big| = \big|\int_{\mathbb{R}} e^{inx}\sum_{i=1}^{m} a_j\mathcal{X}_{E_j}dx\big| = \big|\sum_{i=1}^{m} a_j\int_{\mathbb{R}} e^{inx}\mathcal{X}_{E_j}dx\big| \leq \epsilon$. Therefore for all $n \geq N$, we have

$$\begin{aligned} \Big|\int_{\mathbb{R}} e^{inx}f(x)dx\Big| &\leq \Big|\int_{\mathbb{R}} e^{inx}[f(x)-g(x)]dx\Big| + \Big|\int_{\mathbb{R}} e^{inx}g(x)dx\Big| \\ &\leq \int_{\mathbb{R}} \big|f(x)-g(x)\big|dx + \epsilon \\ &\leq 2\epsilon \end{aligned} \tag{2}$$

Hence $\lim_{n\to\infty}\int_{\mathbb{R}} e^{inx}f(x)dx = 0$. ∎

Problem 2(a).

Proof. Since f_n converges to $f \in L^+$, there exists a sequence of sets $\{E_n\}_{n=1}^{\infty}$ such that $\mu(E_n^c) < \frac{1}{n} = \epsilon$ and $f_n \to f$ uniformly on E_n. Define $E := \cup_{n=1}^{\infty} E_n$, then we have $\mu(E^c) \leq \liminf_{n\to\infty}\mu(E_n^c) = 0$. Then $f_n \to f$ pointwise on E where $\mu(E^c) = 0$. Therefore, $f_n \to f$ a.e. ∎

Problem 2(b).

Proof. For every $\epsilon, \delta > 0$, $\exists$ set E and $N \in \mathbb{N}$ such that $\forall n \geq N, |f_n(x) - f(x)| < \epsilon$ for all $x \in E$ and $\mu(E^c) < \delta$, then we have $\mu(\{x\big||f_n(x)-f(x)| \geq \epsilon\}) \leq \mu(E^c) < \delta$ since $A = \{x\big||f_n(x)-f(x)| \geq \epsilon\} \subseteq E^c$. Therefore by definition $f_n \to f$ in measure. ∎

Problem 3.

Proof. We will prove by contradiction. Suppose $f_n \not\to f$ in L^1. Then $\exists \epsilon > 0$, $\exists$ a subsequence $\{n_k\}$ such that $\int |f_{n_k} - f| \geq \epsilon$ for all k. Since $f_n \to f$ in measure, $f_{n_k} \to f$ in measure and hence there exist a subsequence $\{n_k'\}$ of $\{n_k\}$ such that $f_{n_k'} \to f$ almost everywhere. Also $|f_{n_k'}| \leq g(x) \in L^1$, then by dominated convergence theorem we have $f \in L^1$ and $\lim_{n\to\infty}\int |f_{n_k'} - f| = 0$ which contradicts with the assumption. Therefore we have proven $f_n \to f$ in L^1. ∎

Problem 4.

Proof. By definition, $f(n,m) = \delta_{n,m} - \delta_{n,m+1} = 1$ if $n = m$, equals -1 if $n = m+1$ and equals 0 otherwise. Then we have:

$$\int\int f d\mu d\nu = \int\int f(n,m)d\mu(n)d\nu(m) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(n,m) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\delta_{n,m} - \delta_{n,m+1} = \sum_{m=1}^{\infty}(1-1+0) = 0 \tag{3}$$

$$\int\int f d\nu d\mu = \int\int f(n,m)d\nu(m)d\mu(n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(n,m) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \delta_{n,m} - \delta_{n,m+1} = \sum_{n=1}^{\infty} \mathcal{X}_{\{n=1\}} = 1 \quad (4)$$

Therefore $\int\int f d\mu d\nu \neq \int\int f d\nu d\mu$. The counting measure μ, ν are σ-finite on $\mathbb{N}$, but $f \notin L^+(\mu \times \nu)$ since f can take negative values, and $f \notin L^1(\mu \times \nu)$ since $\int |f| = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |f(n,m)| = \sum_{m=1}^{\infty} 2 = \infty$, so the Fubini's and Tonelli's theorem can not be applied here. ☐

Problem 5.

