MATH 418/544 Assignment 4

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Problem 1(a).

Proof. Let $A_n = \{X \geq \frac{1}{n}\}$, by Markov inequality we have $P(X \geq x) \leq \frac{E(X)}{x}, \forall x > 0$, and $E(X) = 0$ by assumption. Then we have $P(A_n) = P(\{X \geq \frac{1}{n}\}) \leq 0$ for all n. Also, $A_n \uparrow A = \{X > 0\}$. Thus by continuity from below we have $P(A) = P(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n) = 0$, which means $P(\{X > 0\}) = 0$. Given that X is a non-negative random variable, we can conclude that $X = 0$ almost surely. $\square$

Problem 1(b).

Proof. We will prove the statement from either direction:
($\Rightarrow$) Suppose Y is a constant random variable with finite mean μ, then $Y = \mu$ almost surely, $Y^2 = \mu^2$ almost surely. Therefore by definition we have $\sigma_Y^2 = E(Y^2) - [E(Y)]^2 = \mu^2 - \mu^2 = 0$
($\Leftarrow$) $\sigma_Y^2 = 0$, by Chebyshev inequality, we know that $P(|Y - \mu| \geq x) \leq \frac{\sigma_Y^2}{x^2} = 0$ for all $x > 0$. Let $A_n = \{|Y - \mu| \geq \frac{1}{n}\}$, then $P(A_n) \leq 0$ and $A_n \uparrow A = \{|Y - \mu| > 0\}$. By continuity from below we have

$$P(|Y - \mu| > 0) = P(\bigcup_{n=1}^{\infty}\{|Y - \mu| \geq \frac{1}{n}\}) = \lim_{n\to\infty} P(|Y - \mu| \geq \frac{1}{n}) = 0$$

where $|Y - \mu| \geq 0$. Therefore $Y - \mu = 0$ almost surely, thus $Y = \mu$ almost surely. $\square$

Problem 2.

Proof. Since Y is non-negative and $p > 0$, then Y^p is non-negative, by definition of expectation, we have

$$\begin{aligned}
E(Y^p) &= \int_\Omega Y^p(\omega) dP(\omega) = \int_{\mathbb{R}} y^p dP_Y(y) \\
&= \int_{\mathbb{R}} \big[\int_0^y px^{p-1}dx\big] dP_Y(y) \text{ since } y^p = \int_0^y px^{p-1}dx \\
&= \int_0^\infty \big[\int_{\mathbb{R}} 1_{x\leq y} px^{p-1} dP_Y(y)\big] dx \text{ by Fubini's theorem since } x \geq 0, p > 0, \text{ integrand non-negative} \\
&= \int_0^\infty P(Y \geq x) px^{p-1} dx
\end{aligned} \tag{1}$$

$\square$

Problem 3(a).

Proof. Let X be discrete random variable that takes value $-x$ or x (where x is a fixed value) such that $P(X = -x) = P(X = x) = \frac{1}{2}$. Then we have $\mu = E(X) = 0, \sigma^2 = x^2$. Therefore, $P(|X - 0| \geq x) = 1 = \frac{\sigma^2}{x^2}$ holds. $\square$

Problem 3(b).

Proof. Let X be an arbitrary random variable with finite mean μ such that the equality holds in Chebyshev's inequality. Then we have $P(|X - \mu| \geq x) = \frac{\sigma^2}{x^2}$ for all $x > 0$ and by property of probability we know $P(|X - \mu| \geq x) \leq 1$. Suppose $\sigma^2 \neq 0$, then $\sigma^2 = \epsilon > 0$ for some ϵ, we can pick $x^2 = \frac{\epsilon}{2}, x = \sqrt{\frac{\epsilon}{2}} > 0$ such that $P(|X - \mu| \geq x) = \frac{\sigma^2}{x^2} = \epsilon / \frac{\epsilon}{2} = 2$, which leads to contradiction. Therefore $\sigma^2 = 0$, by *part(b) in question 1*, X must be a constant. Therefore there is no non-constant random variable with finite mean so that the ineqaulity holds for all $x > 0$. $\square$

Problem 4.

Proof. Without loss of genearlity, assume $0 < p_1 < p_2 < \infty$. Then to show $||X||_p = \left[\int |X|^p dP\right]^{1/p}$ is a monotone non-decreasing function on $(0, \infty)$, it suffices to show: $\left[\int |X|^{p_2} dP\right]^{1/p_2} \geq \left[\int |X|^{p_1} dP\right]^{1/p_1}$, which is equivalent to $\int |X|^{p_2} dP \geq \left[\int |X|^{p_1} dP\right]^{p_2/p_1}$. By the definition of expectation, this is also same as showing

$$E(|X|^{p_2}) \geq \left[E(|X|^{p_1})\right]^{p_2/p_1} \tag{2}$$

Since $||X||_p$ may be ∞, let $X_n = (|X|^{p_2} \wedge n^{p_2})$ and $Y_n = (|X|^{p_1} \wedge n^{p_1})$, then $X_n, Y_n \geq 0$ for all n are non-decreasing and $X_n \uparrow \left|X\right|^{p_2}, Y_n \uparrow \left|X\right|^{p_1}$. By monotone convergence theorem, we have $\int X_n dP \uparrow \int |X|^{p_2} dP$ and $\int Y_n dP \uparrow \int |X|^{p_1} dP$, in other words

