Segunda entrega
Seccion 5.2

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1. Evaluar cada una de las integrales siguientes si $R = [0,1]\times[0,1]$.

(a) $\int_R (x^3+y^2)\,dA$ $\quad$ $\boxed{\text{(b)}}$ $\int_R ye^{xy}\,dA$

(c) $\int_R (xy)^2\cos x^3\,dA$ $\quad$ (c) $\int_R \ln[(x+1)(y+1)]\,dA$

a. $\int_R (x^3+y^2)\,dA = \int_0^1\int_0^1 x^3+y^2\,dx\,dy = \int_0^1\left(\frac{x^4}{4}+y^2x\Big]_0^1\right)dy = \int_0^1\left(\frac{1}{4}+y^2\right)dy =$

$\frac{y}{4}+\frac{y^3}{3}\Big]_0^1 = \frac{1}{4}+\frac{1}{3} = \frac{7}{12}.$

b. $\int_R ye^{xy}\,dA = \int_0^1\int_0^1 ye^{xy}\,dx\,dy = \int_0^1\left(\frac{y}{y}e^{xy}\Big]_0^1\right)dy = \int_0^1 e^y-1\,dy = e^y-y\Big|_0^1$

$(e-1)-(1) = e-2.$

c. $\int_R (xy)^2\cos x^3\,dA = \int_0^1\int_0^1 x^2y^2\cos x^3\,dy\,dx = \int_0^1\left(x^2\cos x^3\int_0^1 y^2\,dy\right)dx =$

$\int_0^1\left(x^2\cos x^3\left(\frac{y^3}{3}\Big]_0^1\right)\right)dx = \frac{1}{3}\int_0^1 x^2\cos x^3\,dx = \frac{1}{9}\left(\int_0^1\cos u\,du\right) = \frac{1}{9}\operatorname{sen}u\Big]_0^1$

$\frac{1}{9}(\operatorname{sen}1).$

d. $\int_R \ln((x+1)+(y+1))\,dA = \int_0^1\int_0^1 \ln|(x+1)+(y+1))\,dx\,dy = \int_0^1\int_0^1 \ln(x+y+2)\,dx\,dy$

1.
$u = x+y+2\,dx$
$du = dx$

$\int_{y+2}^{y+3}\ln(u) = u\ln(u)-u\Big|_{y+2}^{y+3} = (y+3)\ln(y+3)-(y+3)-(y+2\ln(y+2)-(y+2))$

$((y+3)(\ln(y+3))-(y+2)(\ln(y+2)))-y-3+y+2 = ((y+3)(\ln(y+3))-(y+2)(\ln(y+2)))-1$

2. $\int_0^1 (y+3)(\ln(y+3))-(y+2)(\ln(y+2))-1\,dy =$

$-\int_0^1 y\ln(y+2)-\int_0^1 2\ln(y+2)+\int_0^1 y\ln(y+3)+\int_0^1 3\ln(y+3)-1$

$u=y+2$, $du=dy$ $\quad$ $u=y+2$, $du=dy$ $\quad$ $u=y+3$, $du=1\,dy$

$-\int_2^3(u-2)\ln(u)-2\int_2^3\ln(u)+\int_3^4 u-3\ln(u)+3\int_0^1\ln(u)-1$

$-2\ln(2)-\frac{3}{4}+\frac{3}{2}\ln(3)-6\ln(3)+2+4\ln(2)-8\ln(2)+\frac{5}{4}+\frac{9}{2}\ln(3)-4$

$-6\ln(2)-\frac{3}{2}\ln(3)$

2. Evaluar cada una de las integrales siguientes si $R=[0,1]\times[0,1]$.

(a) $\int_R (x^m y^n)\,dx\,dy$, donde $m,n>0$ $\quad$ $\boxed{\text{(b)}}$ $\int_R (ax+by+c)\,dx\,dy$

(c) $\int_R \operatorname{sen}(x+y)\,dx\,dy$ $\quad$ (d) $\int_R (x^2+2xy+y\sqrt{x})\,dx\,dy$

a. $\int_0^1\int_0^1 x^m y^n\,dx\,dy \quad m,n>0 \quad = \int_0^1 y^n \frac{x^{m+1}}{m+1}\Big]_0^1 dy = \int_0^1 \frac{y^n}{m+1}\,dy =$

$\frac{y^{n+1}}{(n+1)(m+1)}\Big]_0^1 = \frac{1}{(m+1)(n+1)}$

b. $\int_0^1\int_0^1 ax+by+c\,dx\,dy = \int_0^1\left(\frac{ax^2}{2}+byx+cx\Big]_0^1\right)dy$

$\int_0^1 \frac{a}{2}+by+c\,dy = \frac{ay}{2}+\frac{by^2}{2}+cy\Big]_0^1 = \frac{a}{2}+\frac{b}{2}+c$

c. $\int_0^1\int_0^1 \operatorname{sen}(x+y)\,dx\,dy = \int_0^1\left(-\cos(x+y)\Big]_0^1\right)dy$ $\quad$ $u=x+y \quad dx$

$du=dy$

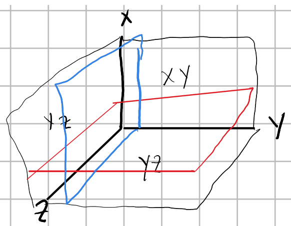
$\int_0^1 -\cos(y+1)+\cos(y)\,dy = -\operatorname{sen}(y+1)+\operatorname{sen}(y)\Big]_0^1 =$

$-\operatorname{sen}(2)+\operatorname{sen}(1)+\operatorname{sen}(1) = -\operatorname{sen}(2)+2\operatorname{sen}(1)$

d. $\int_0^1\int_0^1 x^2+2xy+y\sqrt{x}\,dx\,dy = \int_0^1\left(\frac{x^3}{3}+x^2y+\frac{2yx^{3/2}}{3}\Big]_0^1\right)dy =$

$\int_0^1 \frac{1}{3}+y+\frac{2y}{3}\,dy = \frac{1}{3}y+\frac{y^2}{2}+\frac{y^2}{3}\Big]_0^1 = \frac{1}{3}+\frac{1}{2}+\frac{1}{3} = \frac{2}{3}+\frac{1}{2} = \frac{7}{6}$

4. Calcular el volumen del sólido acotado por el plano xz, el plano yz, el plano xy, los planos $x=1$ y $y=1$, y la superficie $z=x^2+y^4$.



$x=1$

$y=1$

$z=x^2+y^4$

$\int_0^1\int_0^1 x^2+y^4\,dx\,dy$

$\int_0^1\left(\frac{x^3}{3}+y^4x\Big]_0^1\right)dy = \int_0^1 \frac{1}{3}+y^4\,dy = \frac{y}{3}+\frac{y^5}{5}\Big]_0^1 = \frac{1}{3}+\frac{1}{5} = \frac{8}{15}$

Volumen igual a $\frac{8}{15}$

$\boxed{\textbf{5.}}$ Sean f continua en $[a,b]$ y g continua en $[c,d]$. Mostrar que

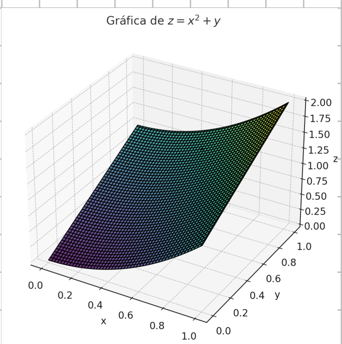
$$\int_R [f(x)g(y)]\,dx\,dy = \left[\int_a^b f(x)\,dx\right]\left[\int_c^d g(y)\,dy\right],$$

donde $R=[a,b]\times[c,d]$.

$$\int_R [f(x)\, g(y)]\, dx\, dy = \int_c^d \left(g(y) \int_a^b f(x)\, dx\right) dy = \left[\int_c^d g(y)\, dy\right]\left[\int_a^b f(x)\, dx\right]$$

- Mover constantes respecto a la variable difereciable.

7. Calcular el volumen del sólido acotado por la gráfica $z = x^2 + y$, el rectángulo $R = [0,1] \times [1,2]$ y los "lados verticales" de R.



$R = [0,1] \times [1,2]$

$z = x^2 + y$

$$\int_1^2 \int_0^1 x^2 + y\, dx\, dy = \int_1^2 \left(\frac{x^3}{3} + xy \Big]_0^1\right) dy = \int_1^2 \frac{1}{3} + y =$$

$$\frac{y}{3} + \frac{y^2}{2}\Big]_1^2 = \frac{2}{3} + \frac{4}{2} - \left(\frac{1}{3} + \frac{1}{2}\right) = \frac{1}{3} + \frac{3}{2} = \frac{11}{6}$$

Seccion 5.3

2. Repetir el ejercicio 1 para las siguientes integrales iteradas:

(a) $\int_{-3}^{2} \int_0^{y^2} (x^2 + y)\, dx\, dy$ $\qquad$ $\boxed{\text{(b)}}$ $\int_{-1}^{1} \int_{-2|x|}^{|x|} e^{x+y}\, dy\, dx$

(c) $\int_0^1 \int_0^{(1-x^2)^{1/2}} dy\, dx$ $\qquad$ (d) $\int_0^{\pi/2} \int_0^{\cos x} y \operatorname{sen} x\, dy\, dx$

$\boxed{\text{(e)}}$ $\int_0^1 \int_{y^2}^{y} (x^n + y^m)\, dx\, dy, \quad m, n > 0$ $\qquad$ (f) $\int_{-1}^{0} \int_0^{2(1-x^2)^{1/2}} x\, dy\, dx$

a. $$\int_{-3}^{2} \int_0^{y^2} x^2 + y\, dx\, dy = \int_{-3}^{2} \frac{x^3}{3} + yx \Big]_0^{y^2} dy = \int_{-3}^{2} \frac{y^6}{3} + y^3\, dy = \frac{y^7}{21} + \frac{y^4}{4}\Big]_{-3}^{2} =$$

$$\frac{2^7}{21} + \frac{2^4}{4} - \left(\frac{(-3)^7}{21} + \frac{3^4}{4}\right) = \frac{128}{21} + 4 + \frac{2187}{21} - \frac{81}{4} = \frac{2315}{21} - \frac{65}{4} = \frac{7895}{84}$$

b. $$\int_{-1}^{1} \int_{-2|x|}^{|x|} e^{x+y}\, dy\, dx = \int_{-1}^{1} e^{x+y}\Big]_{-2|x|}^{|x|} dx = \int_{-1}^{1} e^{x+|x|} - e^{x-2|x|}\, dx$$ Two cases.

