

Invariants of Artin-Schreier curves

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**Joint work with Heidi Goodson, Elisa Lorenzo García, Beth Malmskog, and
Renate Scheidler**

Recognize the Colombian bird species



Recognize the Colombian bird species

(Up to isomorphism)



Zonotrichia capensis



Stilpnia heinei



Penelope argyrosis



Cyanocorax yncas



Thraupis episcopus



Momotus
aequatorialis



Rupornis magnirostris



Turdus serranus



Piranga rubra



Stilpnia heinei

Recognize the Colombian bird species

(Up to isomorphism)



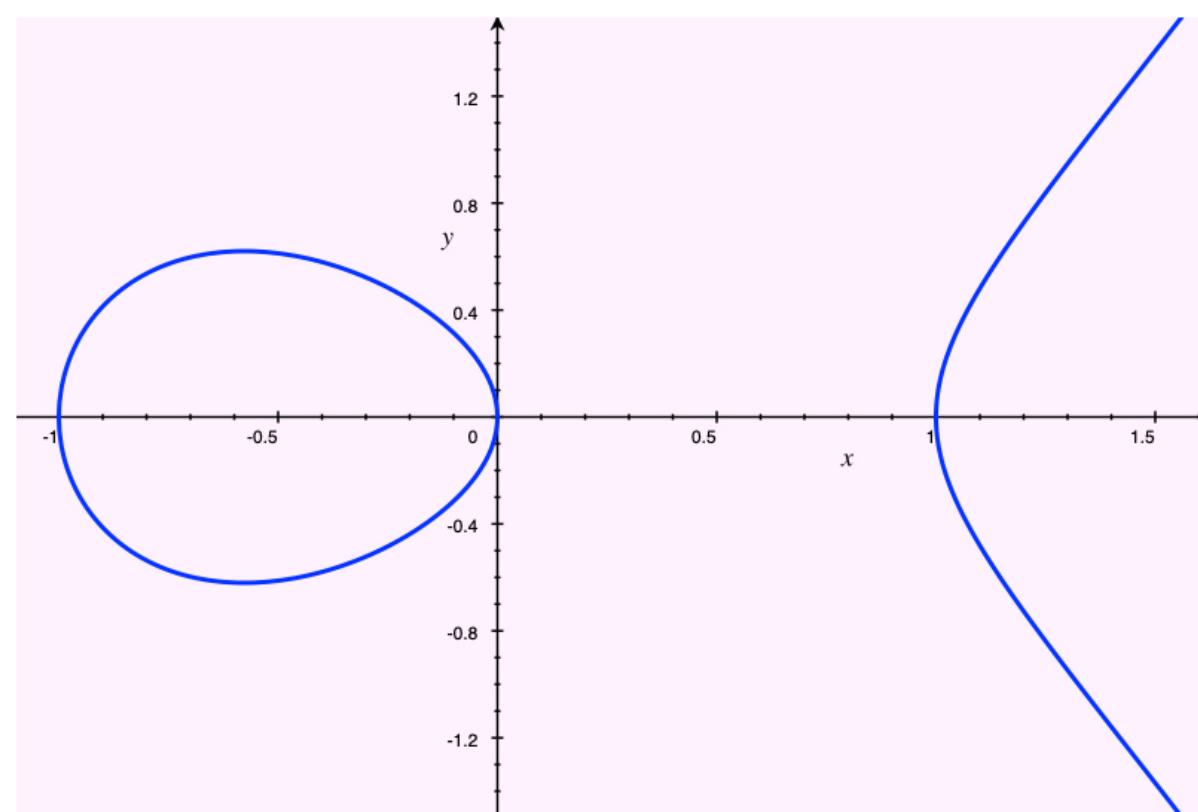
Stilpnia heinei

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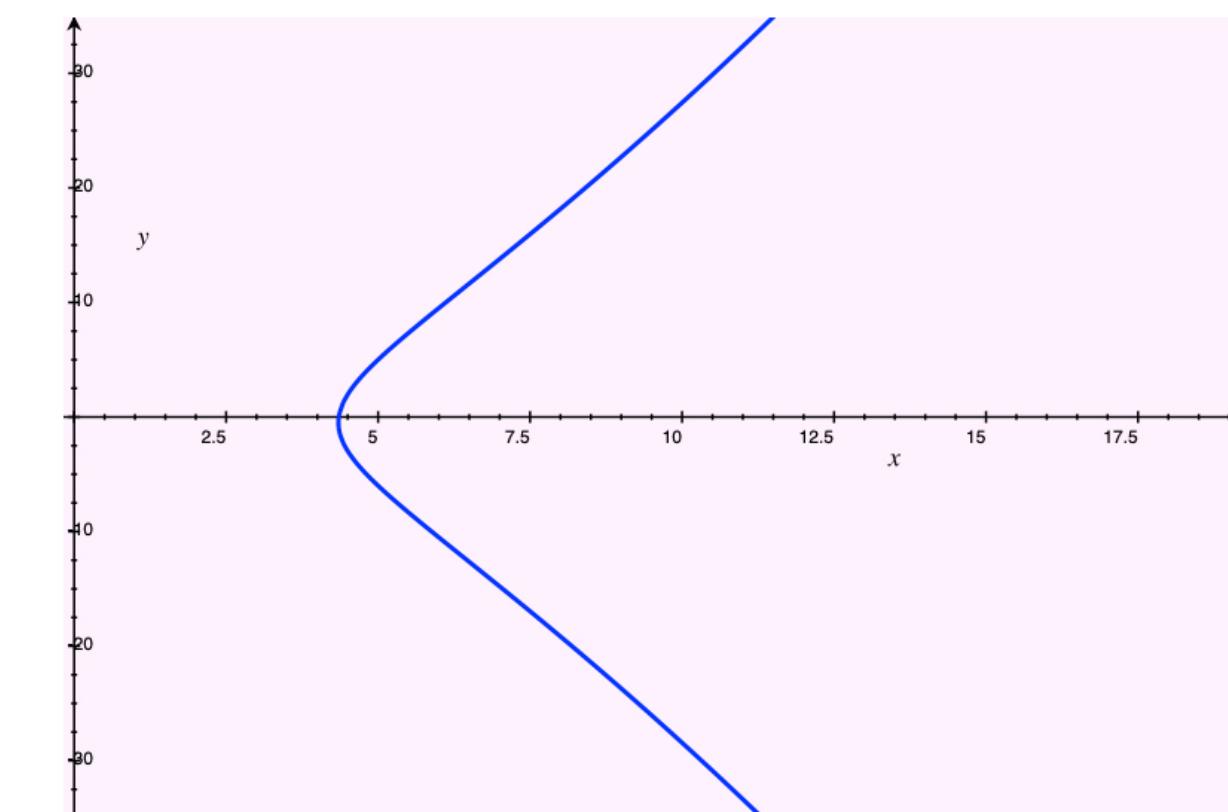


