MATH 418/544 Assignment 6

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Problem 1(a).

Proof. By definition, the characteristic funciton of geometric distribution with $p \in (0,1)$ is

$$\begin{aligned}\phi_X(t) = E[e^{itx}] &= \sum_{n=1}^{\infty} e^{itn} P_X(n) = \sum_{n=1}^{\infty} e^{itn} P(X=n) = \sum_{n=1}^{\infty} e^{itn} p(1-p)^{n-1} \\ &= pe^{it} \sum_{n=1}^{\infty} \left[e^{it}(1-p)\right]^{n-1} = \frac{pe^{it}}{1-(1-p)e^{it}} = \frac{p}{e^{-it}-(1-p)}\end{aligned} \tag{1}$$

□

Problem 1(b).

Proof. The density of the unifrom distribution is: $f_X(x) = \frac{1}{2a} 1_{[-a,a]}(x)$. Therefore by definition, the characteristic funciton of uniform distribution is:

$$\phi_X(t) = E[e^{itx}] = \int_{-a}^{a} \frac{e^{itx}}{2a} dx = \int_{-a}^{a} [\frac{cos(tx)}{2a} + i\frac{sin(tx)}{2a}] dx = \int_{-a}^{a} \frac{cos(tx)}{2a} dx \tag{2}$$

When $t = 0$, we have $\phi_X(t) = \int_{-a}^{a} \frac{1}{2a} dx = 1$ and when $t \neq 0$, we have $\phi_X(t) = \frac{sin(tx)}{2at}|_{-a}^{a} = \frac{sin(at)}{at}$. In summary, the characteristic function is:

$$\phi_X(t) = \begin{cases} \frac{sin(at)}{at} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

□

Problem 2.

Proof. From the question we have

$$\begin{aligned} p_k &= (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-itk} \phi_X(t) dt = (2\pi)^{-1} \int_{-\pi}^{\pi} \int e^{-itk} e^{itx} dP dt \\ &= (2\pi)^{-1} \int \int_{-\pi}^{\pi} e^{it(x-k)} dt dP \text{ by Fubini's since } |e^{it(x-k)}| = 1 \text{ is integrable} \end{aligned} \tag{3}$$

Notice that X is integer valued and $k \in \mathbb{Z}$, so $x - k \in \mathbb{Z}$. Then we have $\int_{-\pi}^{\pi} \frac{e^{it(x-k)}}{2\pi} = 1_{(x=k)}$, and hence

$$p_k = (2\pi)^{-1} \int 1_{(x=k)} dP = P(x=k) \tag{4}$$

which is the the probability of mass function at k by definition. □

Problem 3.

Proof. Let X_i be the round-off of the ith number, then $\{X_n : n \in \mathbb{N}, n \leq 25\}$ are a sequence of i.i.d random variables. Then we have $E(X_i) = 0$, $\sigma^2 = E(X_i^2) - E(X_i)^2 = \int_{-0.5}^{0.5} x^2 dx = \frac{x^3}{3}|_{-0.5}^{0.5} = \frac{1}{12} < \infty$. Denote

$S_n = \sum_{i=1}^{n} X_i$, since the sum of rounded numbers equals to rounded sum of the unrounded numbers, we have:

$$\begin{aligned}
P(-0.5 < S_{25} < 0.5) &= P(\frac{-0.5}{\sqrt{25/12}} < \frac{S_{25}}{\sqrt{\sigma^2 25}} < \frac{0.5}{\sqrt{25/12}}) \\
&\approx P(-\frac{\sqrt{3}}{5} < \frac{S_{25}}{\sqrt{\sigma^2 25}} < \frac{\sqrt{3}}{5}) \\
&\text{where } \frac{S_{25}}{\sqrt{\sigma^2 25}} \sim N(0,1) \text{ by Central Limit Theorem} \\
&\approx P(-0.35 < Z < 0.35) \text{ where } Z = \frac{S_{25}}{\sqrt{\sigma^2 25}} \\
&\approx 0.2736
\end{aligned} \tag{5}$$

□

Problem 4.

Proof. Let $\{X_i : i \in \mathbb{N}\}, \{Y_i : i \in \mathbb{N}\}$ be sequences of i.i.d random variables, where X_i, Y_i is the time to make the ith double decaf skim milk for George and Julia respectively, and $\{X_i\}, \{Y_i\}$ are independent. Then we have $E[X_i] = E[Y_i]$ and same standard deviation $\sigma_{X_i} = \sigma_{Y_i} = 4$. Denote $D_i = Y_i - X_i$, then we have $E[D_i] = E[Y_i] - E[X_i] = 0$ and $\sigma_{D_i}^2 = \sigma_{Y_i}^2 + \sigma_{X_i}^2 = 32$. Also denote $S_n = \sum_{i=1}^{n} D_i$, then the probability that Goerge will get the prize is:

$$\begin{aligned}
P(S_{200} > 80) = P(\sum_{i=1}^{n} D_i > 80) &= P(\frac{S_{200}}{\sqrt{200\sigma_{D_i}^2}} \geq \frac{80}{\sqrt{200\sigma_{D_i}^2}}) = P(\frac{S_{200}}{\sqrt{200\sigma_{D_i}^2}} \geq \frac{80}{\sqrt{200 \times 32}}) \\
&\approx P(Z \geq 1) \text{ by Central limit theorem, where } Z = \frac{S_{200}}{\sqrt{200\sigma_{D_i}^2}} \sim N(0,1) \\
&= \int_1^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx
\end{aligned} \tag{6}$$

□

Problem 5.

Proof. Since $\{X_n\}$ are i.i.d Cauchy random variables and $S_n = \sum_{k=1}^{n} X_k$,

$$\begin{aligned}
\lim_{n\to\infty} \phi_{S_n/n}(t) &= \lim_{n\to\infty} E[e^{it(S_n/n)}] = \lim_{n\to\infty} E[e^{i\frac{t}{n}S_n}] \\
&= \lim_{n\to\infty} \phi_{S_n}(\frac{t}{n}) = [\phi_{X_1}(\frac{t}{n})]^n = (e^{-|t|/n})^n = e^{-|t|} = \phi_{X_1}(t)
\end{aligned} \tag{7}$$

Since $\phi_{S_n/n}(t)$ converges for all t and the limit $\phi_{X_1}(t)$ is continuous at 0, by Levy's continuity theorem we know S_n/n has the same distribution as X_1, and hence the limiting distribution is Cauchy. □