$x > 0$ $$\int_0^1 e^{2x} - e^{-x}\, dx = \frac{1}{2} e^{2x} + e^{-x}\Big]_0^1 = \frac{1}{2} e^2 + e^{-1} - \left(\frac{1}{2} + 1\right) = \frac{1}{2} e^2 + e^{-1} - \frac{3}{2}$$

$x < 0$ $$\int_{-1}^{0} e^0 - e^{3x}\, dx = x - \frac{1}{3} e^{3x}\Big]_{-1}^{0} = -\frac{1}{3} - \left(-1 - \frac{1}{3} e^{-3}\right) = \frac{2}{3} + \frac{1}{3} e^{-3}$$

$$\frac{2e^{-3} + 3e^{2} + 6e^{-1} - 5}{6}$$

c. $$\int_0^1 \int_0^{(1-x)^{1/2}} dy\, dx = \int_0^1 y\Big]_0^{(1-x)^{1/2}} dx = \int_0^1 (1-x)^{1/2} = \frac{-2(1-x)^{3/2}}{3}\Big]_0^1 =$$

$$\frac{2(1)}{3} = \frac{2}{3}$$

d. $$\int_0^{\pi/2} \int_0^{\cos x} y \operatorname{sen} x\, dy\, dx = \int_0^{\pi/2} \operatorname{sen} x \left(\frac{y^2}{2}\Big]_0^{\cos x}\right) dx = \frac{1}{2}\int_0^{\pi/2} \operatorname{sen} x \cos^2 x\, dx$$

$u = \cos x\, dx$

$du = -\operatorname{sen} x\, dx$, $$-\frac{1}{2}\int u^2\, dx = \frac{+\cos x^3}{2.3}\Big]_0^{\pi/2} = +\frac{1}{3.2} = \frac{1}{6}$$

e. $\int_0^1\int_{y^2}^{y} x^n+y^m\,dx\,dy = \int_0^1\left(\frac{x^{n+1}}{n+1}+y^m x\Big]_{y^2}^{y}\right)dy =$ $m,n>0$

$\int_0^1 \frac{y^{n+1}}{n+1}+y^{m+1}-\left(\frac{y^{2n+2}}{n+1}+y^{m+2}\right)dy = \frac{y^{n+2}}{(n+1)(n+2)}+\frac{y^{m+2}}{m+2}-\frac{y^{2n+3}}{(n+1)(2n+3)}-\frac{y^{m+3}}{m+3}\Big|_0^1$

$\frac{1}{(n+1)(n+2)}+\frac{1}{m+2}-\frac{1}{(n+1)(2n+3)}-\frac{1}{m+3}$

f. $\int_{-1}^0\int_0^{2(1-x^2)^{1/2}} x\,dy\,dx = \int_{-1}^0\left(xy\Big]_0^{2(1-x^2)^{1/2}}\right)dx = \int_{-1}^0 x\,2(1-x^2)^{1/2}\,dx$

$u=1-x^2\,dx$
$du=-2x\,dx$

$\int_0^{-1} u^{1/2}\,du = \frac{2(1-x^2)^{3/2}}{3}\Big]_0^{-1} = -\frac{2(1)}{3} = -\frac{2}{3}$

4. **Usar integrales dobles para determinar el área de una elipse con semiejes de longitud *a* y *b*.**

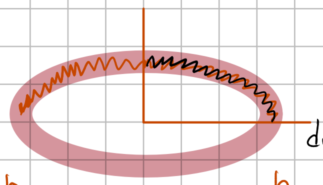
Elipse:

$A=\iint_R dy\,dx$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\quad y=\pm\sqrt{a^2\left(1-\frac{x^2}{b^2}\right)}$

$A=\int_{-b}^{b}\int_{-\sqrt{a^2(1-\frac{x^2}{b^2})}}^{\sqrt{a^2(1-\frac{x^2}{b^2})}} dy\,dx$

$\sqrt{a^2\left(1-\frac{x^2}{b^2}\right)}\le y\le\sqrt{a^2\left(1-\frac{x^2}{b^2}\right)}$
$-b\le x\le b$



$u=\frac{x}{b}dx$
$du=\frac{1}{b}dx$

Par.: b

$A=\int_{-b}^{b} y\Big]_{-\sqrt{a^2(1-\frac{x^2}{b^2})}}^{\sqrt{a^2(1-\frac{x^2}{b^2})}}dx = \int_{-b}^{b}\sqrt{a^2\left(1-\frac{x^2}{b^2}\right)}+\sqrt{a^2\left(1-\frac{x^2}{b^2}\right)}\,dx = \int_{-b}^{b}2\sqrt{a^2\left(1-\frac{x^2}{b^2}\right)}\,dx = 4a\int_0^b\sqrt{1-\frac{x^2}{b^2}}\,dx =$

$4ab\underbrace{\int_0^1\sqrt{1-u^2}\,du}_{} = \int_0^1\cos^2t\,dt=\frac12\int_0^1 1-\cos(2t)\,dt=\frac12\left(t-\frac12\operatorname{sen}(2t)\right)=$

$u=\operatorname{sen}t$ $\quad t=\operatorname{Arcsen}u$ $\quad =\frac12\left(\operatorname{Arcsen}t-\frac12\operatorname{sen}(2\operatorname{Arcsen}t)\right)\Big]_0^1=\frac12\left(\frac{\pi}{2}\right)=\frac{\pi}{4}$
$du=\cos t$

$4ab.\frac{\pi}{4}=ab\pi$

7. **Sea D la región acotada por el eje y y la parábola $x=-4y^2+3$. Calcular**

$$\int_D x^3y\,dx\,dy.$$

$x=-4y^2+3$
$0<x<-4y^2+3$
$-\frac{\sqrt3}{2}<y<\frac{\sqrt3}{2}$

$A=\int_{-\frac{\sqrt3}{2}}^{\frac{\sqrt3}{2}}\int_0^{-4y^2+3}x^3y\,dx\,dy=\int_{-\frac{\sqrt3}{2}}^{\frac{\sqrt3}{2}}\frac{x^4y}{4}\Big]_0^{-4y^2+3} = \int_{-\frac{\sqrt3}{2}}^{\frac{\sqrt3}{2}}\left(\frac{-4y^2+3}{4}\right)^3 y\,dy$

$u=-\frac{4y^2}{3}+1\,dy$
$du=-\frac{8}{3}y\,dy$

$-\frac38\int_{-\frac{\sqrt3}{2}}^{\frac{\sqrt3}{2}} u\,du = -\frac{3}{16}\left(-\frac43y^2+1\right)\Big]_{-\frac{\sqrt3}{2}}^{\frac{\sqrt3}{2}}=0$

Funcion impar respecto a ese eje.

8. **Evaluar $\int_0^1\int_0^{x^2}(x^2+xy-y^2)\,dy\,dx$. Describir esta integral iterada como una integral sobre cierta región D en el plano xy.**

$\int_0^1\int_0^x(x^2+xy-y^2)\,dy\,dx = \int_0^1\int_{-(\frac{x^4}{2}+\frac{x^5}{4}-\frac{x^2}{6})}^{\frac{x^4}{2}+\frac{x^5}{4}-\frac{x^2}{6}}dy\,dx$

$D=\left\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ -\frac{x^4}{2}-\frac{x^5}{4}+\frac{x^2}{6}\le y\le\frac{x^4}{2}+\frac{x^5}{4}-\frac{x^2}{6}\right\}$

$$\int_0^1\int_0^x (x^2+xy-x^2)\,dy\,dx = \iint_D (x^2+xy-y^2)\,dA$$

$$D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ 0\le y\le x\}$$

9. Sea D la región dada como el conjunto de (x,y) donde $1 \le x^2+y^2 \le 2$ y $y \ge 0$. ¿Es D una región elemental? Evaluar $\int_D f(x,y)\,dA$ donde $f(x,y) = 1+xy$.

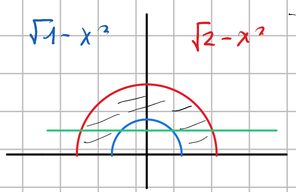
Definición región elemental.

Región elemental tipo x:

$$R=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ g_1(x)\le y\le g_{(2)}(x)\}.$$

$$1 \le x^2+y^2 \le 2 \qquad y\ge 0$$

$$+\sqrt{1-x^2} \le y \le +\sqrt{2-x^2}$$



Tenemos que encontrar un a y b únicos que corten y cumpla con la función.

Trazamos una linea paralela a x, nos damos cuenta que corta en más de dos puntos. Por ende a y b no son únicos para toda la función.

13. Evaluar $\int_D y\,dA$ donde D es el conjunto de puntos (x,y) tales que $0 \le 2x/\pi \le y$, $y \le \operatorname{sen} x$.

$$\int_D y\,dA \qquad D=\left\{(x,y): 0\le \frac{2x}{\pi}\le y,\ y\le \operatorname{Sen} x\right\}$$

Punto de intersección: Punto final del area.

$$\frac{2x}{\pi} = \operatorname{Sen} x \qquad x=0 \ \wedge\ x=\frac{\pi}{2}$$

$$0\le y\le \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}}\int_{\frac{2x}{\pi}}^{\operatorname{sen} x} y\,dy\,dx = \int_0^{\frac{\pi}{2}} \frac{y^2}{2}\Big|_{\frac{2x}{\pi}}^{\operatorname{sen} x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{sen} x^2}{2} - \frac{4x^2}{2\pi^2} = \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \operatorname{sen} x^2 - \frac{4x^2}{\pi^2}\right) =$$

$$\frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{1-\cos(2x)}{2} - \frac{4x^2}{\pi^2}\,dx\right) = \frac{1}{2}\left(\frac{x}{2} - \frac{\operatorname{sen}(2x)}{4} - \frac{4x^3}{\pi^2 3}\Big]_0^{\frac{\pi}{2}}\right) =$$

$$\frac{1}{2}\left(\frac{\pi}{4} - \frac{4\pi^3}{8.3\pi^2}\right) = \frac{\pi}{8} - \frac{\pi}{12} = \frac{3\pi}{24} - \frac{2\pi}{24} = \frac{\pi}{24}$$

Seccion 5.4

1. En las integrales siguientes, cambiar el orden de integración, esbozar las regiones correspondientes y evaluar las integrales de las dos maneras.