Proof. To show $f * g$ is well defined for a.e x, we will show $f(x-y)g(y)$ is integrable with respect to y on $\mathbb{R}$, by definition it suffices to show $\int_{\mathbb{R}} |f(x-y)g(y)|dy \leq \infty$. Denote $\phi(x) = \int_{\mathbb{R}} |f(x-y)g(y)|dy \geq 0$, then it is equivalent to show $\phi(x) < \infty$ a.e. Since any integrable function is finite a.e, it suffices to show $\phi(x)$ is integrable, which by definition is equivalent to show $\int_{\mathbb{R}} |\phi(x)|dx = \int_{\mathbb{R}} \phi(x)dx \leq \infty$. Therefore, it suffices to show:

$$\begin{aligned}
\int_{\mathbb{R}} \phi(x)dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y)g(y)|dydx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y)g(y)|dxdy \text{ by Tonelli's theorem} \\
&= \int_{\mathbb{R}} |g(y)| \int_{\mathbb{R}} |f(x-y)|dxdy \\
&= \int_{\mathbb{R}} |g(y)| \int_{\mathbb{R}} |f(z)|dzdy \text{ by substituion } z = x-y \\
&= \int_{\mathbb{R}} |g(y)|dy \int_{\mathbb{R}} |f(z)|dz \\
&< \infty \text{ since f, g are integrable functions}
\end{aligned} \tag{5}$$

Therefore $f * g$ is well defined, and we've also proved that $f * g$ is integrable since $\int |f * g| = \int_{\mathbb{R}} |\int_{\mathbb{R}} f(x-y)g(y)dy|dx \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y)g(y)|dydx < \infty$. Let $h(x,y) = f(x)$ and $l(x,y) = y$, then the product $(h*l)(x,y)$ is measurable since $h = f(x)$ and $l = y$ are both measurable and with continuous composition, hence $F(x,y) = (h*l)(x-y,y) = f(x-y)g(y)$ is measurable. By applying Fubini's theorem, we have $f*g = \int_{\mathbb{R}} F dy$ is also measurable. Therefore, $f * g$ is well defined a.e, measurable and integrable. ☐

Problem 6.

Proof. We start with some example, suppose: $\alpha = \frac{1}{2}, 1, 2$ then we can get:

Then for a fixed $\alpha \in \mathbb{R}$, the range of the map E is a union of line segments. Each line segments are result from the line $(t, \alpha t)$ intersecting with countable unit length rectangles in $\mathbb{R}^2$, so there are countable number of line segments. Hence E can be written as a countable union of line segments E_i, $E = \bigcup_i^{\infty} E_i$, where each line segment are Borel sets in $\mathbb{R}^2$ and has measure $m^2(E_i) = 0$. Therefore E is also Borel measurable, and $m^2(E) = m^2(\bigcup_i^{\infty} E_i) = 0$ by additivity of measure. (Another method is to show $(t \mod 1)$ and $(\alpha t \mod 1)$ are Borel measurable) ☐

Problem 7.

Proof. $D := \{(x,x)|x \in [0,1]\}$ is a line segment in $[0,1] \times [0,1]$, hence we can find a collection of n rectangles $\{K_{n,j}\}_{j=1}^{n}$ in $\mathbb{R}^2$ each with length $\frac{1}{n}$ that covers D, in other words, $D \subset \bigcup_{j=1}^{n} K_{n,j}$ for all n, so we have $D \subseteq \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{n} K_{n,j}$. The other direction $\bigcap_{n=1}^{\infty} \bigcup_{j=1}^{n} K_{n,j} \subseteq D$ can be shown since if there exist a $x \in \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{n} K_{n,j}$ but not in D, we can find $N \in \mathbb{N}$ such that for all $n \geq N$, the rectangles are small enough that x is not in $\bigcap_{n=1}^{\infty} \bigcup_{j=1}^{n} K_{n,j}$, which leads to a contradiction. Therefore we have $D = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{n} K_{n,j}$. Notice that each rectangle $K_{n,j}$ are in $\mathcal{B}([0,1]) \otimes \mathcal{P}([0,1])$ (We can decompose $K_{n,j} = A_{n,j} \times B_{n,j}$ where $A_{n,j} \in \mathcal{B}([0,1])$ and $B_{n,j} \in \mathcal{P}([0,1])$). As a result, the countable intersection of finite union $D \in \mathcal{B}([0,1]) \otimes \mathcal{P}([0,1])$.
Since $D = \{(x,x)|x \in [0,1]\}$ then $D_x = \{y \in [0,1]|(x,y) \in D\}$ and $D^y = \{x \in [0,1]|(x,y) \in D\}$, we have:

$$\int_X \nu(D_x) d\mu(x) = \int_X 1 d\mu(x) = 1 \tag{6}$$

$$\int_Y \mu(D^y) d\nu(x) = \int_Y 0 d\nu(y) = 0 \tag{7}$$

Also, notice that counting measure ν on $[0,1]$ is ∞, so we have:

$$(\mu \times \nu)(D) = \lim_{n\to\infty} (\mu \times \nu)(\bigcup_{j=1}^{n} K_{n,j}) = \lim_{n\to\infty} n \times (\frac{1}{n} \times \infty) = \infty \tag{8}$$

The Tonelli's theorem cannot be applied here because the counting measure ν is not σ-finite on $\mathcal{P}([0,1])$. $\square$

Problem 8(a).

