Homework 11

Recall the homework guidelines for this course. Do not forget to justify your answers.

1. For $a, b \in \mathbb{Z}$, we say that *a divides b* if there is $k \in \mathbb{Z}$ with $b = ka$. Show that for all $n \in \mathbb{N}$, we have that 8 divides $3^n + 7^n - 2$.

2. The Tower of Hanoi is a game in which you have 3 towers and n disks of different sizes, stacked together in tower 1. The object of the game is to move all the disks over to Tower 3. You can only move one disk at a time and you cannot place a larger disk onto a smaller disk. Find a formula in terms of n for the minimum number of moves required to win the game. Then prove that your formula is correct using induction. Below is an example of the initial state of the game when $n = 3$.

3. Recall that the Fibonacci sequence is defined recursively as

$$F_1 = F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2} \qquad (n > 2).$$

   Show by induction that for all $n \in \mathbb{N}$,

   (a) If $n>2$, then $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}$.

   (b) If $n>2$, then $F_{n-1} \cdot F_{n+1} = F_n^2 - (-1)^{n+1}$.

   (c) If $n \geq 1$, then F_{3n} is even.

4. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence and let $(s_n)_{n\in N}$ be its sequence of partial sums. Show that $(a_n)_{n\in\mathbb{N}}$ is the sequence of difference of $(s_n)_{n\in\mathbb{N}}$.

5. Consider a sequence $(a_n)_{n\in\mathbb{N}}$ and assume that its sequence of difference $(d_n)_{n\in\mathbb{N}}$ is arithmetic.

   (a) Can $(a_n)_{n\in\mathbb{N}}$ be arithmetic?

   (b) Find a closed formula for a_n in terms of a_1, d_1, and the common difference of $(d_n)_{n\in\mathbb{N}}$.

6. We say that a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{Z}$ *diverges to* ∞ if for all $L > 0$ there is $n \in \mathbb{N}$ with $a_n > L$. If $a_n \neq 0$ for all $n \in \mathbb{N}$, we say that $(a_n)_{n\in\mathbb{N}}$ *converges to* 0 if $\left(\frac{1}{a_n}\right)$ diverges to ∞[1].

   Assume that $(a_n)_{n\in\mathbb{N}}$ is a geometric sequence with common ratio $r > 0$ and $a_1 > 0$.

   (a) Show that $(a_n)_{n\in\mathbb{N}}$ diverges to ∞ if and only if $r > 1$.

   (b) Show that $(a_n)_{n\in\mathbb{N}}$ converges to 0 if and only if $r < 1$.

[1] this definition of convergence is non-standard, but it will be enough for us to work with.