Procesos Estocásticos - Cadena de Markov

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ESTIMACIÓN

Consideremos una cadena de Markov $\{X_n; n = 0, 1, \ldots\}$

Maximum-likelihood estimation: Consider a time-homogeneous Markov chain with a finite number, m, of states $(1, 2, \ldots, m)$ and having transition probability matrix $P = (p_{jk})$, $j, k = 1, 2, \ldots, m$. Suppose that the number of observed direct transitions from the state j to the state k is n_{jk}, and that the total number of observations is $(N + 1)$. Put

$$\sum_{k=1}^{m} n_{jk} = n_j \quad \text{and} \quad \sum_{j=1}^{m} n_{jk} = n_k, \qquad j, k = 1, 2, \ldots, m.$$

$$f(n_{jk}) = T(n_{jk}) \frac{\prod_j (n_{j.})!}{\prod_j \prod_k (n_{jk})!} \prod_j \prod_k p_{jk}^{n_{jk}} ;$$

the factor $T(n_{jk})$ which involves the joint distribution of the n_j's is independent of the p_{jk}'s.

The logarithm of the likelihood function can be put as

$$L(p_{jk}) = C + \sum_{j=1}^{m} \sum_{k=1}^{m} n_{jk} \log p_{jk}$$

where C contains all terms independent of p_{jk}.

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Since $\sum_{k} p_{jk} = 1$, (10.2) can be written as

$$L(p_{jk}) = C + \sum_{j=1}^{m}\sum_{k=1}^{m-1} n_{jk} \log p_{jk} + \sum_{j=1}^{m} n_{jm} \log\left(1 - \sum_{k=1}^{m-1} p_{jk}\right).$$

Let r be a specific value of j. The maximum-likelihood estimates $\hat{p}_{rk}$ are given by the the equations

$$\frac{\partial L(p_{rk})}{\partial p_{rk}} = 0, \qquad k = 1, 2, \ldots, (m-1).$$

These equations give

$$\frac{n_{rk}}{p_{rk}} - \frac{n_{rm}}{1 - \sum_{k=1}^{m-1} p_{rk}} = 0, \quad k = 1, 2, \ldots, (m-1).$$

To fix our ideas, let us take a specified value, s, of k. Then

$$\frac{n_{rs}}{p_{rs}} = \frac{n_{rk}}{p_{rk}} = \frac{n_{rm}}{1 - \sum_{k=1}^{m-1} p_{rk}}, \qquad k = 1, 2, \ldots, s, \ldots (m-1).$$

Thus

$$1 - \sum_{k=1}^{m-1} p_{rk} = \frac{n_{rm}}{n_{rs}} p_{rs}$$

and $$p_{rk} = \frac{n_{rk}}{n_{rs}} p_{rs}, \quad k = 1, 2, \ldots, s, \ldots (m-1).$$

Summing (10.6) overall k and adding (10.5), we get

$$1 = \frac{\sum_{k=1}^{m-1} n_{rk}}{n_{rs}} p_{rs}$$

and hence the estimate $\hat{p}_{rs}$ is given by

$$\hat{p}_{rs} = \frac{n_{rs}}{\sum_{k=1}^{m} n_{jk}}$$

Now r, s are two arbitrary values of j, k respectively. Hence, for

$$j, k = 1, 2, \ldots, (m-1)$$

$$\hat{p}_{jk} = \frac{n_{jk}}{\sum_{k=1}^{m} n_{jk}} = \frac{n_{jk}}{n_{j\cdot}}.$$

Consideremos las siguientes observaciones de una cadena de Markov con estados 0 y 1:

$$0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0$$

La tabla de contingencia para las transiciones es

$$\begin{array}{cc|c} 3 & 4 & 7 \\ 4 & 19 & 23 \\ \hline 7 & 23 & 30 \end{array}$$

y usando el estimador dado por (3.46) obtenemos los valores estimados para la matriz de transición

$$\widehat{P} = \begin{pmatrix} 3/7 & 4/7 \\ 4/23 & 19/23 \end{pmatrix}.$$

(*i*) Suppose that one wishes to test the null hypothesis that the observed realisation comes from a Markov chain with a given transition, *i.e.* matrix $P^0 = (p_{jk}^0)$; suppose that the null hypothesis is

$$H_0 : P = P^0.$$

Then, for large N, and for $\hat{p}_{jk}$ given by (10.8), the statistic

$$\sum_{k=1}^{m} \frac{n_{j\cdot}\left(\hat{p}_{jk} - p_{jk}^{0}\right)^{2}}{p_{jk}^{0}}, \quad j = 1, 2, \ldots m,$$

is distributed as χ^2 with $m - 1$ d.f. (degrees of freedom). Here p_{jk}^{0}'s which are equal to zer excluded and the d.f. is reduced by the number of p_{jk}^{0}'s equal to 0. Alternatively, a test fo $\hat{p}_{jk}$ can be obtained by adding overall j, and the statistic

$$\sum_{j=1}^{m}\sum_{k=1}^{m} \frac{n_{j\cdot}\left(\hat{p}_{jk} - p_{jk}^{0}\right)^{2}}{p_{jk}^{0}} \qquad (1$$

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\frac{n_{j\cdot}\left(\hat{p}_{jk}-p_{jk}^{0}\right)^{2}}{p_{jk}^{0}}$$

has an asymptotic χ^2-distribution with $m\,(m-1)$ d.f.
(the number of d.f. being reduced by the number of p_{jk}^{0}'s equal to zero, if any, for j, k m).
The likelihood ratio criterion for H_0 is given by

$$\lambda=\prod_{j}\prod_{k}\left(\frac{\hat{p}_{jk}}{p_{jk}^{0}}\right)^{n_{jk}}.$$

Thus, under the null hypothesis, the statistic

$$-2\log\lambda=2\sum_{j}\sum_{k}n_{jk}\log\frac{n_{jk}}{(n_{j\cdot})\,p_{jk}^{0}}$$

has an asymptotic χ^2-distribution with $m\,(m-1)$ d.f.

(*ii*) The maximum-likelihood estimates could also be used to test the order of a Markov chain. For testing the null hypothesis that the chain is of order zero, *i.e.* H_0: $p_{jk} = p_k$ for all j, against the alternative that the chain is of order 1, the test criterion is

$$\lambda = \prod_j \prod_k \left(\frac{\hat{p}_k}{\hat{p}_{jk}} \right)^{n_{jk}}$$

where

$$\hat{p}_k = n_{.k} \Big/ \left(\sum_j \sum_k n_{jk} \right), \qquad \hat{p}_{jk} = n_{jk} / n_{j}.$$

Under the null hypothesis, the statistic

$$-2 \log \lambda = 2 \sum_j \sum_k n_{jk} \log \frac{n_{kj} \left(\sum \sum n_{jk} \right)}{(n_{.k})(n_{j.})}$$

ıas an asymptotic χ^2-distribution with $(m - 1)^2$ d.f. Similar test can be constructed for testing the null ıypothesis that the chain is of order one against the alternative that it is of order two.