W271-2 – Spring 2016 – **Lab 2**

Juanjo Carin, Kevin Davis, Ashley Levato, Minghu Song

March 7, 2016

Contents

Question 1: Broken Rulers

You have a ruler of length 1 and you choose a place to break it using a uniform probability distribution. Let random variable X represent the length of the left piece of the ruler. X is distributed uniformly in $[0, 1]$. You take the left piece of the ruler and once again choose a place to break it using a uniform probability distribution. Let random variable Y be the length of the left piece from the second break.

1. **Find the conditional expectation of Y given X, $\mathbf{E}(\mathbf{Y}|\mathbf{X})$.**

$f_X = U(0, 1)$ and $f_{Y|X} = U(0, X)$ (because the maximum length of the second left piece cannot be greater than the length of the first left piece). As we know, the probability density function for a variable Z that follows a uniform distribution $U(a, b)$ is:

$$f_Z(z) = \begin{cases} \dfrac{1}{b-a} & a \leq z \leq b \\ 0 & \text{otherwise} \end{cases}$$

So:

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{1}{x} & 0 \leq z \leq x \\ 0 & \text{otherwise} \end{cases}$$

And:

$$\mathbf{E(Y|X)} = \int_{\mathbb{Y}} y \cdot f_{Y|X}(y|x) \cdot dy = \int_{y=0}^{x} y \cdot \frac{1}{x} \cdot dy = \frac{1}{x}\left[\frac{y^2}{2}\right]_{y=0}^{x} = \frac{x^2}{2x} = \mathbf{\frac{x}{2}}$$

We'll make use of some simulations through this Question to confirm the results.

```
simulations <- 1e4 # number of simulations
set.seed(123)
x <- runif(simulations, min=0, max=1) # X ~ U(0,1)
y <- runif(simulations, min=0, max=x) # Y|X ~ U(0,X)
par(mfrow = c(1, 2))
hist(x, main = "pdf(x)", freq = FALSE)
lines(density(x), col = 'red')
hist(y, main = "pdf(y)", freq = FALSE)
lines(density(y), col = 'red')
```

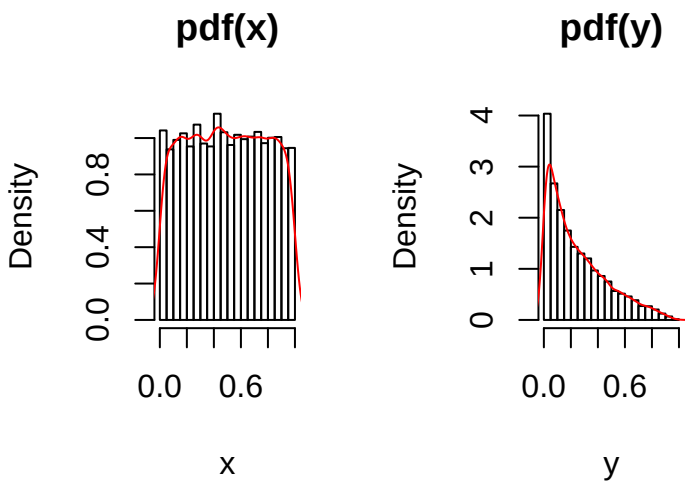


Figure 1: Histogram and approximate pdf of X and Y

```
# y1 <- runif(simulations, min = 0, max = 0.2) # Fix X to 0.2
y1 <- y[x > 0.2 - 1e-2 & x < 0.2 + 1e-2] # Using previous simulation
hist(y1, main = 'pdf(y|x=0.2)', freq = FALSE)
lines(density(y1), xlim = c(0, 1), main = 'pdf(y|x=0.2)', col = 'red')
abline(v = mean(y1), col = 'green', lty = 2, lwd = 4)
```

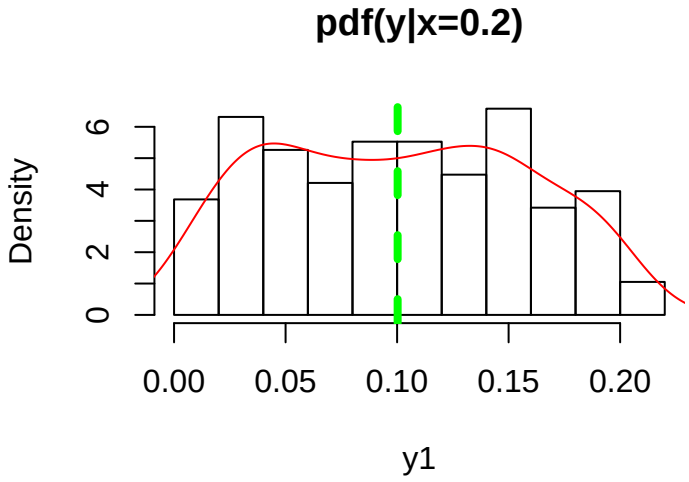


Figure 2: Histogram and approximate pdf of Y conditional on X for a given value of X (0.2)

```
# legend("topright", "E(Y|X=0.2)", lty = 1, bty="n", col = 'red')
```

2. **Find the unconditional expectation of Y. One way to do this is to apply the law of iterated expectations, which states that $\mathbf{E(Y) = E(E(Y|X))}$. The inner expectation is the conditional expectation computed above, which is a function of X. The outer expectation finds the expected value of this function.**

$$\mathbf{E(Y)} = E\left[E(Y|X)\right] = \int_{\mathbb{X}} E(Y|X) \cdot f_X(x) \cdot dx = \int_{x=0}^{1} \frac{x}{2} \cdot 1 \cdot dx = \left[\frac{x^2}{4}\right]_{x=0}^{1} = \mathbf{\frac{1}{4} = 0.25}$$

3. **Write down an expression for the joint probability density function of X and Y, $\mathbf{f_{X,Y}(x,y)}$.**

$$\mathbf{f_{X,Y}(x,y)} = f_{Y|X}(y|x) \cdot f_X(x) = \begin{cases} \mathbf{\frac{1}{x}} & \mathbf{x \in (0,1), y \in (0,x)} \\ \mathbf{0} & \mathbf{\text{otherwise}} \end{cases}$$

Let's check that this is a valid joint *pdf*:

$$\int_{\mathbb{X}} \int_{\mathbb{Y}} f_{X,Y}(x,y) \cdot dx \cdot dy = \int_{x=0}^{1} \int_{y=0}^{x} \frac{1}{x} \cdot dy \cdot dx = \int_{x=0}^{1} \left[\frac{y}{x}\right]_{y=0}^{x} dx = \int_{x=0}^{1} dx = [x]_{x=0}^{1} = 1$$