Stilpnia heinei

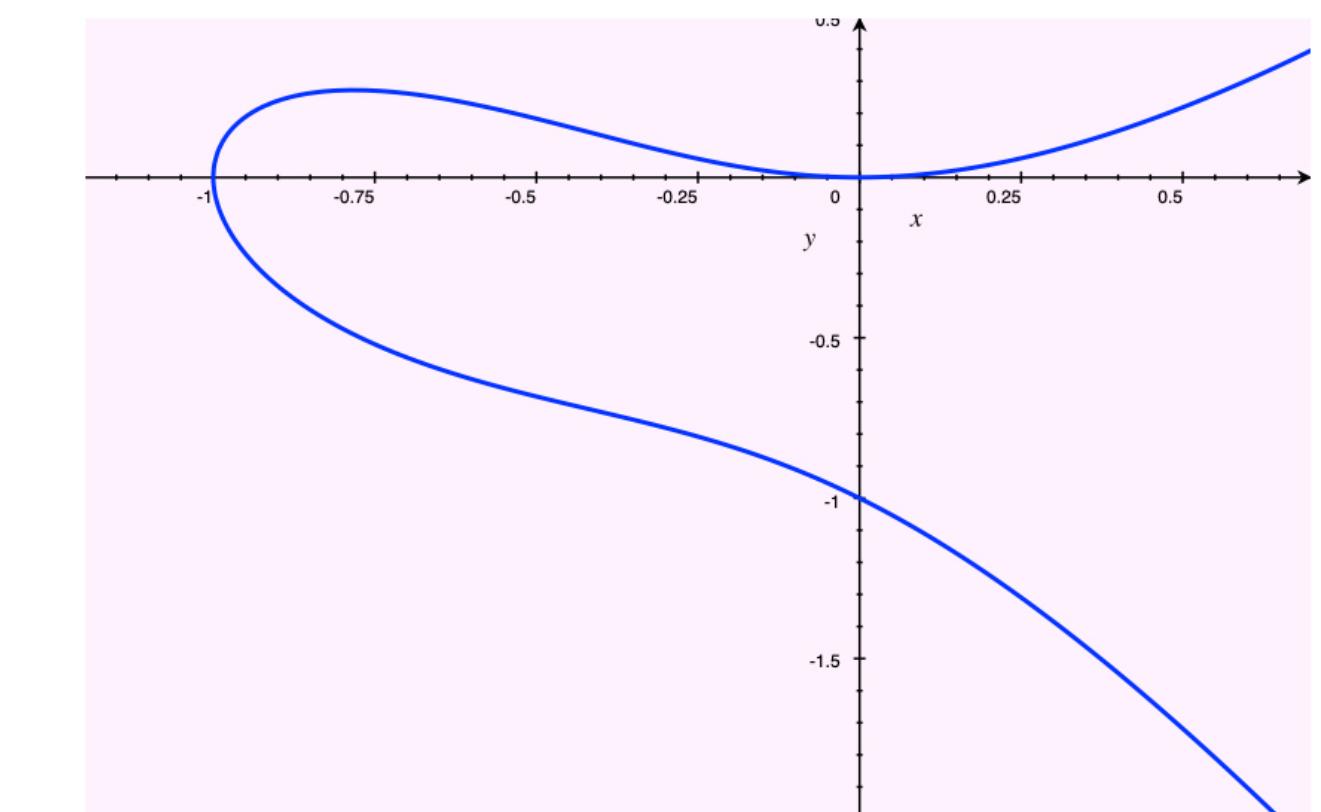
Recognize the elliptic curve



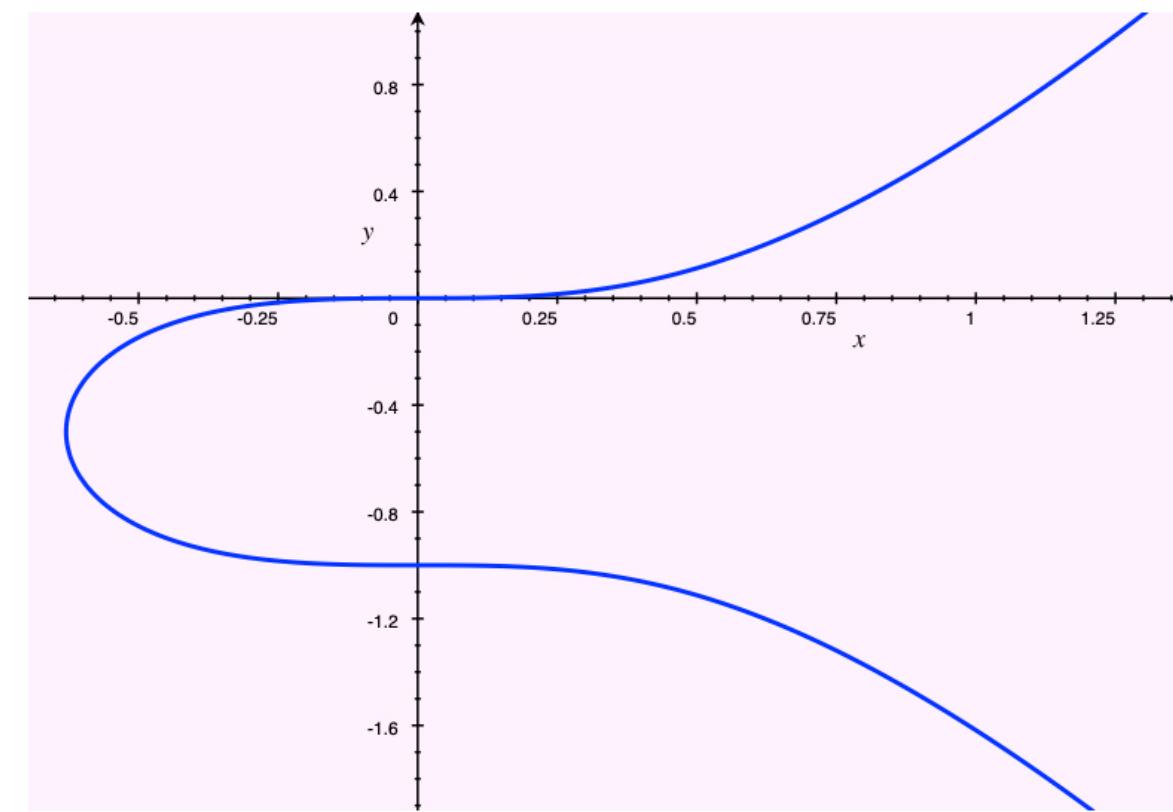
$$y^2 = x^3 - x$$



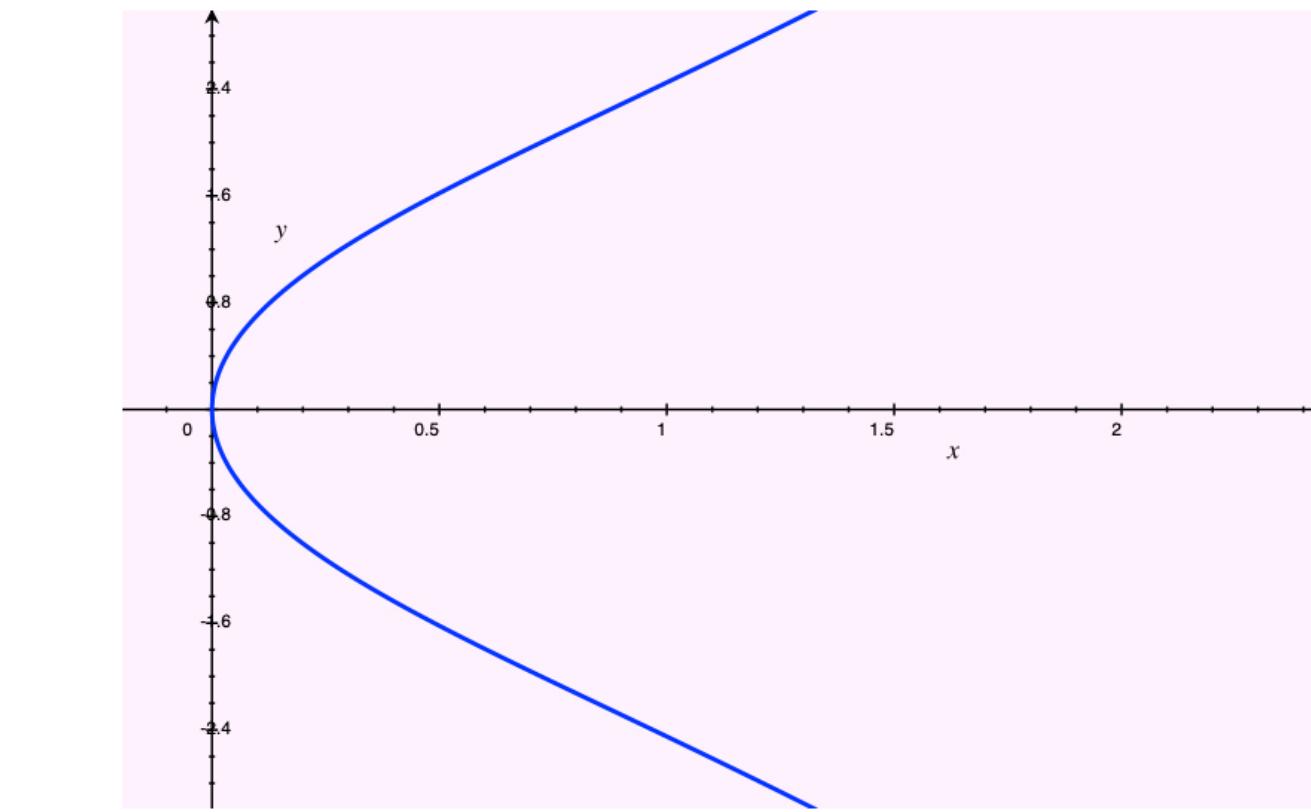
$$y^2 + y = x^3 - x^2 - 10x - 20$$



$$y^2 + xy + y = x^3 + x^2$$



