Fierz-Pauli Field Theory

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This poster presents a theory that explains how massless particles with spin 2 behave. We start by explaining why we choose a field, and why they are related with a spin, and then we move on to discussing the Fierz-Pauli Field Theory, which is a the only ghost-free model for a spin-2 field. We also show how this theory is related to Einstein's theory of gravity and how it can help us better understand the fundamental forces of the universe.

1 Spin Field Theory

Field theory has transformed our understanding of physical interactions by providing a more coherent and consistent explanation compared to action-at-a-distance models. By describing fields as physical quantities that exist throughout space and time and mediate interactions between objects, field theory offers a powerful tool that allows for precise mathematical representations of these interactions. With real-world applications, we can make accurate predictions about how particles will interact under different conditions using field theory.

The behavior of fields is governed by equations that describe how they change as a function of time-space.

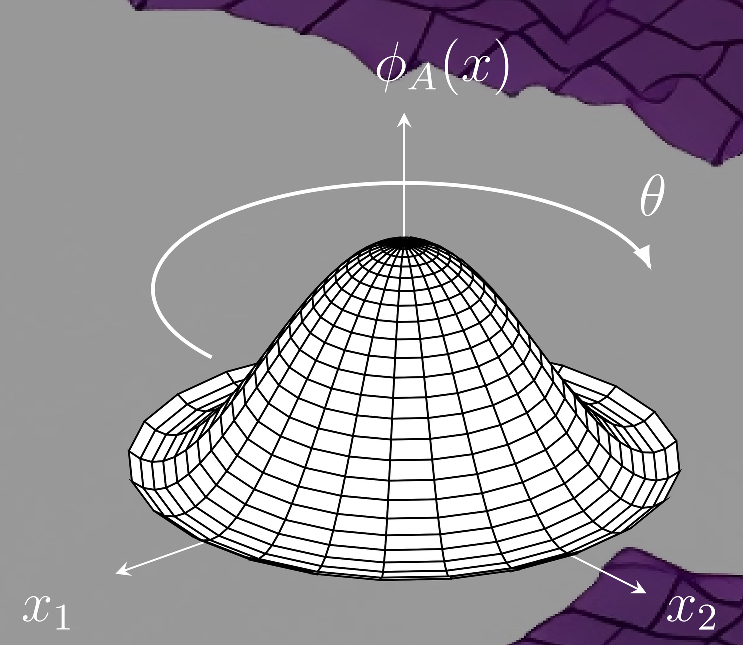
$$\phi_A = \phi_A(x_0, x_1, x_2, x_3) \equiv \phi_A(x). \tag{1}$$

Spin is a fundamental property of fields in physics that measures their intrinsic angular momentum, which must be well-defined for a field's behavior to be consistent in all reference frames. This property is described by a vector representation of the rotation group, characterized by quantum numbers that determine how the field transforms under rotations. Understanding spin is crucial for understanding how fields behave in different reference frames, making it a critical concept in physics.

The helicity h generated by a rotation of a field can be described mathematically through equations, such as:

$$\phi'_A = e^{ih\theta}\phi_A, \tag{2}$$

where ϕ_A and ϕ'_A are the original and rotated field, respectively; h is the helicity of each component of the vector and θ is the rotation angle.



Spin is a fundamental property of all fields that deepens our understanding of the physical world and opens the door for new breakthroughs and discoveries. The study of boson spin field theory, which provides a powerful framework for understanding particles with integer spin, has been crucial in the development of many areas of physics and remains a vibrant and exciting field of research today.

2 Fierz-Pauli Action

The Fierz-Pauli action describes the dynamics of a massless, symmetric, second-order tensor field $h_{\mu\nu}$ with spin-2 in a four-dimensional spacetime. This action must satisfy the principles of locality and relativity. To capture the behavior of this field, we use the most general Lagrangian that satisfies these characteristics, which can be expressed as:

$$\mathcal{L}_{\text{spin}-2} = \frac{1}{4}\partial_\mu h_{\nu\rho}, \partial^\mu h^{\nu\rho} + \lambda_2, \partial_\mu h_{\nu\rho}, \partial^\nu h^{\rho\mu} + \lambda_3, \partial_\mu h, \partial_\rho, h^{\mu\rho} + \lambda_4, \partial_\mu h, \partial^\mu h, \tag{3}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$ and the $\frac{1}{4}$ factor is a result of imposing canonical normalization on the Lagrangian.

