CONTINUOUS TIME MARKOV CHAIN- CTMC

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CTMC

Definition

Let $X = \{X_t; t \geq 0\}$ be a stochastic process with countable state space S. We say that the process is a *continuous-time Markov chain*, if

$$P\left(X_{t_n} = j \mid X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}\right) = P\left(X_{t_n} = j \mid X_{t_{n-1}} = i_{n-1}\right)$$

for all $j, i, \ldots, i_{n-1} \in S$ and for all $0 \leq t_1 < t_2 < \ldots < t_n$.

For Markov chains with discrete-time parameter we saw that the $n-$ step transition matrix can be expressed in terms of the transition matrix raised to the power of n. In the continuous-time case there is no exact analog of the transition matrix P since there is no implicit unit of time. We will see in this section that there exists a matrix Q called the infinitesimal generator of the Markov chain which plays the role of P.

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We say that the continuous-time Markov chain is *homogeneous*, if and only if, the probabilities $P(X_{t+s} = j \mid X_s = i)$ is independent of s for all t.

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Introduction

Let $p_{ij}(t)$ be the probability of transition from state i to state j in an interval of length t. We denote by

$$P(t) = (p_{ij}(t)) \quad \text{for all } i,j \in S\ .$$

We say that $P(t)$ is a transition probability matrix. It is easy to verify that it satisfies the following conditions:

(1) $p_{ij}(0) = \delta_{ij}$

(2) $\lim_{t\to 0^+} p_{ij}(t) = \delta_{ij}$

(3) For any $t \geq 0$, $i,j \in S$, $0 \leq p_{ij}(t) \leq 1$ and $\sum_{k\in S} p_{ik}(t) = 1$ and

(4) For all $i,j \in S$, for any $s,t \geq 0$:

$$p_{ij}(t+s) = \sum_{k\in S} p_{ik}(t).p_{kj}(s)\ . \tag{1}$$

The above equation is called a Chapman-Kolmogorov equation for continuous-time Markov chains.

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1. From part (2) of the above observation we get:

$$\lim_{t \to 0^+} P(t) = I$$

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The following properties of transition probabilities are extremely important for applications of continuous-time Markov chains.

1. $p_{ij}(t)$ is uniformly continuous on $[0, \infty)$.
2. For each $i \in S$ we have:

$$\lim_{t \to 0^+} \frac{1 - p_{ii}(t)}{t} = q_i$$

   exists (but may be equal to $+\infty$).
3. For all $i, j \in S$ with $i \neq j$, we have that the following limit exists:

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DEFINITION

The matrix

$$Q = \begin{bmatrix} -q_0 & q_{01} & q_{02} & \cdots \\ q_{10} & -q_1 & q_{12} & \cdots \\ q_{20} & q_{21} & -q_2 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

is called the *infinitesimal generator* of the Markov chain $\{X_t; t \geq 0\}$.

Since $P(0) = I$, we conclude that

$$Q = P'(0).$$

Suppose that S is finite or countable. The matrix $Q = (q_{ij})_{i,j \in S}$ satisfies the following properties:

1. $q_{ii} \leq 0$ for all i.
2. $q_{ij} \geq 0$ for all $i \neq j$.
3. $\sum_j q_{ij} = 0$ for all i.

The infinitesimal generator Q of the Markov chain $\{X_t; t \geq 0\}$ play an essential role in the theory of continuous time Markov chains.

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1. A state $i \in S$ is called an *absorbing state* if $q_i = 0$.
2. If $q_i < \infty$ and $q_i = \sum_{j \neq i} q_{ij}$, then the state i is called *stable* or *regular*.
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Suppose that $q_i < \infty$ for each $i \in S$, then the transition probabilities $p_{ij}(t)$ are differentiable for all $t \geq 0$ and $i, j \in S$, and satisfy the following equations:

1. (Kolmogorov forward equation)

$$p'_{ij}(t) = -q_j p_{ij}(t) + \sum_{k \neq j} q_{kj} p_{ik}(t)$$

2. (Kolmogorov backward equation)

$$p'_{ij}(t) = -q_i p_{ij}(t) + \sum_{k \neq i} q_{ik} p_{kj}(t)$$

or equivalently:

$$\begin{aligned} P'(t) &= P(t)Q \\ P'(t) &= QP(t) \end{aligned}$$

The initial condition for the both the equations is $P(0) = I$.

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For a given matrix Q we can define a stochastic matrix P as follows:

1. If $q_i \neq 0$,

$$p_{ij} := \begin{cases} \frac{q_{ij}}{q_i}, & i \neq j \\ 0, & i = j \end{cases} . \tag{3}$$

2. If $q_i = 0$ then $P_{ij} := \delta_{ij}$.

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Example

Let

$$Q = \begin{pmatrix} -5 & 3 & 2 \\ 1 & -2 & 1 \\ 4 & 0 & -4 \end{pmatrix}$$

then

$$P = \begin{pmatrix} 0 & \frac{3}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} .$$