(a) $\int_0^1\int_x^1 xy\,dy\,dx$

$\boxed{\text{(b)}}$ $\int_0^{\pi/2}\int_0^{\cos\theta} \cos\theta\,dr\,d\theta$

(c) $\int_0^1\int_1^{2-y} (x+y)^2\,dx\,dy$

a. $$\int_0^1\int_x^1 xy\,dy\,dx = \int_0^1 \frac{xy^2}{2}\Big]_x^1 dx = \int_0^1 \frac{x}{2} - \frac{x^3}{2} = \frac{x^2}{4} - \frac{x^4}{8}\Big]_0^1 = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

$$\int_0^1\int_0^y xy\,dx\,dy = \int_0^1 \frac{x^2y}{2}\Big]_0^y dy = \int_0^1 \frac{y^3}{3}\,dy = \frac{y^4}{8}\Big]_0^1 = \frac{1}{8}$$

b. $$\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} \cos\theta\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\cos\theta\, r\Big]_0^{\cos\theta} d\theta = \int_0^{\frac{\pi}{2}}\cos^2(\theta)\,d\theta = \int_0^{\frac{\pi}{2}}\frac{1+\cos(2\theta)}{2} = \frac{1}{2}\theta + \frac{\text{sen}(2\theta)}{4} = \frac{\pi}{4}$$

$0\le r\le \cos\theta$, $0\le\theta\le \pi/2$ Cambio $0\le r\le 1$, $0\le\theta\le \text{Arccos}\, r$.

$\text{Sen}(\text{Arccos}\,x) = \sqrt{1-x^2}$

$$\int_0^1\int_0^{\text{Arccos}\,r}\cos\theta\,d\theta\,dr = \int_0^1 \text{sen}\,\theta\Big]_0^{\text{Arccos}\,r} dr = \int_0^1 \text{sen}(\text{Arccos}\,r)\,dr = \int_0^1\sqrt{1-r^2}\,dr$$

$r = \text{sen}\,x$, $dr = \cos x$, $x = \text{Arcsen}\,r$

$$\int_0^1 \cos^2 x = \frac{\text{Arcsen}\,x}{2} + \frac{\text{sen}(2\text{Arcsen}\,x)}{4}\Big]_0^1 = \frac{\pi}{4} + 0 = \frac{\pi}{4}$$

c. $$\int_0^1\int_1^{2-y}(x+y)^2\,dx\,dy = \int_0^1 \frac{x^3}{3} + x^2y + xy^2\Big]_1^{2-y} dy = \int_0^1 \frac{(2-y)^3}{3} + (2-y)^2y + (2-y)y^2 + \frac{1}{3} + y + y^2$$

$$-\frac{3(2-y)^4}{12} + \left(2y^2 - \frac{2y^3}{3} + \frac{y^4}{4}\right) + \left(y^3 - \frac{y^4}{4}\right) + \frac{y}{3} + \frac{y^2}{2} + \frac{y^3}{3}\Big]_0^1 = -\frac{y^4}{12} - \frac{1}{3}y^3 + \frac{3y^2}{2} - 2y^2 + \frac{7}{3}y\Big]_0^1$$

$$-\frac{1}{12} - \frac{1}{3} + \frac{3}{2} - 2 + \frac{7}{3} = \frac{17}{12} + \frac{24}{12} - 2 = \frac{41}{12} - \frac{24}{12} = \frac{17}{12}$$

$1\le x\le 2-y$, $0\le y\le 1$ Cambio. $1\le x\le 2$, $0\le y\le 2-x$

$$\int_1^2\int_0^{2-x}(x+y)^2\,dy\,dx = \int_1^2 x^2y + y^2x + \frac{y^3}{3}\Big]_0^{2-x} dx = \int_1^2 x^2(2-x) + (2-x)^2x + \frac{(2-x)^3}{3}$$

$$\frac{2x^3}{3} - \frac{x^4}{4} + 2x^2 - \frac{2x^3}{3} + \frac{x^4}{4} - \frac{3(2-x)^4}{12}\Big]_1^2 = -\frac{2x^3}{3} + 2x^2 - \frac{(2-x)^4}{12}\Big]_1^2$$

$$\frac{-16}{3} + 8 - \left(\frac{-2}{3} + 2 - \frac{1}{12}\right) = \frac{8}{3} - \frac{15}{12} = \frac{17}{12}$$

2. Hallar

(a) $\int_{-1}^{1}\int_{|y|}^{1}(x+y)^2\,dx\,dy$ (b) $\int_{-3}^{1}\int_{-\sqrt{(9-y^2)}}^{\sqrt{(9-y^2)}} x^2\,dx\,dy$

$\boxed{\text{(c)}}$ $\int_0^4\int_{y/2}^2 e^{x^2}\,dx\,dy$ (d) $\int_0^1\int_{\tan^{-1}y}^{\pi/4}(\sec^5 x)\,dx\,dy$

a. $$\int_0^1\int_y^1 x^2 + 2xy + y^2\,dx\,dy = \int_0^1 \frac{x^3}{3} + x^2y + y^2x\Big]_y^1 dy = \int_0^1 \frac{1}{3} + y + y^2 - \frac{y^3}{3} - y^3 - y^3$$

$$\int_0^1 \frac{1}{3} + y + y^2 - \frac{7}{3}y^3 = \frac{1}{3}y + \frac{y^2}{2} + \frac{y^3}{3} - \frac{7y^4}{12}\Big]_0^1 = \frac{1}{3} + \frac{1}{2} + \frac{1}{3} - \frac{7}{12} = \frac{7}{12}$$ $-\frac{1}{3}+\frac{6}{3}$

$$\int_{-1}^0\int_{-y}^1 x^2 + 2xy + y^2\,dx\,dy = \int_{-1}^0 \frac{1}{3} + y + y^2 - \frac{y^3}{3} + y^3 - y^3\,dy = \frac{1}{3}y + \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{12}\Big]_{-1}^0$$

$$-\left(-\frac{1}{3} + \frac{1}{2} - \frac{1}{3} - \frac{1}{12}\right) = \frac{7}{12}$$ Área final $= \frac{7}{6}$

b. $\int_{-3}^{1}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2 \, dx\, dy = \int_{-3}^{1} \frac{x^3}{3}\Big]_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} dy = \int_{-3}^{1} \frac{(9-y^2)^{3/2}}{3} + \frac{(9-y^2)^{3/2}}{3} dy$

$\frac{2}{3}\int_{-3}^{1} (9-y^2)^{3/2} dy = \frac{486 \text{ Arcsen}\left(\frac{1}{3}\right) + 243\pi + 43 \cdot 2^{\frac{5}{2}}}{24}$

C. $\int_0^4 \int_{y/2}^{2} e^{x^2} dx dy = \int_0^2 \int_0^{2x} e^{x^2} dy\, dx = \int_0^2 e^{x^2} 2x \, dx = \int_0^4 e^u du = e^4 - 1$

$U = x^2 dx$
$du = 2x \, dx$

$x = \frac{y}{2} \quad y = 2x$

D. $\int_0^1 \int_{\tan^{-1}(y)}^{\frac{\pi}{4}} \sec^5 x \, dx\, dy = \int_0^{\frac{\pi}{4}} \int_0^{\tan x} \sec^5 x \, dy dx = \int_0^{\frac{\pi}{4}} \sec^5 x \tan x \, dx$

$\int_0^{\frac{\pi}{4}} \sec^4 x \sec x \tan x \, dx = \int u^4 du = \frac{u^5}{5}\Big]_1^{\sqrt{2}} = \frac{4\sqrt{2}}{5} - \frac{1}{5} = \frac{4\sqrt{2}-1}{5}$

$U = \sec x \, dx$

4. Mostrar que $\frac{1}{2}(1-\cos 1) \leq \int_{[0,1]\times[0,1]} \frac{\text{sen}\, x}{1+(xy)^4}\, dx\, dy \leq 1.$

Tenemos que: $\frac{1}{2} - \frac{\cos 1}{2} = \frac{1}{2}\int_0^1 1 - \cos 1 \, dy = \frac{1}{2}\int_0^1\int_0^1 \text{sen}\, x \, dx\, dy$

$1 = \int_0^1\int_0^1 1 \, dx\, dy$

Evaluamos solo los interiores de las integrales:

$\underbrace{\frac{\text{sen}\, x}{2} \leq \frac{\text{sen}\, x}{1+(xy)^4}}_{2 \geq 1 + x^4y^4} \leq 1$

$1 \geq x^4 y^4$ $\quad 0 \leq x \leq 1$, $0 \leq y \leq 1$

Se cumple.

Acotada por izquierda.

$\frac{\text{sen}\, x}{2} \leq \underbrace{\frac{\text{sen}\, x}{1+(xy)^4} \leq 1}_{\text{sen}\, x \leq 1 + x^4 y^4}$

$0 \leq \text{sen} \leq 1$ Max.

$1 \leq 1 + x^4y^4$
$0 \leq x^4 y^4$

Se cumple

Acotada por derecha.

8. Calcular $\int_D f(x,y)\, dA$, donde $f(x,y) = y^2\sqrt{x}$ y D es el conjunto de (x,y) donde $x > 0$, $y > x^2$ y $y < 10 - x^2$.

