Introduction to BTYD (Buy Until You Die) Models

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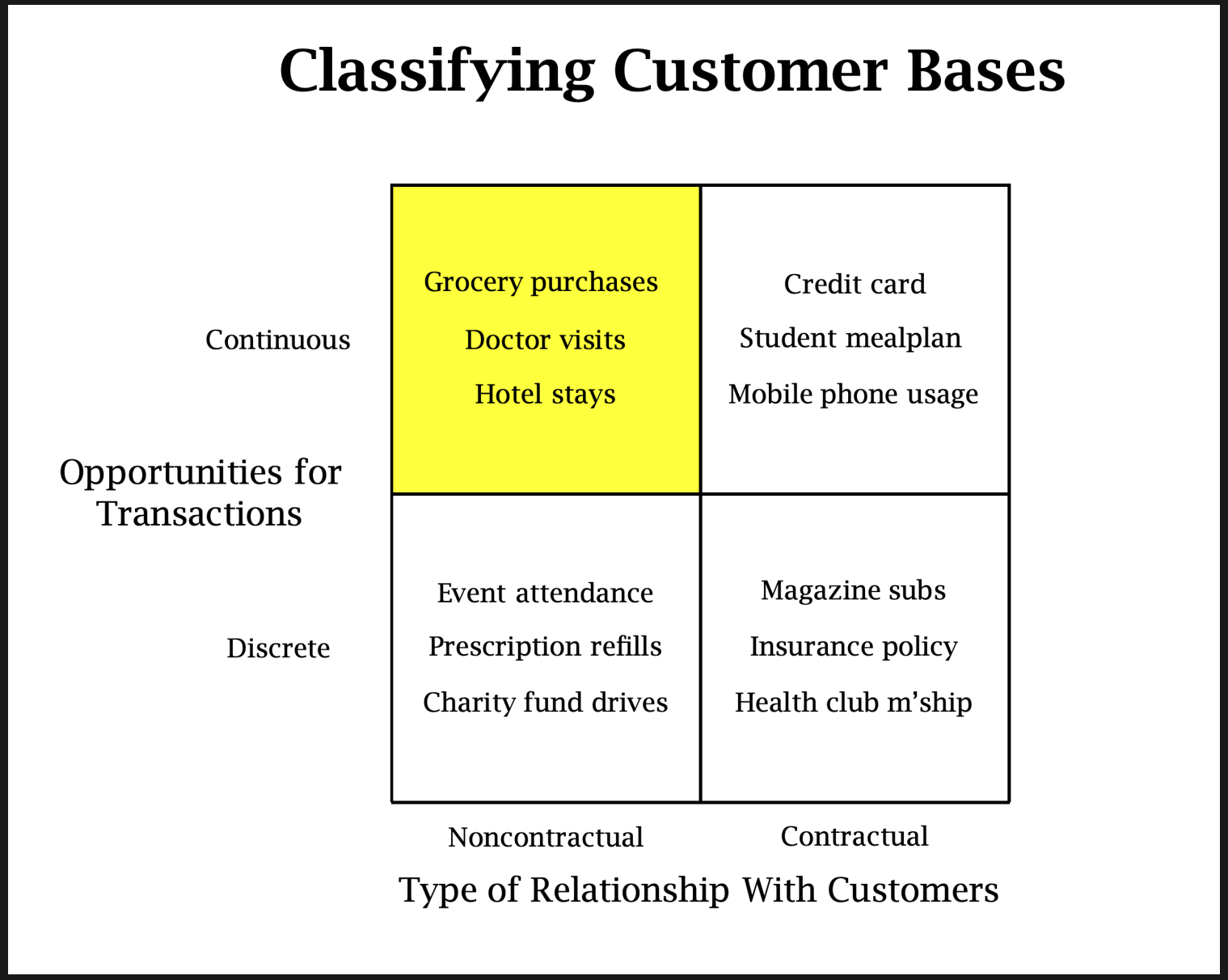
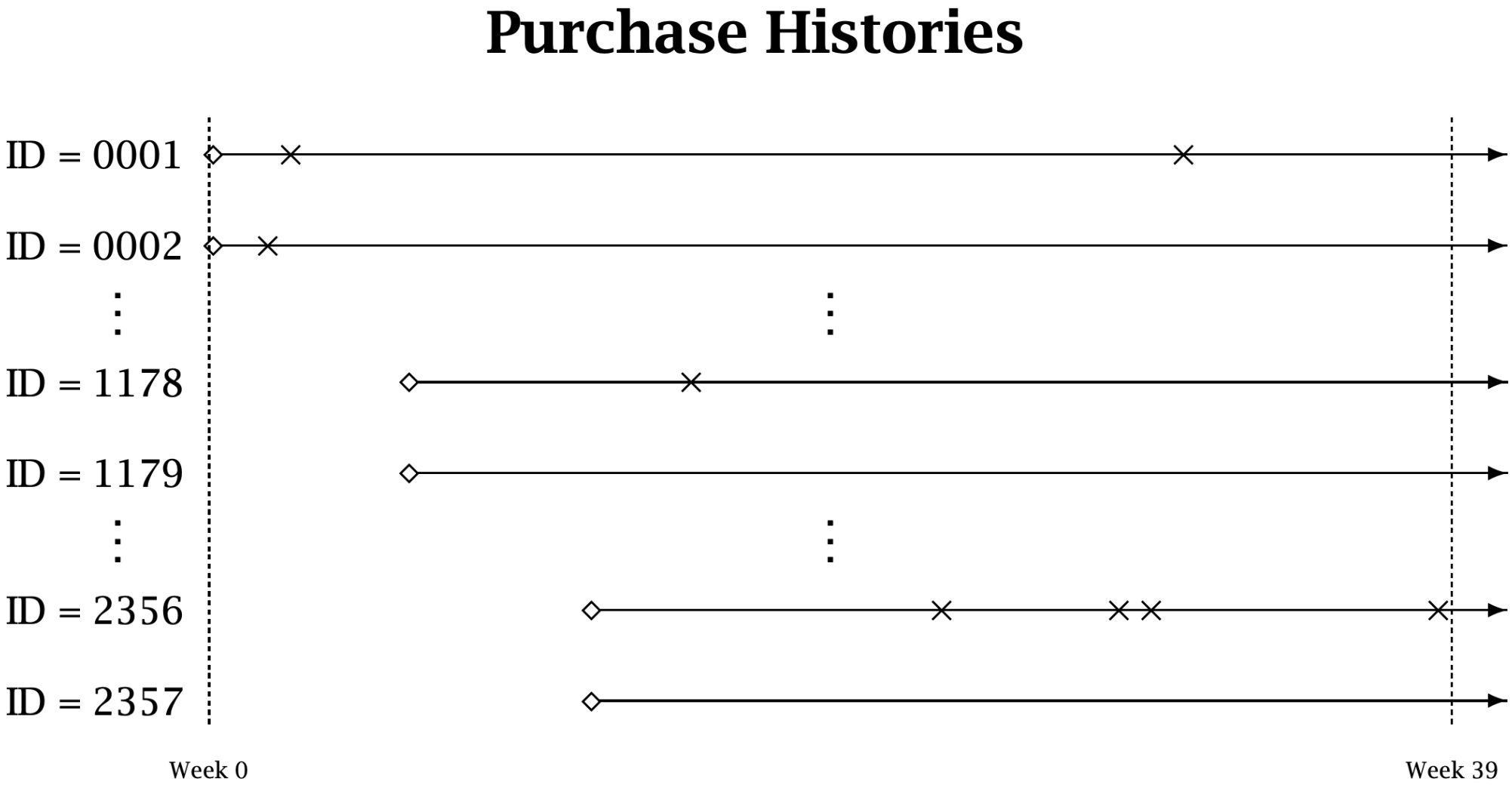
Outline

1. Introduction to BTYD Models
2. BG/NBD Model: Model Specification
3. BG/NBD Model: Maximum Likelihood Estimation (`lifetimes`)
4. BG/NBD Model: Bayesian Estimation (`pymc`)
    - External Regressors
    - Hierarchical Model
5. References and Resources

Classifying Customer Bases



Purchase Histories



The Pareto/NBD Model

The Pareto/NBD Model
(Schmittlein, Morrison and Colombo 1987)