Simulations:

```
pdf_x <- function(x) ifelse(x<1 & x>0, 1, 0) # f(x)
integrate(pdf_x, -Inf, Inf) # integral
```

```
## 1 with absolute error < 4.2e-11
```

```
pdf_y_given_x <- function(x,y) ifelse(y<x & y>0 & x<1 & x>0, 1/x, 0) # f(y|x)
pdf_xy <- function(x,y) pdf_x(x)*pdf_y_given_x(x,y) # f(x,y)
# integral
integrate(function(y) sapply(y, function(y) integrate(function(x)
  pdf_xy(x,y), 0, 1)$value), -Inf, Inf)
```

```
## 1 with absolute error < 9.3e-05
```

```
# Plot f(x,y)
x0 <- y0 <- seq(0, 1, by = 0.01)
grid <- mesh(x0, y0)
z0 <- with(grid, pdf_x(x)*pdf_y_given_x(x,y))
# contour(x0, y0, z0, asp=1)
# par(mfrow = c(1, 2))
persp3D(z = z0, x = x0, y = y0)
```

```
# Confirm that f(x,y) = 1/x
# z2 <- with(grid, ifelse(x<=y | x==0 | y == 0, 0, 1/x))
# persp3D(z = z2, x = x0, y = y0)
```

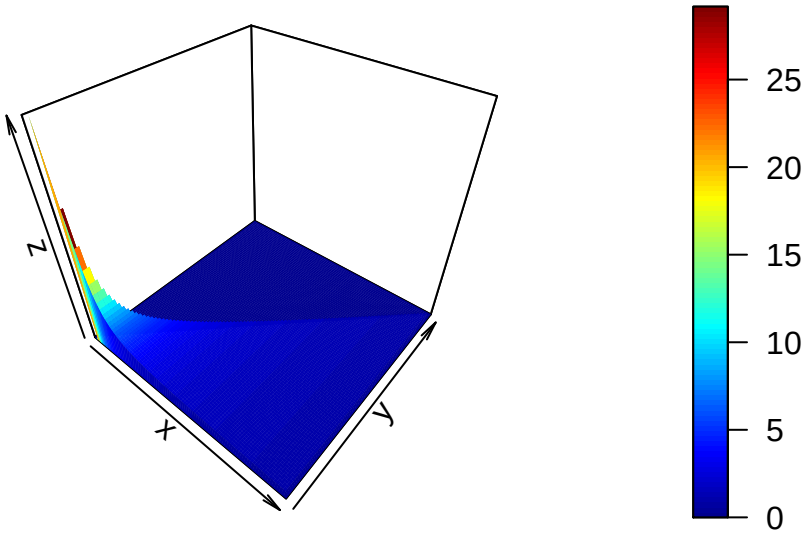


Figure 3: Approximate joint pdf of X and Y

4. **Find the conditional probability density function of X given Y, $\mathbf{f_{X|Y}}$.**

In order to find $f_{X|Y}$ we need the marginal pdf of Y.

$$f_Y(y) = \int_{\mathbb{X}} f_{X,Y}(x,y) \cdot dx = \int_{y=0}^{x} \frac{1}{x} \cdot dx = \int_{x=y}^{1} \frac{dx}{x} = [\log(x)]_{x=y}^{1} dx = -\log(y) = \log\left(\frac{1}{y}\right)$$

This result confirms what the shape of $f_Y(y)$ in Figure 1 suggested.

```
# f(y)
pdf_y <- function(y)
  sapply(y, function(y) integrate(function(x)
    pdf_y_given_x(x,y)*pdf_x(x), 0, 1)$value)
integrate(pdf_y, -Inf, Inf) # integral
```

```
## 1 with absolute error < 9.3e-05
```

```
plot(sort(y), pdf_y(sort(y)), type = 'l', main = 'pdf(y)', xlab = 'y')
# Confirm that f(y) = log(1/y)
lines(sort(y), log(1/sort(y)), type = 'l', main = 'pdf(y)')
```

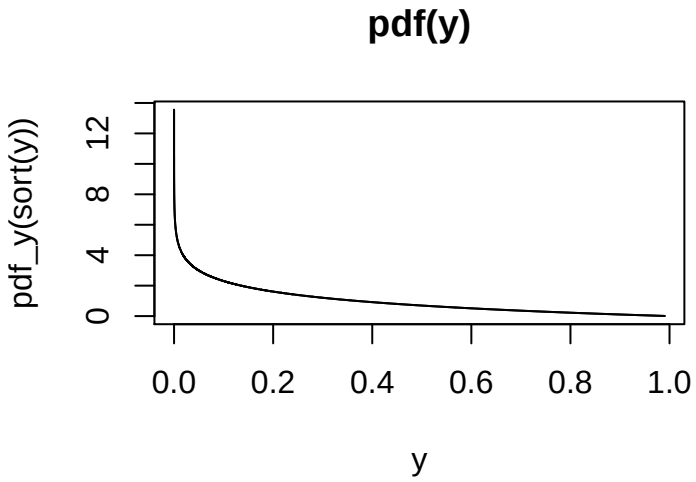


Figure 4: Approximate pdf of Y conditional on X for two values of X

$$\mathbf{f_{X|Y}(x|y)} = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\mathbf{-1}}{\mathbf{x} \cdot \log(\mathbf{y})}$$

5. **Find the expectation of X, given that Y is 1/2, E(X|Y = 1/2).**

$$\mathbf{E(X|Y=1/2)} = \int_{\mathbb{X}} x \cdot f_{X|Y}(x|y=1/2) \cdot dx = \int_{x=1/2}^{1} x \cdot \left(\frac{-1}{x \cdot \log(1/2)} \right) \cdot dx$$
$$= \frac{1}{\log(2)} \int_{x=1/2}^{1} dx = \frac{1}{\log(2)} [x]_{x=1/2}^{1} = \mathbf{\frac{1}{2 \cdot \log(2)} = 0.721}$$

```
# Confirm E(X|Y=0.5) (use values of Y around 0.5 in the previous simulation)
mean(x[y > 0.5 - 1e-2 & y < 0.5 + 1e-2])
```

```
## [1] 0.72847
```

```
1/(2*log(2))
```

```
## [1] 0.7213475
```

Question 2: Investing

Suppose that you are planning an investment in three different companies. The payoff per unit you invest in each company is represented by a random variable. A represents the payoff per unit invested in the first company, B in the second, and C in the third. A, B, and C are independent of each other. Furthermore, Var(A) = 2Var(B) = 3Var(C).

You plan to invest a total of one unit in all three companies. You will invest amount a in the first company, b in the second, and c in the third, where $a, b, c \in [0, 1]$ and $a + b + c = 1$. Find, the values of a, b, and c that minimize the variance of your total payoff.