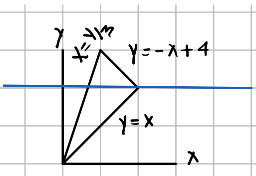
Intersección $10 - x^2 = x^2$

$\int_0^{\sqrt{5}} \int_{x^2}^{10-x^2} y^2 \sqrt{x} \, dy\, dx = \int_0^{\sqrt{5}} \frac{y^3\sqrt{x}}{3}\Big]_{x^2}^{10-x^2} dx = \frac{1}{3}\int_0^{\sqrt{5}} (10-x^2)^3 \sqrt{x} - \frac{x^{\frac{13}{2}}}{3} dx$

$\frac{1}{3}\int_0^{\sqrt{5}} (1000\sqrt{x} - 300x^{3/2} + 30^{5/2} - x^{7/2}) - \frac{x^{13/2}}{3} dx = \frac{1}{3}\left(\frac{2\cdot 1000\, x^{3/2}}{3} - \frac{2\cdot 300\, x^{5/2}}{5} + \frac{2\cdot 30^{7/2}}{7} - \frac{2x^{\frac{9}{2}}}{9}\right) - \frac{2x^{15/2}}{15\cdot 3}$

$\frac{2}{3}\left(\frac{1.000\cdot 5^{3/4}}{3} - \frac{300 \cdot 5^{5/4}}{5} + \frac{30\cdot 5^{7/4}}{7} - \frac{5^{9/4}}{9} - \frac{5^{15/4}}{3}\right) = \frac{78800 \cdot 5^{3/4}}{693}$

10. Evaluar $\iint_D e^{x-y}\,dx\,dy$, donde D es el interior del triángulo con vértices $(0,0)$, $(1,3)$ y $(2,2)$.



$$\int_0^2\int_{y/3}^{y} e^{x-y}\,dx\,dy = \int_0^2 e^{x-y}\Big]_{y/3}^{y} = \int_0^2 e^0 - e^{y/3-y}\,dy = \int_0^2 1 - e^{-2/3\,y}\,dy$$

$$y + \frac{3e^{-\frac{2}{3}y}}{2}\Big]_0^2 = 2 + \frac{3e^{-\frac{4}{3}}}{2} - \frac{3}{2} = \frac{1}{2} + \frac{3e^{-4/3}}{2}$$

$$\int_2^3\int_{y/3}^{-y+4} e^{x-y}\,dx\,dy = \int_2^3 e^{-y+4-y} - e^{y/3-y}\,dy = \int_2^3 e^{-2y+4} - e^{-\frac{2y}{3}}\,dy$$

$$-\frac{1}{2}e^{-2y+4} + \frac{3e^{-\frac{2y}{3}}}{2}\Big]_2^3 = -\frac{1}{2e^2} + \frac{3e^{-2}}{2} + \frac{e^0}{2} - \frac{3e^{-\frac{4}{3}}}{2}$$

$$A = -\frac{1}{2e^2} + \frac{3}{2e^2} + \frac{1}{2} - \frac{3}{2e^{4/3}} = \frac{1}{e^2} + \frac{1}{2} - \frac{3}{2e^{4/3}}$$

11. Evaluar $\iint_D y^3(x^2+y^2)^{-3/2}\,dx\,dy$, donde D es la región determinada por las condiciones $\frac{1}{2} \le y \le 1$ y $x^2+y^2 \le 1$.

$x = r\cos\theta$ $\quad$ $\frac{1}{2} \le r\,\text{sen}\,\theta \le 1$ $\quad$ $dx\,dy = r\,dr\,d\theta$ $\quad$ $r = 1$

$y = r\,\text{sen}\,\theta$ $\quad$ $\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$ $\quad$ $x^2+y^2 \le 1$ $\quad$ $\frac{1}{2\,\text{sen}\,\theta} \le r$

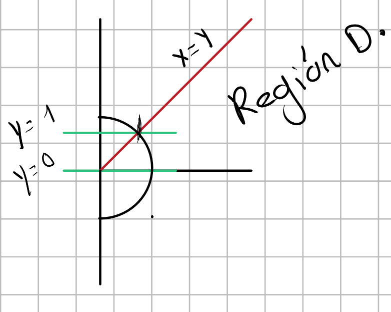
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_{\frac{1}{2\,\text{sen}\,\theta}}^{1} r^3\,\text{sen}^3\theta \cdot r^{-3}\,r\,dr\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_{\frac{1}{2\,\text{sen}\,\theta}}^{1} \text{sen}^3\theta\,r\,dr\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \text{sen}^3\theta\left(\frac{1}{2} - \frac{1}{8\,\text{sen}^2\theta}\right)$$

$$\frac{1}{2}\left(\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1-\cos^2\theta)\,\text{sen}\,\theta\,d\theta - \int\frac{\text{sen}\,\theta}{4}\,d\theta\right) =$$

$$\frac{1}{2}\left(u - \frac{u^3}{3}\Big]_0^{\frac{\sqrt{3}}{2}} + \frac{\cos\theta}{4}\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{3^{3/2}}{3.8}\right) - \frac{\sqrt{3}}{8} = \frac{1}{2}\left(\frac{4\sqrt{3}}{8} - \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8}\right) = \frac{\sqrt{3}}{8}$$

13. Dado que la integral doble $\iint_D f(x,y)\,dx\,dy$ de una función continua positiva f es

$$\iint_D f(x,y)\,dx\,dy = \int_0^1\int_y^{\sqrt{2-y^2}} f(x,y)\,dx\,dy$$



Cambio de orden.

$0 \le x \le 1$ $\qquad$ $1 \le x \le \sqrt{2}$

$0 \le y \le x$ $\qquad$ $0 \le y \le \sqrt{2-x^2}$

$$\int_0^1\int_0^x f(x,y)\,dy\,dx + \int_1^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} f(x,y)\,dy\,dx$$

Seccion 6.1

4. Integrar ze^{x+y} sobre $[0,1]\times[0,1]\times[0,1]$.

$$\int_0^1 z \int_0^1 \int_0^1 e^{x+y} = \int_0^1 \int_0^1 e^{1+y} - e^{y} \, dy\, dz = \int_0^1 z\left(e^2 - e - e^1 + e^0\right) dz$$

$$\frac{z^2}{2}\left(e^2 - 2e + 1\right)\Big]_0^1 = \frac{1e^2}{2} - e + \frac{1}{2}$$

6. Hallar el volumen de la región acotada por $z = x^2 + 3y^2$ y $z = 9 - x^2$.

$$x^2 + 3y^2 = 9 - x^2$$
$$2x^2 + 3y^2 = 9$$
$$\frac{x^2}{\frac{9}{2}} + \frac{y^2}{3}$$

$$\int_0^{2\pi}\int_0^1 9 - \frac{9r^2\cos^2\theta}{2} - \left(9\left(\frac{r^2\cos^2\theta}{2} + r^2 sen^2\theta\right)\right)\frac{3\sqrt{6}}{2} r \, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 (9 - 9r^2)\frac{3\sqrt{6}}{2} r \, dr\, d\theta = \frac{3\sqrt{6}}{2}\int_0^{2\pi} \frac{9}{2} - \frac{9}{4}\, d\theta = 9\pi . \frac{3\sqrt{6}}{2} = \frac{27\sqrt{6}\pi}{2}$$

$$x = \frac{3}{\sqrt{2}} r\cos\theta \qquad y = \sqrt{3}\, r sen\theta$$

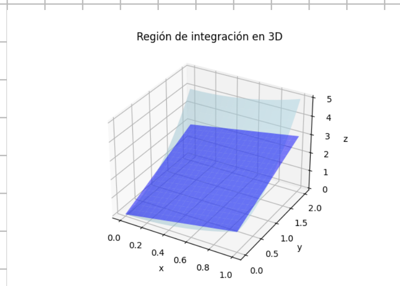
$$\frac{9}{2} r^2\cos^2\theta + 3.3 r^2 sen^2\theta$$

$$9\left(\frac{r^2\cos^2\theta}{2} + r^2 sen^2\theta\right) \qquad 9 - \frac{9r^2\cos^2\theta}{2}$$

$\boxed{\textbf{7.}}$ Evaluar $\int_0^1 \int_0^{2x} \int_{x^2+y^2}^{x+y} dz\, dy\, dx$ y esbozar la región de integración.

$$\int_0^1\int_0^{2x} x + y - (x^2+y^2)\, dy\, dx = \int_0^1 xy + \frac{y^2}{2} - x^2 y - \frac{y^3}{3}\Big]_0^{2x} dx = \int_0^1 2x^2 + 2x^2 - 2x^3 - \frac{8x^3}{3}\, dx =$$

$$\frac{4x^3}{3} - \frac{2x^4}{4} - \frac{8x^4}{12}\Big|_0^1 \qquad \frac{4}{3} - \frac{1}{2} - \frac{2}{3} = \frac{1}{6}$$



8. Hallar el volumen del sólido acotado por las superficies $x^2 + 2y^2 = 2$, $z = 0$ y $x + y + 2z = 2$.

$$x^2 + 2y^2 = 2 \qquad z = 0 \qquad x + y + 2z = 2$$

Cambio a elípticas:

$$x = \sqrt{2}\, r\cos\theta \qquad dA = |\det(J)|\, dr\, d\theta$$
$$y = r\, sen\,\theta$$

$$V = \iiint_W 1\, dV \qquad 0 \le z \le \frac{2 - x - y}{2}$$

$$dA = \begin{pmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{pmatrix} = \begin{pmatrix} \sqrt{2}\cos\theta & -\sqrt{2}\, r\, sen\theta \\ sen\,\theta & r\cos\theta \end{pmatrix}$$

$$V = \int_0^{2\pi}\int_0^1\int_0^{\frac{2 - \sqrt{2} r\cos\theta - r sen\theta}{2}} \sqrt{2}\, r\, dz\, dr\, d\theta$$

$$dA = \sqrt{2}\, r$$

$$(\sqrt{2}\cos\theta . r\cos\theta) - (-\sqrt{2}\, r\, sen^2\theta)$$
$$\sqrt{2}\, r\,(1) = \sqrt{2}\, r\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 \left(\frac{2 - \sqrt{2} r\cos\theta - r sen\theta}{2}\right)\sqrt{2}\, r\, dr\, d\theta =$$

$$\int_0^{2\pi}\int_0^1 \sqrt{2}\, r - \frac{\sqrt{2} r^2\cos\theta}{2} - \frac{\sqrt{2} r^2 sen\theta}{2}\, dr\, d\theta$$

$$\int_0^{2\pi} \frac{\sqrt{2} r^2}{2} - \frac{\sqrt{2} r^3\cos\theta}{6} - \frac{\sqrt{2} r^3 sen\theta}{6}\Big]_0^1 d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2} - \frac{\sqrt{2}\cos\theta}{6} - \frac{\sqrt{2}\, sen\theta}{6}\, d\theta =$$

$$\frac{\sqrt{2}\,\theta}{2} - \frac{\sqrt{2}\, sen\theta}{6} + \frac{\sqrt{2}\cos\theta}{6}\Big]_0^{2\pi} = \sqrt{2}\pi + \frac{\sqrt{2}}{6} - \left(\frac{\sqrt{2}}{6}\right) = \sqrt{2}\pi$$