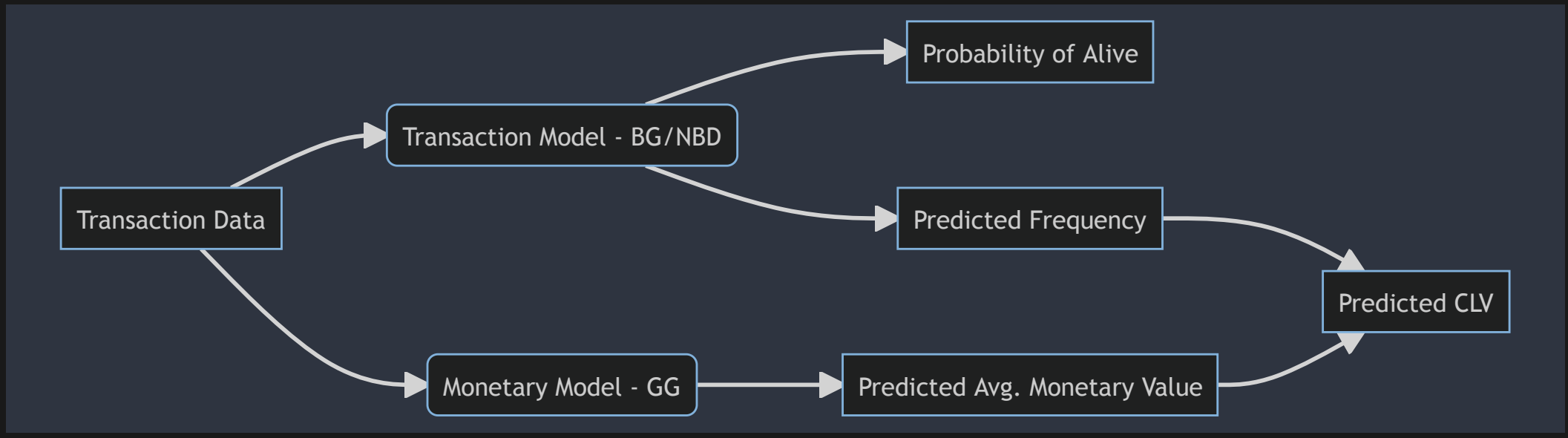
Transaction Process:

- While active, # transactions made by a customer follows a Poisson process with transaction rate λ.
- Heterogeneity in transaction rates across customers is distributed gamma(r, α).

Dropout Process:

- Each customer has an unobserved "lifetime" of length ω, which is distributed exponential with dropout rate μ.
- Heterogeneity in dropout rates across customers is distributed gamma(s, β).

Models Workflow



- **Transaction Model:** Number of transactions per customer.
- **Monetary Model:** Average monetary value of a transaction.

CLV Formula

Discounted cash flow (DCF) method:

$$CLV = \sum_{i=1}^{\infty} \frac{M_i}{(1+\delta)^i}$$

- M_i: cash flow in period i.
- δ: discount rate.

```
for i in steps:
    df["clv"] += (
        (monetary_value * expected_number_of_transactions)
        / (1 + discount_rate) ** i
    )
```

The BG/NBD Transaction Model

The BG/NBD Model
(Fader, Hardie and Lee 2005c)

Purchase Process:

- While active, # transactions made by a customer follows a Poisson process with transaction rate λ.
- Heterogeneity in transaction rates across customers is distributed gamma(r, α).

Dropout Process:

- After any transaction, a customer becomes inactive with probability p.
- Heterogeneity in dropout probabilities across customers is distributed beta(a, b).

Purchase Metrics

- `frequency`: Number of repeat purchases the customer has made. More precisely, It's the count of time periods the customer had a purchase in.
- `T`: Age of the customer in whatever time units chosen. This is equal to the duration between a customer's first purchase and the end of the period under study.
- `recency`: Age of the customer when they made their most recent purchases. This is equal to the duration between a customer's first purchase and their latest purchase.

BG/NBD Assumptions (Frequency)

1. While active, the time between transactions is distributed exponential with transaction rate, i.e.,

$$f(t_j|t_{j-1};\lambda) = \lambda \exp(-\lambda(t_j - t_{j-1})), \quad t_j \geq t_{j-1} \geq 0$$

2. Heterogeneity in λ follows a gamma distribution with pdf

$$f(\lambda|r,\alpha) = \frac{\alpha^r \lambda^{r-1} \exp(-\lambda\alpha)}{\Gamma(r)}, \quad \lambda > 0$$

BG/NBD Assumptions (Dropout)

3. After any transaction, a customer becomes inactive with probability p.
4. Heterogeneity in p follows a beta distribution with pdf

$$f(p|a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1}(1-p)^{b-1}, \quad 0 \leq p \leq 1$$

5. The transaction rate λ and the dropout probability p vary independently across customers.

Likelihood: Easy to Compute!

$$L(a, b, \alpha, r | X = x, t_x, T) = A_1 A_2 (A_3 + \delta_{x>0} A_4)$$

where

$$A_1 = \frac{\Gamma(r+x)\alpha^r}{\Gamma(x)} \qquad A_2 = \frac{\Gamma(a+b)\Gamma(b+x)}{\Gamma(b)\Gamma(a+b+x)}$$

$$A_3 = \left(\frac{1}{\alpha+T}\right)^{r+x} \qquad A_4 = \left(\frac{a}{b+x-1}\right)\left(\frac{1}{\alpha+t_x}\right)^{r+x}$$

Strategy: Write this in `numpy` as pass it through `scipy.optimize.minimize`

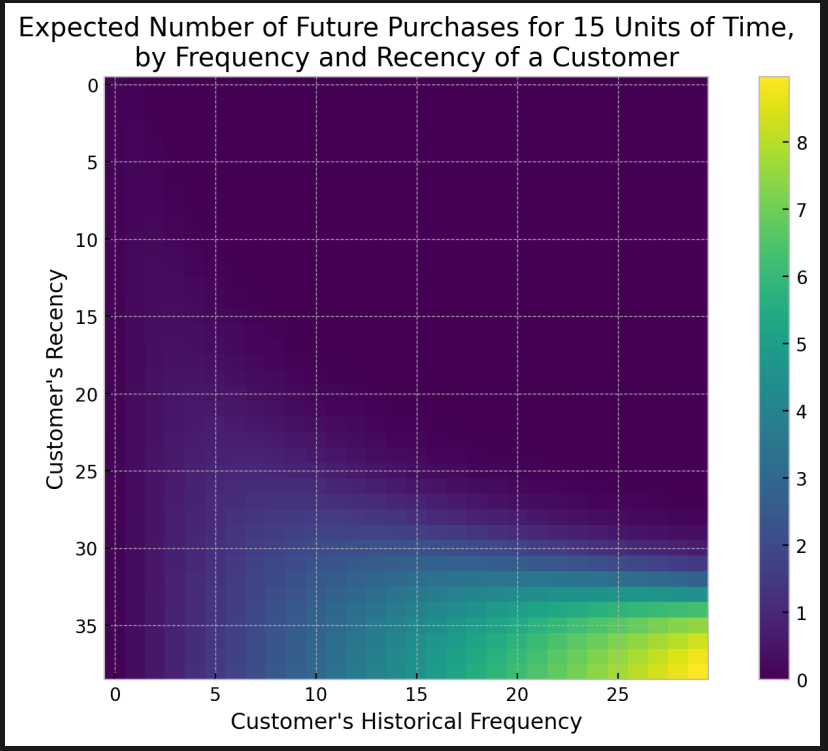
Inference: `lifetimes` Package

The four BG/NBD model parameters can be estimated via the method of **maximum likelihood**.

```python
import numpy as np
import pandas as pd
from lifetimes.datasets import load_cdnow_summary
from lifetimes import BetaGeoFitter

data_df: pd.DataFrame = load_cdnow_summary(index_col=[0])

n = data_df.shape[0]
x = data_df["frequency"].to_numpy()
t_x = data_df["recency"].to_numpy()
T = data_df["T"].to_numpy()

bgf = BetaGeoFitter()
bgf.fit(frequency=x, recency=t_x, T=T)
```

