MATH 418 Assignment 1

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Problem 1.

Proof. Suppose $\omega \in C$:

Then $lim_{n\to\infty}\frac{S_n(\omega)}{n} = \frac{1}{2} \Rightarrow \forall\epsilon \geq 0, \exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1, |\frac{S_n(\omega)}{n} - \frac{1}{2}| \leq \epsilon$

Let $N_2 = N_1$ then $\forall m \geq N_2, m^2 \geq N_2 = N_1 \Rightarrow |\frac{S_{m^2}(\omega)}{m^2} - \frac{1}{2}| \leq \epsilon$

$\therefore lim_{m\to\infty}\frac{S_{m^2}(\omega)}{m^2} = \frac{1}{2}, \omega \in \hat{C}, C \subseteq \hat{C}$

Now suppose $\omega \in \hat{C}$:

Then $lim_{m\to\infty}\frac{S_{m^2}(\omega)}{m^2} = \frac{1}{2}, \Rightarrow \forall\epsilon \geq 0, \exists N_2 \in \mathbb{N}$ s.t. $\forall m \geq N_2, |\frac{S_{m^2}(\omega)}{m^2} - \frac{1}{2}| \leq \epsilon$

From the definition $S_n(w) = \sum_{k=1}^{n} w_k$, we know that $S_n(w) \leq n, \forall n$

$\therefore |S_{n_1}(w) - S_{n_2}(w)| \leq |S_{n_1}(w) - n_1| + |n_1 - n_2| + |S_{n_2}(w) - n_2| \leq |n_1 - n_2|, \forall n_1, n_2 \in \mathbb{N}$

$\therefore |\frac{S_n(w)}{n} - \frac{S_{m^2}(w)}{m^2}| \leq |\frac{S_n(w)}{n} - \frac{S_{m^2}(w)}{n}| + |\frac{S_{m^2}(w)}{n} - \frac{S_{m^2}(w)}{m^2}| \leq \frac{1}{n}|S_n(w) - S_{m^2}(w)| + S_{m^2}(w)|\frac{1}{n} - \frac{1}{m^2}|$

Let $m = \lfloor\sqrt{n}\rfloor \in \mathbb{N}$, so $m^2 \leq n < (m+1)^2$, then:

$\frac{1}{n}|S_n(w) - S_{m^2}(w)| + S_{m^2}(w)|\frac{1}{n} - \frac{1}{m^2}| \leq \frac{n-m^2}{n} + m^2|\frac{m^2-n}{nm^2}| \leq \frac{n-m^2}{n} + |\frac{m^2-n}{n}|$

$\leq \frac{(m+1)^2-m^2}{m^2} + |\frac{(m+1)^2-m^2}{m^2}| = \frac{4m+2}{m^2} \leq \frac{4+\frac{2}{m}}{m} \leq \frac{6}{m} \to 0$ as $m \to \infty$

$\therefore lim_{n\to\infty}\frac{S_n(\omega)}{n} = \frac{1}{2}, w \in C, \hat{C} \subseteq C$

Therefore we've proved that $C = \hat{C}$ $\square$

Problem 2(a).

Proof. Since $\mathcal{F}_n$ is a increasing sequence of fields:

Then $\emptyset \in \mathcal{F}_n, \forall n \Rightarrow \emptyset \in \cup_{n=1}^{\infty}\mathcal{F}_n$

Let $A \in \cup_{n=1}^{\infty}\mathcal{F}_n$, then $\exists k \in \mathbb{N}$ s.t. $A \in \mathcal{F}_k$

Since $\mathcal{F}_k$ is an field, then we could also know that $A^c \in \mathcal{F}_k \Rightarrow \therefore A^c \in \cup_{n=1}^{\infty}\mathcal{F}_n$

Suppose $\{A_i\}_i^m$ is a finite sequence of sets such that $A_i \in \cup_{n=1}^{\infty}\mathcal{F}_n, \forall i$

Then $\forall i, \exists n_i \in \mathbb{N}, A_i \in \mathcal{F}_{n_i} \Rightarrow \cup_{i=1}^{m}A_i \in \cup_{n_i}\mathcal{F}_{n_i} \Rightarrow \cup_{i=1}^{m}A_i \subseteq \cup_{n=1}^{\infty}\mathcal{F}_n$

$\therefore \cup_{n=1}^{\infty}\mathcal{F}_n$ is also a field $\square$

Problem 2(b).

