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XVI. *On Yule's Method of Investigating Periodicities of Disturbed Series. The Motion of a Pendulum in a Turbulent Fluid.*

By T. E. W. Schumann, Ph.D.*

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1. *Introduction.*

The harmonic analysis of a time series by Schuster's method has in general yielded disappointing results. For example, in the analysis of records of atmospheric pressure, numerous authors have detected many "significant" periodicities, but most of the periodicities thus discovered are not persistent and hence have proved to be of little, if any, practical value.

In an important paper Yule (1927) approached the problem from a new angle by introducing the idea of "disturbed" harmonic oscillations, which he visualises in the following words :—"If we observe at short equal intervals of time the departures of a simple harmonic pendulum from its position of rest, errors of observation will cause *superposed* fluctuations. . . . . The recording apparatus is left to itself, and unfortunately boys get into the room and start pelting the pendulum with peas, sometimes from one side, and sometimes from the other. The motion is now affected, not by *superposed fluctuations*, but by true *disturbances*, and the effect on the graph (showing the motion of the pendulum) will be of an entirely different kind. The graph will remain surprisingly smooth, but amplitude and phase will vary continually."

Yule then proceeded to investigate the motion of a pendulum subject to such random disturbances and applied his results to the analysis of Wolfer's sunspot numbers. His work was later extended by Sir Gilbert Walker (1931) and others, but thus far only the case where the disturbances are entirely random has been dealt with, and if any fundamental advance is to be made on Yule's original research, it becomes imperative to deal with the more general case of disturbances which are interrelated. The object of this paper then is to discuss the general case of disturbed harmonic motion, where the time series constituting the disturbances is not entirely haphazard, but shows finite coefficients of auto-correlation for finite time-lags.

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With this object in view, it seems advisable to deviate somewhat from Yule's rather formal treatment of the problem and to lay more stress on its dynamical aspects. Not only will the solution of the problem thereby be facilitated, but we shall also more readily be led to the correct interpretation of the statistical results.

2. *Statement of the Problem.*

Suppose we have a simple pendulum, subject to a certain amount of damping, which at *equal* intervals of time is bombarded in its line of motion by pellets all of equal mass, the velocities of the pellets being distributed according to the normal (Gaussian) or to any other symmetrical distribution law. Also suppose that the velocities of the pellets are not entirely random with respect to the time, but have given auto-correlations for given time-lags. Our problem is to investigate the motion of the pendulum in the statistical sense, *i. e.*, to express the parameters defining the average motion of the pendulum in terms of the parameters by which the motion of the pellets are described.

For this purpose we make use of the following notation :—

t=time.

ξ=time-lag.

h=interval of time between one impact and the next.

M=mass of the simple pendulum.

mh=mass of a single pellet.

$$D=\frac{M-mh}{M+mh}.$$

$$E=\frac{mh}{M+mh}.$$

$y_0 \ldots . y_n$=displacement of the pendulum from its zero position at times $t=0, h, 2h \ldots . nh$.

$u_0' \ldots . u_n'$=the corresponding velocities of M just *before* impact.

$u_0 \ldots . u_n$= ,, ,, ,, *after* ,,

$2\pi/\alpha$=the (damped) period of the pendulum.

l=its damping factor.

$\lambda=l+m/M$.

$Q=e^{-lh}$.

$v_1 \ldots . v_n$=the velocities of the first to the nth pellet.

$R_1 \ldots . R_s$=coefficients of auto-correlation of the v's after time-lags of $h \ldots . sh$.

$r_1 \ldots . r_s$=the corresponding auto-correlations of the y's.

$M^2\eta^2=\frac{m^2}{n}\Sigma v_r^2=$ the sum of the squares of the momenta of the pellets striking the pendulum in unit time.