11. Hallar el volumen acotado por el paraboloide $z = 2x^2 + y^2$ y el cilindro $z = 4 - y^2$.

Cambio a coordenadas polares

$2x^2 + y^2 = 4 - y^2$

$x^2 + y^2 = 2$

$x = r\cos\theta \qquad r \in [0, \sqrt{2}]$

$y = r\,\text{sen}\,\theta \qquad \theta \in [0, 2\pi]$

Paraboloide

$2r^2\cos^2\theta + r^2\text{sen}^2\theta$

$r^2(2\cos^2\theta + \text{sen}^2\theta)$

Cilindro

$4 - r^2\text{sen}^2\theta$

$$V = \int_0^{2\pi}\int_0^{\sqrt{2}} 4 - r^2\text{sen}^2\theta - r^2(2\cos^2\theta) + \text{sen}^2\theta)\ r\ dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} r(4 - 2r^2)\,dr\,d\theta = \int_0^{2\pi} 2r^2 - \frac{r^4}{2}\Big]_0^{\sqrt{2}}$$

$$\int_0^{2\pi} 4 - \frac{4}{2}\ d\theta = 2.2\pi = 4\pi$$

12. Evaluar $\int_0^{\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \text{sen}\,2\phi\,d\rho\,d\phi\,d\theta$.

$$\int_0^{\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \text{sen}\,2\phi\ d\rho\,d\phi\,d\theta = \int_0^{\pi}\int_0^{\frac{\pi}{4}} \rho\,\text{sen}(2\phi)\Big|_0^{\sec\phi} d\phi\,d\theta = \int_0^{\pi}\int_0^{\frac{\pi}{4}} \sec\phi\ \text{sen}(2\phi)\ d\phi\ d\theta =$$

$$\int_0^{\pi}\int_0^{\frac{\pi}{4}} \sec\phi\ 2\cos\phi\ \text{sen}\,\phi\ d\phi\,d\theta = \int_0^{\pi} -2\cos\phi\Big]_0^{\frac{\pi}{4}} d\theta = \int_0^{\pi} -\frac{2\sqrt{2}}{2} - (-2)\ d\theta = \int_0^{\pi} -\sqrt{2} + 2\ d\theta$$

$$-\sqrt{2}\,\theta + 2\theta\Big]_0^{\pi} = -\sqrt{2}\pi + 2\pi = \pi(-\sqrt{2} + 2)$$

13. Calcular la integral de la función $f(x, y, z) = z$ sobre la región W en el primer octante de $\mathbf{R}^3$ acotada por los planos $y = 0$, $z = 0$, $x + y = 2$, $2y + x = 6$ y el cilindro $y^2 + z^2 = 4$.

$\iiint_W f(x,y,z)\,dv$, donde $f(x,y,z) = z \qquad x, y, z \geq 0$

$y = 0 \quad z = 0 \quad x + y = 2 \quad 2y + x = 6 \quad y^2 + z^2 = 4$

Analisis en XY

$x + y = 2$

$2y + x = 6$

y cuando $y = 2$ $y^2 = 4$

Intersecciones

$x = 0\ y = 2 \qquad y = 0\ x = 2$

$y = 3\ x = 0 \qquad x = 6\ y = 0$

$y = 2$

Región en xy

$x + y = 2$

$2y + x = 6$

$0 \leq y \leq 2$

Limites para z. y x.

$z = \sqrt{4 - y^2}$

$x = -y + 2$

$$\int_0^2\int_0^{-y+2}\int_0^{\sqrt{4-y^2}} z\ dz\,dx\,dy = \int_0^2\int_0^{-y+2} \frac{z^2}{2}\Big]_0^{\sqrt{4-y^2}} =$$

$$\int_0^2\int_0^{-y+2} \frac{4 - y^2}{2}\ dx\,dy = \int_0^2 2x - \frac{y^2x}{2}\Big]_0^{-y+2} dy = \int_0^2 -2y + 4 - \frac{y^2(-y+2)}{2}\ dy =$$

$$\int_0^2 -2y + 4 + \frac{y^3 - 2y^2}{2}\ dy = -y^2 + 4y + \frac{y^4}{8} - \frac{2y^3}{6}\Big]_0^2 = -4 + 8 + 2 - \frac{8}{3} = \frac{10}{3}$$

15. Sea W un conjunto acotado cuya frontera está formada por gráficas de funciones continuas. Supongamos que W es simétrica en el plano xy: $(x, y, z) \in W$ implica que $(x, y, -z) \in W$. Suponer que f es una función continua acotada en W y $f(x, y, z) = -f(x, y, -z)$. Probar que $\int_W f(x, y, z)\, dV = 0$.

F es continua acotada en W y $f(x,y,z) = -f(x,y,-z)$.

Plantearemos la integral en dos partes cuando $z \geq 0$ "z^+" y cuando $z \leq 0$ "z^-"

$$\int_W f(x,y,z)\, dv = \int_{z^+} f(x,y,z)\, dv + \int_{z^-} f(x,y,z)\, dv$$

Por hipotesis $z = -z'$

$$\int_{z^-} f(x,y,z)\, dv = \int_{z^+} f(x,y,-z')(-dz') = -\int_{z^+} f(x,y,z)\, dv$$

$$\int_{z^-} f(x,y,z)\, dv = -\int_{z^+} f(x,y,z)\, dv$$

$$\int_W f(x,y,z)\, dv = \int_{z^+} f(x,y,z)\, dv - \int_{z^+} f(x,y,z)\, dv = 0$$

16. Usar el resultado del ejercicio 15 para probar que $\int_W (1 + x + y)\, dV = 4\pi/3$, donde W, la bola unitaria, es el conjunto de (x, y, z) con $x^2 + y^2 + z^2 \leq 1$.

$$\int_W (1+x+y)\, dv = \frac{4\pi}{3} \qquad W: (x,y,z) : x^2+y^2+z^2 \leq 1$$

$$\int_W 1\, dv + \underbrace{\int_W x\, dv + \int_W y\, dv}_{} = \frac{4\pi}{3}$$

Nos damos cuenta que.
$(x,y,z) \in W \longleftrightarrow (-x,y,z) \in W$
$(x,y,z) \in W \longleftrightarrow (x,-y,z) \in W$
Simetria.

$$\int_W x\, dv = 0 \qquad \int_W y\, dv = 0$$

$x^2+y^2+z^2 \leq r^2$
$r = 1$

Volumen de la esfera radio 1:

$$\frac{4\pi r^3}{3} = \frac{4\pi}{3}$$

En integrales triples:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} z\, dz\, dy\, dx$$

$$\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} z\, dz = 0$$

$$\iiint_W z\, dv = 0$$

Caso analogo con $\int_W y = 0 \wedge \int_W x = 0$

18. Sea B la región determinada por las condiciones $0 \leq x \leq 1$, $0 \leq y \leq 1$ y $0 \leq z \leq xy$.

(a) Hallar el volumen de B (b) Evaluar $\iiint_B x\, dx\, dy\, dz$.

a.

$$B = \int_0^1\int_0^1\int_0^{xy} dz\, dy\, dx = \int_0^1\int_0^1 xy\, dy\, dx = \int_0^1 \frac{xy^2}{2}\Big]_0^1 dx = \int_0^1 \frac{x}{2}\, dx = \frac{x^2}{4}\Big]_0^1 = \frac{1}{4}$$

b. Cambio en el orden.

$0 \leq x \leq 1 \quad 0 \leq y \leq 1 \quad 0 \leq z \leq xy$

$0 \leq x \leq 1 \quad 0 \leq z \leq 1 \quad \frac{z}{y} \leq x \leq 1$

$$\int_0^1\int_0^1\int_{\frac{z}{y}}^1 x\, dx\, dy\, dz = \int_0^1\int_0^1 \frac{x^2}{2}\Big]_{\frac{z}{y}}^1 dy\, dz = \int_0^1\int_0^1 \frac{1}{2} - \frac{z^2}{2y^2}\, dy\, dz = \int_0^1 \frac{1}{2}y + \frac{z^2}{2y}\Big]_0^1 dz = \int_0^1 \frac{1}{2} + \frac{z^2}{2}\, dz$$

$$\frac{1}{2}\left(z + \frac{z^3}{3}\right)\Big]_0^1 = \frac{4}{6} = \frac{2}{3}$$

Posibilidad de divergencia en z/y cerca a y = 0

Seccion 6.3

1. Sea D el círculo unitario. Evaluar

$$\int_D \exp(x^2+y^2)\,dx\,dy$$

haciendo un cambio de variables a coordenadas polares.

Jacobiano: $\begin{pmatrix} \cos\theta & -r\operatorname{sen}\theta \\ \operatorname{sen}\theta & r\cos\theta \end{pmatrix}$ $r\,dr\,d\theta = dx\,dy$

$x = r\cos\theta \quad y = r\operatorname{sen}\theta$

$x^2+y^2 = r^2$

$$\int_D e^{(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \cdot \int_0^1 e^{r^2}\,r\,dr$$

$\theta\Big]_0^{2\pi} = 2\pi \qquad = \frac{1}{2}(e-1)$

$2\pi \cdot \frac{1}{2}(e-1) = \pi(e-1)$

2. Sea D la región $0 \le y \le x$ y $0 \le x \le 1$. Evaluar

$$\int_D (x+y)\,dx\,dy$$

haciendo el cambio de variables $x = u+v$, $y = u-v$. Verificar la respuesta obtenida evaluando directamente la integral, usando una integral iterada.