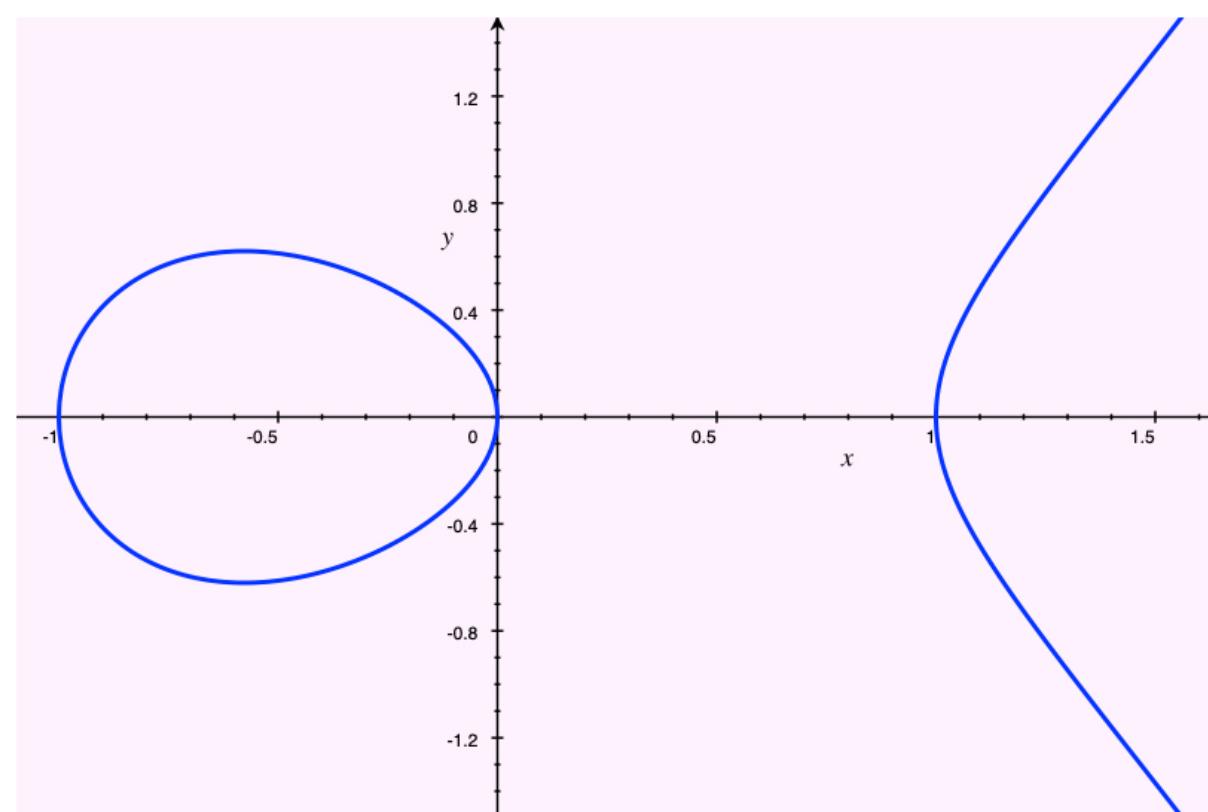
$$y^2 + y = x^3$$



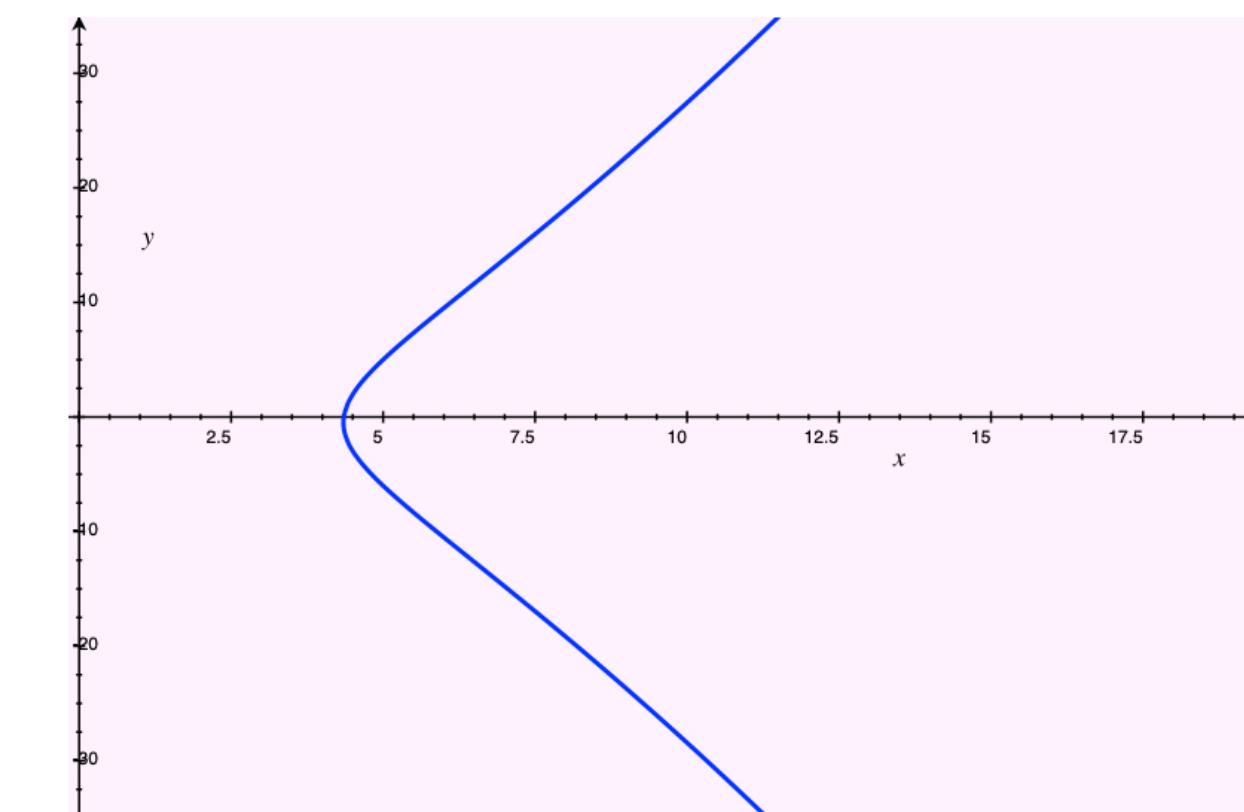
$$y^2 = x^3 + 5x$$

Recognize the elliptic curve

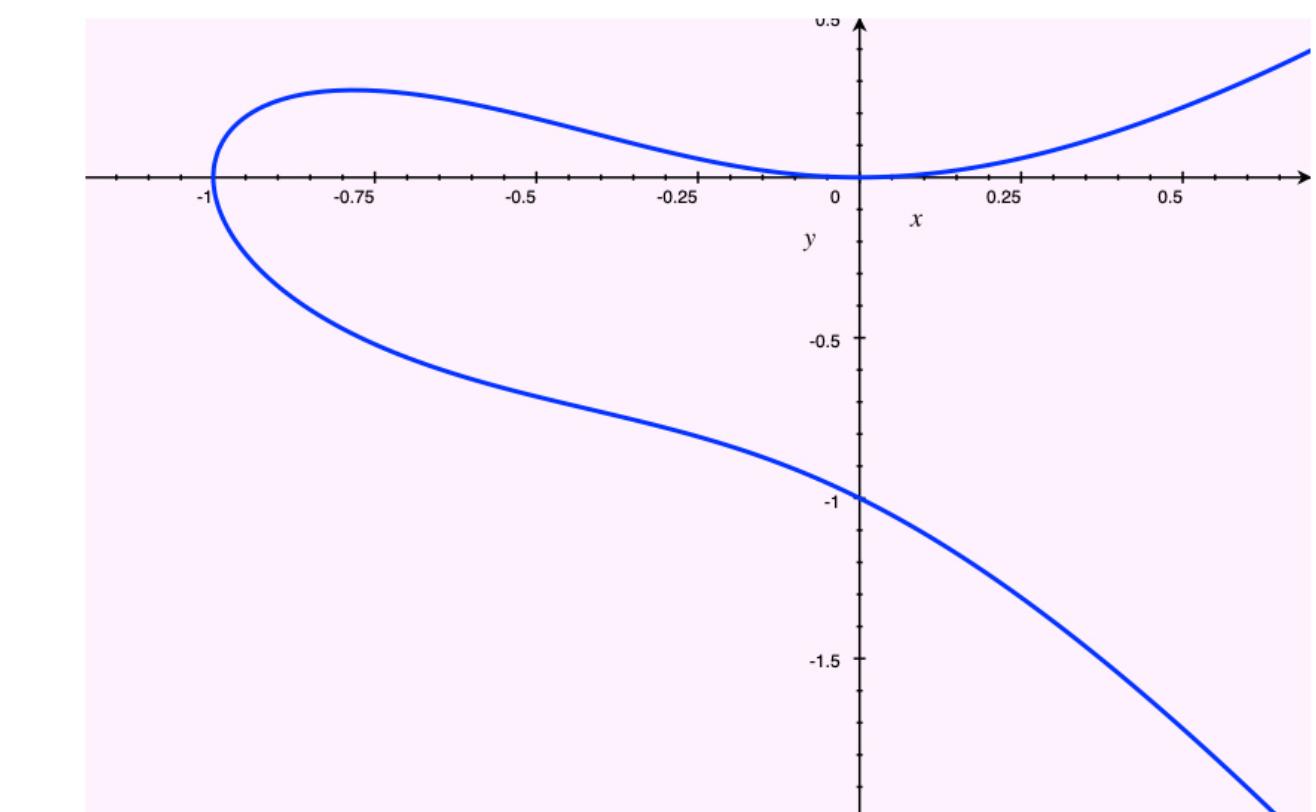
(Up to isomorphism over $\overline{\mathbb{Q}}$)