Predicted Future Purchases

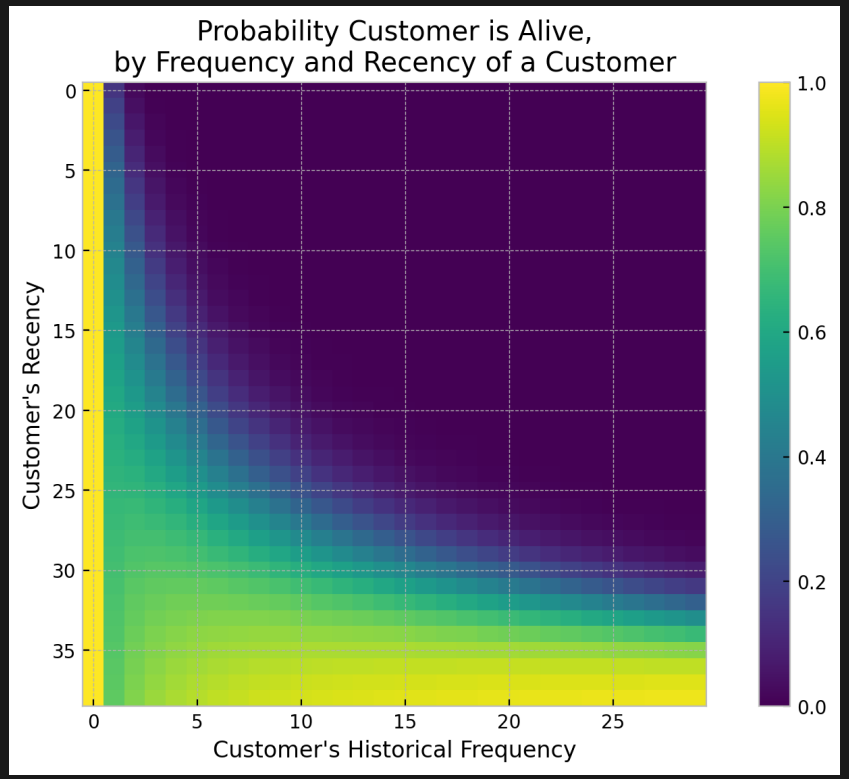
```
1  from lifetimes.plotting import plot_frequency_recency_matrix
2
3  ax = plot_frequency_recency_matrix(model=bgf, T=15)
```

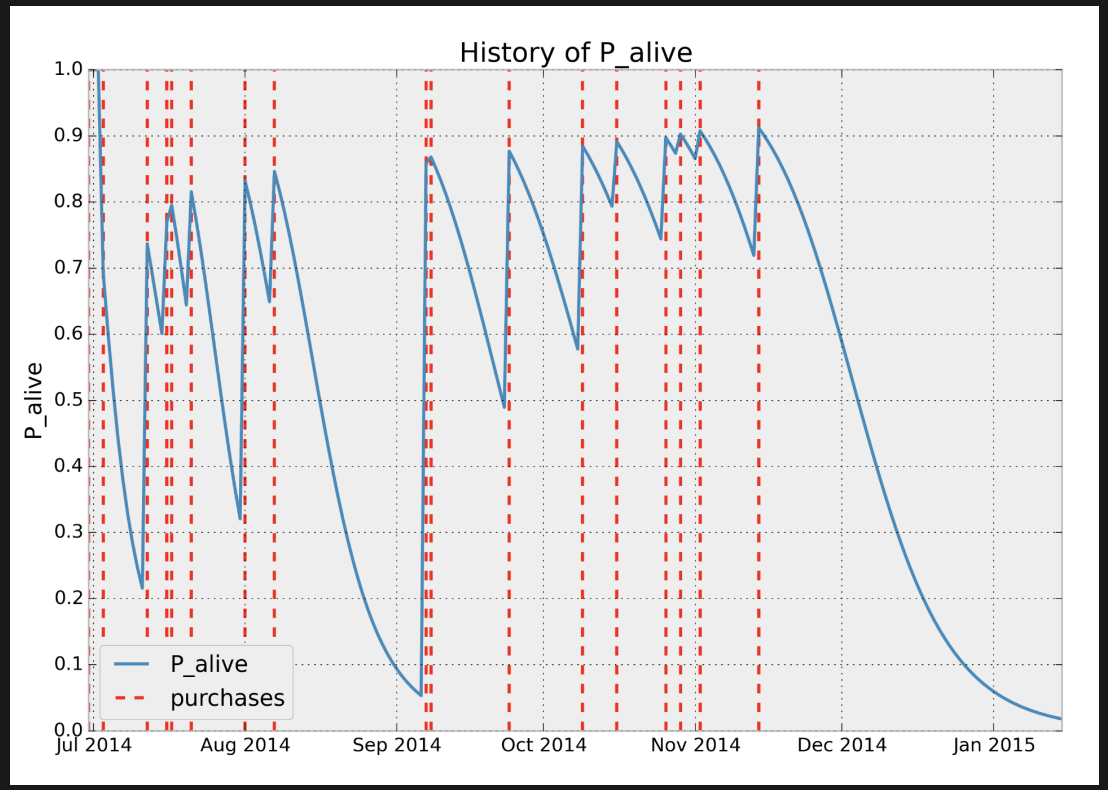


$$E[Y(t)|X = x, t_x, T, r, \alpha, a, b]$$

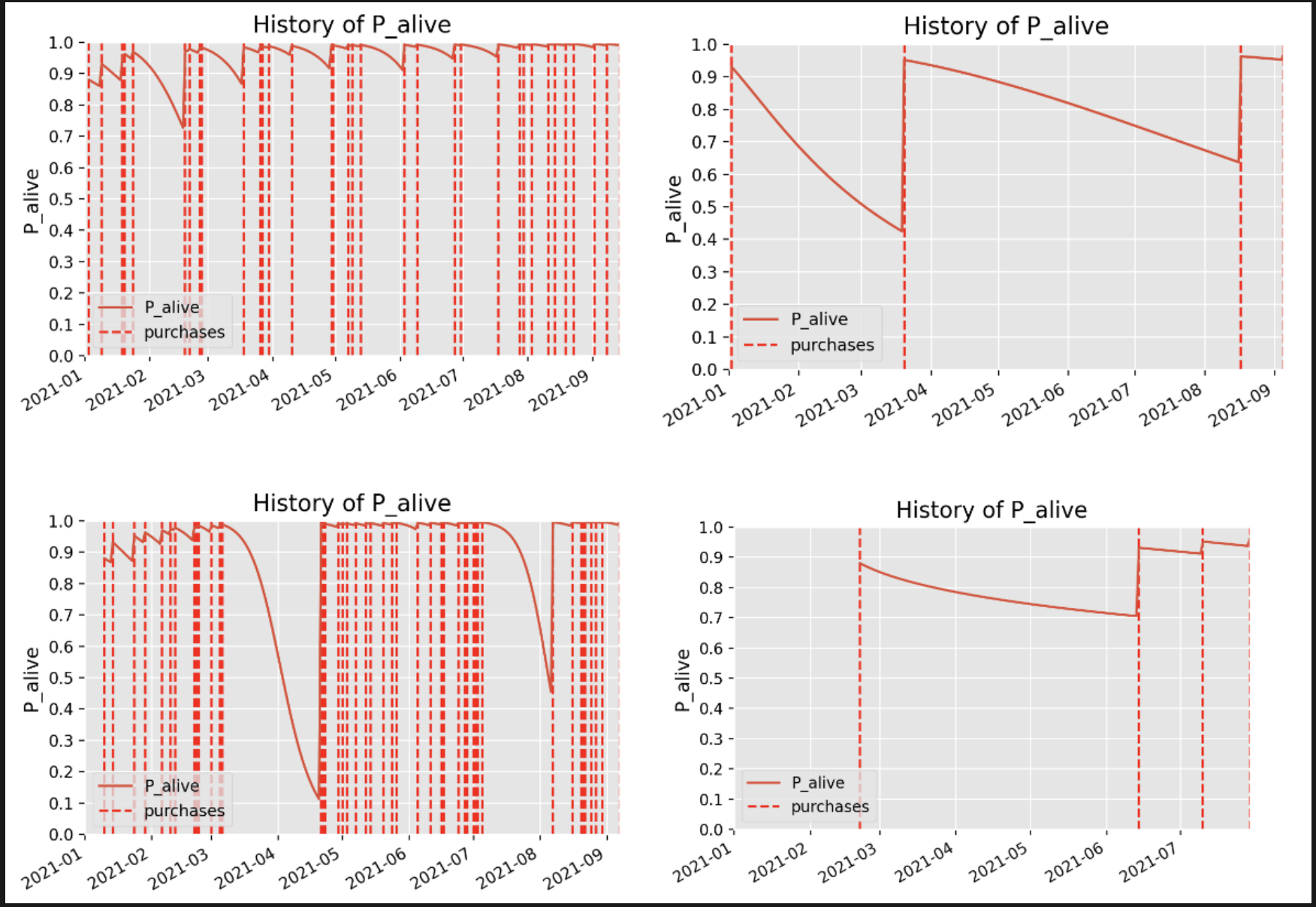
Probability of Alive

```
1  from lifetimes.plotting import plot_probability_alive_matrix
2
3  ax = plot_probability_alive_matrix(model=bgf)
```

