MATH 420/507 Assignment 3

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Problem 1(a).

Proof. For all $x \in \mathbb{R}$ and $\epsilon > 0$, there exist a N so that $N > \frac{1}{x}$ then for all $n > N, \frac{1}{n} < \frac{1}{x}$. Therefore $|f_n - 0| = |n^p \mathcal{X}_{(0,\frac{1}{n})}(x)| = 0 < \epsilon$. $\square$

Problem 1(b).

Proof. $\lim_{n\to\infty} \int f_n dm = \lim_{n\to\infty} \int n^p \mathcal{X}_{(0,\frac{1}{n})}(x)dm$. Since f_n are simple functions defined on a finite set, then we have

$$\lim_{n\to\infty} \int n^p \mathcal{X}_{(0,\frac{1}{n})}(x)dm = \lim_{n\to\infty} n^p m((0,\frac{1}{n})) = \lim_{n\to\infty} n^{p-1}$$

The limit goes to 0 for $p \in [0,1)$. $\square$

Problem 1(c).

Proof. Let $g(x) = \frac{1}{x^p}$ for $x \in (0,1)$ and 0 elsewhere, then $g(x) \geq |f_n(x)|$. $g(x) \in L^1(m)$ if and only if $p \in [0,1)$ since $\int_{\mathbb{R}} |g(x)| = \int_{(0,1)} \frac{1}{x^p} dm < \infty$ only when $p < 1$. Then from part(a) we have already shown that $f_n \to 0$ pointwise, and $|f_n| \leq g(x)$, we know that when $p \in [0,1), g \in L^1(m)$ and thus by dominated convergence theorem $\lim_{n\to\infty} \int f_n dm = 0$, which is the result of (b). $\square$

Problem 2(a).

Proof. $X = \mathbb{N}, \mu =$counting measure. $\int_{\mathbb{N}} |f| d\mu < \infty \Rightarrow \sum_{n=1}^{\infty} |f(n)| < \infty$, therefore $\sum_{n=1}^{\infty} f(n)$ has to converge absolutely where $\{f(n)\}_{n=1}^{\infty}$ is a sequence of numbers. Thus, $L^1(\mu) = \{f : \sum_{n=1}^{\infty} |f(n)| < \infty\}$, and $\int_{\mathbb{N}} f d\mu = \sum_{n=1}^{\infty} f(n)$ $\square$

Problem 2(b).

Proof. $X = \mathbb{R}, \mu =$counting measure. $\int_{\mathbb{R}} |f| d\mu < \infty \Rightarrow L^1(\mu) = \{f :$ there exist a countable set $E \subseteq \mathbb{R}$ such that $\forall x \in E^c, f(x) = 0$ and $\sum_{x\in E} |f(x)| < \infty\}$, and $\int_{\mathbb{R}} f d\mu = \sum_{x\in E} f(x)$ $\square$

Problem 2(c).

Proof. $X = \mathbb{R}, \mu = \mu_F$ with $F = \lfloor x \rfloor$. $\int_{\mathbb{R}} |f| d\mu_F < \infty \Rightarrow$ By monotone convergence theorem we have $\int_{\mathbb{R}} |f| d\mu_F = \lim\limits_{N\to\infty} \sum\limits_{n=-N}^{N} \int_{[n,n+1)} |f| d\mu_F = \lim\limits_{N\to\infty} \sum\limits_{n=-N}^{N} |f(n)| = \sum\limits_{n\in\mathbb{Z}} |f(n)|$ since $F(x) = \lfloor x \rfloor$ has a jump at $x \in \mathbb{Z}$ and therefore μ_F only has measure 1 at $x \in \mathbb{Z}$ and 0 elsewhere. Thus, $L^1(\mu) = \{f : \sum\limits_{n\in\mathbb{Z}} |f(n)| < \infty\}$ and $\int_{\mathbb{R}} f d\mu_F = \sum\limits_{n\in\mathbb{Z}} f(n)$. $\square$

Problem 3(a).

