MA 293

Homework 7

Due Mar 21 at 4 pm

Recall the homework guidelines for this course.

In this homework set, for integers $0 \leq k \leq n$, we define the set $B_{n,k}$ to be the set of subsets of $\{1, \ldots, n\}$ containing k elements (so $B_{n,k} \subseteq \mathscr{P}(\{1, \ldots, n\})$). For example,

$$B_{4,2} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\} \subseteq \mathscr{P}(\{1,2,3,4\}).$$

1. Let's do some examples first.

   (a) Compute $B_{4,k}$ for all $k \in \{0,1,2,3,4\}$.

   (b) Give bijective functions from $B_{4,0}$ to $B_{4,4}$ and from $B_{4,1}$ to $B_{4,3}$. Do not forget to explain why they are bijective functions.

   (c) Compute $B_{5,k}$ for all $k \in \{0,1,2,3,4,5\}$.

   (d) Give bijective functions from $B_{5,0}$ to $B_{5,5}$, from $B_{5,1}$ to $B_{5,4}$, and from $B_{5,2}$ to $B_{5,3}$. Do not forget to explain why they are bijective functions.

2. Now we can generalize the examples.

   (a) Given any positive integers n and k with $k \leq n$, give a bijective function between $B_{n,k}$ and $B_{n,n-k}$. That is, define the function and prove that it is bijective. *Hint: use the complement of the subset.*

   (b) Prove that $|B_{n,k}| = |B_{n,n-k}|$ for any $0 \leq k \leq n$.

3. Another interesting property of $|B_{n,k}|$: for $k \geq 1$, prove that $|B_{n,k}| = |B_{n-1,k}| + |B_{n-1,k-1}|$.

4. Now let's compute all of the possible values.

   (a) Show that for all $n \geq 0$, we have $|B_{n,0}| = |B_{n,n}| = 1$.

   (b) Fill out the following triangle:

$$\begin{array}{ccccccccccc}
 & & & & & |B_{0,0}| & & & & & \\
 & & & & |B_{1,0}| & & |B_{1,1}| & & & & \\
 & & & |B_{2,0}| & & |B_{2,1}| & & |B_{2,2}| & & & \\
 & & |B_{3,0}| & & |B_{3,1}| & & |B_{3,2}| & & |B_{3,3}| & & \\
 & |B_{4,0}| & & |B_{4,1}| & & |B_{4,2}| & & |B_{4,3}| & & |B_{4,4}| & \\
|B_{5,0}| & & |B_{5,1}| & & |B_{5,2}| & & |B_{5,3}| & & |B_{5,4}| & & |B_{5,5}|
\end{array}$$

   Hint: the triangle should look familiar.

   (c) How would you use what you have learned to compute $|B_{100,30}|$ without writing the set explicitly? This asks for an explanation of a procedure and **not** a number.

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5. Now we will find an identity that relates $|\mathscr{P}(\{1,\dots,n\})|$ with the sum[1]

$$|B_{n,0}| + |B_{n,1}| + \cdots + |B_{n,n-1}| + |B_{n,n}|.$$

   (a) Show that if $A \in \mathscr{P}(\{1,\dots,n\})$, then A belongs to $B_{n,k}$ for a unique $k \in \{0,\dots,n\}$.

   (b) Conclude that $|\mathscr{P}(\{1,\dots,n\})| = |B_{n,0}| + |B_{n,1}| + \cdots + |B_{n,n-1}| + |B_{n,n}|$.

6. Now let's find a different pattern

   (a) Compute the sum of all entries in each row from the completed triangle in Problem 4. What pattern do you observe?

   (b) How many elements does $\mathscr{P}(\{1,2,\dots,n\})$ have? Justify your answer using the fact that each element of $\{1,2,\dots,n\}$ can either be included in a subset or not.

   (c) Prove that

$$2^n = |B_{n,0}| + |B_{n,1}| + \cdots + |B_{n,n-1}| + |B_{n,n}|.$$

[1] Note that this is the sum of the rows of the triangle.