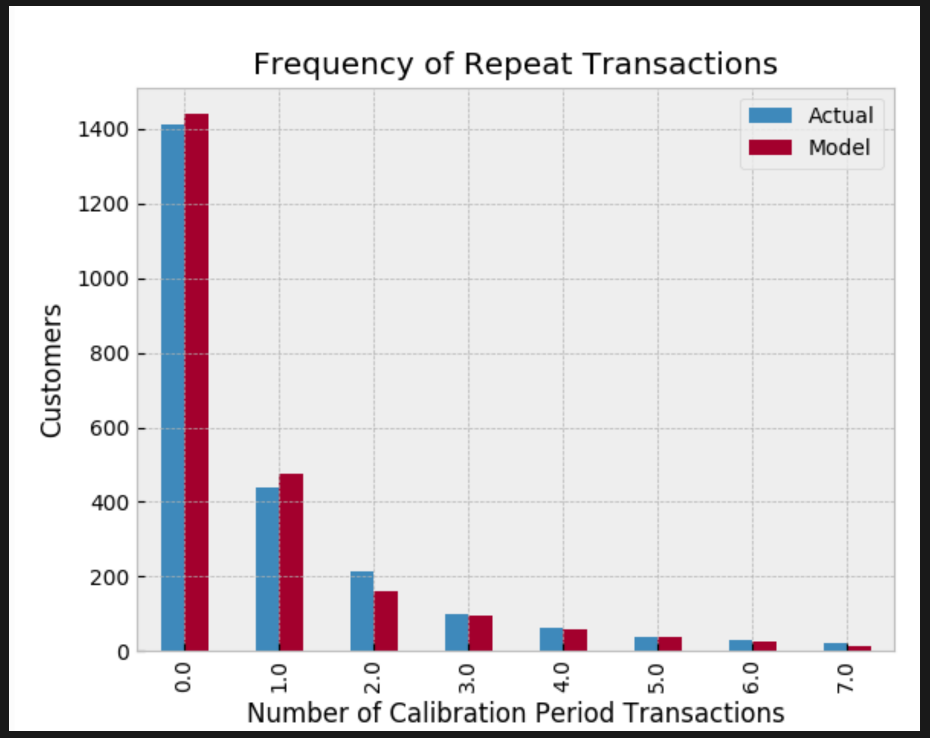


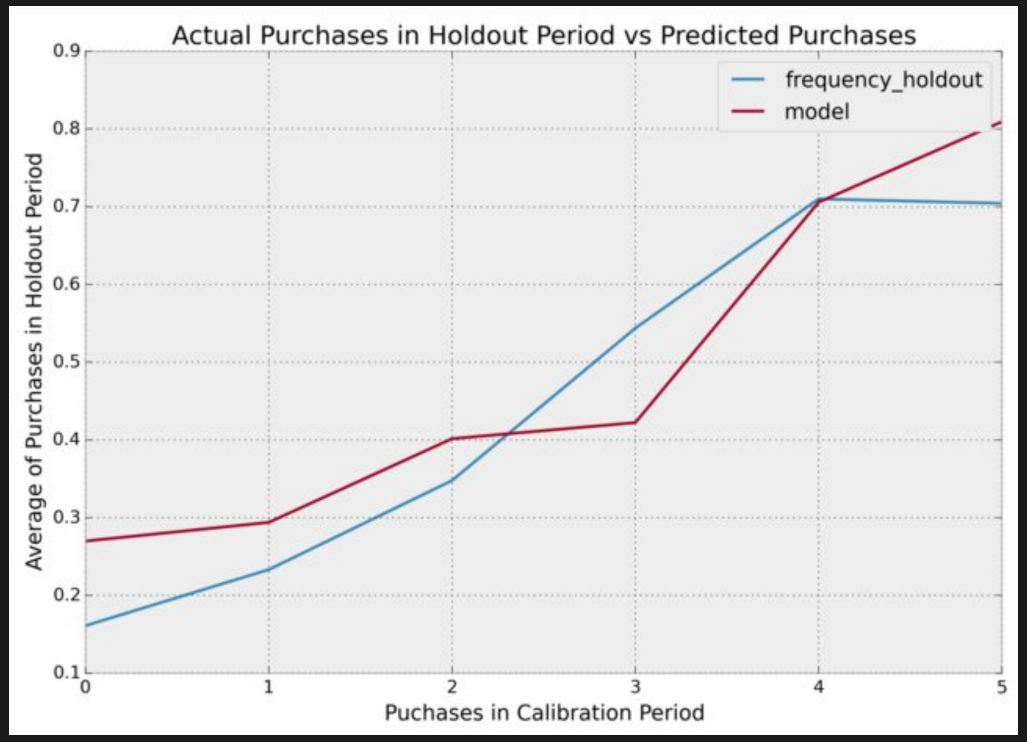


Probability of Alive



Model Evaluation

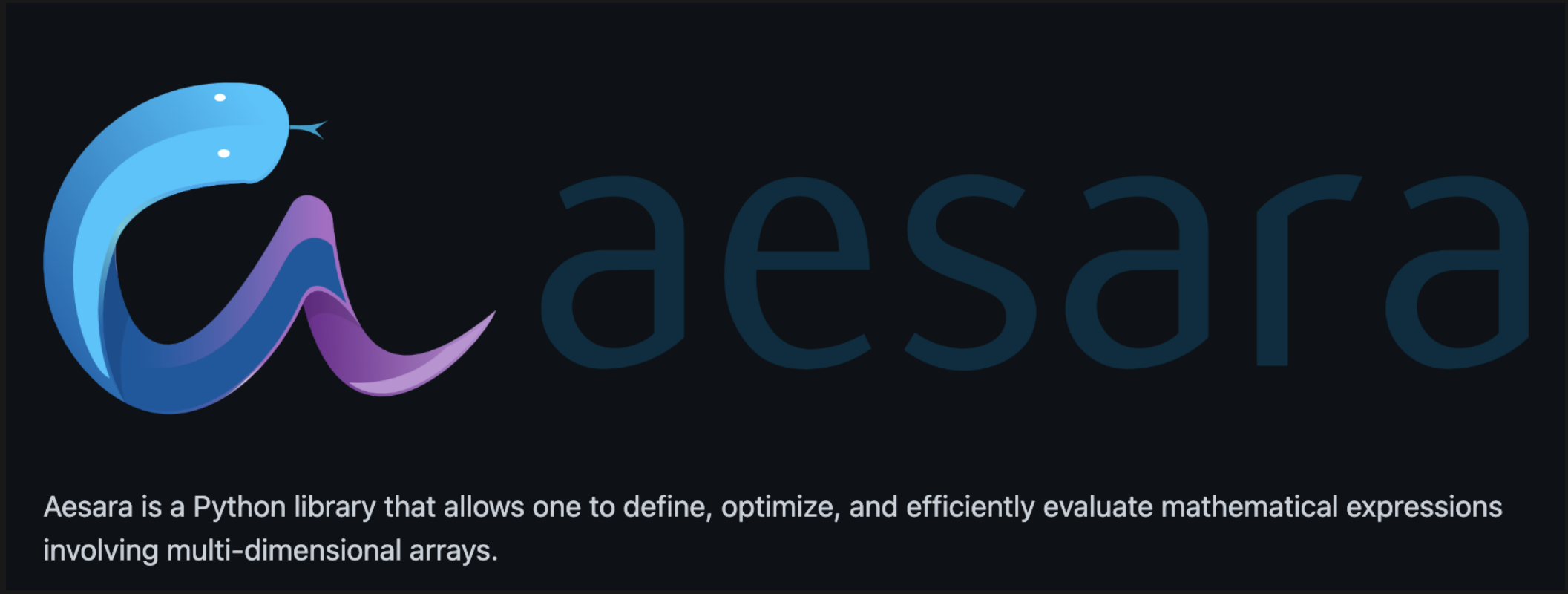





```
summary["model_predictions"] = model.conditional_expected_number_of_purchas
    duration_holdout, summary["frequency_cal"], summary["recency_cal"], sum
)

summary.groupby("frequency_cal")[["frequency_holdout", "model_predictions"]
```

From Numpy to Aesara

We can re-write the `numpy` likelihood implementation in `aesara`.