Let's call P the total payoff:

$$Var(P) = Var(aA + bB + cC) = a^2 Var(A) + b^2 Var(B) + c^2 Var(C)$$

because A, B, and C are independent of each other. And since $Var(A) = 2Var(B) = 3Var(C)$, we can derive that:

$$Var(P) = Var(A)\left(a^2 + \frac{b^2}{2} + \frac{c^2}{3}\right)$$

We want to:

$$\begin{array}{ll} \text{minimize} & P(a, b, c) \\ \text{subject to} & g(a, b, c) = 0 \end{array}$$

where $g(a, b, c) = a + b + c - 1$, so $g(a, b, c) = 0$ is our constraint.

Using the Lagrange multiplier method, we can define:

$$\mathcal{L}(a, b, c, \lambda) = P(a, b, c) - \lambda \cdot g(a, b, c)$$

So to find our local minima we need to solve:

$$\nabla_{a,b,c,\lambda} \mathcal{L} = 0$$

$$\left(\frac{\partial \mathcal{L}}{\partial a}, \frac{\partial \mathcal{L}}{\partial b}, \frac{\partial \mathcal{L}}{\partial c}, \frac{\partial \mathcal{L}}{\partial \lambda}\right) = \left(2a - \lambda, b - \lambda, \frac{2}{3}c - \lambda, -(a + b + c - 1)\right) = \mathbf{0}$$

$$\Rightarrow \begin{cases} 2a - \lambda = 0 \\ b - \lambda = 0 \\ 2c/3 - \lambda = 0 \\ a + b + c - 1 = 0 \end{cases} \Rightarrow \begin{cases} a = \lambda/2 \\ b = \lambda \\ c = 3\lambda/2 \\ \frac{\lambda}{2} + \lambda + \frac{3\lambda}{2} = 3\lambda = 1 \end{cases} \Rightarrow \begin{cases} \mathbf{a = \frac{1}{6} = 0.1667} \\ \mathbf{b = \frac{1}{3} = 0.3333} \\ \mathbf{c = \frac{1}{2} = 0.5} \end{cases}$$

Let's prove the result in R:

```
payoff <- function(x) {
  a <- x[1]
  b <- x[2]
  c <- x[3]
  a^2 + b^2/2 + c^2/3
}
gradient_payoff <- function(x) {
  a <- x[1]
  b <- x[2]
  c <- x[3]
  c(2*a, b, 2*c/3)
}
sol <- constrOptim(theta = c(.3, .3, .4), f = payoff, grad = gradient_payoff,
                   ui = rbind(c(1, 0, 0), c(0, 1, 0), c(0, 0, 1),
                              c(-1, 0, 0), c(0, -1, 0), c(0, 0, -1),
                              c(1, 1, 1), c(-1, -1, -1)),
                   ci = c(0, 0, 0, -1, -1, -1, 1-1e-6, -1-1e-6))
sol$par
```

```
## [1] 0.1666989 0.3333673 0.4999327
```

Question 3: Turtles

Next, suppose that the lifespan of a species of turtle follows a uniform distribution over $[\mathbf{0}, \theta]$. Here, parameter θ represents the unknown maximum lifespan. You have a random sample of n individuals, and measure the lifespan of each individual i to be $\mathbf{y_i}$.

1. **Write down the likelihood function, $\mathbf{l}(\theta)$ in terms of $\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_n}$.**

$$l(\theta; y_1, \ldots, y_n) = f(y_1, \ldots, y_n|\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \begin{cases} \prod_{i=1}^{n} \frac{1}{\theta} = \theta^{-n} & 0 \leq y_i \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

2. **Based on the previous result, what is the maximum-likelihood estimator for θ?**

The MLE of θ must be a value of θ for which $\theta \geq y_i$ for $i = 1, \ldots, n$ and which maximizes $1/\theta^n$ among all such values. I.e., the maximum value of y_i within the sample.

$$\hat{\theta}_{ml} = \arg\max_{\theta \in \Theta} \; \hat{l}(\theta; y_1, \ldots, y_n) = \max\{y_1, \ldots, y_n\}$$

3. **Let $\hat{\theta}_{\mathbf{ml}}$ be the maximum likelihood estimator above. For the simple case that $\mathbf{n = 1}$, what is the expectation of $\hat{\theta}_{\mathbf{ml}}$, given θ?**

$$E\left(\hat{\theta}_{ml} \mid \theta\right) = E(y_1) = E(y) = \int_{y=0}^{\theta} \frac{y}{\theta} \cdot dy = \left[\frac{y^2}{2\theta}\right]_{y=0}^{\theta} = \frac{\theta}{2}$$

4. **Is the maximum likelihood estimator biased?**

Yes, it is:

$$E\left(\hat{\theta}_{ml}\right) - \theta = \frac{\theta}{2} \neq 0$$

5. **For the more general case that $\mathbf{n \geq 1}$, what is the expectation of $\hat{\theta}_{\mathbf{ml}}$?**

Without loss of generality, let's suppose the individual n is the one with the maximum lifespan among the sample, i.e., $y_n \geq y_i$ for $i = 1, \ldots, n-1$. Call z that maximum value of y_i.

$$E[max\{y_1, \ldots, y_n\}] = E(y_n) = E(z) = \int_{z=0}^{\theta} z \cdot f(z) dz$$

But what is the distribution of z?

$$F(z) = \Pr(y_n \leq z) = \Pr(y_1 \leq z \cap \cdots \cap y_n \leq z) \overset{\text{i.i.d}}{=} \prod_{i=1}^{n} \Pr(y_i \leq z) = \left(\frac{z}{\theta}\right)^n \Rightarrow f(z) = \frac{n z^{n-1}}{\theta^n}$$

$$E[max\{y_1, \ldots, y_n\}] = \frac{n}{\theta^n} \int_{z=0}^{\theta} z^n dz = \frac{n}{\theta^n} \left[\frac{z^{n+1}}{n+1}\right]_{z=0}^{\theta} = \frac{n}{n+1} \cdot \theta$$

(Which confirms the previous result, for $n = 1$.)