Eventually we shall have occasion to consider the limiting case, where the interval of time h becomes indefinitely small, in which case the kinetic energy of the pellets striking the pendulum would become infinitely large if the mass of a single pellet remained finite. This is the reason why we have assumed the mass of the pellet proportional to h, for then the energy of the pellets striking the pendulum per unit time remains constant even though h is varied at will.

Specifically our problem is to derive relationships between the motion of the pendulum and the original disturbances producing that motion.

3. *Derivation of the Fundamental Equations.*

In the first place we can obtain an expression between u_r' and u_r, the velocities of the pendulum just before and just after the rth impact, by applying the laws of the conservation of momentum and of energy and supposing the impacts to be perfectly elastic. The relation is

$$u_r=\frac{M-mh}{M+mh}u_r'+\frac{2mh}{M+mh}v_r,$$

$$=Du_r'+2Ev_r. \qquad (1)$$

Between impacts the equation of motion of the pendulum is

$$\frac{d^2y}{dt^2}-2l\frac{dy}{dt}+(d^2+l^2)y=0, \qquad (2)$$

and the solution of this equation may be written

$$y=Ae^{-lt}\sin(\alpha t+\omega)$$

and

$$u=-lAe^{-lt}\sin(\alpha t+\omega)+\alpha Ae^{-lt}\cos(\alpha t+\omega).$$

In particular we may write

$$y_0=A_1\sin\omega_1. \qquad (3)$$

$$u_0=-lA_1\sin\omega_1+\alpha A_1\cos\omega_1. \qquad (4)$$

$$y_1=A_1Q\sin(\alpha h+\omega_1). \qquad (5)$$

$$u_1'=-lA_1Q\sin(\alpha h+\omega_1)+\alpha A_1Q\cos(\alpha h+\omega_1). \qquad (6)$$

Now, due to the impact at the time $t=h$, both the amplitude and the phase are changed to A_2 and ω_2 respectively (say). Accordingly, since y_1 is not instantaneously changed by the impact, we may again write

$$y_1=A_2\sin\omega_2. \qquad (7)$$

$$u_1=-lA_2\sin\omega_2+\alpha A\cos\omega_2. \qquad (8)$$

$$y_2=A_2Q\sin(\alpha h+\omega_2).$$

But according to (1)

$$u_1 = \mathrm{D}u_1' + 2\mathrm{E}v_1$$
$$= -\mathrm{D}l\mathrm{A}_1\mathrm{Q}\sin(\alpha h+\omega_1)+\mathrm{D}\alpha\mathrm{A}_1\mathrm{Q}\cos(\alpha h+\omega_1)+2\mathrm{E}v_1. \quad . \quad . \quad (9)$$

Between equations (3) to (9) it is possible to eliminate the six constants A_1, A_2, ω_1, ω_2, u_0 and u_1, the result being

$$y_2 = kqy_1 - q^2y_0 + \epsilon_2,$$

or, in general,

$$y_r = kqy_{r-1} - q^2y_{r-2} + \epsilon_r, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where we have introduced the constants k, q, and ϵ_r, their values being

$$kq = \mathrm{Q}(\mathrm{D}+1)\cos\alpha h + \frac{\mathrm{Q}l}{\alpha}(1-\mathrm{D})\sin\alpha h. \quad . \quad . \quad . \quad . \quad (11)$$

$$q^2 = \mathrm{DQ}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\epsilon_r = 2\mathrm{QE}v_{r-1}\frac{\sin\alpha h}{\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

We may also write down the corresponding equation for u_r, derived in exactly the same way :

$$u_r = kqu_{r-1} - q^2u_{r-1} + \delta_r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where

$$\delta_r = 2\mathrm{E}\left\{v_r - \mathrm{Q}v_{r-1}\left(\frac{l}{\alpha}\sin\alpha h + \cos\alpha h\right)\right\}. \quad . \quad . \quad . \quad . \quad (15)$$