Since,

$$q_1 = \sum_{j \neq 1} q_{1j} = 5$$

we have for example,

$$p_{12} = \frac{q_{12}}{q_1} = \frac{3}{5}. \qquad \blacktriangle$$

Example

Consider a 2 unit system. Unit A has a failure rate λ_A and unit B has failure rate λ_B. There is one repairman and the repair rate of each of the units is μ. When both the machines fail, the system comes to a stop. In this case, $\{X_t; t \geq 0\}$ is a continuous time Markov chain with state space $S = \{0, 1_A, 1_B, 2\}$ where 0 denotes both units failed, 1_A denotes unit A working and unit B has failed, 1_B denotes unit A has failed and unit B working and 2 denotes both units working. The corresponding infinitesimal generator matrix is given by

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda_A & -(\lambda_A + \mu) & 0 & \mu \\ \lambda_B & 0 & -(\lambda_B + \mu) & \mu \\ 0 & \lambda_B & \lambda_A & -(\lambda_A + \lambda_B) \end{pmatrix}.$$

The state 0 is an absorbing state. The state transition diagram for this Markov chain is shown in Figure:

Then the transition probability matrix P is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\lambda_A}{\lambda_A+\mu} & 0 & 0 & \frac{\mu}{\lambda_A+\mu} \\ \frac{\lambda_B}{\lambda_B+\mu} & 0 & 0 & \frac{\mu}{\lambda_B+\mu} \\ 0 & \frac{\lambda_B}{\lambda_A+\lambda_B} & \frac{\lambda_A}{\lambda_A+\lambda_B} & 0 \end{pmatrix}.$$

Definition

Let $\{X_t; t \geq 0\}$ be continuous-time Markov chain with transition probability matrix $(P(t))_{t\geq 0}$. A measure μ defined over the state space S, is called an *invariant measure* for $\{X_t; t \geq 0\}$, if and only if, for all $t \geq 0$, μ satisfies

$$\mu = \mu P(t)$$

that is, for each $j \in S$, μ satisfies

$$\mu(j) = \sum_{i \in S} \mu(i) p_{ij}(t).$$

If $\sum_{j\in S} \mu(j) = 1$, then μ is called a *stationary distribution*.

Example

Let $\{X_t; t \geq 0\}$ be a continuous-time Markov chain with state space $S = \{0, 1\}$ and transition matrix given by

$$P(t) = \begin{pmatrix} \frac{2}{3} + \frac{1}{3}e^{-3t} & \frac{1}{3} - \frac{1}{3}e^{-3t} \\ \frac{2}{3} - \frac{2}{3}e^{-3t} & \frac{1}{3} + \frac{2}{3}e^{-3t} \end{pmatrix}.$$

It is easy to verify that $\mu = \left(\frac{2}{3}, \frac{1}{3}\right)$ is a stationary distribution for $\{X_t; t \geq 0\}$.

A *birth and death process (BDP)* is a continuous-time Markov chain $\{X_t; t \geq 0\}$ with state space $S = \mathbb{N}$ such that the elements $q_{i,i-1}, q_{ii}$ and $q_{i,i+1}$ of the intensity matrix Q are the only ones that can be different from zero. Let

$$\lambda_i := q_{i,i+1} \quad \text{and} \quad \mu_i := q_{i,i-1}$$

be the birth and death rates respectively, as they are known. The matrix Q is given by:

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{pmatrix}.$$

It is clear that $\lambda_i h + o(h)$ represents the probability of a birth in the interval of infinitesimal length $(t, t+h)$ given that $X_t = i$. Similarly $\mu_i h + o(h)$ represents the probability of a death in the interval of infinitesimal length $(t, t+h)$ given that $X_t = i$. From the Kolmogorov backward equations, we obtain:

$$\begin{aligned} p'_{0j}(t) &= -\lambda_0 p_{0j}(t) + \lambda_0 p_{1j}(t), \quad j = 0, 1, 2, \ldots \\ p'_{ij}(t) &= -(\lambda_i + \mu_i) p_{ij}(t) + \lambda_i p_{i+1,j}(t) + \mu_i p_{i-1,j}(t) \ \text{ for } \ i \geq 1 \end{aligned}$$

Similarly for the forward Kolmogorov equations, we obtain:

$$\begin{aligned} p'_{i0}(t) &= -\lambda_0 p_{i0}(t) + \mu_1 p_{i1}(t), \quad i \in S \\ p'_{ij}(t) &= -(\lambda_j + \mu_j) p_{ij}(t) + \lambda_{j-1} p_{i,j-1}(t) + \mu_{j+1} p_{i,j+1}(t) \text{ for } j \geq 1 \ . \end{aligned}$$

These equations can be solved explicitly for some special cases.

Next we will suppose that the state space S is finite and that $\lambda_i > 0, \mu_i > 0$ for $i \in S$. The embedded Markov chain is irreducible and positive recurrent. Hence there exists a stationary distribution for $\{X_t; t \geq 0\}$, say $\pi = (\pi_0, \pi_1, \dots, \pi_m)$. π is the solution of the system $\pi Q = 0$, which is given by:

$$\pi_i = \frac{\lambda_0 \lambda_1 \dots \lambda_{i-1}}{\mu_1 \mu_2 \dots \mu_i} \pi_0, \qquad i = 1, 2, \dots, m \,.$$

Also $\sum_{i=0}^{m} \pi_i = 1$, then

$$\pi_0 = \frac{1}{1 + \sum_{i=1}^{m} \frac{\lambda_0 \lambda_1 \dots \lambda_{i-1}}{\mu_1 \mu_2 \dots \mu_i}} \,.$$