6. **Is the maximum likelihood estimator consistent?**

It is:

$$\Pr\left(|\hat{\theta}_{ml} - \theta| > \varepsilon\right) = \Pr\left(\frac{\theta}{n+1} > \varepsilon\right) \to 0 \text{ as } n \to \infty$$

```
simulations <- 1e3 # number of simulations
theta <- 100 # an arbitrary value of theta
y <- runif(n = simulations, min = 0, max = theta) # Y ~ U(0,theta)
# any(y == theta); all(y < theta) # FALSE and TRUE, respectively
# No matter how large is the sample, Yi is always lower than 1
set.seed(1)
num_simulations <- sort(c(1, sample(c(2:simulations), 49)))
theta_mle <- unlist(lapply(num_simulations, function(n)
  mean(max(runif(n = n, min = 0, max = theta)))))
plot(num_simulations, theta_mle, ylim = c(floor(min(theta_mle)), theta),
     xlab = "Number of simulations", ylab = "MLE of theta", pch = '*')
lines(num_simulations, theta_mle, lwd = 0.5, col = 'blue')
```

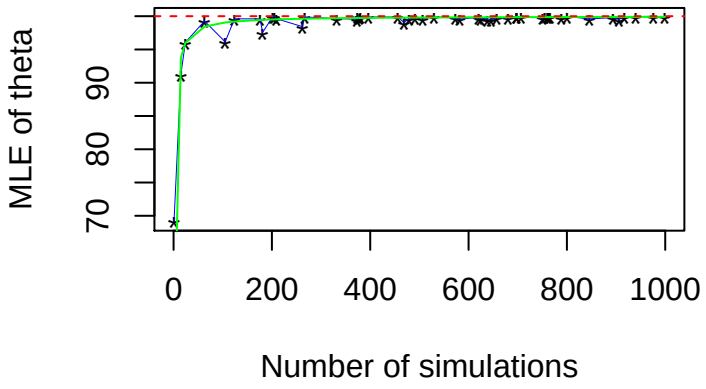


Figure 5: Trend of the MLE of θ depending on the sample size

Question 4: CLM 1

Background

The file `WageData2.csv` contains a dataset that has been used to quantify the impact of education on wage. One of the reasons we are proving another wage-equation exercise is that this area by far has the most (and most well-known) applications of instrumental variable techniques, the endogenity problem is obvious in this context, and the datasets are easy to obtain.

The Data

You are given a sample of 1000 individuals with their wage, education level, age, working experience, race (as an indicator), father's and mother's education level, whether the person lived in a rural area, whether the person lived in a city, IQ score, and two potential instruments, called z1 and z2.

The dependent variable of interest is `wage` (or its transformation), and we are interested in measuring "return" to education, where return is measured in the increase (hopefully) in wage with an additional year of education.

Question 4.1

Conduct an univariate analysis (using tables, graphs, and descriptive statistics found in the last 7 lectures) of all of the variables in the dataset.

Also, create two variables: (1) natural log of wage (name it `logWage`) (2) square of experience (name it `experienceSquare`)

Table 1: Summary Statistics of Wage Data

Statistic	N	Mean	St. Dev.	Min	Pctl(25)	Median	Pctl(75)	Max
X	1,000	1,466.678	866.535	5	715.5	1,431.5	2,212	3,009
wage	1,000	578.783	266.569	127	400	543	702.5	2,404
education	1,000	13.219	2.729	2	12	12	16	18
experience	1,000	8.788	4.221	0	6	8	11	23
age	1,000	28.007	3.118	24	25	27	30	34
raceColor	1,000	0.238	0.426	0	0	0	0	1
dad_education	761	10.181	3.748	0	8	11	12	18
mom_education	872	10.451	3.126	0	8	12	12	18
rural	1,000	0.391	0.488	0	0	0	1	1
city	1,000	0.712	0.453	0	0	1	1	1
z1	1,000	0.440	0.497	0	0	0	1	1
z2	1,000	0.686	0.464	0	0	1	1	1
IQscore	684	102.273	15.843	50	93	103	113	144
logWage	1,000	6.263	0.447	4.844	5.991	6.297	6.555	7.785
experienceSquare	1,000	95.030	86.786	0	36	64	121	529

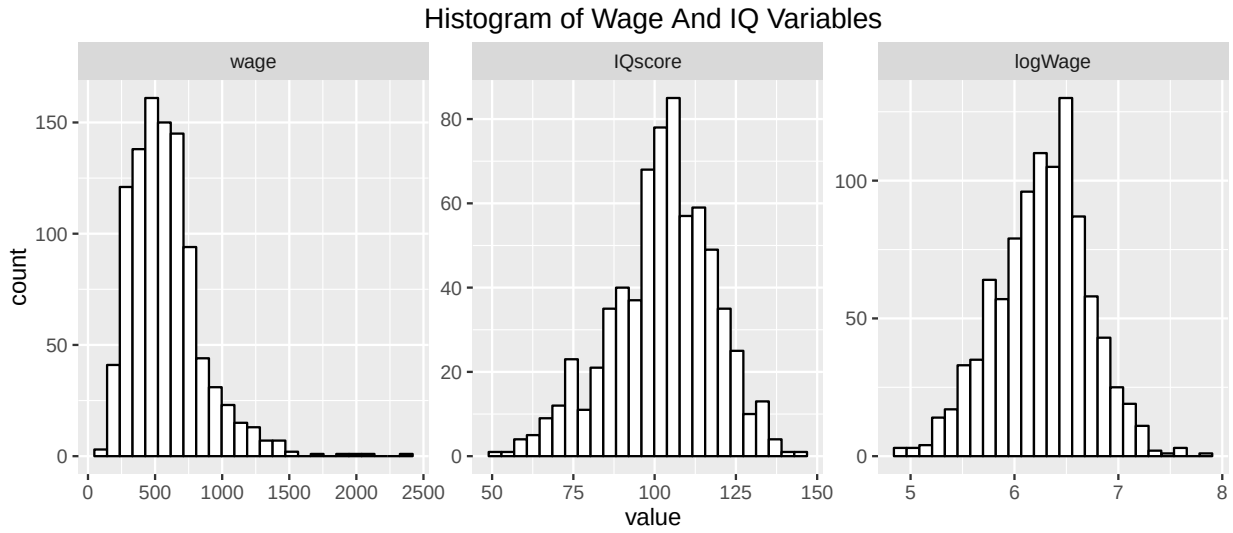


Figure 6: Histogram of Wage , IQ Score, and log(Wage)