As we shall, in all our subsequent work, consider n to be very large, the series being practically infinite in extent, we may without loss of generality put y_0 equal to zero, in which case it follows from (10) that

$$y_1 = \epsilon_1,$$
$$y_2 = kq\epsilon_1 + \epsilon_2,$$
$$y_3 = (k^2q^2 - q^2)\epsilon_1 + kq\epsilon_2 + \epsilon_r, \text{ etc.}$$

In general these equations may be written

$$y_r = \phi_0\epsilon_r + \phi_1\epsilon_{r-1} + \ldots . \phi_{r-1}\epsilon_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where we have introduced the function ϕ_r, whose value is given by

$$\phi_0 = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\phi_1 = kq, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and

$$\phi_{r+1} = kq\phi_r - q^2\phi_{r-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The last is a difference equation which we shall encounter again in the course of this work. Accordingly, we briefly show how its solution may be obtained ;

Subtract $p\phi_r$ from both sides of (19)

$$\phi_{r+1}-p\phi_r=(kq-p)\left(\phi_r-\frac{q^2}{kq-p}\phi_{r-1}\right). \quad . \quad . \quad . \quad . \quad (20)$$

Now choose p such that $\quad p^2-kqp+q^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$

Two cases arise according to whether k is less or greater than 2. If $k>2$ the motion of the pendulum corresponds to the highly damped aperiodic case. For the time being we leave this case out of consideration and confine our attention to the motion when $k<2$, in which case a solution of the quadratic (21) is

$$p=qe^{-i\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where

$$k=2\cos\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

It thus follows from (20) that

$$\phi_{r+1}-p\phi_r=\frac{q^2}{p}(\phi_r-p\phi_{r-1})$$

$$=qe^{i\theta}(\phi_r-p\phi_{r-1}).$$

Reducing the order of the right-hand side step by step, we finally find

$$\phi_{r+1}-e^{-i\theta}\phi_r=q^re^{ir\theta}(\phi_1-e^{-i\theta}\phi_0). \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Equating the imaginary terms of (24) we obtain the result

$$\phi_r\sin\theta=q^{r-1}\phi_1\sin r\theta-q^r\phi_0\sin(r+1)\theta. \quad . \quad . \quad . \quad . \quad (25)$$

This is the general solution of the difference equation (19), but in our particular case, where $\phi_0=1$ and $\phi_1=kq=2q\cos\theta$, the value of ϕ_r reduces to

$$\phi_r=q^r\sin(r+1)\theta/\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In equation (16), which may be re-written in the form

$$y_r=\sum_{s=0}^{r}\phi_s\epsilon_{r-s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

the deviations $y_0 \ldots . y_n$ are expressed in terms of the disturbances $\epsilon_1 \ldots . \epsilon_n$ and of the functions ϕ_r, defined in (26). Equations (26) and (27) may therefore be regarded as fundamental to the further development of the theory.

4. *The Computation of the Auto-correlation Coefficients.*

We can now proceed to calculate the standard deviation of the y's, as well as the auto-correlation coefficients r_1, r_2, r_3, etc., and for this purpose we employ the symbols

$$\sigma_0^2=\frac{1}{n}\sum_0^n y_r^2,$$

$$\sigma_s^2=\frac{1}{n-s}\sum_0^{n-s}y_ry_{r+s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

We also use the symbol σ, without subscript, to denote the standard deviation of the disturbances

$$\sigma^2 = \frac{1}{n}\sum_0^n \epsilon_r^2. \qquad (29)$$

By ordinary algebra it is now possible to determine the value of σ_s; *e. g.*, from (15) it is evident that

$$y_r y_{r+s} = (\phi_0\epsilon_r + \ldots \phi_r\epsilon_0)(\phi_0\epsilon_{r+s} + \ldots \phi_{r+s}\epsilon_0).$$