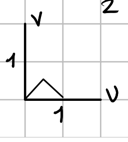
$$\int_0^1\int_0^x (x+y)\,dy\,dx = \frac{1}{2}$$

$u+v = x$

$u-v = y$

Jacobiano: $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = |-2| = 2$

$v = \frac{x-y}{2} \quad u = \frac{x+y}{2}$ Dado esto



Cambio de variable

$$4\int_0^{1/2}\int_v^{1-v} u\,du\,dv = 2\int_0^{\frac{1}{2}} 1-2v+v^2-v^2\,dv = 2\int_0^{\frac{1}{2}} 1-2v\,dv = 2\left[v-v^2\right]_0^{\frac{1}{2}} = 2\cdot\frac{1}{4} = \frac{1}{2}$$

3. Sea $T(u,v) = (x(u,v), y(u,v))$ la función definida por $T(u,v) = (4u, 2u+3v)$. Sea D^* el rectángulo $[0,1]\times[1,2]$. Hallar $D = T(D^*)$ y evaluar

(a) $\int_D xy\,dx\,dy$ (b) $\int_D (x-y)\,dx\,dy$

haciendo un cambio de variables para evaluarlas como integrales sobre D^*.

1. Determinar la región D.

$D = T(D^*) \quad T(u,v)$

$D^* \quad [0,1]\times[1,2] \qquad u \in [0,1]$

$v \in [1,2]$

$D : [0,4]\times[3,8]$

$x \in [0,4]$

$y = 2u+3v \qquad u=0 \wedge u=1$

$y\in[3,6] \vee y\in[5,8] \rightarrow y\in[3,8]$

a. $\int_D xy\,dx\,dy$

$x = 4u \quad y = 2u+3v$

$\int 4u(2u+3v)\,12\,du\,dv$

Jacobiano $\begin{pmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix} = 12$

$$48\int_0^1\int_1^2 2u^2+3vu\,du\,dv = \int_0^1 \frac{2u^3}{3}+\frac{3vu^2}{2}\Big]_1^2 dv =$$

$$48\int_0^1 \frac{16}{3}+6v-\frac{2}{3}-\frac{3v}{2} = 48\int_0^1 \frac{14}{3}+\frac{9v}{2}\,dv$$

$$48\ \frac{14}{3}v+\frac{9v^2}{4}\Big]_0^1 = 48\left(\frac{56}{12}+\frac{27}{12}\right) = \frac{83}{12}\cdot 48$$

$83\times 4 = 332$

b. $\int_D (x-y)\,dx\,dy$

$$12\int_0^1\int_1^2 2u-3v\,du\,dv = 12\int_0^1 u^2-3vu\Big]_1^2 dv = 12\int_0^1 (4-6v-1+3v)\,dv = 12\int_0^1 3-3v\,dv = 12\left(3-\frac{3}{2}\right) = 18$$

4. Repetir el ejercicio 3 para $T(u,v) = (u, v(1+u))$.

$T(u,v) = (u, v+uv)$ Jacobiano

$x = u \quad y = v+uv$

$\begin{pmatrix} 1 & 0 \\ v & 1+u \end{pmatrix} = 1+u\ du\,dv$

$$\int_0^1\int_1^2 (uv+u^2v)(1+u)\,du\,dv = \int_0^1\int_1^2 uv+u^2v+u^2v+u^3v\,du\,dv = \int_0^1 \frac{u^2v}{2}+\frac{2u^3v}{3}+\frac{u^4v}{4}\Big]_1^2 dv = \int_0^1 2v+\frac{16v}{3}+4v-\left(\frac{1v}{2}+\frac{2v}{3}+\frac{1v}{4}\right)dv$$

$$\int_0^1 6v + \frac{47}{12}v \, dv = 3v^2 + \frac{47v^2}{24}\Big]_0^1 = 3 + \frac{47}{24} = \frac{119}{24}$$

$$\int_0^1\int_1^2 (u-v-uv)(1+u)\,du\,dv = \int_0^1\int_1^2 u - v - uv + u^2 - uv - u^2v \, du\,dv = \int_0^1 \frac{u^2}{2} - uv - \frac{u^2v}{2} + \frac{u^3}{3} - \frac{u^2v}{2} - \frac{u^3v}{3}\Big]_1^2 dv$$

$$\int_0^1 2 - 2v - 2v + \frac{8}{3} - 2v - \frac{8v}{3} - \left(\frac{1}{2} - v - \frac{v}{2} + \frac{1}{3} - \frac{v}{2} - \frac{v}{3}\right) dv = \int_0^1 \frac{-19}{3} + \frac{25}{6} = \frac{-19}{6} + \frac{23}{6} = \frac{4}{6} = \frac{2}{3}$$

5. Evaluar

$$\int_D \frac{dx\,dy}{\sqrt{1+x+2y}}$$

donde $D = [0,1] \times [0,1]$, haciendo $T(u,v) = (u, v/2)$ y evaluando una integral sobre D^*, donde $T(D^*) = D$.

$\int_D \frac{dx\,dy}{\sqrt{1+x+2y}}$ $D = [0,1] \times [0,1]$ $T(u,v) = (u, v/2)$ $T(D^*) = D$

$D^* = T^{-1}(D)$

$x = u$ $y = v/2$

$[0,1] \times [0,2] = T^{-1}(D)$

Jacobiano $\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} = \frac{1}{2}$

$$\int_{D^*} \frac{1}{\sqrt{1+u+v}}\,dv\,du = \int_0^1\int_0^2 \frac{1}{\sqrt{1+u+v}}\,\frac{1}{2}\,dv\,du = \frac{1}{2}\int_0^1\int_{1+u}^{3+u} \frac{1}{\sqrt{w}}\,dw\,du = \int_0^1 \sqrt{3+u} - \sqrt{1+u}\,du =$$

$w = 1+u+v$

$$-\left(\int_0^1 \sqrt{1+u} - \int_0^1 \sqrt{3+u}\right) = -\frac{2(1+u)^{3/2}}{3} + \frac{2(3+u)^{3/2}}{3}\Big]_0^1 = -\frac{2(2)^{3/2}}{3} + \frac{2(4)^{3/2}}{3} + \frac{2}{3} - \frac{2(3)^{3/2}}{3} = \frac{-2\,2^{3/2} - 2(4)^{3/2} - 2 + 2(3)^{3/2}}{3}$$

$$\frac{2(-2^{3/2} - 3^{3/2} + 9)}{3}$$

6. Definir $T(u,v) = (u^2 - v^2, 2uv)$. Sea D^* el conjunto de (u,v) con $u^2 + v^2 \leq 1$, $u \geq 0$, $v \geq 0$. Hallar $T(D^*) = D$. Evaluar $\int dx\,dy$.

$T(u,v) = (u^2 - v^2, 2uv)$

$u \geq 0,\ v \geq 0$ $D^* = u^2 + v^2 \leq 1$

$T(D^*) = D$

$T(D^*) = [0,1] \times [0,1]$

$x = u^2 - v^2$

$y = 2uv$

Jacobiano $\begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = (2u)^2 - (-4v^2) = 4u^2 + 4v^2\,du\,dv$

$$\int_D dx\,dy = \int_{D^*} 4u^2 + 4v^2\,du\,dv = 4\int_{D^*} u^2 + v^2\,du\,dv = 4\int_0^{\frac{\pi}{2}}\int_0^1 r^2\sin^2\theta + r^2\cos^2\theta\; r\,dr\,d\theta = 4\int_0^{\frac{\pi}{2}}\int_0^1 r^3\,dr\,d\theta$$

A polares: $u = r\,\text{sen}\,\theta$, $v = r\cos\theta$, $du\,dv = r\,dr\,d\theta$

$\int_0^{\frac{\pi}{2}} 1 = \frac{\pi}{2}$

7. Sea $T(u,v)$ como en el ejercicio 6. Haciendo ese cambio de variables, evaluar

$$\int_D \frac{dx\,dy}{\sqrt{x^2+y^2}}.$$

$\int_D \frac{dx\,dy}{\sqrt{x^2+y^2}}$

D D^*

$u = r\cos\theta$

$v = r\,\text{sen}\,\theta$

$r \in [0,1]$

$\theta \in [0, \frac{\pi}{2}]$

$T(D^*) = D$

$x = T((r\cos\theta)^2 - (r\,\text{sen}\,\theta)^2) = r^2\cos(2\theta)$

$y = T(2(r\cos\theta)(r\,\text{sen}\,\theta)) = r^2\,\text{sen}(2\theta)$

$r \in [0,1]$

$\theta \in [0,\pi]$

Jacobiano

$r\,dr\,d\theta$

$$\int_0^{\frac{\pi}{2}}\int_0^1 \frac{1}{\sqrt{r^4\cos(\theta)^4 + r^4\,\text{sen}(\theta)^4}}\,4r^2\,r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^1 \frac{1}{r^2}\,4r^2\,r\,dr\,d\theta = 2\left(\frac{\pi}{2}\right) = \pi$$

8. Calcular $\int_R \frac{1}{x+y}\,dy\,dx$, donde R es la región acotada por $x = 0$, $y = 0$, $x + y = 1$, $x + y = 4$, usando la función $T(u,v) = (u - uv, uv)$.

$$\int_R \frac{1}{x+y}\,dy\,dx \qquad \int_0^4\int_0^{4-x} \frac{1}{x+y}\,dy\,dx - \int_0^1\int_0^{1-x}\frac{1}{x+y}\,dy\,dx = A$$

$T(u,v) = (u - uv, uv)$

$x = u - uv$

$y = uv$

Jacobiano $\begin{pmatrix} 1-v & -u \\ v & u \end{pmatrix} = u - uv + uv = u\,du\,dv$