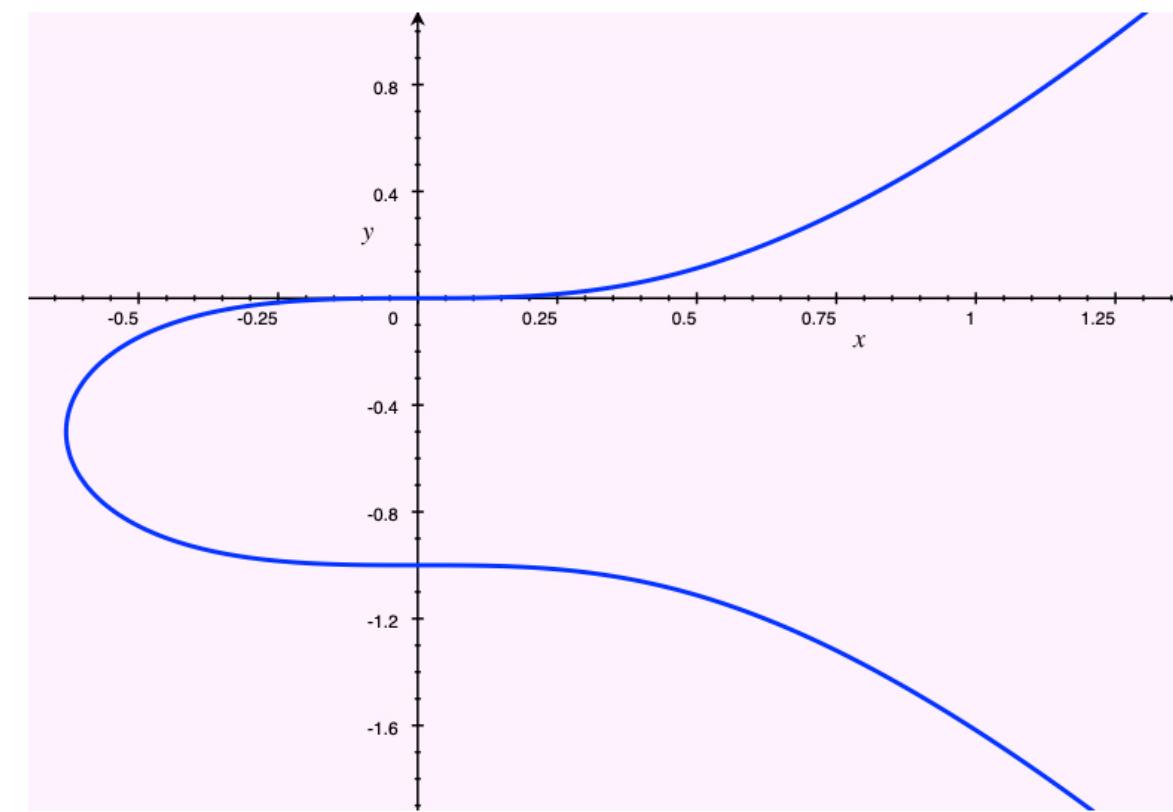
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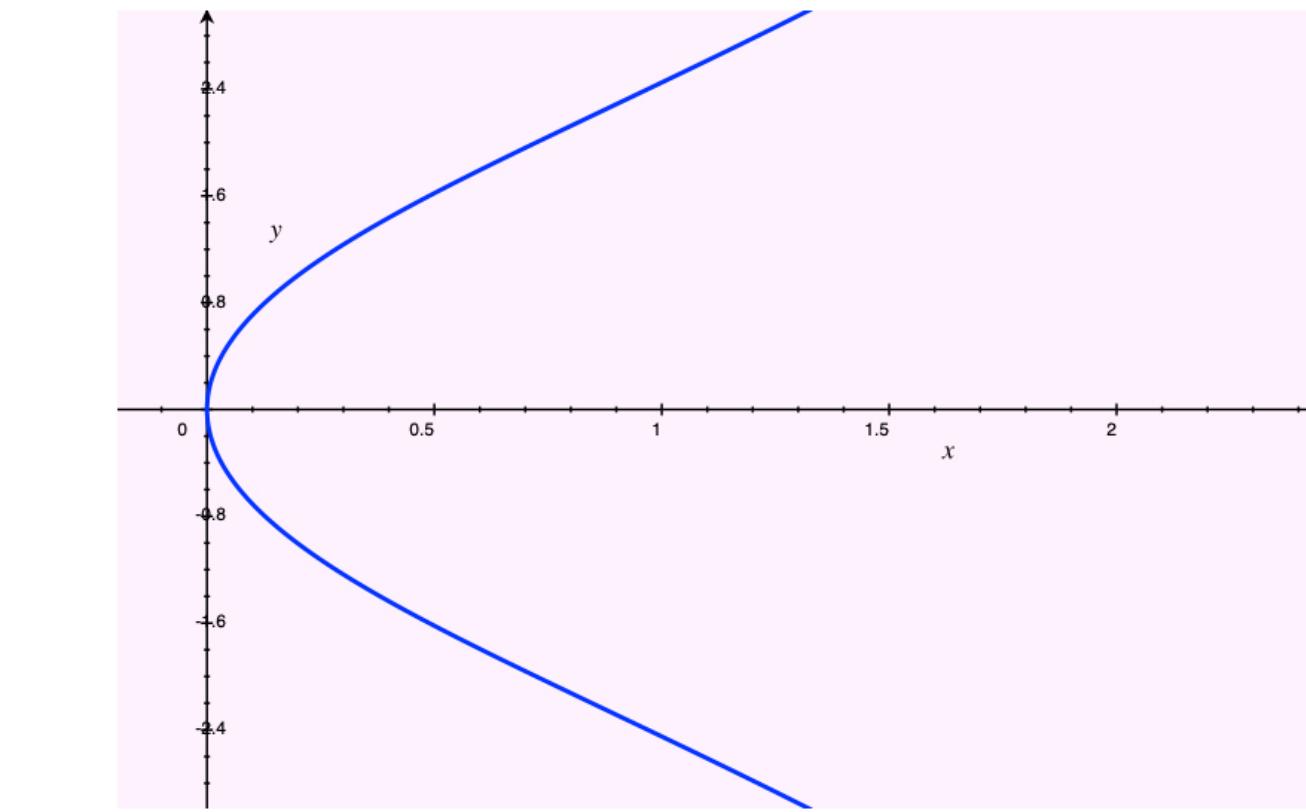
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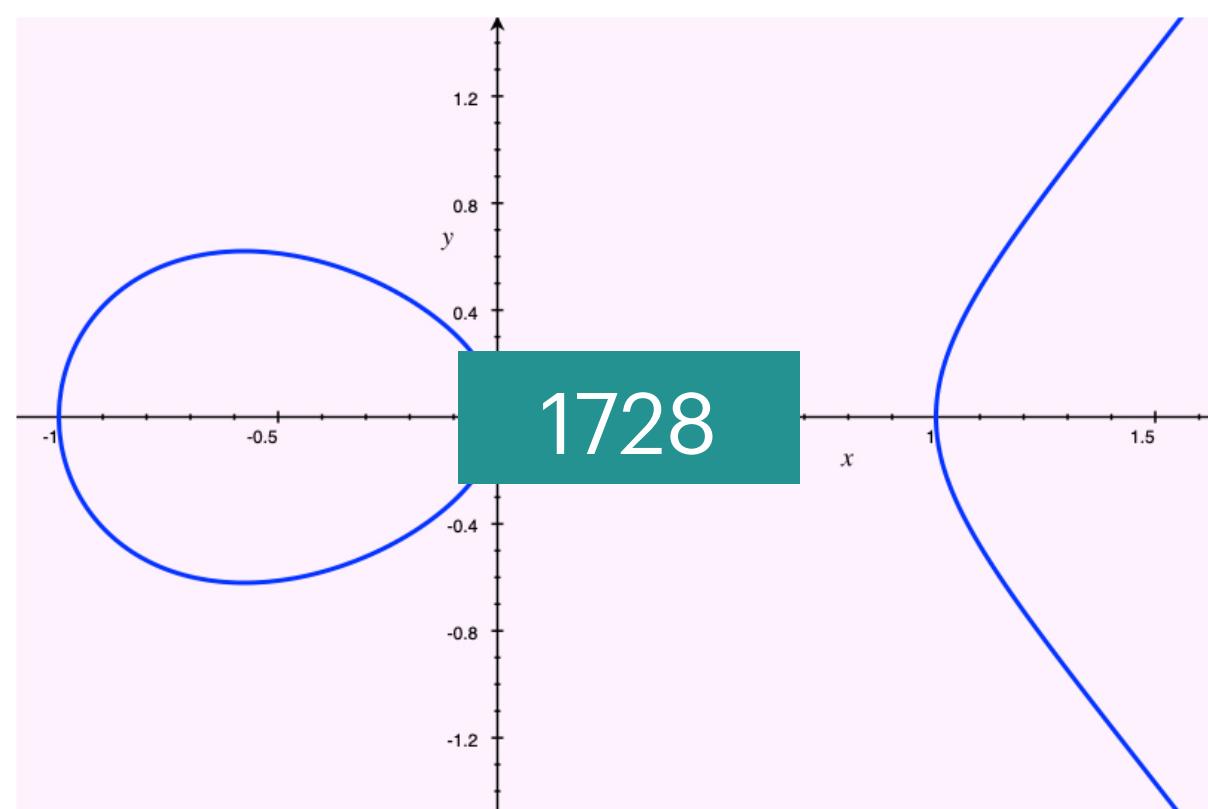
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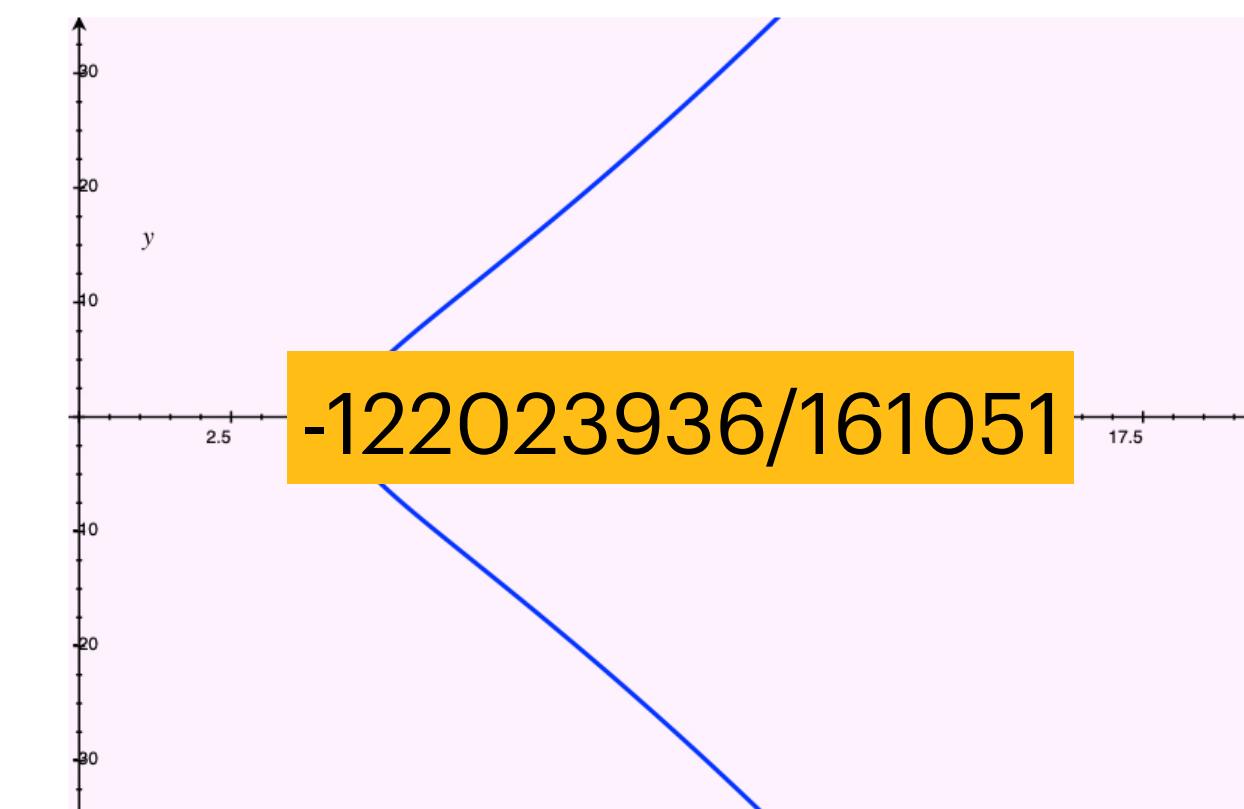
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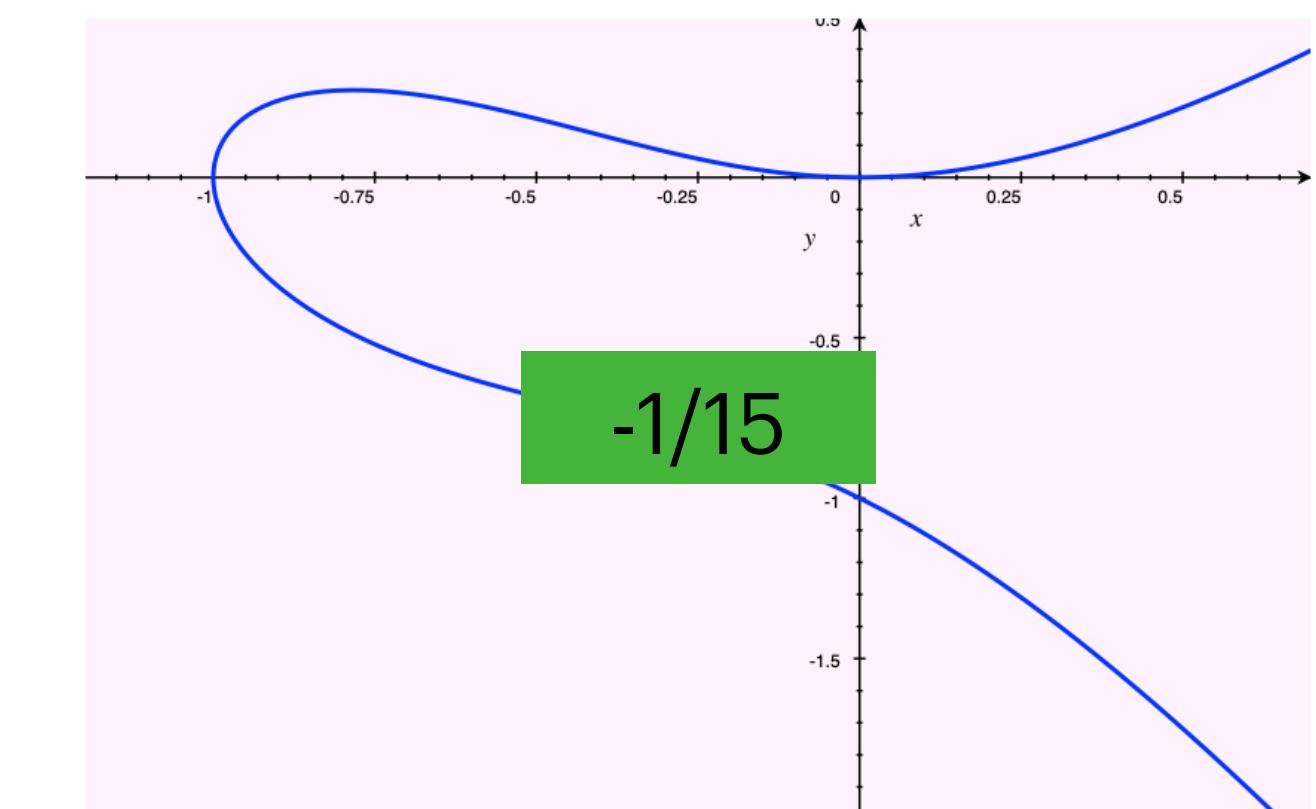