If the multiplication is carried out and the values of $y_r y_{r+s}$ for all values of r are added together, the different terms being properly grouped and due regard being paid to the fact that the ϕ's form a convergent series and that the series is practically infinite in extent, it is possible to demonstrate that

$$\sigma_0^2 = \sigma^2(\chi_0 + 2R_1\chi_1 + 2R_2\chi_2 + \ldots),$$

and, in general,

$$\sigma_s^2 = \sigma^2\{\chi_s + R_1(\chi_{s-1} + \chi_{s+1}) + \ldots R_s(\chi_0 + \chi_{2s}) + R_{s+1}(\chi_1 + \chi_{2s+1}) + \ldots\}, \qquad (30)$$

where the R's are the autocorrelations of the ϵ's as previously stated, and where the functions χ_r are defined thus :

$$\left.\begin{aligned} \chi_0 &= \sum_0^\infty \phi_r^2, \\ \text{and} \qquad \chi_s &= \sum_0^\infty \phi_r\phi_{r+s} \end{aligned}\right\} \qquad (31)$$

Our next step is to determine the values of the χ's, for which purpose we make use of the following identity, an expression which may readily be derived by using the properties of complex quantities :

$$\sum_{x=0}^{\chi} a^x \sin(\omega x + \beta) = \frac{\sin\beta - a\sin(\beta-\omega) - a^{\chi+1}\sin(\omega\chi+\omega+\beta) + a^{\chi+2}\sin(\omega\chi+\beta)}{1+a^2-2a\cos\omega}. \qquad (32)$$

If this equation is applied, the following results can be derived from (31) without much difficulty :

$$\chi_0 = \frac{1+q^2}{(1-q^2)L^2}, \qquad (33)$$

and

$$\chi_1 = \frac{kq}{(1-q^2)L^2}, \qquad (34)$$

where

$$L^2 = (1+q^2)^2 - k^2q^2. \qquad (35)$$

Furthermore, we note that

$$\chi_s = kq\chi_{s-1} - q^2\chi_{s-2}. \qquad (36)$$

This is the same difference equation we encountered in (19), whose general solution is contained in (25). Thus, the final value of χ_s, after suitable transformation, is found to be

$$\chi_s=\frac{q^s \sin (s\theta+\gamma)}{(1-q^2)\mathrm{L} \sin \theta}, \quad . \quad . \quad . \quad . \quad (37)$$

where $$\sin \gamma=\frac{1+q^2}{\mathrm{L}}\sin \theta \quad \text{and} \quad \cos \gamma=\frac{1-q^2}{\mathrm{L}}\cos \theta. \quad . \quad . \quad . \quad (38)$$

Equation (30) may now be written in the form

$$r_s=\frac{\sum\limits_{r=0}^{\infty}(\mathrm{R}_s\chi_{r+s}+\mathrm{R}_{r+s}\chi_r)+\sum\limits_{r=0}^{s-2}\mathrm{R}_{r+1}\chi_{s-r-1}}{2\sum\limits_{0}^{\infty}\mathrm{R}_r\chi_r-\chi_0}. \quad . \quad . \quad . \quad (39)$$

where $$\sigma_0{}^2/\sigma^2=2\sum_0^{\infty}\mathrm{R}_r\chi_r-\chi_0. \quad . \quad . \quad . \quad . \quad (40)$$

In (39) and (40) we have expressions for the standard deviation and the auto-correlations of the y's, and our problem has thus been formally solved. However, the result still remains in the form of a series, and if r_s is to be expressed in simpler analytic form, it is essential that R should also be given in analytic form, and thus we now proceed to discuss the particular case where R is an exponential function of the time. At the same time it should be pointed out that it is quite possible also to treat other cases, *e. g.*, where R itself is a damped simple harmonic function.