A ghost field is a field that has negative kinetic energy and is thus unstable, leading to negative probabilities and causality violations. The presence of a ghost field can lead to instability, negative probabilities, and causality violations in Lagrangians. For the case of a spin-2 field, the phantom field can be well observed when we perform a decomposition along the lines of: $h_{\mu\nu} = b_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu$ (see a detailed information in [2]).

To fix this issue, we can use the specific values of λ_2, λ_3, and λ_4 given by the equation:

$$\lambda_2 = -\frac{1}{2},\ \lambda_3 = \frac{1}{2},\ \lambda_4 = -\frac{1}{4}. \tag{4}$$

With these values, the ghost field is removed from the Lagrangian. As a consequence, we obtain the Fierz-Pauli action, which describes the dynamics of the spin-2 field and is gauge invariance with transformations of the field. The Fierz-Pauli action is given by:

$$S_{FP} = \int_{M_4} dx^4 \left(\frac{1}{4}\partial_\mu h_{\nu\rho}\,\partial^\mu h^{\nu\rho} - \frac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\rho\mu} + \frac{1}{2}\partial_\mu h\,\partial_\rho h^{\mu\rho} - \frac{1}{4}\partial_\mu h\,\partial^\mu h.\right), \tag{5}$$

$$\therefore S_{FP}\left(h_{\mu\nu}\right) = S_{FP}\left(h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\right), \tag{6}$$

where M_4 is the four-dimensional Minkowski space-time and ξ_μ is an arbitrary vector field. The gauge transformation leaves the action S_{FP} in the same form, meaning that it is invariant under such transformations.

This characteristic is important because it allows us to choose a particular gauge that simplifies calculations.That means that the gauge can be exploited to simplify calculations. One common choice of gauge is the harmonic gauge, which is defined by the condition

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0. \tag{7}$$

Furthermore, the choice of a full gauge allows the reduction of the degrees of freedom of the field, to such an extent that it matches the values expected by the Wigner classification.

To obtain the equation of motion, we vary the action with respect to $h_{\mu\nu}$

$$\frac{\delta S_{FP}}{\delta h^{\mu\nu}} = 0 \wedge (7) \implies \partial_\alpha\partial^\alpha h_{\mu\nu} = 0. \tag{8}$$

This traceless solution for $h_{\mu\nu}$ is in accordance with a relativistic wave equation.

3 Linear General Relativity

The action of Einstein-Hilbert is a mathematical representation of the fundamental principles of general relativity (see more in [1]. It provides a way to describe the dynamics of spacetime in terms of the geometry of the universe. The action is defined as:

$$S_{EH}\left(g_{\mu\nu}\right) = \frac{1}{16\pi G}\int_{\mathcal{M}_4} \mathrm{d}^4x\sqrt{|det(g_{\mu\nu})|}R\left(g_{\mu\nu}\right), \tag{9}$$

where $g_{\mu\nu}$ is the metric tensor, R is the scalar curvature, and G is the gravitational constant. Einstein's field equation is derived from the system's action - the Einstein-Hilbert action and the material action - using the Hamiltonian principle. It can be written as:

$$\frac{\delta\left(S_{EH} + S_{\text{material}}\right)}{\delta g^{\mu\nu}} = 0 \implies R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}. \tag{10}$$

Here, S_{material} represents the action of the rest of the physical system and $T_{\mu\nu}$ its stress-energy tensor, and $R_{\mu\nu}$ is the Ricci tensor. This equation describes the distribution of matter and energy in the universe, providing a way to describe the relationship between the geometry of spacetime and the distribution of matter and energy within it.