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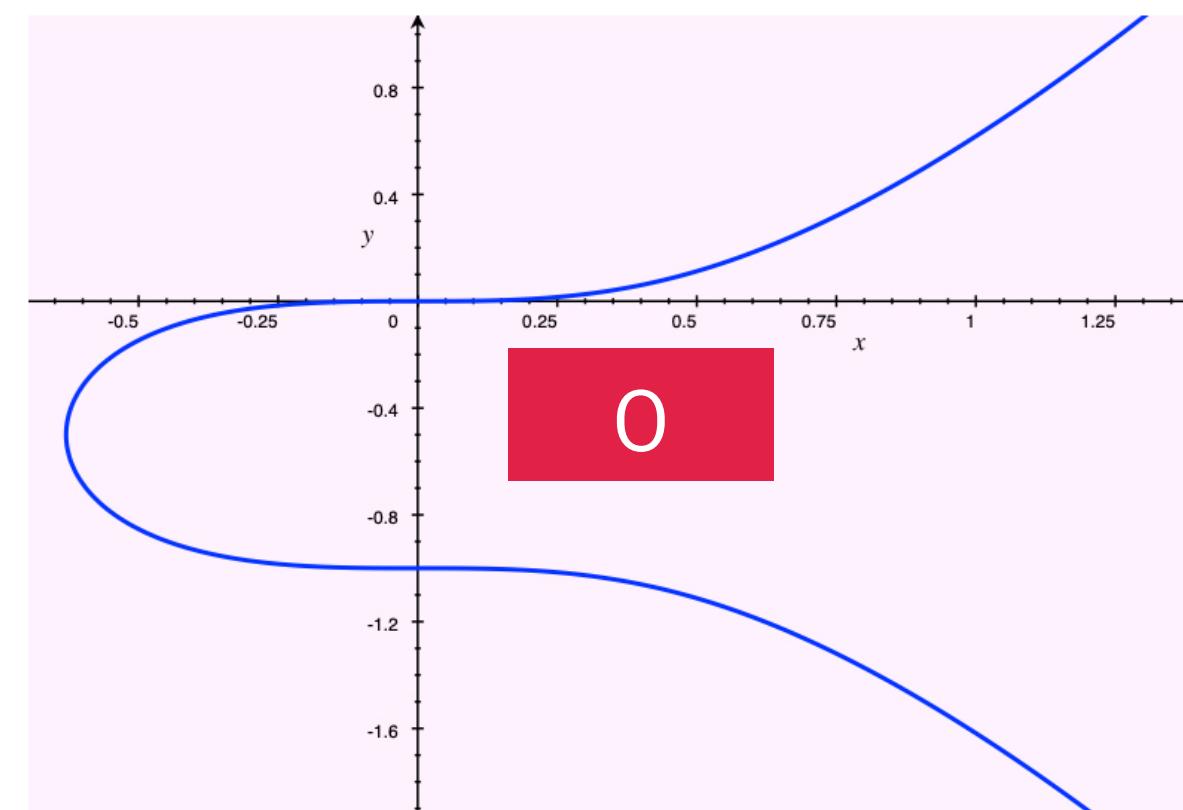
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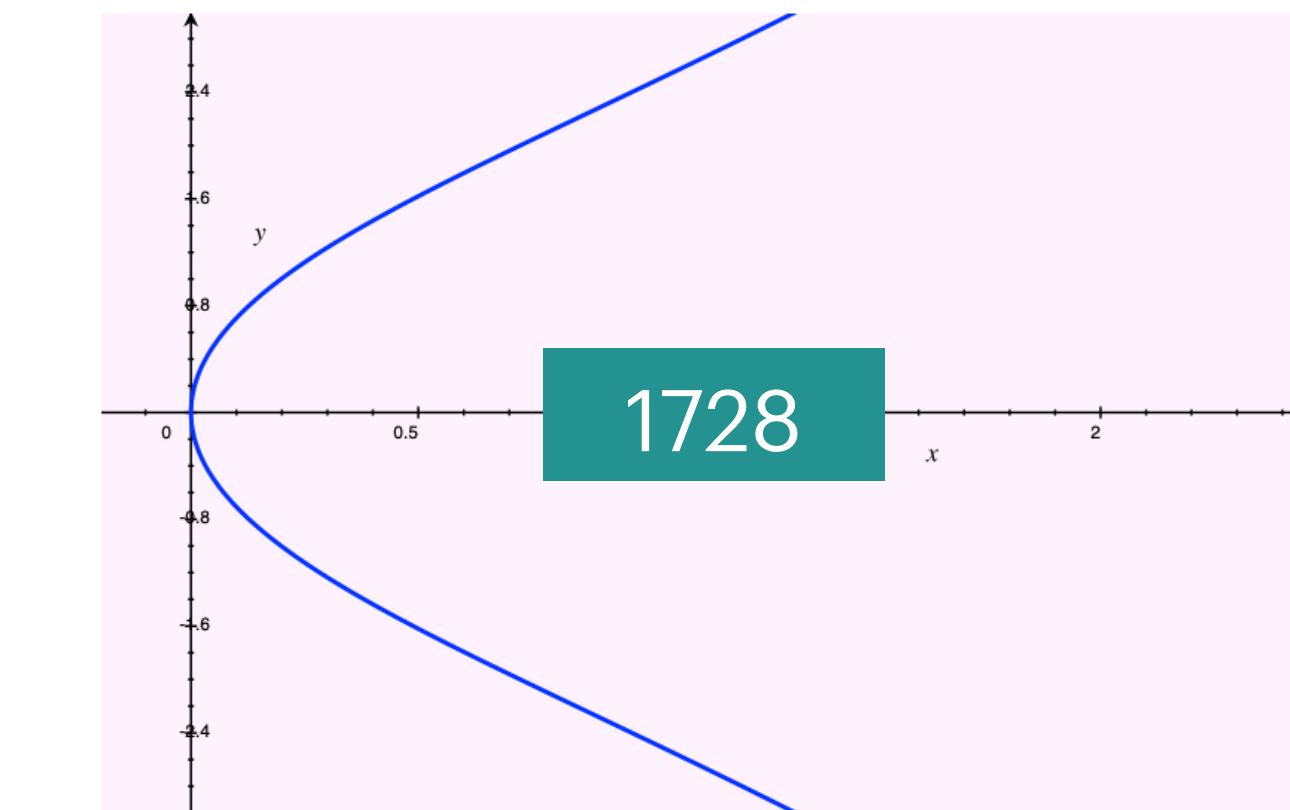
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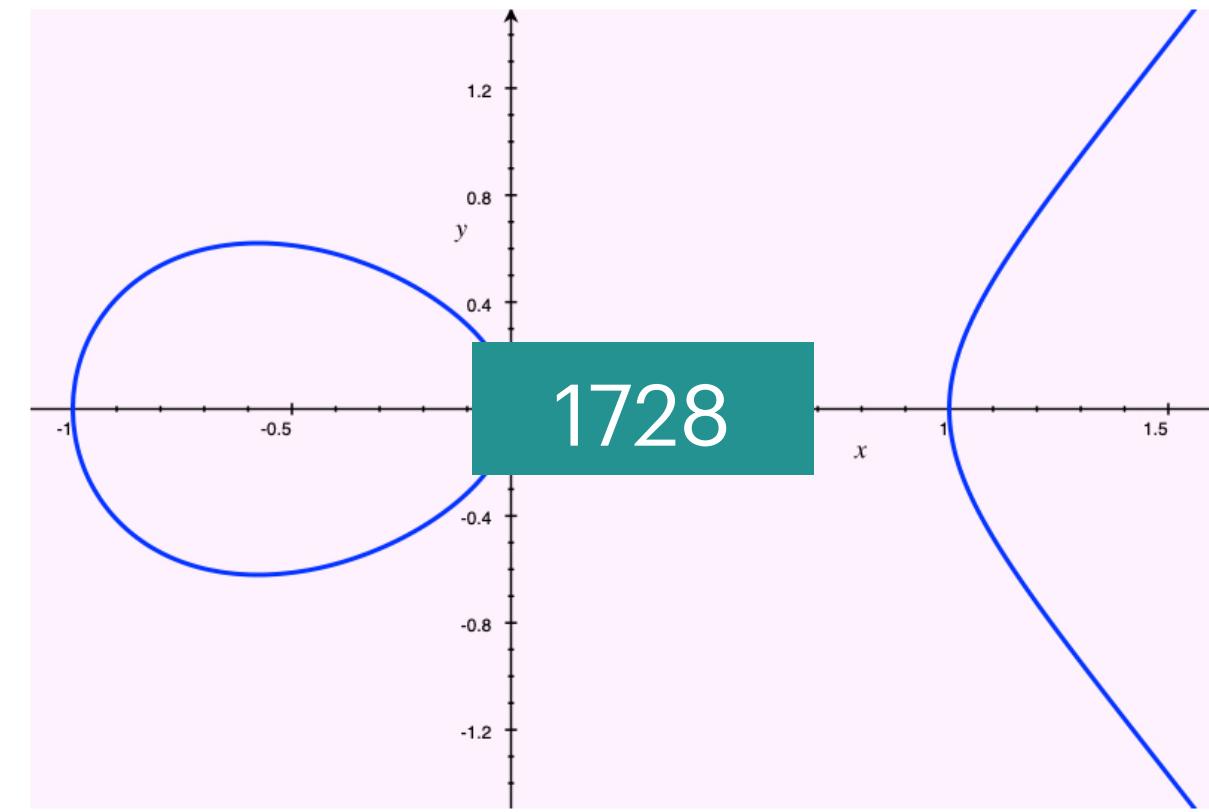
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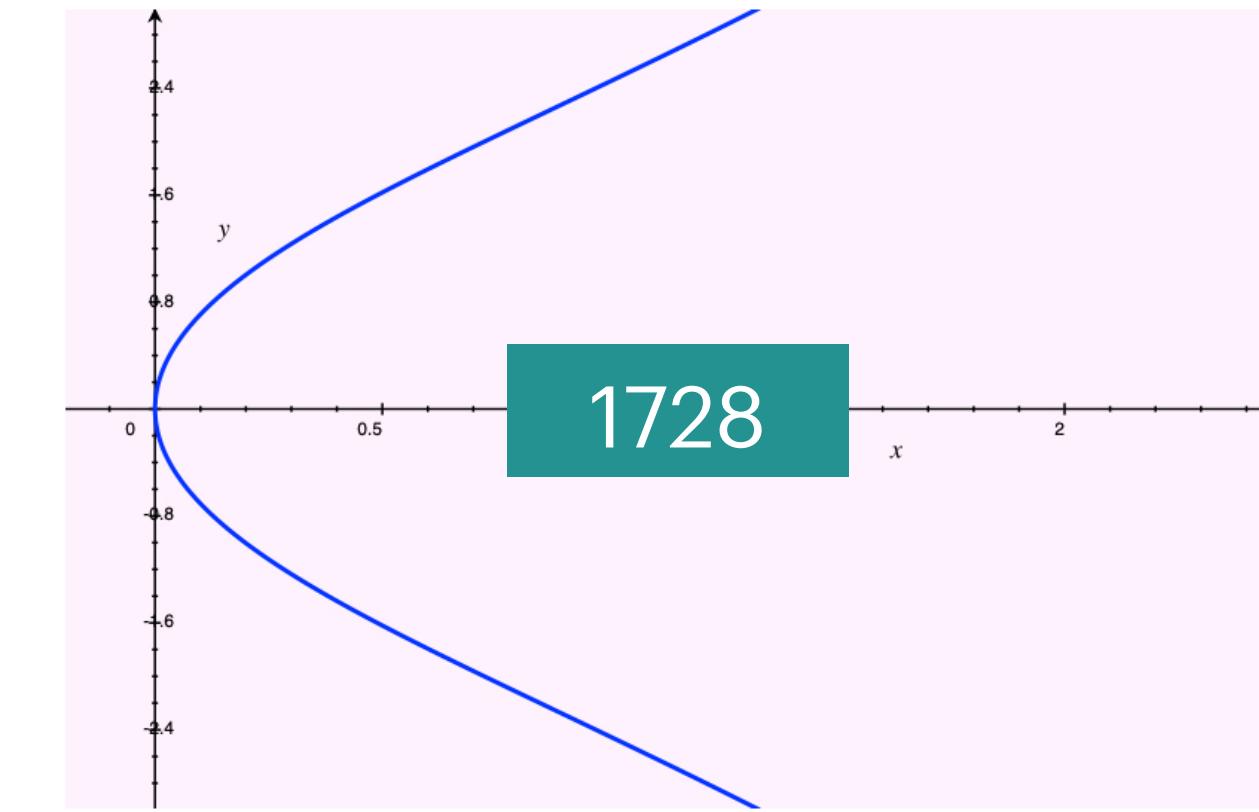
$$y^2 = x^3 + 5x$$