5. *Particular Case* : R *an Exponential Function of the Time-Lag.*

As shown in a previous publication by the present author (1941), we have good reason to devote special attention to the case where the auto-correlation coefficient decreases exponentially with the time. Accordingly we put

$$\mathrm{R}_1=e^{-\mu h}. \quad . \quad . \quad . \quad . \quad . \quad (41)$$

and $$\mathrm{R}_s=e^{-\mu sh}=\mathrm{R}_1^s. \quad . \quad . \quad . \quad . \quad . \quad (42)$$

From (39) and (40) it follows that

$$\sigma_s{}^2(1-q^2)\mathrm{L}\sin\theta/\sigma^2=q^s\sum_0^{\infty}\mathrm{R}_1{}^rq^r\sin(r\theta+s\theta+\gamma)+\mathrm{R}_1{}^s\sum_0^{\infty}\mathrm{R}_1{}^rq^r\sin(r\theta+\gamma)$$
$$-\mathrm{R}_1q^{s-1}\sum_0^{s-2}\mathrm{R}_1{}^rq^{-r}\sin(r\theta+\theta-s\theta-\gamma). \quad (43)$$

The evaluation of this expression by the aid of (32) is fairly laborious, but it involves only straightforward algebra, and leads to the result

$$\frac{\mathrm{L}^2\mathrm{M}^2\mathrm{N}^2\sigma_s{}^2}{\sigma^2}=\frac{\mathrm{R}_1(1+q^2)(2-k^2)+kq(1+\mathrm{R}_1{}^2)}{(4-k^2)^{\frac{1}{2}}}(1-\mathrm{R}_1{}^2)q^{s+1}\sin s\theta$$

$$+\frac{q(1+R_1^2)(1+q^2)-R_1k(1+q^4)}{(1-q^2)}(1-R_1^2)q^{s+1}\cos s\theta$$

$$+N^2R_1^{s+2}, \quad \ldots \quad (44)$$

where
$$M^2=1+R_1^2q^2-R_1kq, \quad \ldots \quad (45)$$

$$N^2=R_1^2+q^2-R_1kq, \quad \ldots \quad (46)$$

and where L has been previously defined in (35).

This rather involved expression, which may be written in the form

$$\sigma_s^2=Be^{-\lambda s}\cos(\theta s+\omega)+Ce^{-\mu s}, \quad \ldots \quad (47)$$

is given for the sake of completeness. From it the values of the correlations r_s may be obtained directly. However, in nature the disturbances operating on an oscillating system can hardly be regarded as discrete impulses : in general the disturbances will be in the nature of a continuous fluctuating force acting on the system, and our next step will be to consider this case. Suppose, *e. g.*, that a stream of turbulent air flows at right-angles to the line of motion of the pendulum, then the nature of the problem remains essentially the same, with the exception that the discrete impulses imparted by the pellets is replaced by the continuous fluctuating force exerted by the turbulent component of the stream of air. This case of a continuous fluctuating force must be regarded as being of far more practical importance than the case just treated.

6. *The Limiting Case of a Continuous Fluctuating Disturbance.*

In the limiting case, when the time interval h approaches zero, we can develop the different functions in power series of h, retaining only such terms as ultimately prove to be significant. Thus, putting

$$\lambda=l+m/M, \quad \ldots \quad (48)$$

$$\beta^2=\alpha^2+l^2-\lambda^2, \quad \ldots \quad (49)$$

$$\beta_1^2=\beta^2+\lambda^2=\alpha^2+l^2, \quad \ldots \quad (50)$$

we find that

$$\left.\begin{aligned} &q=e^{-\lambda h}=1-\lambda h+\tfrac{1}{2}\lambda^2h^2,\\ &\theta=\beta h,\\ &k=2-h^2\beta^2,\\ &L=4h^2(\beta^2+\lambda^2)=4h^2\beta_1^2,\\ &\sigma^2=4\eta^2h^2 \end{aligned}\right\} \quad \ldots \quad (51)$$