Proof. $(\Rightarrow)$: E is v-null, then $\forall F \subseteq E, v(F) = 0$. Suppose $|v|(E) = v^+(E) + v^-(E) > 0$, let $X = N \cup P$ be an Hahn Decomposition, then $v(E) = v^+(E) - v^-(E) = 0 \Rightarrow v^+(E) = v^-(E)$. Hence we have $|v|(E) = 2v^+(E) = 2v^-(E) > 0$, so $v^+(E), v^-(E) > 0$. Thus $v^+(E \cap P) = v^+(E \cap N) + v^+(E \cap P) = v^+(E) > 0$, and $v^-(E \cap P) = 0$ since $E \cap P \subset P$. Therefore $v(E \cap P) = v^+(E \cap P) - v^-(E \cap P) > 0$ which is a contradiction since $E \cap P \subset E$, we should have $v(E \cap P) = 0$. Hence $|v|(E) = 0$.
$(\Leftarrow)$: For any set $F \subseteq E$, we have $|v|(F) = 0 \Rightarrow v^+(F) + v^-(F) = 0 \Rightarrow v^+(F) = v^-(F) = 0 \Rightarrow v(F) = 0$. Therefore E is v-null. $\square$

Problem 8(b).

Proof. $(v \perp \mu \Rightarrow |v| \perp \mu)$: suppose $v \perp \mu$ then we have $A, B \subset X$ where $A \cap B = \emptyset$ and $A \cup B = X$, A is v-null and B is μ-null. Then we have for all $F \subseteq A$, $|v|(F) = v^+(F) + v^-(F) = v(F \cap P) + v(F \cap N) = 0 + 0 = 0$ since $F \cap P \subseteq A \cap P, F \cap N \subseteq A \cap N$. Therefore A is also v-null, we can conclude that $|v| \perp \mu$.

$(|v| \perp \mu \Rightarrow v^+ \perp \mu, v^- \perp \mu)$: similar from above, let A be $|v|$-null and B be μ-null where $A \cap B = \emptyset$ and $A \cup B = X$. For any set $F \subseteq A$, we have $|v|(F) = v^+(F) + v^-(F) = 0$, so $v^+(F) = v^-(F) = 0$, therefore A is v^+-null and v^--null, so we can conclude that $v^+ \perp \mu, v^- \perp \mu$.

$(v^+ \perp \mu, v^- \perp \mu \Rightarrow v \perp \mu)$: with Hahn decomposition, denote A_1 be v^+-null, B_1 be μ-null where $A_1 \cap B_1 = \emptyset$ and $A_1 \cup B_1 = X$. Also denote A_2 be v^--null, B_2 be μ-null with $A_2 \cap B_2 = \emptyset$ and $A_2 \cup B_2 = X$. Then we have $A = A_1 \cap A_2$ is v-null and $B = B_1 \cup B_2$ is μ-null since $\forall F \subseteq A, F \subseteq A_1$ and $A_2, v^+(F) = v^-(F) = 0$ so $v(F) = 0$, and $\forall F' \subseteq B, F' \subseteq B_1$ or $B_2, \mu(F') = 0$. Also notice that $A \cap B = (A_1 \cap A_2 \cap B_1) \cup (A_1 \cap A_2 \cap B_2) = \emptyset \cup \emptyset = \emptyset$ and $A \cup B = (B_1 \cup B_2 \cup A_1) \cap (B_1 \cup B_2 \cup A_2) = X \cap X = X$. $\square$

Problem 8(c).

Proof. $v_1 << \mu$ and $v_2 << \mu$, then $\forall E \subset X, \mu(E) = 0 \Rightarrow v_1(E) = v_2(E) = 0$ by definition, so $(v_1+v_2)(E) = v_1(E) + v_2(E) = 0$, therefore $v_1 + v_2 << \mu$. $\square$

Problem 8(d).

Proof. $v_1 \perp \mu$ and $v_2 \perp \mu$, denote A_1, A_2 be v_1-null,v_2-null repsectively and B_1, B_2 be μ-null.
Let $A = A_1 \cap A_2$ and $B = B_1 \cup B_2$, for any $E \subset A$, $E \subset A_1, E \subset A_2$, hence $(v_1+v_2)(E) = v_1(E)+v_2(E) = 0$. For any $E' \subset B$, $E' \subset B_1$, or $E' \subset B_2$, then $\mu(E') = 0$. Also, $A \cap B = (A_1 \cap A_2 \cap B_1) \cup (A_1 \cap A_2 \cap B_2) = \emptyset \cup \emptyset = \emptyset$ and $A \cup B = (B_1 \cup B_2 \cup A_1) \cap (B_1 \cup B_2 \cup A_2) = X \cap X = X$. Therefore $v_1 + v_2 \perp \mu$. $\square$