Proof. $\mu^{(g)}(\emptyset) = \int \mathcal{X}_{\emptyset} g d\mu = 0$ by definition of indicator function, $\mu^{(g)}(E) = \int \mathcal{X}_E g d\mu \geq 0$ since $g \in L^+, \mathcal{X}_E$ are always non-negative. To show subadditivity, suppose $A \subset B$, then we have

$$\mu^{(g)}(A) = \int \mathcal{X}_A g d\mu = \mathcal{X}_A \int g d\mu \leq \mathcal{X}_B \int g d\mu = \int \mathcal{X}_B g d\mu = \mu^{(g)}(B)$$

therefore $\mu^{(g)}(A) \leq \mu^{(g)}(B)$. Next we'll show countable additivity, let $\{A_i\}_i^{\infty}$ be a collection of disjoint sets, then

$$\mu^{(g)}(\bigcup_{i=1}^{\infty} A_i) = \int \mathcal{X}_{\bigcup\limits_{i=1}^{\infty} Ai} d\mu = \int \lim_{n\to\infty} \sum_{i=1}^{n} \mathcal{X}_{A_i} d\mu = \lim_{n\to\infty} \int \sum_{i=1}^{n} \mathcal{X}_{A_i} d\mu = \lim_{n\to\infty} \sum_{i=1}^{n} \int \mathcal{X}_{A_i} d\mu = \sum_{i=1}^{\infty} \int \mathcal{X}_{A_i} d\mu$$

by monotone convergence theorem. Thus $\mu^{(g)}$ defines a measure on $(X, \mathcal{M})$. $\square$

Problem 3(b).

Proof. Without loss of generality, let $E = [a, b), a \leq b$. Then $m_F(E) = m_F([a, b)) = F(b) - F(a)$ by definition, and $m^{(F')}(E) = \int_E F' dm = F(b) - F(a)$ since F' is non-negative and continuous on E and by foundamental theorem of calculus (F is continuously differentiable and increasing). Therefore we have $m_F(E) = m^{(F')}(E)$. Since $B_{\mathbb{R}}$ is a σ-algebra generated by half intervals, and both measures are σ-finite on $(\mathbb{R}, B_{\mathbb{R}})$. (We can show this by rewriting $\mathbb{R} = \cup_n [n, n+1)$ where on each subset the measure is finite), hence $m_F(E) = m^{(F')}(E)$ on $(\mathbb{R}, B_{\mathbb{R}})$. $\square$

Problem 4(a).

Proof. $\{f_n\}_{n=1}^{\infty} \subset L^+ \Rightarrow \int_E \liminf f_n \leq \liminf \int_E f_n$ by Fatou's lemma, then $\int_E f \leq \liminf \int_E f_n$ since $f_n \to f$ pointwise. Similarly, we have $\int_{X \setminus E} f \leq \liminf \int_{X \setminus E} f_n$ since $\int_E f \leq \liminf \int_E f_n$ and $\lim\limits_{n\to\infty} \int f_n = \int f$. Therefore, we have:

$$\begin{aligned}
\int_E f = \int_X f - \int_{X\setminus E} f &\geq \int_X f - \liminf \int_{X\setminus E} f_n = \lim \int_X f_n - \liminf \int_{X\setminus E} f_n \\
&= \limsup \int_X f_n - \liminf \int_{X\setminus E} f_n = \limsup \int_X f_n + \limsup \int_{X\setminus E} -f_n \\
&\geq \limsup(\int_X f_n + \int_{X\setminus E} -f_n) = \limsup \int_E f_n
\end{aligned} \tag{1}$$

Thus we have $\int_E f \leq \liminf \int_E f_n$ and $\int_E f \geq \limsup \int_E f_n$. Hence we can conclude that $\int_E f = \liminf \int_E f_n = \limsup \int_E f_n = \lim\limits_{n\to\infty} \int_E f_n$ $\square$

Problem 4(b).

Proof. Let $f_n = n\mathcal{X}_{(0,\frac{1}{n})} + \mathcal{X}_{(1,\infty)}, \mathcal{M} = \mathcal{B}_{\mathbb{R}}$, then $f_n \to f = \mathcal{X}_{(1,\infty)}$ where $\int f = \infty$. Let $E = (0,1) \in \mathcal{B}_{\mathbb{R}}$, then we have $\int_{(0,1)} f = \int_{(0,1)} \mathcal{X}_{(1,\infty)} = 0$ but $\int_{(0,1)} f_n = \int_{(0,1)} n\mathcal{X}_{(0,\frac{1}{n})} = 1 \neq 0$ for all n, therefore $\int_E f = \lim\limits_{n\to\infty} \int_E f_n$ does not hold. $\square$

Problem 5.