In proceeding to the limit, we also find that (37) and (38) reduce to

$$\chi_s=\frac{e^{-\lambda sh}\sin(\beta sh+\gamma)}{4\lambda\beta\beta_1h^3}, \quad \ldots \quad (52)$$

where
$$\sin\gamma-\beta/\beta_1. \quad \ldots \quad (53)$$

If we now denote the time-lag sh by x, h may be regarded as the increment dx, and it follows from (51) and (52) that

$$\sigma^2\chi_x=\frac{\eta^2e^{-\lambda x}\sin(\beta x+\gamma)}{\alpha\beta\beta_1}dx. \qquad (54)$$

We may now apply this result to (39), replacing the summation by integral signs. However, it is advisable first to re-write (39) and adjust the limits of integration thus :

$$\frac{\sigma_\xi^2\,dx}{\sigma^2}=\int_0^\infty(R_x\chi_{x+\xi}+R_{x+\xi}\chi_x)\,dx+\int_0^\xi R_x\chi_{\xi-x}\,dx$$

$$=\int_0^\infty(R_{x+\xi}+R_{x-\xi})\chi_x\,dx+\int_0^\xi(R_{\xi-x}-R_{x-\xi})\chi_x\,dx. \qquad (55)$$

Hence, by virtue of (54)

$$\sigma_0^2=\frac{2\eta^2}{\lambda\beta\beta_1}\int_0^\infty R_xe^{-\lambda x}\sin(\beta x+\gamma)\,dx \qquad (56)$$

and

$$r_\xi=\frac{1}{2\int_0^\infty R_xe^{-\lambda x}\sin(\beta x+\gamma)}\left[\int_0^\infty(R_{x+\xi}+R_{x-\xi})e^{-\lambda x}\sin(\beta x+\gamma)\,dx\right.$$

$$\left.+\int_0^\xi(R_{\xi-x}-R_{x-\xi})e^{-\lambda x}\sin(\beta x+\gamma)\,dx\right]. \qquad (57)$$

This equation constitutes the general solution of our problem, and it must be regarded as an equation of far-reaching importance, for therein the parameters defining the statistical properties of the disturbed motion are described in terms of the parameters η^2 and R of the disturbing agency. If R is defined in any way whatsoever, the corresponding values of σ_0 and r can be evaluated either by ordinary or by numerical integration.

Suppose, *e. g.*, that R is a simple harmonic function $\cos b\xi$, which means that the disturbing force is also of the simple harmonic type. Here we have the well-known case of forced simple harmonic motion in which the pendulum eventually swings with the same period as that of the disturbing force, and hence the value of r should also be $\cos b\xi$. If, now, $R_\xi=\cos b\xi$ be substituted in (57), we do indeed find this to be the case.

In the *particular case* where R is an exponential function of the time-lag, $R_x=e^{\mu x}$, the integration can be performed directly, and leads to the result

$$\sigma_0^2=\frac{2\eta^2(\mu+2\lambda)}{\lambda\beta_1^2\{\beta^2+(\mu+\lambda)^2\}} \qquad (58)$$

and $$r_\xi=Be^{-\lambda\xi}\cos(\beta\xi+\omega)+Ce^{-\mu\xi}, \qquad (59)$$

where
$$B=\frac{\mu\beta_1}{\beta(\mu+2\lambda)}\left\{\frac{\beta^2+(\mu+\lambda)^2}{\beta^2+(\mu-\lambda)^2}\right\}^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

$$C=\frac{2\lambda\beta_1^2}{(\mu+2\lambda)\{\beta^2+(\mu-\lambda)^2\}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

and
$$\cos\omega=\frac{\beta(\beta^2+\mu^2-3\lambda^2)}{\beta_1\{(\beta^2+\mu^2+\lambda^2)^2-4\mu^2\lambda^2\}^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad (62)$$

When we consider the velocities u_r instead of the displacements y_r of the pendulum, it is clear that the auto-correlation coefficient remains the same. From (15) it is readily seen that, in the limit, when h approaches zero,

$$\Sigma\delta_r^2=\mu^2\Sigma\epsilon_r^2,$$

and hence the standard deviation of the velocity is given by

$$\frac{1}{n}\Sigma u_r^2=\mu^2\sigma_0^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

From (58) it is evident that the standard deviation σ_0 becomes infinite when $\lambda=0$, or when $q=1$, but this only happens when the pendulum is undamped, *i. e.*, when $l=0$ and when m/M is infinitely small. In Yule's original work he devotes special attention to this case, but apparently he did not realize that the amplitude of the pendulum then becomes infinitely large.