j-invariant!

An isomorphic pair



~



$$y^2 = x^3 - x$$

$$y^2 = x^3 + 5x$$

(x, y)

→

$(\alpha^2 x, \alpha^3 y)$

Where α is a root of $t^4 + 5 = 0$.

Elliptic curves and the j -invariant

An **elliptic curve** over any field K is a curve of the form

$$E: y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

for $a_1, a_2, a_3, a_4, a_6 \in K$.

The j -invariant of E is

$$\begin{aligned} & (a_1^{12} + 24a_1^{10}a_2 - 72a_1^9a_3 + 240a_1^8a_2^2 - 144a_1^8a_4 - 1152a_1^7a_2a_3 + 1280a_1^6a_2^3 - 2304a_1^6a_2a_4 + 1728a_1^6a_3^2 \\ & - 6912a_1^5a_2^2a_3 + 6912a_1^5a_3a_4 + 3840a_1^4a_2^4 - 13824a_1^4a_2^2a_4 + 13824a_1^4a_2a_3^2 + 6912a_1^4a_4^2 - 18432a_1^3a_2^3a_3 \\ & + 55296a_1^3a_2a_3a_4 - 13824a_1^3a_3^3 + 6144a_1^2a_2^5 - 36864a_1^2a_2^3a_4 + 27648a_1^2a_2^2a_3^2 + 55296a_1^2a_2a_4^2 - 82944a_1^2a_3^2a_4) \\ j(E) = & -18432a_1a_2^4a_3 + 110592a_1a_2^2a_3a_4 - 165888a_1a_3a_4^2 + 4096a_2^6 - 36864a_2^4a_4 + 110592a_2^2a_4^2 - 110592a_4^3) \\ & (-a_1^6a_6 + a_1^5a_3a_4 - a_1^4a_2a_3^2 - 12a_1^4a_2a_6 + a_1^4a_4^2 + 8a_1^3a_2a_3a_4 + 9a_1^3a_3^4 - 8a_1^3a_3^3 + 36a_1^3a_3a_6 - 8a_1^2a_2^2a_3^2 \\ & - 48a_1^2a_2^2a_6 + 8a_1^2a_2a_4^2 + 18a_1^2a_3^3a_4 - 48a_1^2a_3^2a_4 + 72a_1^2a_4a_6 + 16a_1a_2^2a_3a_4 + 36a_1a_2a_3^4 + 144a_1a_2a_3a_6 \\ & - 96a_1a_3a_4^2 - 16a_2^3a_3^2 - 64a_2^3a_6 + 16a_2^2a_4^2 + 72a_2a_3^3a_4 + 288a_2a_4a_6 - 27a_3^6 - 216a_3^3a_6 - 64a_4^3 - 432a_6^2)^{-1} \end{aligned}$$