Log-Likelihood in Aesara

The translation bewtween `numpy` and `aesara` is very easy (replace `np` by `at`)!

```
import aesara.tensor as at


def logp(x, t_x, T, x_zero):
    a1 = at.gammaln(r + x) - at.gammaln(r) + r * at.log(alpha)
    a2 = (
        at.gammaln(a + b)
        + at.gammaln(b + x)
        - at.gammaln(b)
        - at.gammaln(a + b + x)
    )
    a3 = -(r + x) * at.log(alpha + T)
    a4 = (
        at.log(a) - at.log(b + at.maximum(x, 1) - 1) - (r + x) * at.log(t_x
    )
    max_a3_a4 = at.maximum(a3, a4)
    ll_1 = a1 + a2
    ll_2 = (
```

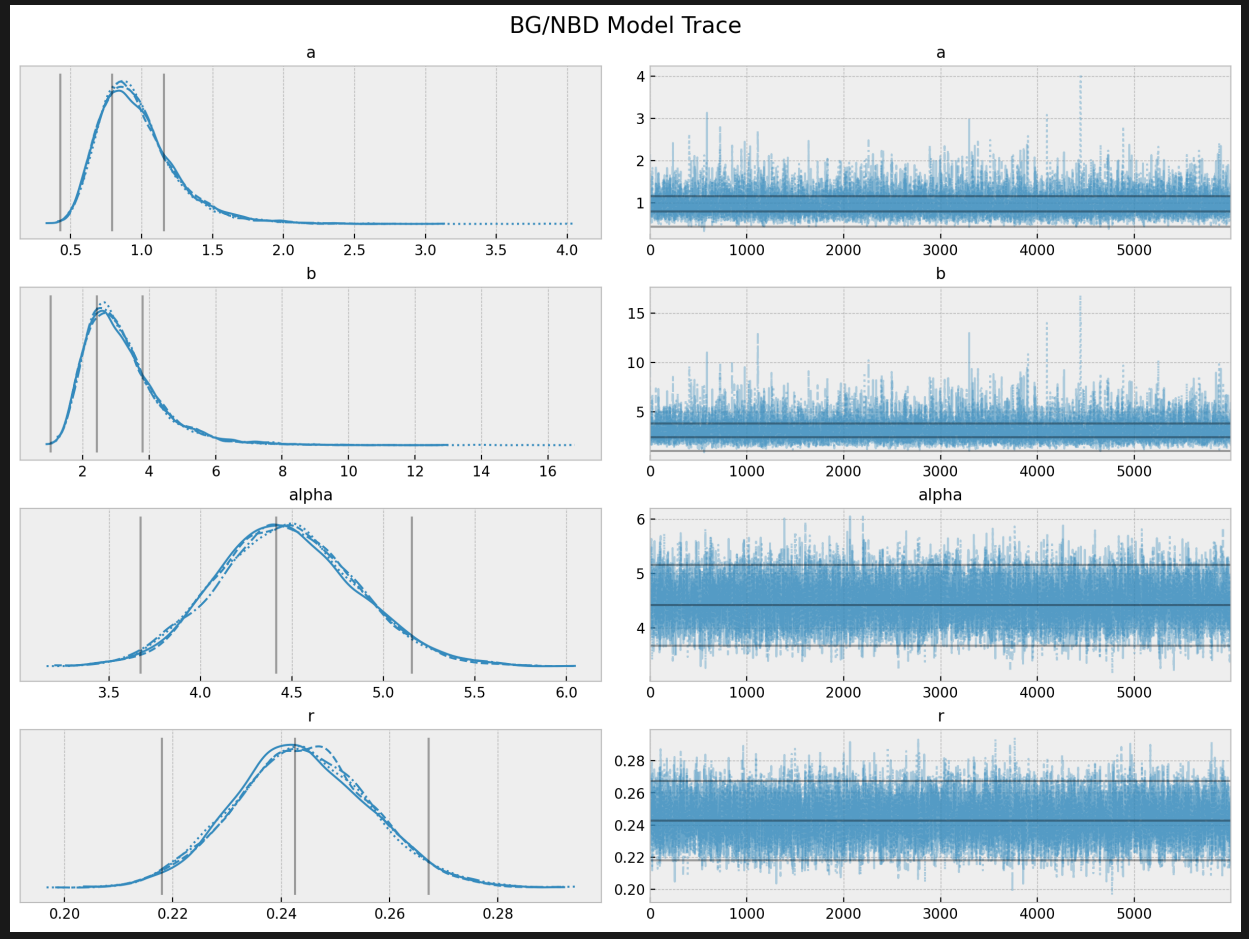
BG/NBD PyMC Model

Model Structure

```
import pymc as pm
import pymc.sampling_jax


with pm.Model() as model:

    a = pm.HalfNormal(name="a", sigma=10)
    b = pm.HalfNormal(name="b", sigma=10)

    alpha = pm.HalfNormal(name="alpha", sigma=10)
    r = pm.HalfNormal(name="r", sigma=10)

    def logp(x, t_x, T, x_zero):
        ...

    likelihood = pm.Potential(
        name="likelihood",
        var=logp(x=x, t_x=t_x, T=T, x_zero=x_zero),
```

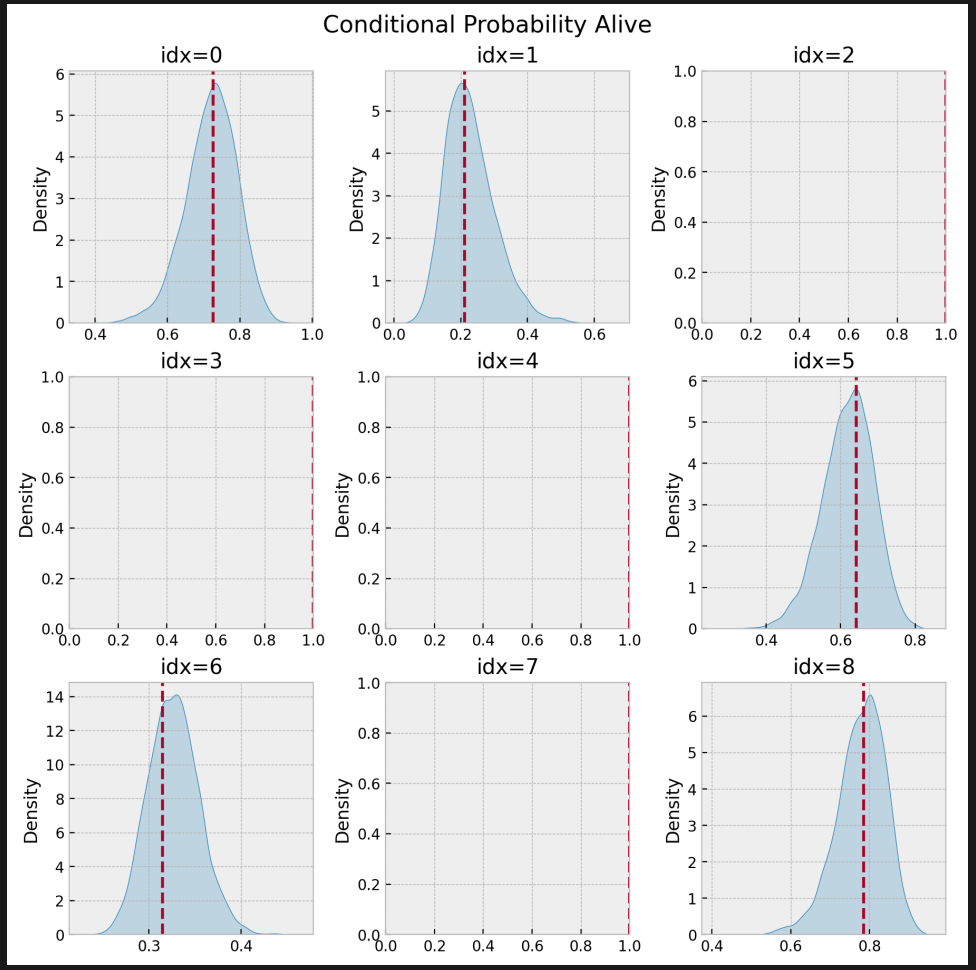
Model Sampling

```
with model:
    trace = pm.sampling_jax.sample_numpyro_nuts(
        tune=3000, draws=6000, chains=4,target_accept=0.95
)
```