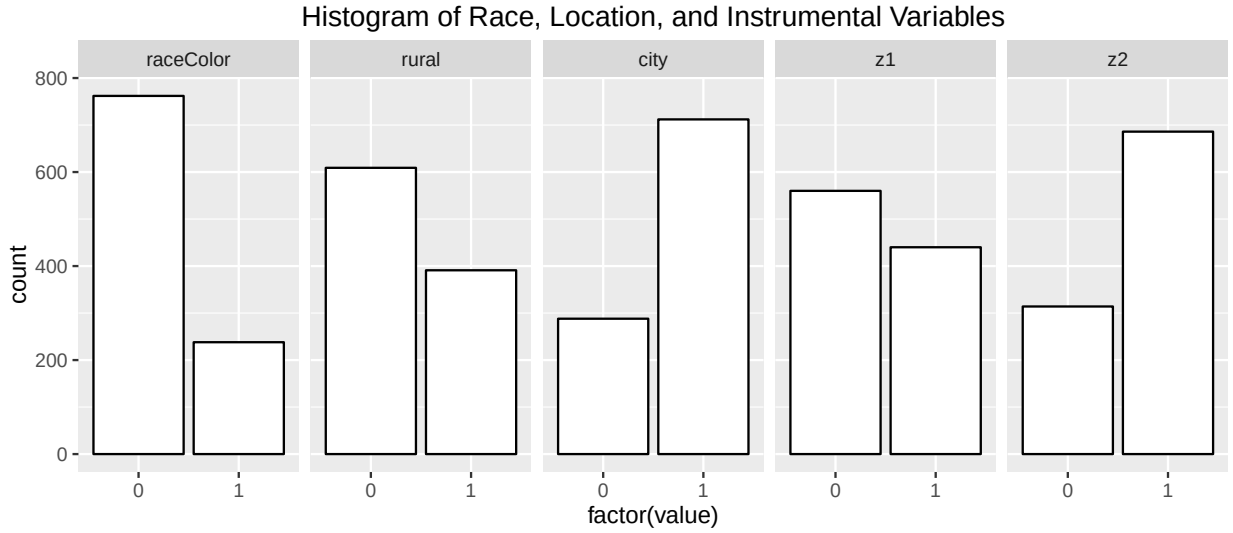


Figure 7: Histogram of Race , Rural, City, Z1, and Z2

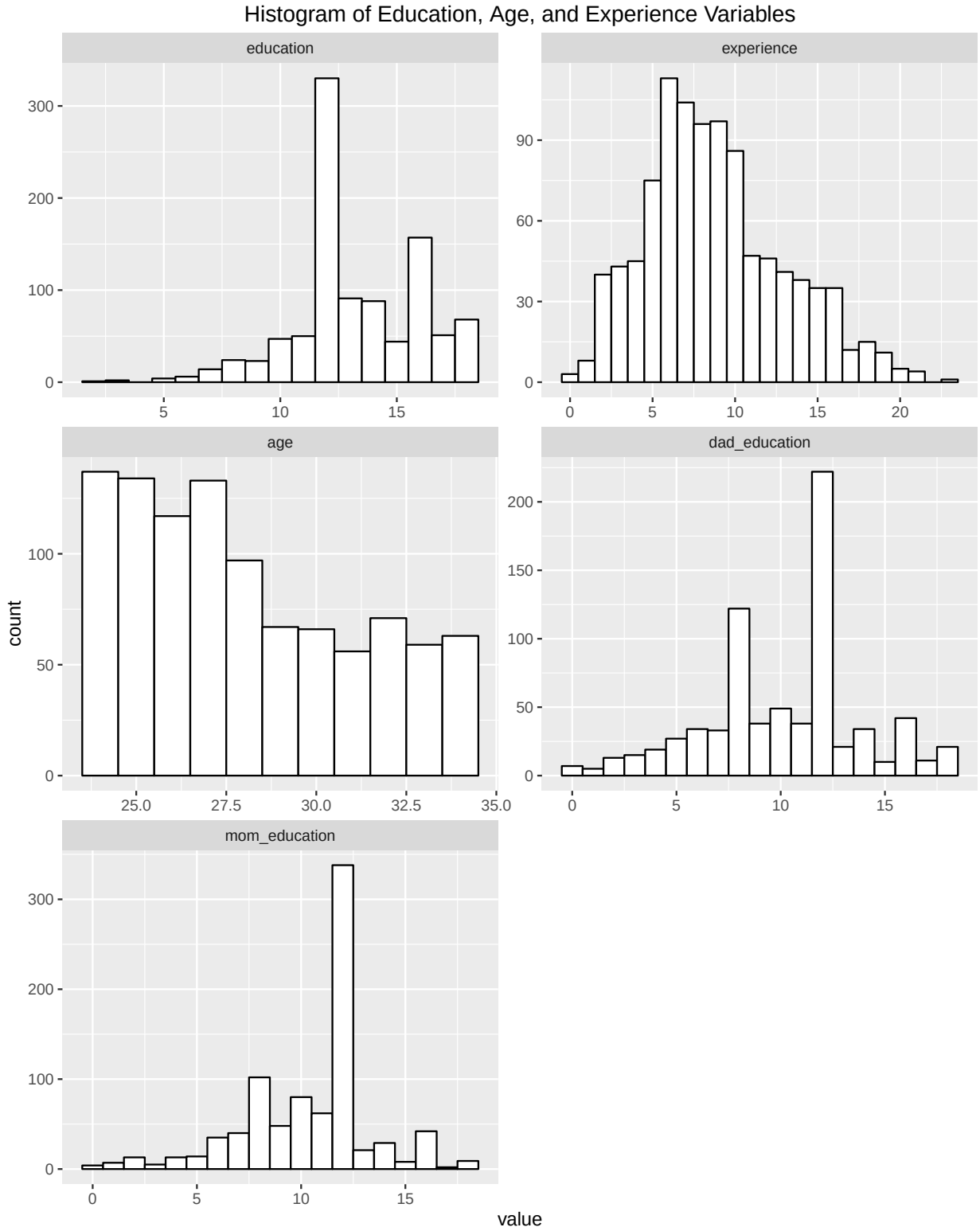


Figure 8: Histogram of Education, Expereince, Age, Father's Education, and Mother's Education

Question 4.2

Conduct a bivariate analysis (using tables, graphs, descriptive statistics found in the last 7 lectures) of `wage` and `logWage` and all the other variables in the datasets.

