MATH 418/544 Assignment 2

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Problem 1.

Proof. $\mu(E) \geq 0, \forall E \in \mathcal{F}_0$ by definition, and $\mu(\emptyset) = 0$ since $\nexists [x, \infty) \subset \emptyset, \forall x > 0$. Also, since $[1, \infty) \subset \mathbb{R}$, we have $\mu(R) = 1$. Let $\{A_j\}_{j=1}^N$ be a finite sequence of arbitrary disjoint sets in $\mathcal{F}_0$ (without loss of generality). Then at most one of the $A_1, A_2, ...A_N$ has a measure 1 since if more than two sets has measure 1, say $\mu(A_1) = \mu(A_2) = 1$. This means:

$$\exists x_1 > 0, [x_1, \infty) \subset A_1, \exists x_2 > 0, [x_2, \infty) \subset A_2$$

Without loss of genearality, suppose $x_1 \geq x_2$, then $[x_1, \infty) \subset [x_2, \infty) \subset A_2$, which means A_1, A_2 are not disjoint, this leads to a contradiction. Therefore there can only be two cases:

(i) $\exists k, \exists x, [x, \infty) \subset A_k$, therefore $\exists x, [x, \infty) \subset \bigcup\limits_{j=1}^{N} A_j$. Then $\sum\limits_{j=1}^{\infty} \mu(A_j) = 1$, and $\mu(\bigcup\limits_{j=1}^{N} A_j) = 1$

(ii) $\forall j, \nexists x, [x, \infty) \subset A_j$, therefore $\nexists x, [x, \infty) \not\subset \bigcup\limits_{j=1}^{N} A_j$. Then $\sum\limits_{j=1}^{N} \mu(A_j) = \mu(\bigcup\limits_{j=1}^{N} A_j) = 0$

Hence in either case $\mu(\bigcup\limits_{j=1}^{N} A_j) = \sum\limits_{j=1}^{N} \mu(A_j)$, so we've shown μ is a finitely additive probability

Let $A_n = (n, n+1]$, then $\{A_n\}_{n=1}^\infty$ is a countable sequence of sets in $\mathcal{F}_0$, where each of them by definition $\mu(A_n) = 0$ since $\forall n, \nexists x, [x, \infty) \subset (n, n+1]$. But:

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \mu([1, \infty)) = 1 \neq \sum_{n=1}^{\infty} \mu(A_n) = 0$$

Therefore countable additivity does not hold, and hence μ is not a probability measure on $(\mathbb{R}, \mathcal{F}_0)$ □

Problem 2.

Proof. Let $a = (m_1, m_2, ..., m_i, ..., m_d)$ and $b = (m_1, m_2, ..., m_i', ..., m_d)$ be two d-timensional vectors where $a, b \in \mathbb{R}^d$ and $m_i \leq m_i'$. Without loss of generality (we can take $i = 1, 2, ..., d$), to show $F(x_1, ..., x_d)$ is non-decreasing is equivalent to show $F(a) \leq F(b)$. By definition of distribution function and *theorem 1.21*, there exist a unique probability measure P s.t. $P((-\infty, x]) = F(x)$. Hence it suffcies to show:

$$P((-\infty, a]) \leq P((-\infty, b])$$

For a d-dimensional vector $x = (x_1, ..., x_d) \in (-\infty, a], x \in (-\infty, b]$ hence $(-\infty, a] \subseteq (-\infty, b]$, and from the monotonicity property of probability measure P, we know that $P((-\infty, a]) \leq P((-\infty, b])$, so F is non-decreasing in each variable x_i □

Problem 3.