$$E(X_n) \uparrow E(|X|^{p_2}), E(Y_n) \uparrow E(|X|^{p_1}) \tag{3}$$

By Jensen's inequality,

$$E(X_n) = E(Y_n^{p_2/p_1}) \geq [E(Y_n)]^{p_2/p_1} \tag{4}$$

for all n since Y_n is intergable and $\phi(z) = z^{p_2/p_1}$ is a convex function on $(0, \infty)$ where $P((0, \infty)) = 1$. (The convexity can be shown since for a twice differentiable function of a single variable $\phi(z)$, the second derivative $\frac{p2}{p1}(\frac{p2}{p1} - 1)z^{\frac{p2}{p1}-2} \geq 0$ for all z on $(0, \infty)$) Then, by (3),(4) we have $E(|X|^{p_2}) \geq [E(|X|^{p_1})]^{p_2/p_1}$, which is inequality (2). Therefore, we can conclude that $p \to ||X||_p$ is a monotone non-decreasing function on $(0, \infty)$. □

Problem 5(a).

Proof. Since X, Y are random variables on with uniform distribution on $[0, 2]$, then by definition we have $f_X(x) = 1_{[0,2]}(x)\frac{1}{2}$, $f_Y(y) = 1_{[0,2]}(y)\frac{1}{2}$. Also X, Y are independent and by *theorem 2.3(a)*, we have $f_{X,Y}(x, y) = f_X(x)f_Y(y) = \frac{1}{4}1_{[0,2]}(x)1_{[0,2]}(y)$, in other words, we can write as $f_{X,Y}(x, y) = \frac{1}{4}1_{[0,2]\times[0,2]}(x, y)$, which is the joint probability distribution function of (X, Y). □

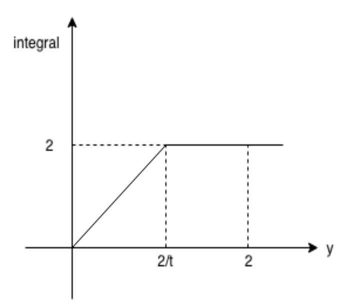
Problem 5(b).

Proof.

$$\begin{aligned}
P(X/Y \leq t) = E(1_{\{X/Y \leq t\}}) &= \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{(x/y \leq t)} dP_X(x) dP_Y(y) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{(x/y \leq t)} f_X(x) f_Y(y) dx dy \\
&= \frac{1}{4} \int_0^2 \int_0^2 1_{(x/y \leq t)} dx dy \text{ by 5(a)}
\end{aligned} \tag{5}$$

Case 1: When $t < 0$, since X, Y are uniform distributed on $[0, 2]$, then $1_{(x/y \leq t)} = 0$, $P(X/Y \leq t) = 0$

Case 2: When $t \geq 0$, $\frac{1}{4}\int_0^2 \int_0^2 1_{(x/y \leq t)} dx dy = \frac{1}{4}\int_0^2 \int_0^{ty \wedge 2} dx dy$. If $t \leq 1$, we have $ty \leq 2$, then $ty \wedge 2 = ty$, the integral becomes $\frac{1}{4}\int_0^2 \int_0^{ty} dx dy = \frac{t}{2}$. Otherwise $t > 1$, in this case the integral would be the area of the trapezoid in the figure.



Hence $\frac{1}{4}\int_0^2\int_0^{ty\wedge 2} dxdy = \frac{1}{4}\times 2\times(2+2-2/t)/2 = 1-\frac{1}{2t}$. Therefore the distribution function for X/Y is:

$$F_{X/Y}(t) = \begin{cases} 0 & \text{if } t<0 \\ \frac{t}{2} & \text{if } 0\le t\le 1 \\ 1-\frac{1}{2t} & \text{if } t>1 \end{cases}$$

and $F_{X/Y}(t) = \int_{-\infty}^t f(t)dt$, then the probability density function of X/Y would be:

$$f_{X/Y}(t) = \begin{cases} \frac{1}{2} & \text{if } 0\le t\le 1 \\ \frac{1}{2t^2} & \text{if } t>1 \\ 0 & t<0 \end{cases}$$

□

Problem 6.

Proof. Let $(\Omega,\mathcal{F},P) = ([0,1],\mathcal{B}([0,1]),m)$ where m is the Lebesgue measure on $[0,1]$. Also let $A_1 = [0,\frac{1}{2}], A_2 = [\frac{1}{8},\frac{5}{8}], A_3 = [\frac{3}{8},\frac{7}{8}]$, therefore $P(A_1) = P(A_2) = P(A_3) = \frac{1}{2}$ and $P(A_1\cap A_2\cap A_3) = P([\frac{3}{8},\frac{1}{2}]) = \frac{1}{8} = P(A_1)P(A_2)P(A_3)$. However A_1, A_2 are not independent because $P(A_1\cap A_2) = \frac{3}{8} \neq P(A_1)P(A_2) = \frac{1}{4}$. □