Table 2: Correlations for Wage and log(Wage)

	Wage	log(Wage)
wage	1	0.946
education	0.310	0.332
experience	-0.006	-0.029
age	0.264	0.251
raceColor	-0.301	-0.341
dad_education	0.190	0.189
mom_education	0.198	0.210
rural	-0.222	-0.250
city	0.220	0.236
z1	0.101	0.087
z2	0.171	0.177
IQscore	0.186	0.201
logWage	0.946	1
experienceSquare	-0.043	-0.065

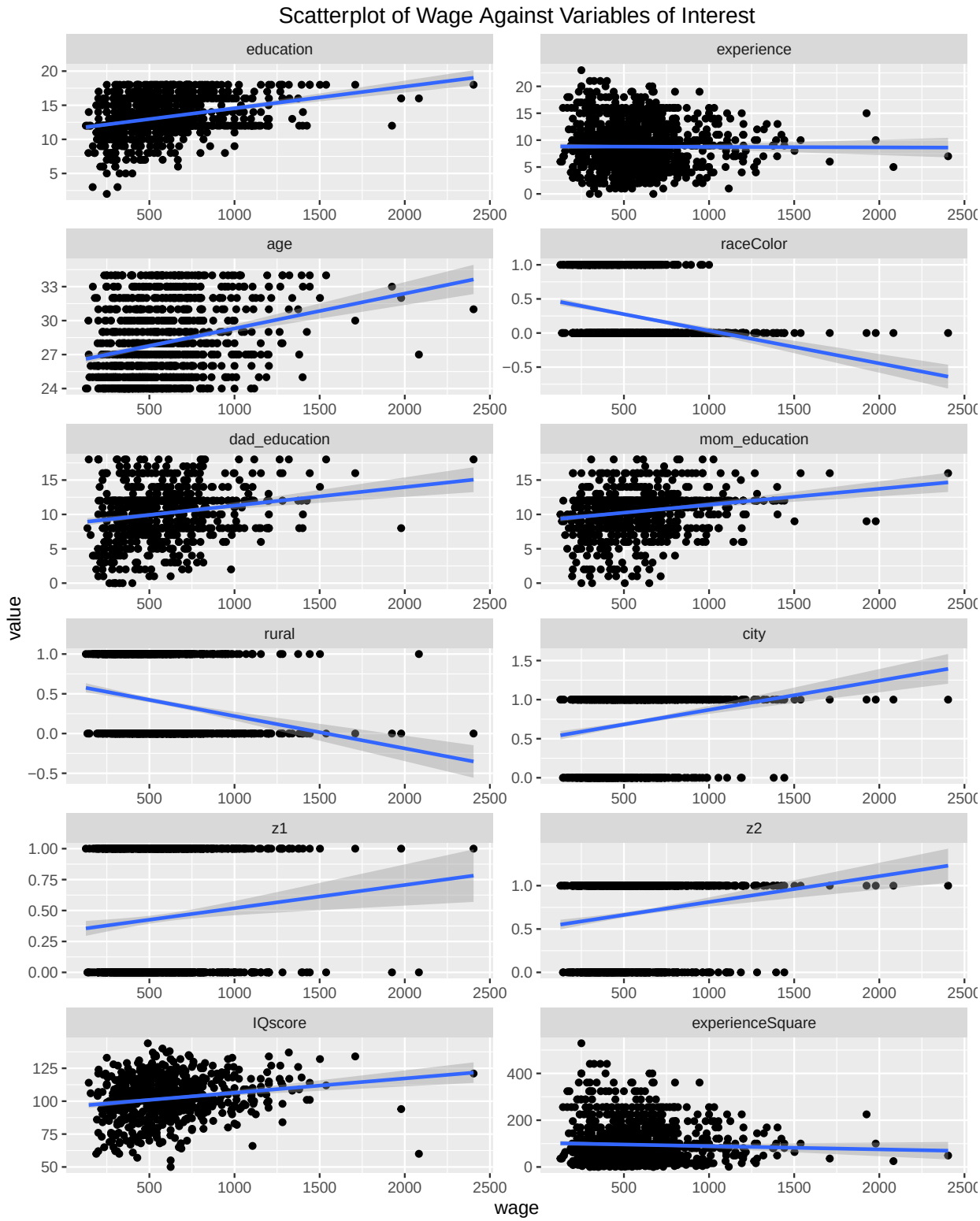


Figure 9: Scatterplot of Wage Against Variables of Interest

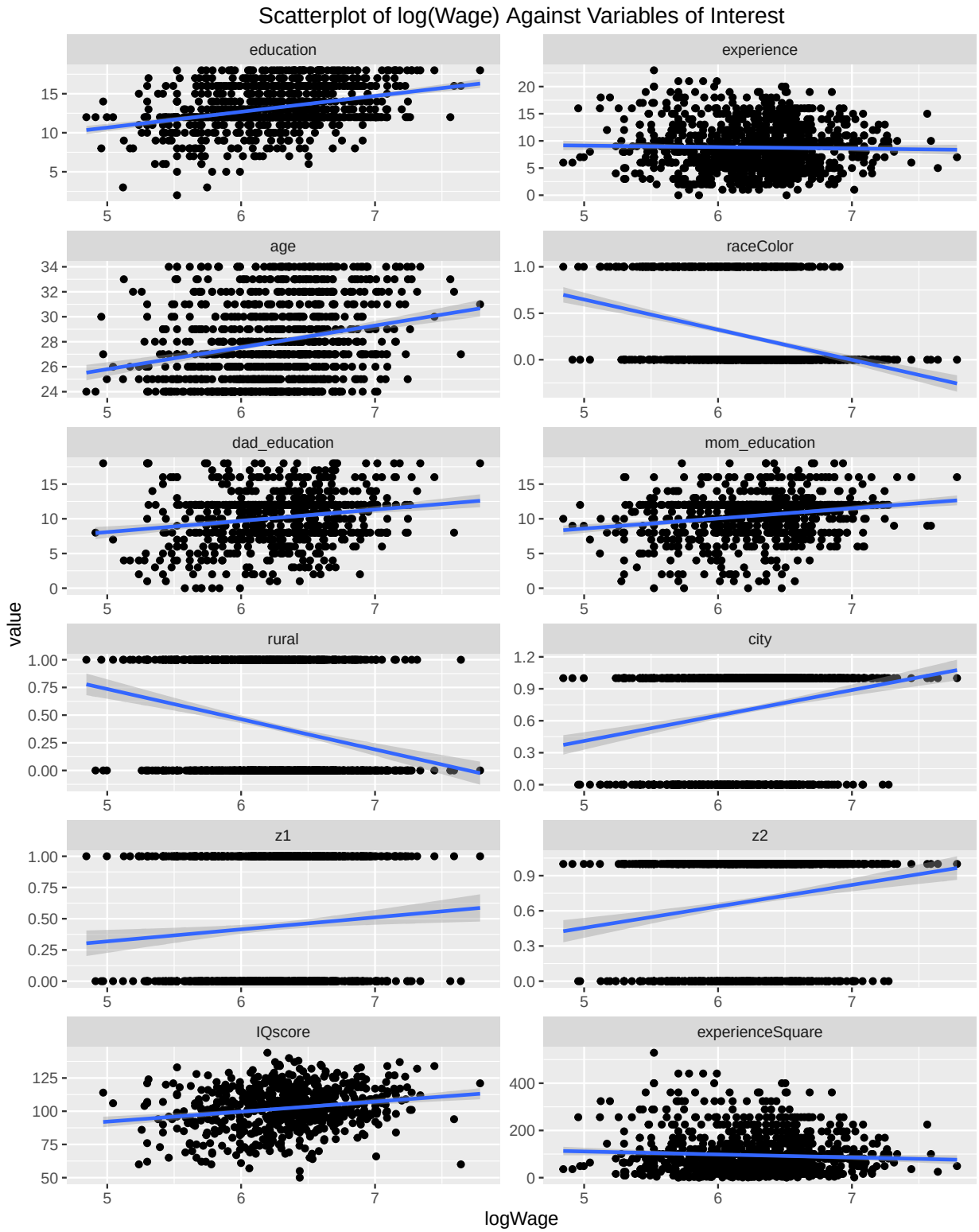


Figure 10: Scatterplot of log(Wage) Against Variables of Interest

Question 4.3

Regress *log(wage)* on education, experience, age, and raceColor.