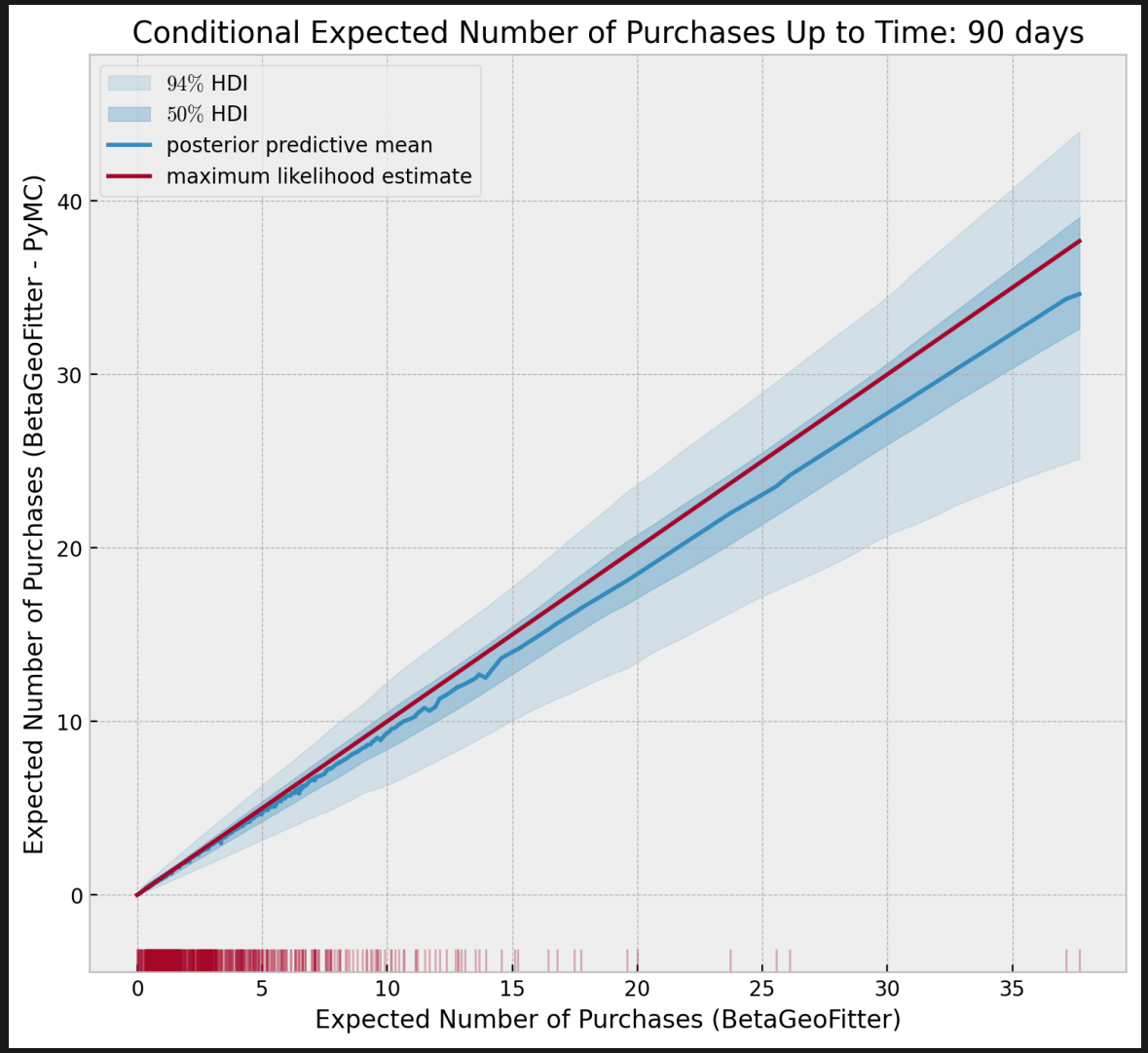
Probability of Active

How? Simply use the posterior samples (`xarray`) and broadcast the `numpy` expressions.



Future Purchases

Similarty, using the analytical expressions:



Time-Independent Covariates?

- Allow covariates z_1 and z_2 to explain the cross-sectional heterogeneity in the purchasing process and cross-sectional heterogeneity in the dropout process respectively.
- The likelihood and quantities of interests computed by computing expectations, remain almost the same. One only has to replace:

$$\alpha \longmapsto \alpha_0 \exp(-\gamma_1^T z_1)$$
$$a \longmapsto a_0 \exp(\gamma_2^T z_2)$$
$$b \longmapsto b_0 \exp(\gamma_3^T z_2)$$

Simulated Example

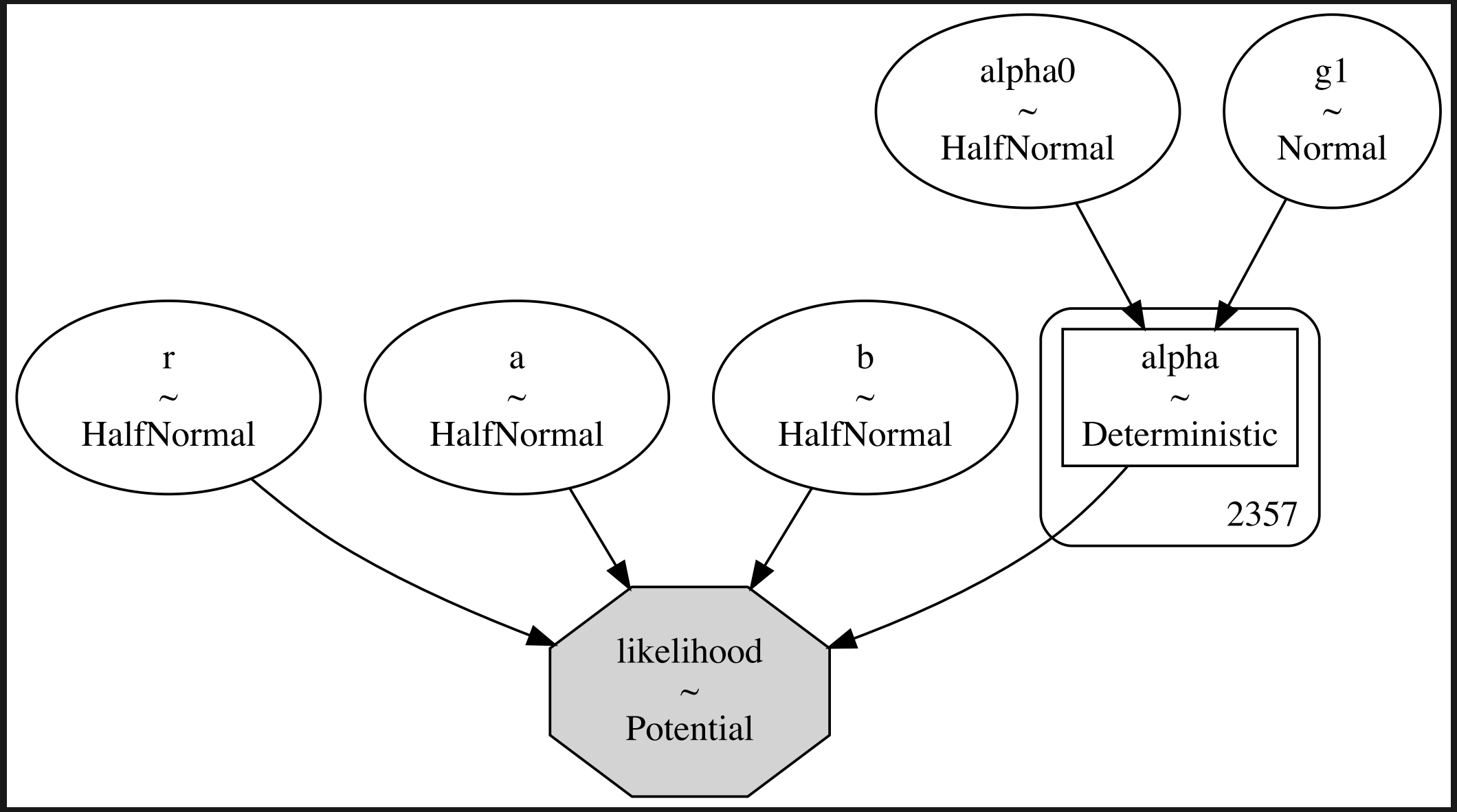
```
np.random.seed(42)

# construct covariate
mu = 0.4
rho = 0.7
z = np.random.binomial(n=1, p=mu, size=x.size)

# change frequency values by reducing it the values where z !=0
x_z = np.floor(x * (1 - (rho * z)))

# make sure the recency is zero whenever the frequency is zero
t_x_z = t_x.copy()
t_x_z[np.argwhere(x_z == 0).flatten()] = 0

# sanity checks
assert x_z.min() == 0
assert np.all(t_x_z[np.argwhere(x_z == 0).flatten()] == 0)
```

$z = 1 \Rightarrow$ the frequency is reduced by 70%.