Proof. An example that satisfies $(a) - (c)$ but not a two-dimensional distribution function would be:

$$F(x, y) = \begin{cases} 1 & \text{if } x, y \geq 1 \\ 1 & \text{if } x \geq 1 \text{ and } 0 \leq y < 1 \\ 1 & \text{if } 0 \leq x < 1 \text{ and } y \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

where $F(x,y)$ is non-decreasing in x and y, which satisfies $3(a)$.
However, when $a=(0,0), b=(1,1)$ then

$$\Delta_{(a,b]} = (F(1,1)-F(0,1)) - (F(1,0)-F(0,0)) = (1-1)-(1-0) = -1 < 0$$

which violates $2(i)$ and also says that it is not a distribution function. We can also verify that $\lim\limits_{(x',y')\downarrow(x,y)} F(x',y') = F(x,y)$ for $x' \geq x, y' \geq y$, since the function at region below $x \leq 1, y \leq 1$ is always 0 and always 1 when $x > 1, y > 1$. At the frontier of $x=1$ or $y=1$, $F(x,y)=1$ and $F(x',y')$ approaching from above is also 1. Besides, $\lim\limits_{x\to\infty, y\to\infty} F(x,y)=1$, and $\lim\limits_{x\downarrow-\infty} F(x,y)=0$ since $x \leq 0, F(x,y)=0$, and $\lim\limits_{y\downarrow-\infty} F(x,y)=0$ since $y \leq 0, F(x,y)=0$. So $F(x,y)$ satisfies $3(b), 3(c)$. In conclusion $F(x,y)$ satisfies $(a)-(c)$ but is not a two-dimensional distribution function □

Problem 4(a).

Proof. $F_X \prec F_Y$ by definition is equivalent to

$$F_X(x) \geq F_Y(x), \forall x \in \mathbb{R} \iff P(Y \leq x) \leq P(X \leq x), \forall x \in \mathbb{R}$$

By subadditivity,

$$P(Y \leq x) = P(\{X \leq Y\} \cap \{Y \leq x\}) + P(\{X > Y\} \cap \{Y \leq x\})$$

By monotonicity,

$$\leq P(\{X \leq Y\} \cap \{Y \leq x\}) + P(\{X > Y\})$$

Since $\{X \leq Y\} \cap \{Y \leq x\} = \{X \leq x\}$ and

$$\begin{aligned} P(X > Y) &= P(\{X > x\} \cap \{Y \leq x\}) = 1 - P(\{X \leq x\} \cap \{Y > x\}) \\ &\leq 1 - P(\{X \leq x\} \cap \{Y \geq x\}) = 1 - P(X \leq Y) = 0 \end{aligned} \tag{1}$$

$\therefore P(Y \leq x) \leq P(\{X \leq Y\} \cap \{Y \leq x\}) + P(\{X > Y\}) \leq P(X \leq x), \forall x \in \mathbb{R}$
$\therefore F_X \prec F_Y$ □

Problem 4(b).

Proof. Let $\Omega = (0,1)$, $\mathcal{F} = \mathcal{B}(0,1)$, $P =$ Lebesgue measure

$$F \prec G \iff F_X \prec F_Y \iff F_X(x) \geq F_Y(x), \forall x$$

Consider X, Y: $X(\omega) = \sup\{x : F_X(x) \leq \omega\} \in \mathcal{S}$ and $Y(\omega) = \sup\{x : F_Y(x) \leq \omega\} \in \mathcal{S}$ where both are random variable defined on $\mathcal{F}$. Therefore $X - Y$ is also a random variable defined on $\mathcal{F}$ and $\{X - Y \leq 0\}$ is also in $\mathcal{F}$. Let $x' \in \{x : F_X(x) \leq \omega\}$, which means $F_X(x') \leq \omega$, so $F_Y(x') \leq F_X(x') \leq \omega$, then $x' \in \{x : F_Y(x) \leq \omega\}$. This shows $\{x : F_X(x) \leq \omega\} \subset \{x : F_Y(x) \leq \omega\}$, so we know $X(\omega) \leq Y(\omega)$ by the property of supremum. So $P(X \leq Y) = P(\{\omega \in (0,1) : X(\omega) \leq Y(\omega)\}) = P((0,1)) = 1$. Also, for $\omega \in \Omega = (0,1)$, $\lim\limits_{x\to\infty} P(\{\omega : X(\omega) \leq x\}) = P(\Omega) = 1$ and $\lim\limits_{x\to-\infty} P(\{\omega : X(\omega) \leq x\}) = P(\emptyset) = 0$
Therefore X, Y are random variables on some probability space $((0,1), \mathcal{B}(0,1), m)$ with $P(X \leq Y) = 1$ □