1. **Report all the estimated coefficients, their standard errors, t-statistics, F-statistic of the regression, R^2, R^2_{adj} , and degrees of freedom.**

Table 3: Regression Summary

	Dependent variable:
	logWage
education	0.080***
	(0.006)
	t = 12.486
experience	0.035***
	(0.004)
	t = 8.869
age	
raceColor	−0.261***
	(0.030)
	t = −8.564
Constant	4.962***
	(0.113)
	t = 43.774
Observations	1,000
R^2	0.236
Adjusted R^2	0.234
Residual Std. Error	0.392 (df = 996)
F Statistic	102.582*** (df = 3; 996)
Note:	*p<0.1; **p<0.05; ***p<0.01

2. **Explain why the degrees of freedom takes on the specific value you observe in the regression output.**

The overall degrees of freedom (F(3, 996) is one smaller than we might expect otherwise. This is because the parameter for age is a linearly dependent combination of the other parameters and it's coefficient cannot be estimated. R automatically removes this variable from the model, and thus our regression output reflects the degrees of freedom for a model with 3 parameters rather than 4. Thus estimating this model is equivalent to estimating $log(wage) = education + experience + raceColor$.

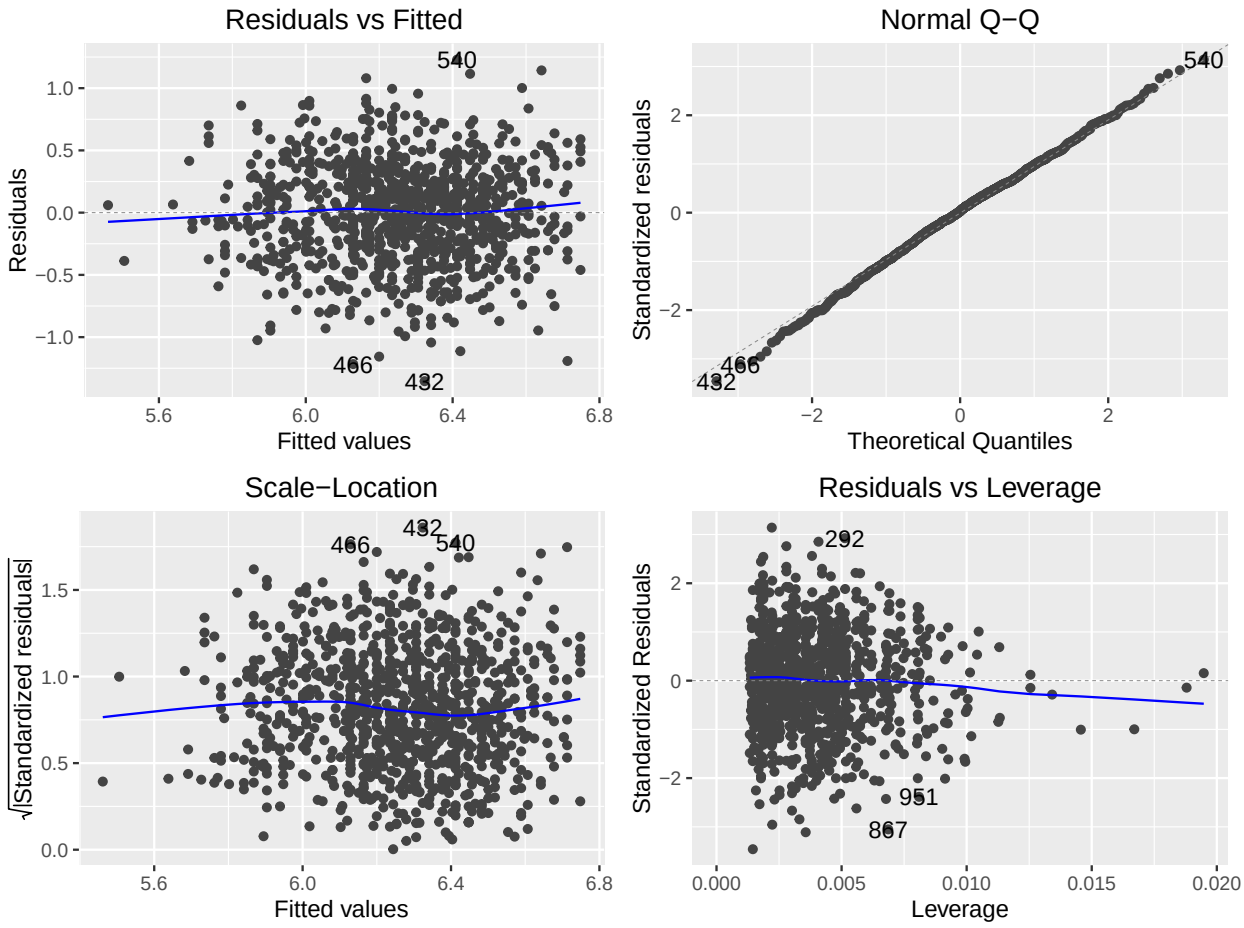


Figure 11: Regression Diagnostic Plots for Question 4.3

Table 4: Regression summary

	Dependent variable:
	log(Wages)
Education	0.080***
	(0.006)
Experience	0.035***
	(0.004)
Race (White or Non-white)	−0.261***
	(0.030)
Constant	4.962***
	(0.115)
F Statistic	100.278***
df	3; 996
Observations	1,000
R^2	0.236
Adjusted R^2	0.234
Residual Std. Error	0.392

·p<0.1; *p<0.05; **p<0.01; ***p<0.001

3. **Describe any unexpected results from your regression and how you would resolve them (if the intent is to estimate return to education, condition on race and experience).**

The inability to estimate the age coefficient is an unexpected result. Looking at the correlation matrix from above, we don't see that any variable is perfectly correlated with $log(wage)$. However, if we take the correlation of *age* with the sum of *education* and *experience*, we see that *age* is perfectly correlated with the two variables. Since the model already incorporates all of the information contained in the age variable, we can simply remove it from the model and not lose anything.