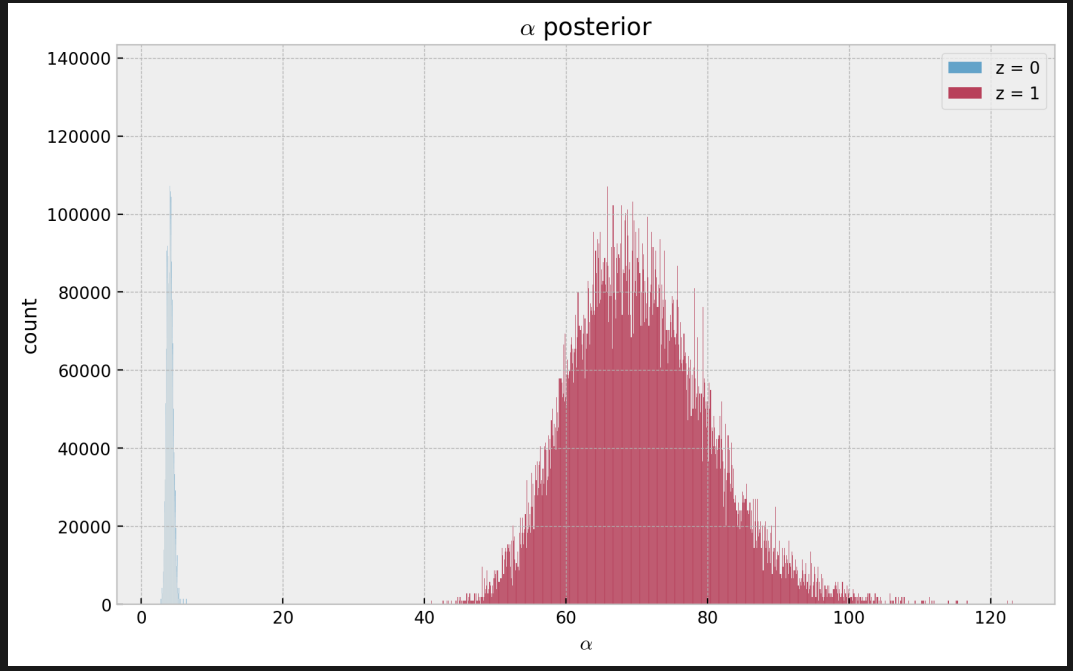
PyMC Model

```
with pm.Model() as model_cov:

    a = pm.HalfNormal(name="a", sigma=10)
    b = pm.HalfNormal(name="b", sigma=10)

    alpha0 = pm.HalfNormal(name="alpha0", sigma=10)
    g1 = pm.Normal(name="g1", mu=0, sigma=10)
    alpha = pm.Deterministic(name="alpha", var=alpha0 * at.exp(- g1 * z))

    r = pm.HalfNormal(name="r", sigma=10)

    def logp(x, t_x, T, x_zero):
        ...

    likelihood = pm.Potential(
        name="likelihood",
        var=logp(x=x_z, t_x=t_x_z, T=T, x_zero=x_zero_z),
    )
```

Model Parameters



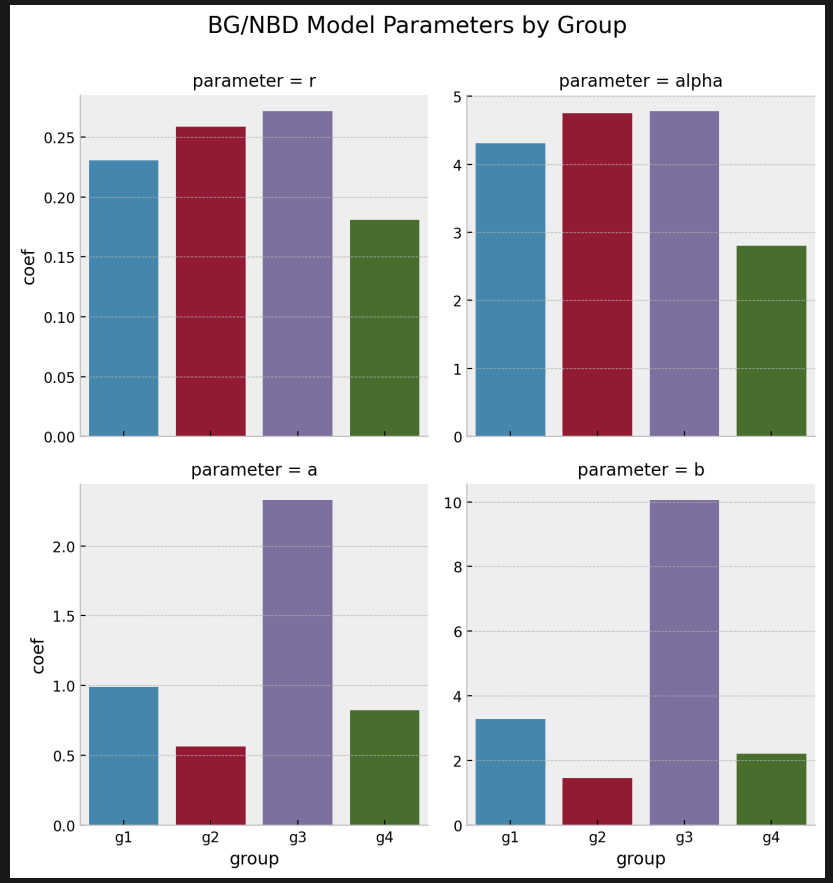
Effect in Parameters



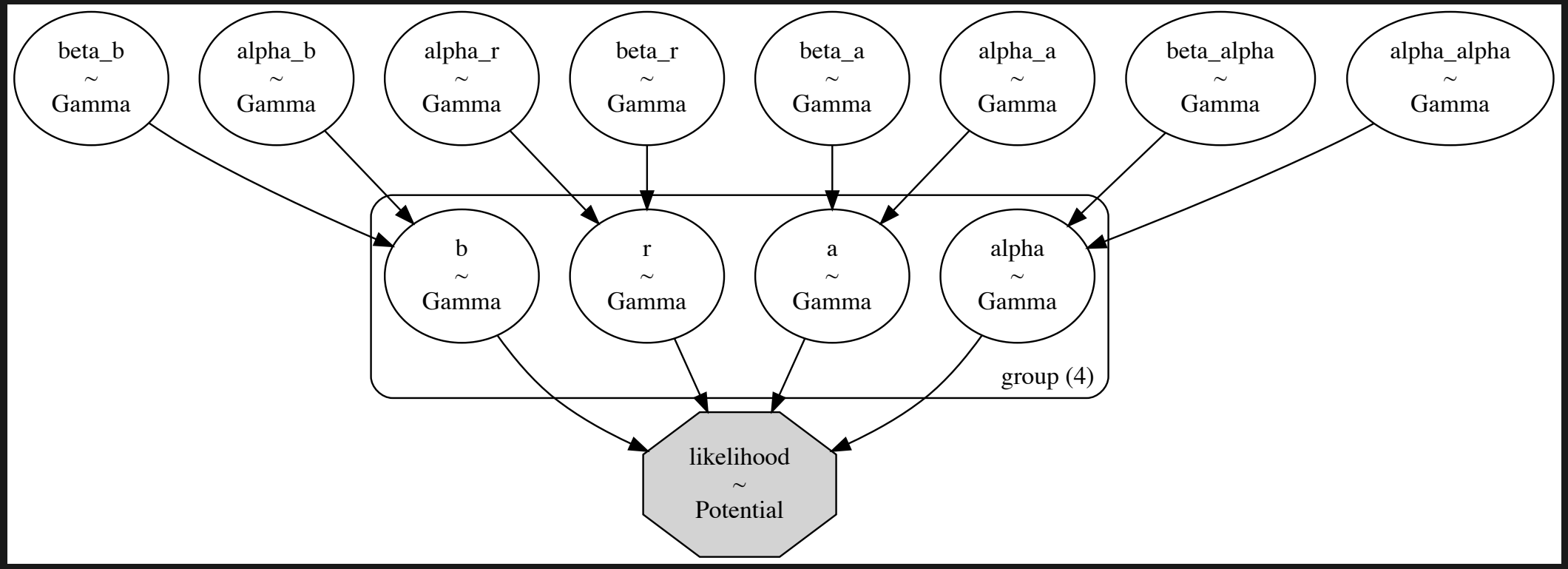
- Recall $\lambda \sim \text{Gamma}(r, \alpha)$, which has expected value r/α.
- Hence, as $g_1 < 0$, then $\alpha(z = 1) > \alpha(z = 0)$ which is consistent with the data generation process where $z = 1$ implied a lower frequency.