Proof. $\Omega = \mathbb{N}$, let $\mathcal{F}_n = \{A \subseteq \mathbb{N} : A \in 2^{\{0,1,\ldots,n\}} \text{ or } A^c \in 2^{\{0,1,\ldots,n\}}\}$

We can verify that $\mathcal{F}_n$ is an increasing sequence of sigma fields:

If $A \in \mathcal{F}_n$ then either $A \in 2^{\{0,1,\ldots,n\}}$ or $A^c \in 2^{\{0,1,\ldots,n\}}$, hence either $A \in 2^{\{0,1,\ldots,n+1\}}$ or $A^c \in 2^{\{0,1,\ldots,n+1\}}$, so $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$

(i) $\emptyset \in \mathcal{F}_n$ since $\emptyset \in 2^{\{0,1,\ldots,n\}}$

(ii) $A \in \mathcal{F}_n \Rightarrow A^c \in \mathcal{F}_n$ by the construction of $\mathcal{F}_n$

(iii) $\{A_i\}_i^\infty \in \mathcal{F}_n \Rightarrow$ for all i, either $A_i \in 2^{\{0,1,\ldots,n\}}$ (we relabel these A_i as A_j), or $A_i^c \in 2^{\{0,1,\ldots,n\}}$(we relabel these A_i as A_k), then $\cup_{i=1}^\infty A_i = (\cup_{j=1}^\infty A_j) \cup (\cup_{k=1}^\infty A_k)$

$A_j \in 2^{\{0,1,\ldots,n\}} \Rightarrow (\cup_{j=1}^\infty A_j) \in 2^{\{0,1,\ldots,n\}} \Rightarrow (\cup_{j=1}^\infty A_j) \in \mathcal{F}_n$

$A_k^c \in 2^{\{0,1,\ldots,n\}} \Rightarrow (\cap_{k=1}^\infty A_k^c) \in 2^{\{0,1,\ldots,n\}} \Rightarrow (\cup_{k=1}^\infty A_k)^c \in 2^{\{0,1,\ldots,n\}} \Rightarrow (\cup_{k=1}^\infty A_k)^c \in \mathcal{F}_n \Rightarrow (\cup_{k=1}^\infty A_k) \in \mathcal{F}_n$ by the construction of $\mathcal{F}_n$

$\therefore \cup_{i=1}^\infty A_i \in \mathcal{F}_n$, hence we've shown that $\mathcal{F}_n$ is an increasing sequence of sigma fields

Let $A_i = \{2i\}$, then $\forall i, \exists n_i \in \mathbb{N}, A_i \in \mathcal{F}_{n_i}$, then $A_i \in \cup_{n=1}^\infty \mathcal{F}_n$

Then $B = \cup_i^\infty A_i$ ={all even numbers}, $B^c = (\cup_i^\infty A_i)^c = \{\text{all odd numbers}\}$

where we know $\nexists n \in \mathbb{N}$ s.t. B, or $B^c \in 2^{\{0,1,\ldots,n\}}$, in other words, $B, B^c \notin \mathcal{F}_n, \forall n \in \mathbb{N}$

Since full union $\cup_i^\infty A_i$ must be in one of the $\mathcal{F}_n$ in order to be in $\cup_{n=1}^\infty \mathcal{F}_n$

Therefore $\cup_i^\infty A_i \notin \cup_{n=1}^\infty \mathcal{F}_n$, $\cup_{n=1}^\infty \mathcal{F}_n$ is not a σ-field □

Problem 3(a).