Elliptic curves and the j -invariant

An **elliptic curve** over any field K is a curve of the form

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The j -invariant of E is

$$\begin{aligned} j(E) &= \frac{(a_1^{12} + 24a_1^{10}a_2 - 72a_1^9a_3 + 240a_1^8a_2^2 - 144a_1^8a_4 - 1152a_1^7a_2a_3 + 1280a_1^6a_2^3 - 2304a_1^6a_2a_4 + 1728a_1^6a_3^2 - 6912a_1^5a_2^2a_3 + 6912a_1^5a_3a_4 + 3840a_1^4a_2^4 - 13824a_1^4a_2^2a_4 + 13824a_1^4a_2a_3^2 + 6912a_1^4a_4^2 - 18432a_1^3a_2^3a_3 + 55296a_1^3a_2a_3a_4 - 13824a_1^3a_3^3 + 6144a_1^2a_2^5 - 36864a_1^2a_2^2a_4 + 27648a_1^2a_2^2a_3^2 + 55296a_1^2a_2a_4^2 - 82944a_1^2a_3^2a_4 - 18432a_1a_2^4a_3 + 110592a_1a_2^2a_3a_4 - 165888a_1a_3a_4^2 + 4096a_2^6 - 36864a_2^4a_4 + 110592a_2^2a_4^2 - 110592a_4^3) (-a_1^6a_6 + a_1^5a_3a_4 - a_1^4a_2a_3^2 - 12a_1^4a_2a_6 + a_1^4a_4^2 + 8a_1^3a_2a_3a_4 + 9a_1^3a_3^4 - 8a_1^3a_3^3 + 36a_1^3a_3a_6 - 8a_1^2a_2^2a_3^2 - 48a_1^2a_2^2a_6 + 8a_1^2a_2a_4^2 + 18a_1^2a_3^3a_4 - 48a_1^2a_3^2a_4 + 72a_1^2a_4a_6 + 16a_1a_2^2a_3a_4 + 36a_1a_2a_3^4 + 144a_1a_2a_3a_6 - 96a_1a_3a_4^2 - 16a_2^3a_3^2 - 64a_2^3a_6 + 16a_2^2a_4^2 + 72a_2a_3^3a_4 + 288a_2a_4a_6 - 27a_3^6 - 216a_3^3a_6 - 64a_4^3 - 432a_6^2)^{-1}}{a_1^{12} + 24a_1^{10}a_2 - 72a_1^9a_3 + 240a_1^8a_2^2 - 144a_1^8a_4 - 1152a_1^7a_2a_3 + 1280a_1^6a_2^3 - 2304a_1^6a_2a_4 + 1728a_1^6a_3^2 - 6912a_1^5a_2^2a_3 + 6912a_1^5a_3a_4 + 3840a_1^4a_2^4 - 13824a_1^4a_2^2a_4 + 13824a_1^4a_2a_3^2 + 6912a_1^4a_4^2 - 18432a_1^3a_2^3a_3 + 55296a_1^3a_2a_3a_4 - 13824a_1^3a_3^3 + 6144a_1^2a_2^5 - 36864a_1^2a_2^2a_4 + 27648a_1^2a_2^2a_3^2 + 55296a_1^2a_2a_4^2 - 82944a_1^2a_3^2a_4 - 18432a_1a_2^4a_3 + 110592a_1a_2^2a_3a_4 - 165888a_1a_3a_4^2 + 4096a_2^6 - 36864a_2^4a_4 + 110592a_2^2a_4^2 - 110592a_4^3) (-a_1^6a_6 + a_1^5a_3a_4 - a_1^4a_2a_3^2 - 12a_1^4a_2a_6 + a_1^4a_4^2 + 8a_1^3a_2a_3a_4 + 9a_1^3a_3^4 - 8a_1^3a_3^3 + 36a_1^3a_3a_6 - 8a_1^2a_2^2a_3^2 - 48a_1^2a_2^2a_6 + 8a_1^2a_2a_4^2 + 18a_1^2a_3^3a_4 - 48a_1^2a_3^2a_4 + 72a_1^2a_4a_6 + 16a_1a_2^2a_3a_4 + 36a_1a_2a_3^4 + 144a_1a_2a_3a_6 - 96a_1a_3a_4^2 - 16a_2^3a_3^2 - 64a_2^3a_6 + 16a_2^2a_4^2 + 72a_2a_3^3a_4 + 288a_2a_4a_6 - 27a_3^6 - 216a_3^3a_6 - 64a_4^3 - 432a_6^2)^{-1}} \end{aligned}$$

Theorem. Two elliptic curves E_1 and E_2 are isomorphic over \overline{K} if and only if $j(E_1) = j(E_2)$.

Elliptic curves and the j -invariant

An **elliptic curve** over any field K of characteristic not 2 or 3 is a curve of the form

$$E: y^2 = x^3 + ax + b,$$

for $a, b \in K$.

The j -invariant of E is

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

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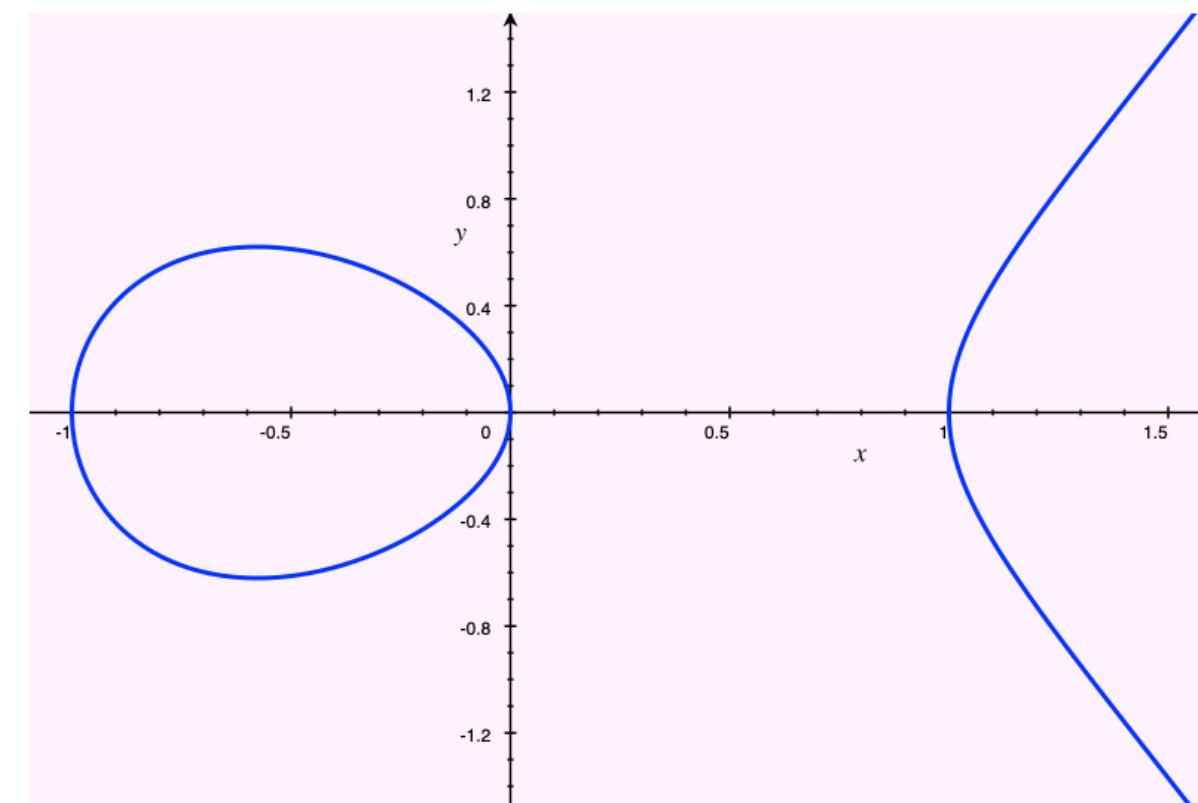
Elliptic curves and the j -invariant

The value $j(E)$ is a **reconstructing invariant**: for all $j(E) \in \overline{\mathbb{Q}}$, there is an elliptic curve with this j -invariant.

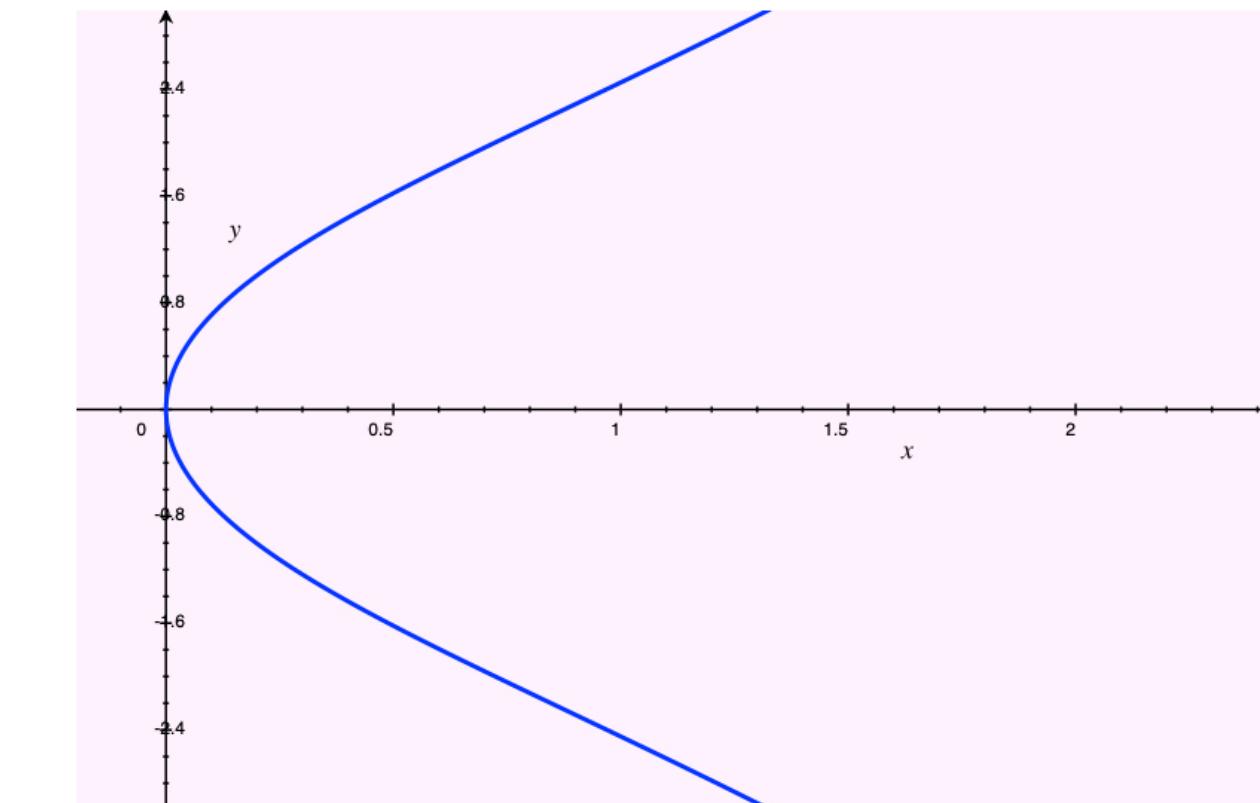
$$E: y^2 = x^3 + ax + b,$$

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

Example. For $j(E) = 1728$, we can pick



\approx



$$\begin{aligned} a &= -1 \text{ and } b = 0 \\ y^2 &= x^3 - x \end{aligned}$$

$$\begin{aligned} a &= 5 \text{ and } b = 0 \\ y^2 &= x^3 + 5x \end{aligned}$$

Invariants for curves

- Elliptic curves (genus 1): j -invariant.
- Genus 2 curves over \mathbb{Q} : Igusa-Clebsch invariants [Igusa '60].
- Genus 3 non-hyperelliptic curves [Ohno '07].
- Genus 3 hyperelliptic curves [Lercier-Ritzenthaler '12].

