Homework 6

Recall the homework guidelines for this course.

1. Let A, B, and C be sets. Consider functions $f: A \to B$ and $g: B \to C$, and their composition $g \circ f$. For each of the following statements, determine whether it is true or false. If true, provide a proof; if false, give a counterexample.
   (a) If g is surjective, then $g \circ f$ is also surjective.
   (b) If $g \circ f$ is surjective, then g is surjective.
   (c) If f and g are injective, then $g \circ f$ is injective.
   (d) The function $g \circ f$ is injective if and only if both f and g are injective.

2. In each part, provide an example of a function that holds the desired properties. Do not forget to prove that each function holds the properties you want.
   (a) $g: A \to B$ with $|A| = |B|$ and g not bijective.
   (b) $f: A \to A$ with A an infinite set and f injective but not surjective.
   (c) $h: A \to A$ with A an infinite set and f surjective but not injective.

In problems 3 and 4, you will prove the following theorem.

Theorem A. Let A and B be sets (not necessarily finite). A function $f: A \to B$ is bijective if and only if there exists a function $f^{-1}: B \to A$ such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.

3. Assume $f: A \to B$ is bijective. Define $f^{-1}: B \to A$ as
$$f^{-1}(b) = a, \text{ where } a \in A \text{ is such that } f(a) = b.$$
   (a) Prove that f^{-1} is a function.
   (b) Prove that $(f \circ f^{-1})(b) = b$ for all $b \in B$.
   (c) Prove that $(f^{-1} \circ f)(a) = a$ for all $a \in A$.
   (d) Conclude that the forward direction of **Theorem A** holds.

4. Let $f: A \to B$ be a function.
   (a) Prove that if there is a function $g: B \to A$ such that $g \circ f = \mathrm{id}_A$, then f is injective.
   (b) Prove that if there is a function $g: B \to A$ such that $f \circ g = \mathrm{id}_B$, then f is surjective.
   (c) Conclude that the backward direction of **Theorem A** holds.

5. Let $f: A \to B$, $g: B \to A$ be functions. Prove that if $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$, then f and g are bijective and that $f^{-1} = g$ and $g^{-1} = f$. *Hint: start by using Problem 4.*

6. Let $\{0,1\}^{\mathbb{N}}$ be the infinite set
$$\{0,1\}^{\mathbb{N}} = \{(a_1, a_2, a_3, \ldots) : a_i \in \{0,1\} \text{ for all } i \in \mathbb{N}\}.$$
Thus, the elements of $\{0,1\}^{\mathbb{N}}$ are infinite ordered lists of 0s and 1s. Prove by contradiction that there is no bijective function $f: \mathbb{N} \to \{0,1\}^{\mathbb{N}}$. *Hint: Consider constructing an element of* $\{0,1\}^{\mathbb{N}}$ *of the form* $(a_1, a_2, \ldots)$*, where* $a_i = 0$ *if* $f(i)$ *has a* 1 *in the* i*-th position and* $a_i = 1$ *if* $f(i)$ *has a* 0 *in the* i*-th position.*