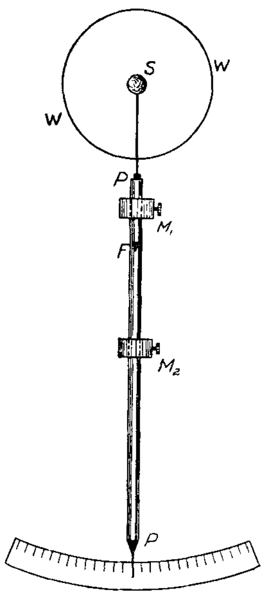
The practical application of the results obtained to experimental time series, *e. g.*, Wolfer's sunspot numbers, must be deferred to a subsequent publication, but there is another practical application which merits discussion. The relations established in (56) and (57) naturally lead to the question whether it is not possible to use a pendulum for the experimental investigation of the turbulent flow of a fluid. If the motion of a pendulum, subject to the turbulent force exerted on it by a fluid, be observed, then these equations should enable us to deduce the turbulent flow of the fluid itself, and the next paragraph will be devoted to the consideration of this question.

7. *The Experimental Determination of Turbulence.*

In the theory of turbulence, G. I. Taylor (1921) introduced the important concept of R_ξ, which he defines as the correlation coefficient of the turbulent velocity of a fluid at a given point and its velocity at the same point after a time ξ.

Now, for the further development of the theory of turbulence, it is of the greatest importance to determine R_ξ experimentally, but its determination by means of a hot-wire anemometer (Taylor, 1935) presents fairly serious experimental difficulties, and a relatively simple mechanical appliance must be considered as a very welcome addition to the equipment of the experimentalist, who concerns himself with the practical study of turbulence.

The proposed apparatus consists essentially of a rigid pendulum, as shown schematically in the accompanying figure. The straight rod PP is suspended at F, its mode of suspension being such that it is constrained to move in the plane of the paper. A thin steel needle projects upwards through a narrow slit in the cylindrical wind-tunnel WW and terminates in the sphere S. M_1 and M_2 are movable weights by which the moment of inertia and the period of the pendulum can be adjusted at will.



Pendulum for measuring the turbulence of an air-stream.

Furthermore, vanes (not shown in the diagram) can be attached to the pendulum in order to regulate the damping, and if a high degree of damping is desired, one of the vanes may be made to move within a fluid. A pointer at the lower-end of the pendulum moves over a suitable scale, and the position of the pendulum can thus be read off at any instant of time. Although the observation can be done by eye, it would, in general, be advisable to employ an automatic recording apparatus, such as a movie-camera, which would give a permanent record of the motion of the pendulum.

When a stream of air is allowed to pass through the wind-tunnel the sphere S is subjected to the turbulent force of the air, and the pendulum will start moving irregularly, and the departure y of the pointer from its zero position is recorded at small equal intervals of time. Having determined $y_1 \ldots . y_n$ for a large number of time-intervals, it is now possible to calculate r_ξ for different values of ξ.

Various ways of using this instrument for investigating the turbulent flow of a fluid suggest themselves, but for the present we shall confine our attention exclusively to the experimental determination of R_ξ, since this is a matter of such paramount importance in the theory of turbulence.