$\int_{R^*} \frac{1}{u}\,u\,du\,dv$

$x + y = u$

$0 \leq x \leq 4$

$0 \leq y \leq 4 - x$

$x + y = u$

$y = 0$ $x = u$

$y = 4 - x$ $4 = u$

$u \in [0,4]$

$x = u(1-v)$

$y \leq 4 - u(1-v)$

$y \leq 4 - u + uv$

$y \leq u$

$v = \frac{y}{u}$

$v \in [0,1]$

$$\int_0^1\int_0^4 du\,dv = 4 \qquad \int_0^1\int_0^1 du\,dv = 1 \qquad 4-1 = 3$$

9. Evaluar $\int_D (x^2+y^2)^{3/2}\,dx\,dy$ donde D es el disco $x^2+y^2 \le 4$.

$$\int_D (x^2+y^2)^{3/2} \qquad x^2+y^2 \le 4 \qquad r = 2$$

$x = r\,\text{sen}\,\theta$
$y = r\cos\theta$

$$\int_0^{2\pi}\int_0^2 (r^2)^{3/2}\,r\,dr\,d\theta = \int_0^{2\pi} \left.\frac{r^5}{5}\right]_0^2 d\theta = \int_0^{2\pi} \frac{32}{5}\,d\theta = \frac{32\theta}{5} = \frac{64\pi}{5}$$

11. Usar integrales dobles para hallar el área dentro de la curva $r = 1+\text{sen}\,\theta$.

$$A = \int_{\theta_1}^{\theta_2}\int_0^{r(\theta)} r\,dr\,d\theta \longrightarrow \int_0^{2\pi}\int_0^{1+\text{sen}(\theta)} r\,dr\,d\theta = \int_0^{2\pi} \left.\frac{r^2}{2}\right]_0^{1+\text{sen}\,\theta} d\theta = \int_0^{2\pi} \frac{1}{2} + \text{sen}\,\theta + \frac{\text{sen}(\theta)^2}{2}\,d\theta$$

$1+\text{sen}\,\theta$

$\theta \in [0, 2\pi]$

$$\left.\frac{1}{8}\left(6x - \text{sen}(2x) - 8\cos(x)\right)\right]_0^{2\pi} = \left(12\pi - \text{sen}(4\pi) - 8\cos(2\pi)\right)\frac{1}{8}$$

$$\frac{3}{2}\pi - 1 - (-1) = \frac{3\pi}{2}$$

15. Usando coordenadas polares, hallar el área acotada por la *lemniscata* $(x^2+y^2)^2 = 2a^2(x^2-y^2)$.

$$(x^2+y^2)^2 = 2a^2(x^2-y^2)$$

Cambio a polares
$x = r\cos\theta \quad y = r\,\text{sen}\,\theta$

$\cos 2\theta > 0$
$\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$

Lemniscata simetría en ambos ejes

$$(r^2) = 2a^2(\cos^2\theta - \text{sen}^2\theta)$$
$$r = \sqrt{2a^2\cos(2\theta)}$$

Sustituimos

$$A = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2a^2\cos(2\theta)\,d\theta = a^2\left(\left.\frac{\text{sen}(2\theta)}{2}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{a^2}{2}\left(\text{sen}\left(\frac{\pi}{2}\right) - \text{sen}\left(-\frac{\pi}{4}\right)\right)\right.$$

Área en coordenadas polares

$$A = \frac{1}{2}\int_2^{w} r^2\,d\theta$$

$$a^2\left(\frac{1}{2}+\frac{1}{2}\right) = a^2$$

Dado que integramos la leminisca en una cuarta parte, el área total es $4a^2$

23. Evaluar $\int_S \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$, donde S es el sólido acotado por las dos esferas $x^2+y^2+z^2 = a^2$ y $x^2+y^2+z^2 = b^2$, donde $0 < b < a$.

$$\int_S \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$$

Cambio de variable

$x = r\,\text{sen}\,\theta\cos\phi$
$y = r\,\text{sen}\,\theta\,\text{sen}\,\theta$
$z = r\cos\theta$

$$\int_b^a\int_0^{\pi}\int_0^{2\pi} \frac{r^2\,\text{sen}\,\theta}{r^3}\,d\phi\,d\theta\,dr \qquad dv = r^2\,\text{sen}\,\theta\,dr\,d\theta\,d\phi$$

$$\int_b^a\int_0^{\pi}\int_0^{2\pi} \frac{\text{sen}\,\theta}{r}\,d\phi\,d\theta\,dr = 2\pi\int_b^a\int_0^{\pi} \frac{\text{sen}\,\theta}{r}\,d\theta\,dr = 4\pi\int_b^a \frac{1}{r}\,dr = 4\pi\ln\left(\frac{a}{b}\right)$$

27. Evaluar $\iint_B (x+y)\,dx\,dy$ donde B es el rectángulo en el plano xy con vértices en $(0,1)$, $(1,0)$, $(3,4)$ y $(4,3)$.

$$\iint_B x+y\,dx\,dy = \int_1^3\int_{y-1}^{y+1} x+y\,dx\,dy + \int_0^1\int_{1-y}^{y+1} x+y\,dx\,dy + \int_3^4\int_{y-1}^{7-y} x+y\,dx\,dy$$

$$\int_1^3 \left.\frac{x^2}{2} + yx\right]_{y-1}^{y+1} = \int_1^3\left(\frac{(y+1)^2}{2} + y^2+y\right) - \left(\frac{(y-1)^2}{2} + y^2-y\right) = \int_1^3 2y\,dy = \left.y^2\right]_1^3 = 9-1 = 8$$

$$\int_0^1 \left.\frac{x^2}{2} + yx\right]_{1-y}^{y+1} = \int_0^1\left(\frac{(y+1)^2}{2} + y^2+y\right) - \left(\frac{(1-y)^2}{3} + y-y^2\right) = \int_0^1 2y^2+2y = \left.\frac{2y^3}{3} + y^2\right]_0^1 = \frac{5}{3}$$

$$\int_3^4 \left.\frac{x^2}{2} + yx\right]_{y-1}^{7-y} = \int_3^4\left(\frac{(7-y)^2}{2} + 7y-y^2\right) - \left(\frac{(y-1)^2}{3} + y^2-y\right) = \int_3^4 -2y^2+2y+24\,dy = \left.-\frac{2y^3}{3} + y^2 + 24y\right]_3^4$$

$$8 + \frac{5}{3} + \frac{19}{3} = 16$$

Seccion 6.4

3. Hallar el centro de masa de la región entre $y = x^2$ y $y = x$ si la densidad es $x + y$.

Masa: $\int_0^1 \int_{x^2}^{x} x+y \, dy\, dx = \int_0^1 xy + \frac{y^2}{2} \Big]_{x^2}^{x} dx = \int_0^1 \frac{3x^2}{2} - x^3 - \frac{x^4}{2}\, dx = \frac{x^3}{2} - \frac{x^4}{4} - \frac{x^5}{10}\Big|_0^1$

$\frac{1}{2} - \frac{1}{4} - \frac{1}{10} = \frac{10}{20} - \frac{5}{20} - \frac{2}{20} = \frac{3}{20} = m$

$M_y = \int_0^1 \int_{x^2}^{x} x^2 + yx \, dy\, dx = \int_0^1 x^2 y + \frac{y^2 x}{2} \Big]_{x^2}^{x} = \int_0^1 x^3 + \frac{x^3}{2} - \left(x^4 + \frac{x^5}{2}\right) dx$

$\int_0^1 \frac{3x^3}{2} - x^4 - \frac{x^5}{2} = \frac{3x^4}{8} - \frac{x^5}{5} - \frac{x^6}{12}\Big]_0^1 = \frac{3}{8} - \frac{1}{5} - \frac{1}{12} = \frac{3}{8} + \frac{-12-5}{60} = \frac{3}{8} - \frac{17}{60} = \frac{11}{120}$

$M_x = \int_0^1 \int_{x^2}^{x} xy + y^2 \, dy\, dx = \int_0^1 \frac{xy^2}{2} + \frac{y^3}{3} \Big]_{x^2}^{x} dx = \int_0^1 \frac{x^3}{2} + \frac{x^3}{3} - \left(\frac{x^5}{2} + \frac{x^6}{3}\right) dx$

$\int_0^1 \frac{5x^3}{6} - \frac{x^5}{2} - \frac{x^6}{3}\, dx = \frac{5x^4}{24} - \frac{x^6}{12} - \frac{x^7}{21}\Big]_0^1 = \frac{5}{24} - \frac{2}{24} - \frac{1}{21} = \frac{1}{8} - \frac{1}{21} = \frac{13}{168}$

$\vec{X} = \frac{\frac{11}{120}}{\frac{3}{20}} = \frac{\cancel{220}^{11}}{\cancel{360}_{18}} = \frac{11}{18}$ $\qquad \vec{Y} = \frac{\frac{13}{168}}{\frac{3}{20}} = \frac{\cancel{260}^{130}}{\cancel{504}_{252}} = \frac{65}{126}$

Centro de masa $\left(\frac{11}{18}, \frac{65}{126}\right)$

4. Hallar el centro de masa de la región entre $y = 0$ y $y = x^2$, donde $0 \leq x \leq \frac{1}{2}$.

$M = \int_0^{\frac{1}{2}} \int_0^{x^2} dy\, dx = \int_0^{\frac{1}{2}} x^2 = \frac{x^3}{3}\Big]_0^{\frac{1}{2}} = \frac{1}{24}$

$M_y = \int_0^{\frac{1}{2}} \int_0^{x^2} x \, dy\, dx = \int_0^{\frac{1}{2}} xy\Big]_0^{x^2} dx = \int_0^{\frac{1}{2}} x^3\, dx = \frac{x^4}{4}\Big]_0^{\frac{1}{2}} = \frac{1}{64}$

$M_x = \int_0^{\frac{1}{2}} \int_0^{x^2} y \, dy\, dx = \int_0^{\frac{1}{2}} \frac{y^2}{2}\Big]_0^{x^2} dx = \int^{\frac{1}{2}} \frac{x^4}{2}\, dx = \frac{x^5}{10}\Big]^{\frac{1}{2}} = \frac{1}{10 \cdot 32}$

$x = \frac{M_y}{m}$ $\quad y = \frac{M_x}{m}$ $\quad x = \frac{\frac{1}{64}}{\frac{1}{24}} = \frac{\cancel{24}^{\cancel{12}\,\cancel{6}\,3}}{\cancel{64}_{32\,16\,8}} = \frac{3}{8}$ $\quad y = \frac{\frac{1}{320}}{\frac{1}{24}} = \frac{\cancel{24}^{\cancel{12}\,\cancel{6}\,3}}{\cancel{320}_{\cancel{160}\,\cancel{80}\,40}} = \frac{3}{40}$

Centro de masa:

$\left(\frac{3}{8}, \frac{3}{40}\right)$

5. Una placa de oro grabada D está definida por $0 \leq x \leq 2\pi$ y $0 \leq y \leq \pi$ (centímetros) y tiene una densidad de masa $\rho(x, y) = y^2 \operatorname{sen}^2 4x + 2$ (gramos por centímetro cuadrado). Si el oro cuesta 7 dls por gramo, ¿cuánto vale el oro en la placa?