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Today: Invariants for Artin-Schreier curves.

Artin-Schreier curves

Fix a prime p . An **Artin-Schreier curve** over $\bar{\mathbb{F}}_p$ is a curve of the form

$$C_p: y^p - y = f(x),$$

where $f(x) \in \bar{\mathbb{F}}_p(x)$ and $f(x) \neq z^p - z$ for any $z \in \bar{\mathbb{F}}_p(x)$.

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Examples. Artin-Schreier curves over $\bar{\mathbb{F}}_3$ of genus 3:

$$y^3 - y = x^4 + x^2 + 1 \quad \text{and} \quad y^3 - y = \frac{x^3 - x^2 + 1}{x}.$$

Artin-Schreier curves

Theorem [DR-Goodson-Lorenzo García-Malmskog-Scheidler '24]. There is an explicit set of reconstructing invariants for all Artin-Schreier curves of genus 3 and 4 in odd characteristic.

Isomorphisms

Lemma. Any isomorphism between Artin-Schreier curves is given by a map of the form

$$(x, y) \mapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}, \lambda y + h(x) \right),$$

where

$$\lambda \in \mathbb{F}_p^\times, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\bar{\mathbb{F}}_p), \quad h(x) \in \bar{\mathbb{F}}_p(x).$$

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Example. $y^3 - y = x^4 - x^3 - x^2 + x$ and $y^3 - y = x^4 - x^2$ are isomorphic via

$$(x, y) \mapsto (-x + 1, y - \epsilon), \text{ where } \epsilon \text{ is a root of } t^3 - t - 1 = 0.$$

Standard form

Theorem [Farnell '10, DR-Goodson-Lorenzo García-Malmskog-Scheidler '24].

Let p be an odd prime and $C_f: y^p - y = f(x)$ be an Artin-Schreier $\bar{\mathbb{F}}_p$ -curve with $f(x)$ having one pole of order d . Then C_f is isomorphic to an Artin-Schreier curve

$$C: y^p - y = x^d + Q(x),$$

where $Q(x) \in \bar{\mathbb{F}}_p[x]$ is a multiple of x^2 and no monomial appearing in $Q(x)$ has an exponent that is divisible by p .

Standard form

Theorem [Farnell '10, DR-Goodson-Lorenzo García-Malmskog-Scheidler '24]. Let p be an odd prime and $C_f: y^p - y = f(x)$ be an Artin-Schreier $\bar{\mathbb{F}}_p$ -curve with $f(x)$ having 3 poles of respective orders $d_1 \geq d_2 \geq d_3$. Then C_f is isomorphic to

$$C_g: y^p - y = g(x),$$

where $g(x) \in \bar{\mathbb{F}}_p(x)$ is given by

$$g(x) = F(x) + G\left(\frac{1}{x}\right) + H\left(\frac{1}{x-1}\right),$$

for $F(x), G(x), H(x) \in \bar{\mathbb{F}}_p[x]$, $\deg(F) = d_1$, $\deg(G) = d_2$, $\deg(H) = d_3$, and no monomial appearing in $F(x), G(x), H(x)$, has an exponent that is divisible by p .

Example: $\mathcal{AS}_{3,0}$

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Every Artin-Schreier $\bar{\mathbb{F}}_3$ -curve with $f(x)$ having only one pole of order 4 is isomorphic to a curve of the form

$$C: y^3 - y = x^4 + ax^2$$

where $a \in \bar{\mathbb{F}}_3$.

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To describe the space $\mathcal{AS}_{3,0}$ of Artin-Schreier $\bar{\mathbb{F}}_3$ -curves with one pole of order 4, it is enough to give $a \in \bar{\mathbb{F}}_p$!

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$\mathcal{AS}_{3,0}$ denotes the moduli space of Artin-Schreier $\bar{\mathbb{F}}_3$ -curves of genus 3 and p -rank 0.

Example: $\mathcal{AS}_{3,0}$

$$C: y^3 - y = x^4 + ax^2$$

Proposition. Isomorphisms between curves in standard form with $a \neq 0$ are given, up to composition with powers of σ : $(x, y) \mapsto (x, y + 1)$, by

$$(x, y) \mapsto (\alpha x, \lambda y),$$

where $\lambda \in \mathbb{F}_3^\times$ and $\alpha \in \bar{\mathbb{F}}_3$ with $\alpha^4 = \lambda$.

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Now we have a finite group G acting on $\bar{\mathbb{F}}_3[a]$!

Example: $\mathcal{AS}_{3,0}$

$$C: y^3 - y = x^4 + ax^2$$

$(x, y) \mapsto (\alpha x, \lambda y)$, $\lambda \in \mathbb{F}_3^\times$ and $\alpha \in \bar{\mathbb{F}}_3$ with $\alpha^4 = \lambda$.

λ	α	\tilde{C}	a
1	ζ_8^2	$y^3 - y = x^4 + \zeta_8^2 ax^2$	$\zeta_8^2 a$
1	ζ_8^4	$y^3 - y = x^4 + \zeta_8^4 ax^2$	$\zeta_8^4 a$
1	ζ_8^6	$y^3 - y = x^4 + \zeta_8^6 ax^2$	$\zeta_8^6 a$
1	ζ_8^8	$y^3 - y = x^4 + ax^2$	$\zeta_8^8 a$

λ	α	\tilde{C}	a
-1	ζ_8	$y^3 - y = x^4 - \zeta_8 ax^2$	$\zeta_8^5 a$
-1	ζ_8^3	$y^3 - y = x^4 - \zeta_8^3 ax^2$	$\zeta_8^7 a$
-1	ζ_8^5	$y^3 - y = x^4 - \zeta_8^5 ax^2$	$\zeta_8 a$
-1	ζ_8^7	$y^3 - y = x^4 - \zeta_8^7 ax^2$	$\zeta_8^3 a$

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$$C: y^3 - y = x^4 + ax^2$$

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$$\{a\}^G = \{\zeta_8^2 a, \zeta_8^4 a, \zeta_8^6 a, \zeta_8^8 a, \zeta_8^5 a, \zeta_8^7 a, \zeta_8 a, \zeta_8^3 a\}$$

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Corollary. The element a^8 is a reconstructing invariant and generates the ring of invariants for $\mathcal{AS}_{3,0}$.

Example: $\mathcal{AS}_{3,0}$

If we start with a model

$$C: y^3 - y = \frac{ax^4 + bx^3 + cx^2 + dx + e}{(x - \tau)^4},$$

the reconstructing invariant can be chosen to be

$$I(C) = \frac{c^8}{(a\tau^4 + b\tau^3 + c\tau^2 + d\tau + e)^4}.$$

Example: $\mathcal{AS}_{3,0}$

If we start with a model

$$C: y^3 - y = \frac{ax^4 + bx^3 + cx^2 + dx + e}{(x - \tau)^4},$$

the reconstructing invariant can be chosen to be

$$I(C) = \frac{c^8}{(a\tau^4 + b\tau^3 + c\tau^2 + d\tau + e)^4}.$$

Example.

$$\begin{aligned} I(y^3 - y = x^4 - x^3 - x^2 + x) &= I(y^3 - y = x^4 - x^2) &= 1 \\ y^3 - y = x^4 - x^3 - x^2 + x &\simeq y^3 - y = x^4 - x^2 \\ (x, y) &\mapsto (-x + 1, y - \epsilon) & \epsilon^3 - \epsilon - 1 = 0. \end{aligned}$$

Theorem [DR-Goodson-Lorenzo García-Malmskog-Scheidler '24]. A system of reconstructing invariants for all Artin-Schreier curves of genus $g = 3, 4$ in characteristic $p > 2$ is:

g	p	s	Standard form	Set of Reconstructing invariants over $\overline{\mathbb{F}}_p$
3	3	0	$y^3 - y = x^4 + ax^2$	$\{a^8\}$
3	3	2	$y^3 - y = x^2 + ax + \frac{b}{x}$	$\{a^4, ab, b^4\}$
3	7	0	$y^7 - y = x^2$	\emptyset
4	3	0	$y^3 - y = x^5 + cx^4 + dx^2$	$\{(c^3 + d)^{10}, (-cd - \epsilon^2)^5, (c^3 + d)^2(-cd - \epsilon^2)\}$ where $\epsilon^3 = c$
4	3	2	$y^3 - y = x^2 + ax + \frac{b}{x} + \frac{c}{x^2}$	$\{c, ab, a^4c^2 - b^4\}$
4	3	4	$y^3 - y = x^2 + ax + \frac{b}{x} + \frac{c}{x-1}$	$\{(abc)^2, (abc)(a - b - c), ab + ac - bc\}$
4	5	0	$y^5 - y = x^3 + ax^2$	$\{a^{12}\}$
4	5	1	$y^5 - y = x + \frac{a}{x}$	$\{a^2\}$

The general algorithm

Algorithm [DR-Lorenzo García-Malmskog-Scheidler '25+].

1. Write a general curve C in standard form, with variables a_1, \dots, a_n .
2. Find the possible transformations of C that produce a new curve in standard form.
They are composition of the following:
 - a. Swap poles that are not distinguished and have the same pole order.
 - b. Pick new distinguished poles (keeping the order of the poles the same).
 - c. Act with $\lambda \in \bar{\mathbb{F}}_p$ as $(x, y) \mapsto (x, \lambda y)$.
3. Collect the possible images of a_1, \dots, a_n under the transformations from part 2.
4. Compute invariants of this representation.



“por los bellos países donde el verde es de todos los colores”

Aurelio Arturo