Proof. Let $C = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$, define two distinct probabilities on $(\Omega, \mathcal{F})$:

$P_1(\{1\}) = \frac{2}{12}, P_1(\{2\}) = \frac{4}{12}, P_1(\{3\}) = \frac{2}{12}, P_1(\{4\}) = \frac{4}{12}$

$P_2(\{1\}) = \frac{4}{12}, P_2(\{2\}) = \frac{2}{12}, P_2(\{3\}) = \frac{4}{12}, P_2(\{4\}) = \frac{2}{12}$

where $P_1(\Omega) = P_2(\Omega) = 1$

$\therefore P_1(\{1,2\}) = P_2(\{1,2\}) = \frac{1}{2}, P_1(\{2,3\}) = P_2(\{2,3\}) = \frac{1}{2}$

$P_1(\{3,4\}) = P_2(\{3,4\}) = \frac{1}{2}, P_1(\{1,4\}) = P_2(\{1,4\}) = \frac{1}{2}$

$\therefore P_1, P_2$ agrees on C

Then it suffices to show that $\sigma(C) = \mathcal{F}$:

$\sigma(C)$ must be closed under countable unions of subsets, which includes all subsets of three and four elements (i.e. unions of elements in C): $\{1,2,3\},\{1,2,4\},\{2,3,4\},\{1,3,4\},\{1,2,3,4\}$

Meanwhile, it must contain the complements of these subsets, which includes all the singleton and the empty set: $\{1\}, \{2\}, \{3\}, \{4\}, \emptyset$, and also their unions: $\{1,3\}, \{2,4\}$

$\therefore \sigma(C) = 2^{\{1,2,3,4\}} = \mathcal{F}$ $\square$

Problem 3(b).

Proof. Let $\hat{C} = \{A : A^c \in C\}$, then we can show that $\hat{C}$ is a π-system:

$\forall A, B \in \hat{C} \Rightarrow A^c, B^c \in C \Rightarrow A^c \cup B^c \in C \Rightarrow (A \cap B)^c \in C \Rightarrow A \cap B \in \hat{C}$ by the definition of $C, \hat{C}$

Then if we know two probabilites P_1, P_2 agree on $\hat{C}$, it would agree on $\sigma(\hat{C})$ by $\pi - \lambda$ theorem

Since we know P_1, P_2 agree on C,

$\forall A \in C \subset \mathcal{F}, P_1(A) = P_2(A) \Rightarrow P_1(A^c) = P_2(A^c) = 1 - P_1(A) = 1 - P_2(A)$, since $(\Omega, \mathcal{F})$ is a measurable space, where A^c is an arbitrary set in $\hat{C}$

$(\forall A^c \in \hat{C}, \exists A \in C$ from the construction of $\hat{C})$

Then P_1, P_2 agree on $\hat{C}$, hence agree on $\sigma(\hat{C})$, thus it suffices to show that $\sigma(\hat{C}) = \mathcal{F}$

Let $A \in \sigma(C) = \mathcal{F} \Rightarrow A^c \in \sigma(C)$, and $A^c \in \hat{C}$ by definition

Then $A^c \in \sigma(\hat{C}), (A^c)^c \in \sigma(\hat{C}) \Rightarrow A \in \sigma(\hat{C})$

$\therefore \mathcal{F} = \sigma(C) \subseteq \sigma(\hat{C})$, and since $\sigma(\hat{C}) \subseteq \mathcal{F}$ for the reason that it is the smallest unique σ-field containing $\hat{C}$, we've showed that $\sigma(\hat{C}) = \mathcal{F}$

Thus P_1, P_2 agree on $\mathcal{F}$ $\square$

Problem 3(c).

Proof. No, use the same counterexample from 3(a):

$C = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}\} \subset \mathcal{F}$ is closed under complementation and it has been shown that $\sigma(C) = \mathcal{F}$

However, we've shown that P_1, P_2 agree on C but they do not agree on $\mathcal{F}$ $\square$