Homework 5

Due Feb 21 at 4 pm

Recall the homework guidelines for this course.

0. Fill out this anonymous survey.

1. Let A and B be finite sets, and $f : A \to B$ be a function.

   (a) Show that if f is injective, then $|A| \leq |B|$.

   (b) Show that if f is surjective, then $|A| \geq |B|$.

   (c) Deduce that if f is bijective, then $|A| = |B|$.

2. Let $f\colon A \to B$ be a function between finite sets, where $|A| = |B|$. Prove that f is injective if and only if it is surjective.

3. Let A and B be sets. Recall that $A \times B$ is the set of ordered pairs (a, b), where $a \in A$ and $b \in B$, i.e., $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. A set-theoretical definition of the cartesian product represents the ordered pair (a, b) as the set $\{\{a\}, \{a, b\}\}$. In this problem, you will show that the definitions are equivalent.

   Show that the map

$$\begin{array}{rccc} \varphi\colon & A \times B & \to & \{\{\{a\}, \{a, b\}\} : a \in A \text{ and } b \in B\} \\ & (a, b) & \mapsto & \{\{a\}, \{a, b\}\} \end{array}$$

   is a bijection.

4. For each of the following functions, determine their domain, range, and image. Also, determine whether they are injective, surjective, and/or bijective, providing proofs for your claims.

   (a) $\begin{array}{rccc} f\colon & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & x^2 \end{array}$.

   (b) Let A be any set, and let B be a nonempty set. Fix an element $b \in B$ and consider the function $\iota_b : A \to A \times B$ defined by $\iota(a) = (a, b)$ for all $a \in A$.

5. Let U be a set. Define the function $\pi\colon \mathscr{P}(U) \to \mathscr{P}(U)$ as $\pi(A) = \overline{A}$ for all $A \in \mathscr{P}(U)$ (where the universe of $A \in \mathscr{P}(U)$ is U).

   (a) Prove that for $A, B \in \mathscr{P}(U)$, $A = B$ if and only if $\overline{A} = \overline{B}$.

   (b) Prove that for all $A \in \mathscr{P}(U)$, $\overline{\overline{A}} = A$ (i.e. the complement of the complement of A equals A).

   (c) Show that π is bijective.

6. Give examples of the following. Please, identify the domain and range of the functions, and justify every claim.

   (a) A function that is injective but not surjective.

(b) A function that is surjective but not injective.

(c) A function with range $\mathbb{Q}$ and image $\mathbb{Z}$.

(d) For all sets A, a function with domain A and range $\mathscr{P}(A)$.

(e) For all sets A in a universe U with $A \neq U$, a function with domain A and range $\overline{A}$.