Consider a small fluctuation $h_{\mu\nu}$ around the Minkowski metric $\eta_{\mu\nu}$ (deeper development in [4]), which describes the flat spacetime of special relativity. We can think of this fluctuation as a small disturbance to the spacetime fabric. To study the effects of this perturbation, we express the metric tensor $g_{\mu\nu}$ as:

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu} \quad \downarrow \quad \varepsilon^3 \simeq 0, \tag{11}$$

where ε is a small parameter and h is the trace of the metric perturbation $h_{\mu\nu}$. Using this expression, we can re-express the elements of the Einstein-Hilbert action in terms of $h_{\mu\nu}$. Specifically, we find that

$$g^{\mu\nu} = \eta^{\mu\nu} - \varepsilon h^{\mu\nu} + \varepsilon^2 h^{\mu\lambda}h^\nu_\lambda + \mathcal{O}(\varepsilon^3), \tag{12}$$

$$\sqrt{|det(g_{\mu\nu})|} = 1 + \frac{\varepsilon}{2}h + \frac{\varepsilon}{8}h^2 - \frac{\varepsilon^2}{4}h_{\mu\nu}h^{\mu\nu} + \mathcal{O}(\varepsilon^3), \tag{13}$$

$$R\left(g_{\mu\nu}\right) = \frac{1}{2}\left(\varepsilon R^{(1)} + \varepsilon^2 R^{(2)}\right) + \mathcal{O}(\varepsilon^3), \tag{14}$$

where the quantities $R^{(1)}$ and $R^{(2)}$ are the first and second order corrections to $R\left(g_{\mu\nu}\right)$ in terms of ε, respectively. Their expression are:

$$R^{(1)}(\varepsilon) = (\eta^{\mu\nu} - \varepsilon h^{\mu\nu})\left(\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \partial_\alpha\partial^\alpha h_{\mu\nu} - \partial_\mu\partial_\nu h\right),$$

$$R^{(2)}(\varepsilon^2) = \frac{1}{2}\partial_\rho h\partial^\rho h - \partial_\lambda h\partial_\rho h^{\rho\lambda} + \partial_\rho h_{\sigma\lambda}\partial^\sigma h^{\rho\lambda} - \frac{1}{2}\partial_\rho h_{\sigma\lambda}\partial^\rho h^{\sigma\lambda}.$$

Now, let's consider how the Einstein-Hilbert action S_{EH} changes under this perturbation. Using the expressions above, we can write the perturbative action as:

$$\mathcal{L}_{EH}\left(\eta_{\mu\nu} + \varepsilon h_{\mu\nu}\right) \simeq \tag{15}$$
$$\simeq \varepsilon^2\left(\frac{1}{4}\partial_\mu h_{\nu\rho}\,\partial^\mu h^{\nu\rho} - \frac{1}{2}\partial_\mu h_{\nu\rho}\,\partial^\nu h^{\rho\mu} + \frac{1}{2}\partial_\mu h\,\partial_\rho h^{\mu\rho} - \frac{1}{4}\partial_\mu h\,\partial^\mu h\right),$$

$$\therefore S_{EH}\left(\eta_{\mu\nu} + \varepsilon h_{\mu\nu}\right) \simeq \kappa^2 S_{FP}, \quad \kappa^2 = \frac{\varepsilon^2}{16\pi G}.$$

In the previous section, we discussed the gauge-invariant Lagrangian for the spin-2 field coupled with matter. By choosing the harmonic gauge (7), the equation of motion for the field takes the form:

$$\partial_\alpha\partial^\alpha h_{\mu\nu} = -\frac{1}{\kappa^2}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right). \tag{16}$$

In the non-relativistic limit, the energy of the system is associated with a static particle of mass M, which can be described by the energy-momentum tensor:

$$T_{\mu\nu} = \varepsilon^2 M\,\delta^0_\mu\,\delta^0_\nu\,\delta(\vec{x}), \tag{17}$$

where the first two deltas are Kronecker deltas, and the third one is a Dirac delta for the spatial coordinates. By solving the equation of motion, see the development in [3], we obtain:

$$h_{00} = \frac{1}{4\kappa^2}\frac{\varepsilon^2 M}{r} = \frac{2GM}{r}, \tag{18}$$

which is the gravitational potential of Newton's Law.

Looking to the future, there are exciting directions for further research, including exploring the implications of this theory for cosmology and developing a more complete understanding of gravity at the quantum level. These efforts could be crucial in advancing our understanding of the fundamental laws of the universe.

References

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