Hierarchical BG/NBD Model - Data

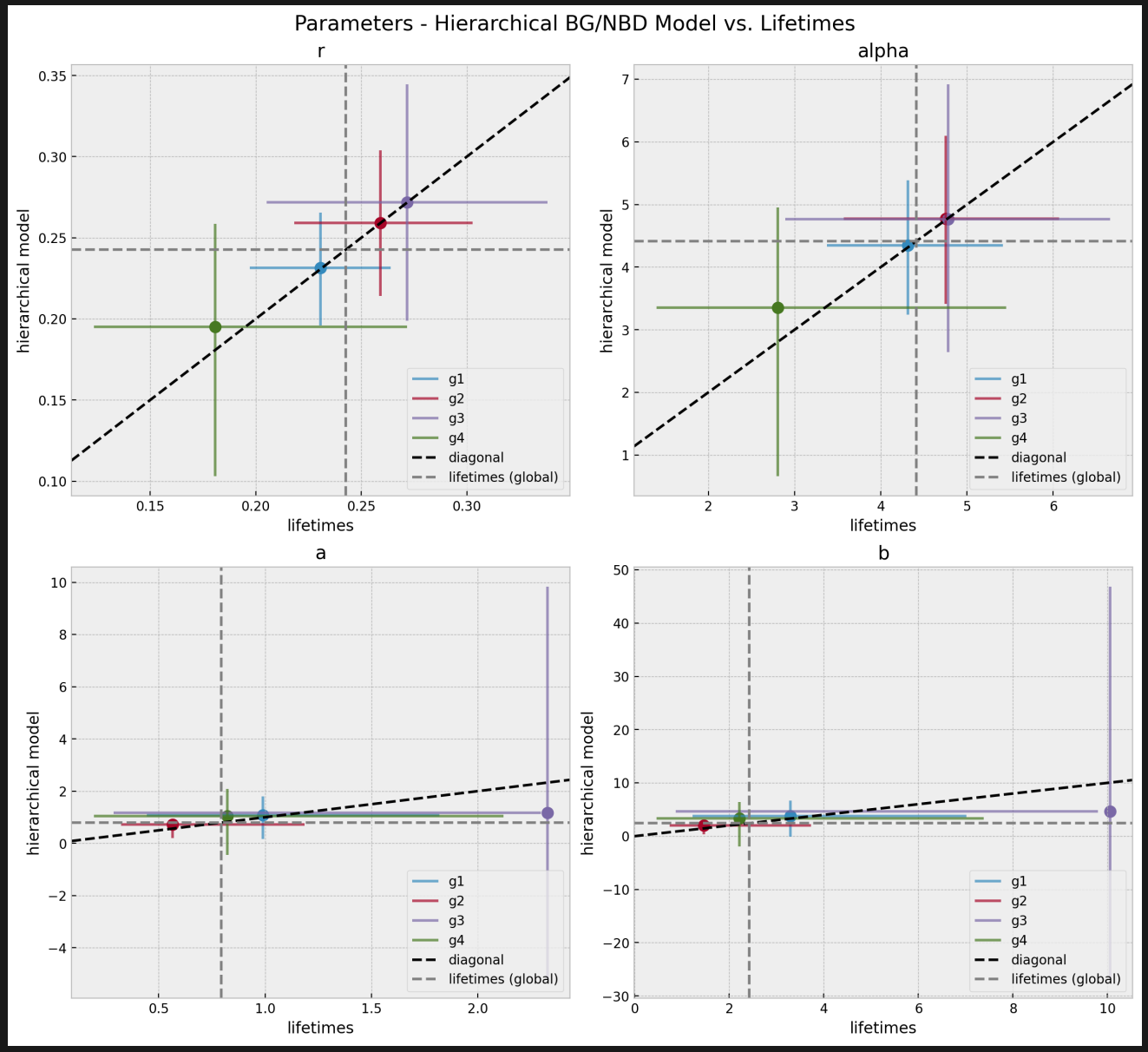
```
groups = ["g1", "g2", "g3", "g4"]

data_df["group"] = rng.choice(
    a=groups, p=[0.45, 0.35, 0.15, 0.05], size=n_obs
)
```



Hierarchical BG/NBD Model Structure



Hierarchical BG/NBD Model Results

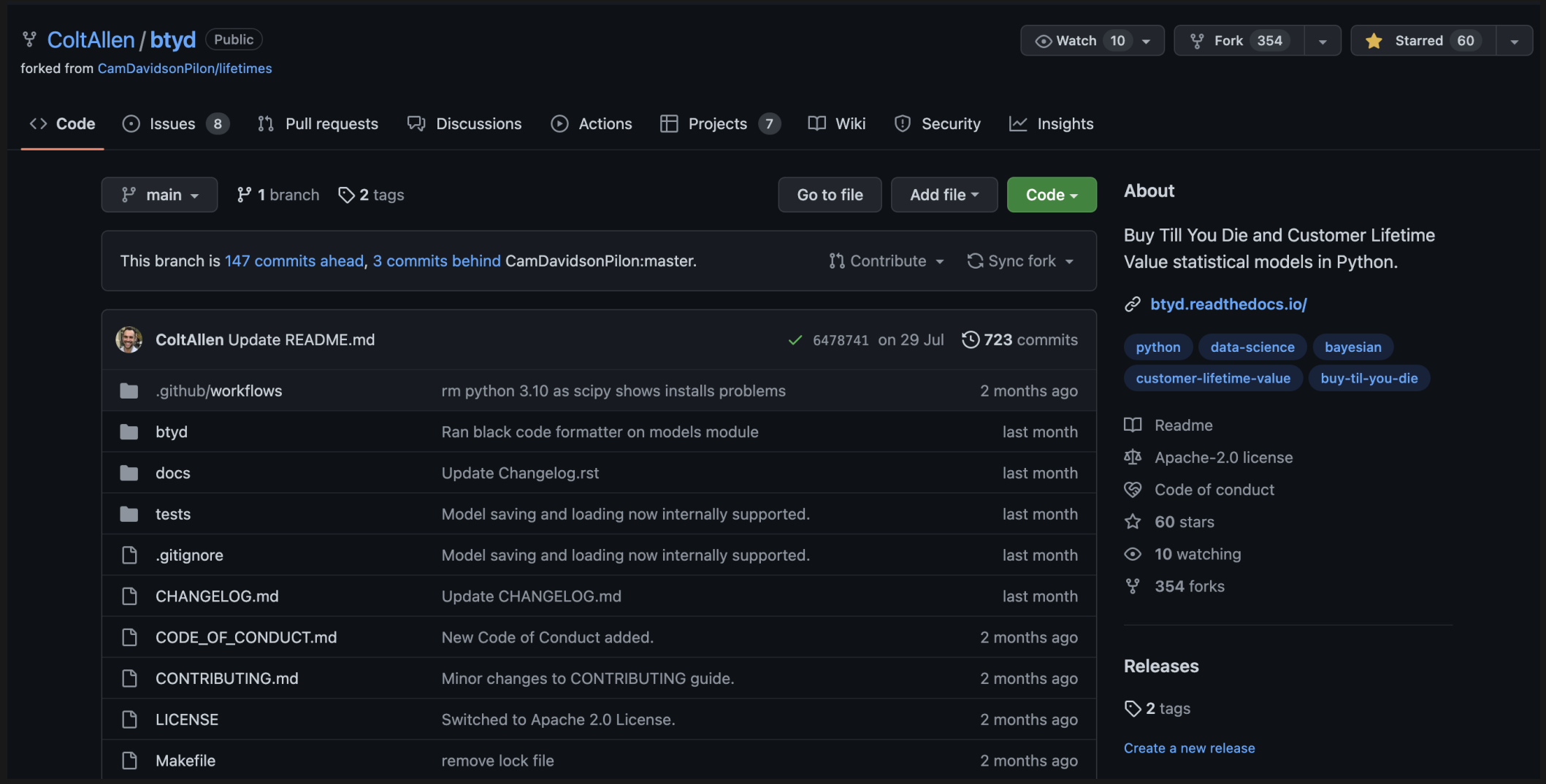


Bayesian BG/NBD Model

It opens many posibilities and oportunities!

- Time-Invariant Covariates
- Hierarchical Models
- ...

A sucessor package of lifetimes



Contributors needed!

References

Blog Posts

- BG/NBD Model in PyMC
- Gamma-Gamma Model of Monetary Value in PyMC

Papers

- Probability Models for Customer-Base Analysis
- Counting Your Customers: Who Are They and What Will They Do Next?
- “Counting Your Customers” the Easy Way: An Alternative to the Pareto/NBD Model
- Computing P(alive) Using the BG/NBD Model

Videos

- New Perspectives on CLV and E-Commerce Buying Patterns, Peter Fader

References

Software

Python

- `lifetimes`
- A sucessor package `btyd` of `lifetimes`.
- PyMC

R

- `BTYD`
- `BTYDplus`

Thank you!

juanitorduz.github.io