Having determined r_ξ as a function of ξ experimentally, we must now deduce the value of R_ξ. For this purpose we have available (57), which, however, is an integral equation and thus not directly adaptable for our purpose. It is possible, however, to derive an equation in a more suitable form by differentiating (57) directly, but before doing this it is advisable once more to adjust the limits of integration and write (57) in the form

$$r_\xi = \frac{1}{2\int_0^\infty R_x \chi_x \, dx} \left[\int_0^\infty R_x(\chi_{x+\xi} + \chi_{x-\xi})\, dx + \int_0^\infty R_x(\chi_{\xi-x} - \chi_{x-\xi})\, dx \right].$$

If the differentiation of r_ξ is carried out four times in succession, and the necessary eliminations are effected between the four equations thus obtained and the original equation, one readily arrives at the following expression :

$$\frac{4\eta^2}{\sigma_0^2} R_\xi = \beta_1^4 r + 2(\beta_1^2 - 2\lambda^2)\frac{d^2 r}{d\xi^2} + \frac{d^4 r}{d\xi^4}. \quad . \quad . \quad . \quad . \quad (64)$$

The same result may also be obtained by making use of equation (10),

$$\epsilon_r = y_r - kg\, y_{r-1} + q^2 y_{r-2},$$

and computing
$$\frac{1}{n}\Sigma \epsilon_r \epsilon_{r+s},$$

which is equal to $\sigma^2 R_s$, in the limiting case when h approaches zero.

Since $R_\xi = r_\xi = 1$ when $\xi = 0$, equation (64) may also be written in the form

$$R_\xi = \frac{\beta_1^4 r + 2(\beta_1^2 - 2\lambda^2)\dfrac{d^2 r}{d\xi^2} + \dfrac{d^4 r}{d\xi^4}}{\beta_1^4 + 2(\beta_1^2 - 2\lambda^2)\left(\dfrac{d^2 r}{d\xi^2}\right)_0 + \left(\dfrac{d^4 r}{d\xi^4}\right)_0}. \quad . \quad . \quad . \quad . \quad (65)$$

Since r has been determined experimentally, its second and fourth derivatives with respect to ξ can also be computed, and in the above expression it remains to be seen whether β_1 and λ can also be determined experimentally. Both the period and the logarithmic decrement can be found in the ordinary way, and hence also β_1 according to (50). On the other hand, $\lambda = l + m/M$ in the case of a simple pendulum, but as we are now dealing with a rigid pendulum, the moment of inertia enters into the original equation instead of the mass M, and accordingly we should write

$$\lambda = l + C/I, \qquad (66)$$

in which I is the moment of inertia of the pendulum about its axis of rotation and C is a constant. If this value be substituted for λ in (65), the latter may be regarded as containing the two unknowns R_ξ and C, since I can also be found experimentally. It thus becomes necessary to perform at least two experiments, keeping the turbulent flow of the fluid unaltered, but varying the moment of inertia of the pendulum.

Once the value of C has been found, equation (65) allows us to determine R_ξ for all values of ξ, and thus possibly to express it in analytic form.

Experimental work on these lines is proceeding at present, and the results will be reported at a later stage.

Summary.

With Yule's method of treating " disturbed " simple harmonic motion as a starting point, there is here developed the theory of the motion of a pendulum which is subject to a continuous fluctuating force. The theory leads to expressions whereby the statistical characteristics of the motion of the pendulum are expressed in terms of the statistical characteristics of the disturbing force.

A method is indicated whereby the character of the turbulent motion of a fluid can be experimentally determined by means of a suitably constructed rigid pendulum.

References.

G. U. Yule, Phil. Trans. Roy. Soc. A, vol. ccxxvi. p. 267 (1927).
G. T. Walker, Proc. Roy. Soc. Series A, vol. cxxxi. p. 518 (1931).
T. E. W. Schumann, Phil Mag. (1941).
G. I. Taylor, Proc. London Math. Soc. vol. xx. p. 196 (1921).
G. I. Taylor, Proc. Roy. Soc. Series A, vol. cli. p. 421 (1935).