$M = \int_0^{\pi}\int_0^{2\pi} y^2 \operatorname{sen}^2(4x) + 2 \, dx\, dy = \int_0^{\pi} y^2\, dy \cdot \int_0^{2\pi} \operatorname{sen}^2(4x) + 2\, dx$

$\frac{1}{4}\int_0^{2\pi} \operatorname{sen}^2(u) = \frac{1}{4}\left(\frac{4x}{2} - \frac{\operatorname{sen}(8x)}{4}\right)\Big]_0^{2\pi} = \frac{1}{2}x - \frac{\operatorname{sen}(8x)}{4}\Big]_0^{2\pi} = \pi$

$\int_0^{\pi} y^2 \pi = \frac{y^3 \pi}{3}\Big]_0^{\pi} = \frac{\pi^4}{3}$ $\qquad \frac{\pi^4}{3} + \int_0^{\pi}\int_0^{2\pi} 2\, dx\, dy = \frac{\pi^4}{3} + 4\pi \cdot \pi$

$7\left(\frac{\pi^4}{3} + 4\pi^2\right) = \left(\frac{7\pi^4}{3} + 28\pi^2\right)$ dls.

9. Hallar el centro de masa de la región acotada por $x+y+z=2$, $x=0$, $y=0$ y $z=0$.

$$M=\int_0^2\int_0^{2-x}\int_0^{2-x-y} dz\,dy\,dx = \int_0^2\int_0^{2-x} 2-x-y \; dy\,dx = \int_0^2 (2-x)y-\frac{y^2}{2}\Big]_0^{2-x} dx =$$

$$\int_0^2 (2-x)^2-\frac{(2-x)^2}{2}\,dx = \int_0^2 2-2x+\frac{x^2}{2} = 2x-x^2+\frac{x^3}{6}\Big]_0^2 = \frac{8}{6}=\frac{4}{3}$$

$$M_{zy} = \int_0^2 2x-2x^2+\frac{x^3}{2} = x^2-\frac{2x^3}{3}+\frac{x^4}{8}\Big]_0^2 = 4-\frac{16}{3}+2 = \frac{18-16}{3} = \frac{2}{3}$$

$$M_{zx} = \int_0^2\int_0^{2-x} 2y-xy-y^2\,dy\,dx = \int_0^2 y^2-\frac{xy^2}{2}-\frac{y^3}{3}\Big]_0^{2-x} dx = \int_0^2 (2-x)^2-\frac{x(2-x)^2}{2}-\frac{(2-x)^3}{3}\,dx = \frac{2}{3}$$

$$M_{xy} = \int_0^2\int_0^{2-x}\int_0^{2-x-y} z\,dz\,dy\,dx = \int_0^2\int_0^{2-x} \frac{z^2}{2}\Big]^{2-x-y} dy\,dx = \int_0^2\int_0^{2-x}\frac{(2-x-y)^2}{2}\,dy\,dx = \frac{2}{3}$$

$$\overrightarrow{x\;y\;z} = \frac{\frac{2}{3}}{\frac{4}{3}} = \frac{6}{12} = \frac{1}{2}$$

$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ Centro de masa.

10. Hallar el centro de masa del cilindro $x^2+y^2\le 1$, $1\le z\le 2$, si la densidad es $\rho=(x^2+y^2)z^2$.

Simetría Cilindro y densidad.

$$\int_1^2\int_0^1\int_0^{2\pi} r^3z^2\,d\theta\,dr\,dz = \int_0^{2\pi}\int_1^2\int_0^1 r^3z^2\,dr\,dz\,d\theta = \int_0^{2\pi}\int_1^2 \frac{z^2}{4}\,dz\,d\theta = \int_0^{2\pi}\frac{z^3}{12}\Big]_1^2 d\theta =$$

$$\int_0^{2\pi}\frac{8}{12}-\frac{1}{12}\,dz = \frac{7}{12}z\Big]_0^{2\pi} = \frac{7\pi}{6}$$

$$M_{xy} = \int_0^{2\pi}\int_1^2\int_0^1 r^3z^3\,dr\,dz\,d\theta = \int_0^{2\pi}\int_1^2 \frac{r^4z^3}{4}\Big]_0^1 dr\,dz = \int_0^{2\pi}\frac{z^4}{16}\Big]_1^2 = \frac{15}{16}z\Big]_0^{2\pi} = \frac{15\pi}{8}$$

$$\vec{z} = \frac{\frac{15}{8}}{\frac{7}{6}} = \frac{105}{56} = \frac{15}{8}$$

Centro de masa $\left(0,0,\frac{15}{8}\right)$

$$P = x^2z^2+y^2z^2$$

$\vec{x}=0$

$\vec{y}=0$

Muestra simetría y estamos en un cilindro simétrico en x y en y

12. Hallar el valor promedio de e^{-z} sobre la bola $\underbrace{x^2+y^2+z^2\le 1}_{\text{Volumen esfera radio 1.}}$.

$$V_{prom} = \frac{1}{vol(V)}\iiint_V e^{-z}\,dV$$

Cambio a esféricas

$$\frac{1}{\frac{4\pi}{3}}\int_0^{2\pi} d\theta\int_0^{\pi} \operatorname{sen}\phi\,d\phi\;\underbrace{\int_0^1 e^{-r\cos\phi}\,r^2\,dr}$$

$$\frac{1}{\frac{4}{3}}(2\pi)\int_0^{\pi}\operatorname{sen}\phi\,d\phi\int_0^1 e^{-r\cos\phi}\,r^2\,dr.$$ No hay antiderivada elemental.

- Usando software y operando la integral completa, nos da: 1.811

13. Un sólido con densidad constante está acotado por arriba por el plano $z = a$ y por debajo por el cono descrito en coordenadas esféricas por $\phi = k$, donde k es una constante $0 < k < \pi/2$. Dar una integral para su momento de inercia alrededor del eje z.

- Formula del momento de inercia alrededor del eje z:

$$I_z = \delta \int_v (x^2+y^2)\, dv$$

d densidad constante

$\theta \in [0, 2\pi]$
$\phi \in [0, k]$
$\rho \in [0, A/\cos\phi]$

$x = \rho \operatorname{sen}\phi \cos\theta$
$y = \rho \operatorname{sen}\phi \operatorname{sen}\theta$
$z = \rho\cos\phi$

$x^2+y^2 = \rho^2 \operatorname{sen}^2\phi$
$dv = \rho^2 \operatorname{sen}\phi \, d\rho\, d\phi\, d\theta$

$z = a \rightarrow \phi = k$
$z = \rho\cos\phi = A$
$\rho = \frac{A}{\cos\phi}$

$$d\int_0^{2\pi}\int_0^{k}\int_0^{\frac{a}{\cos\phi}} \rho^4 \operatorname{sen}^3\phi \, d\rho\, d\phi\, d\theta = d\int_0^{2\pi} d\theta \int_0^{k} \operatorname{sen}^3\phi\, d\phi \int_0^{\frac{a}{\cos\phi}} \rho^4 \, d\rho$$

14. Hallar el momento de inercia alrededor del eje y para la bola $x^2+y^2+z^2 \le R^2$ si la densidad de masa es una constante ρ.

$x^2+y^2+z^2 \le R^2$
$x = r \operatorname{sen}\phi \cos\theta$
$y = r \operatorname{sen}\phi \cos\phi$
$z = r\cos\phi$

- Formula inercia en y

$$I_y = d\int_A x^2+z^2 \, dA$$

$$I_y = \rho \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2(\operatorname{sen}^2\theta\cos^2\phi + \cos^2\theta)\, r^2 \operatorname{sen}\theta \, dr\, d\theta\, d\phi$$

$$\int_0^{2\pi} \cos^2\phi\, d\phi = \frac{1}{2}x + \frac{\operatorname{sen}(2\theta)}{4}\Big]_0^{2\pi} = \pi$$

$$\int_0^{\pi} \operatorname{sen}^3\theta\, d\theta = \int_0^{\pi}(1-\cos^2\theta)\operatorname{sen}\theta\, d\theta = -\int_1^{-1} 1-u^2 \, du = u - \frac{u^3}{3}\Big]_{-1}^{1} = \left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right) = \frac{4}{3}$$

$u = \cos\theta\, d\theta$
$du = -\operatorname{sen}\theta\, d\theta$

$$\int_0^{\pi}\cos^2\theta \operatorname{sen}\theta\, d\theta = \int_{-1}^{1} u^2\, du = \frac{u^3}{3}\Big]_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

$u = \cos\theta\, d\theta$
$du = -\operatorname{sen}\theta\, d\theta$

$$\int_0^{R} r^4\, dr = \frac{R^5}{5}$$

$r^4 \operatorname{sen}^3\theta\cos^2\phi + r^4\cos^2\theta\operatorname{sen}\theta$

$$I_y = \rho\left(\frac{R^5}{5}\left(\frac{4}{3}\pi\right) + \frac{R^5}{5}\left(\frac{2}{3}\right)2\pi\right)$$

$$\rho\left(\frac{4R^5\pi}{15}\right)2 = \frac{\rho\, 8 R^5 \pi}{15}$$