Proof. If f is bounded, then $\exists M, |f| \leq M$. For every $\epsilon > 0$, pick $\delta = \frac{\epsilon}{M}$, then we have $\mu(E) < \frac{\epsilon}{M}$, thus $\int_E |f| d\mu < \int_E M d\mu = M\mu(E) < M\frac{\epsilon}{M} = \epsilon$. If f is unbounded, let $f_M = \min(|f|, M)$ which is a bounded non-negative function such that $f_M \uparrow |f|$ as $M \to \infty$. Since $|f_M| \leq |f|$, so $\int_E |f_M| d\mu \leq \int_E |f| d\mu \leq \infty \Rightarrow f_M \in L^1(\mu)$, then we know $f_M \in L^+$. By the monotone convergence theorem, we have $\lim\limits_{M\to\infty} \int_E f_M d\mu = \int_E \lim\limits_{M\to\infty} f_M d\mu = \int_E |f| d\mu$. Then for every $\epsilon > 0$, (i) we can pick N such that

$$\forall M \geq N, \int_E |f| d\mu - \int_E f_M d\mu = \int_E (|f| - f_M) d\mu \leq \frac{\epsilon}{2}$$

since $|f| - f_M$ are non-negative and the integral is finite. (ii) Then we can pick $\delta = \frac{\epsilon}{2M}$ such that $\int_E f_M d\mu \leq M\frac{\epsilon}{2M} \leq \frac{\epsilon}{2}$ since f_M is bounded. By $(i), (ii)$, we have

$$\int_E |f| d\mu = \int_E (|f| - f_M) d\mu + \int_E f_M d\mu \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon$$

since all functions that we're integrating are non-negative integrable functions. $\square$

Problem 6(a).

Proof. Since ϕ is bounded, there exist a $M > 0$ such that $|\phi(x)| < M$ for all $x \in \mathbb{R}$. Also $\psi \in L^1(m) \Rightarrow \int |\psi| dm < \infty$. Therefore we have $|\phi(\frac{x}{n})\psi(x)| \leq M|\psi(x)|$, and $\int M|\psi(x)|dm = M \int |\psi(x)|dm < \infty$, denote $g(x) = M|\psi(x)|, g(x) \in L^1(m)$. Denote $f_n = \phi(\frac{x}{n})\psi(x)$ then $f_n \to f = \phi(0)\psi(x)$ by the continuity of ϕ, and $|f_n| \leq g$ for all x. By the dominated convergence theorem, we know that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi(\frac{x}{n})\psi(x)dm = \int_{\mathbb{R}} \phi(0)\psi(x)dm = \phi(0) \int_{\mathbb{R}} \psi(x)dm$$

□

Problem 6(b).

Proof. Let $f_n = \phi(nx)\psi(x)$, then $|f_n| \leq M|\psi| = g$ for some $M > 0$ since ϕ is continuous and compactly supported, therefore it is bounded on the closed set on $\mathbb{R}$ and zero elsewhere. Also we have $f_n \to f = 0$ since for every x we can find an N such that for $n \geq N$, $\phi(nx) = 0$ by compactly supported. Since $\psi \in L^1(m)$, we know that $\int |\psi| dm < \infty$ and hence $\int M|\psi|dm < \infty \Rightarrow g \in L^1(m)$. By the dominated convergence theorem, we know that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi(nx)\psi(x)dm = \int_{-\infty}^{\infty} 0dm = 0$$

□

Problem 6(c).

Proof. Denote $\phi(x) = \frac{1}{x}sin(x)$ where $\phi(x)$ is continuous (by defining $\phi(0) = \lim\limits_{x\to 0} \frac{1}{x}sin(x) = 1$) and bounded by 1. Also denote $\psi(x) = e^{-|x|}$, then $\int |e^{-|x|}|dm = \int e^{-|x|}dm = 2\int_0^{\infty} e^{-|x|}dm = 2 < \infty \Rightarrow \psi(x) \in L^1(m)$. Then by the part(a), we have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{n}{x} sin(x/n)e^{-|x|}dm = \lim_{n\to\infty} \int_{-\infty}^{\infty} \phi(\frac{x}{n})\psi(x)dm = \phi(0) \int_{\mathbb{R}} \psi(x)dm = 2$$

□

Problem 6(d).

Proof. Let $f_n = \frac{1+nx^2}{(1+x^2)^n} \geq 0$ and $(1+x^2)^n = \sum_{k=0}^{n} \binom{n}{k} x^{2k} \geq 1 + nx^2$, so $f_n \leq 1$. For $x \in (0,1]$, we have $0 \leq \frac{1+nx^2}{(1+x^2)^n} \leq \frac{1+nx^2}{1+nx^2+\binom{n}{2}x^4}$ and by squeeze theorem we can conclude that $f_n = \frac{1+nx^2}{(1+x^2)^n} \to 0$. For $x = 0$, we have $f_n = 1$. Hence we know that $f_n \to 0$ almost everywhere. Let $g = 1 \geq |f_n|, g \in L^1(m)$. By the dominated convergence theorem, we know that

$$\lim_{n\to\infty} \int_{[0,1]} \frac{1+nx^2}{(1+x^2)^n} dm = \int_{[0,1]} 0dm = 0$$

□