4. **Interpret the coefficient estimate associated with education.**

Holding other covariates constant, a one year increase in *education* was associated with a statistically significant increase in $log(wage)$, ($\beta = 0.080$, std. error = 0.006, t = 12.394, p < .001). This effect also represents a practically significant wage return on education.

5. **Interpret the coefficient estimate associated with experience.**

Holding other covariates constant, a one year increase in *experience* was associated with a statistically significant increase in $log(wage)$, ($\beta = 0.035$, std. error = 0.004, t = 8.81, p < .001). This effect also represents a practically significant wage return on experience.

Question 4.4

Regress *log(wage)* on education, experience, experienceSquare, and race-Color.

1. **Plot a graph of the estimated effect of experience on wage.**

2. **What is the estimated effect of experience on wage when experience is 10 years?**

The estimated effect of experience at 10 years on wages is 10 · 0.092 + 10^2 · -0.003 = 0.637

Table 5: Regression summary

	Dependent variable:
	log(Wages)
Education	0.079***
	(0.006)
Experience	0.092***
	(0.011)
Expereince^2	−0.003***
	(0.001)
Race (White or Non-white)	−0.263***
	(0.030)
Constant	4.736***
	(0.120)
F Statistic	84.960***
df	4; 995
Observations	1,000
R^2	0.257
Adjusted R^2	0.254
Residual Std. Error	0.387

·p<0.1; *p<0.05; **p<0.01; ***p<0.001

Question 4.5

Regress `logWage` on education, experience, `experienceSquare`, `raceColor`, `dad_education`, `mom_education`, rural, city.

1. **What are the number of observations used in this regression? Are missing values a problem? Analyze the missing values, if any, and see if there is any discernible pattern with wage, education, experience, and raceColor.**

There are 723 observations in the regression. A comparison of the original data and the observations in the regression reveals that the missing observations tend to have lower wages and education and are less likely to live in the city. The missing observations have more experience, and are more likely to be non-white and live in rural areas.

2. **Do you just want to "throw away" these observations?**

The clear pattern of missing values representing more rural, non-white observations with lower education indicates that these values should not simply be discarded from the model. In order to accurately assess the effect of covariates these values are important for the regression.

3. **How about blindly replace all of the missing values with the average of the observed values of the corresponding variable? Rerun the original regression using all of the observations?**

```
data7 <- data
data7$dad_education <- na.fill(data7$dad_education, mean(data7$dad_education,
                                                         na.rm=T))
data7$mom_education <- na.fill(data7$mom_education, mean(data7$mom_education,
```

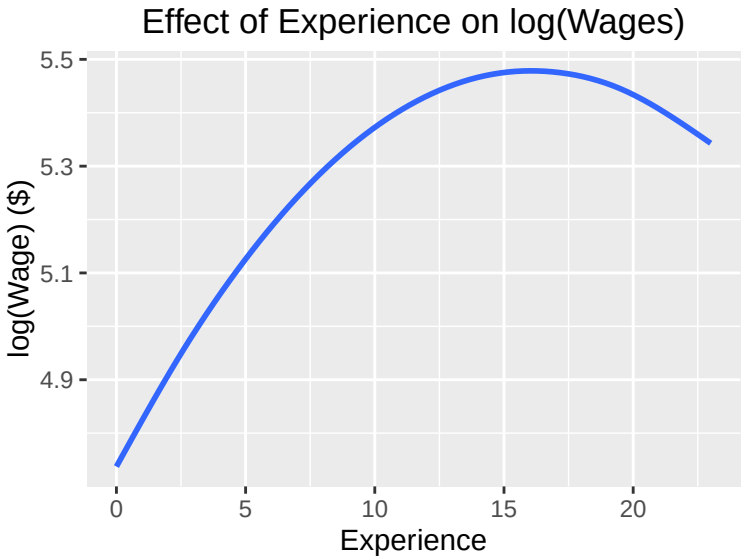


Figure 12: Estimated Effect of Experience on Wages

Table 6: Regression summary

	Dependent variable:
	log(Wages)
Education	0.068***
	(0.008)
Experience	0.097***
	(0.013)
Expereince^2	−0.003***
	(0.001)
Race (White or Non-white)	−0.213***
	(0.041)
Father's Education	−0.001
	(0.006)
Mother's Education	0.011
	(0.007)
Rural (Yes or No)	−0.092**
	(0.032)
City (Yes or No)	0.178***
	(0.032)
Constant	4.642***
	(0.150)
F Statistic	32.592***
df	8; 714
Observations	723
R^2	0.275
Adjusted R^2	0.267
Residual Std. Error	0.379

·p<0.1; *p<0.05; **p<0.01; ***p<0.001

Table 7: All Observations

Statistic	N	Mean	St. Dev.	Min	Pctl(25)	Median	Pctl(75)	Max
X	1,000	1,466.678	866.535	5	715.5	1,431.5	2,212	3,009
wage	1,000	578.783	266.569	127	400	543	702.5	2,404
education	1,000	13.219	2.729	2	12	12	16	18
experience	1,000	8.788	4.221	0	6	8	11	23
age	1,000	28.007	3.118	24	25	27	30	34
raceColor	1,000	0.238	0.426	0	0	0	0	1
dad_education	761	10.181	3.748	0	8	11	12	18
mom_education	872	10.451	3.126	0	8	12	12	18
rural	1,000	0.391	0.488	0	0	0	1	1
city	1,000	0.712	0.453	0	0	1	1	1
z1	1,000	0.440	0.497	0	0	0	1	1
z2	1,000	0.686	0.464	0	0	1	1	1
IQscore	684	102.273	15.843	50	93	103	113	144
logWage	1,000	6.263	0.447	4.844	5.991	6.297	6.555	7.785
experienceSquare	1,000	95.030	86.786	0	36	64	121	529

Table 8: Missing Father's Education

Statistic	N	Mean	St. Dev.	Min	Pctl(25)	Median	Pctl(75)	Max
X	239	1,675.259	864.761	15	921.5	1,712	2,515.5	3,009
wage	239	529.531	267.216	127	351.5	465	640.5	2,083
education	239	12.121	2.696	3	11	12	13	18
experience	239	10.527	4.223	1	7	10	14	23
age	239	28.649	3.090	24	26	28	31	34
raceColor	239	0.464	0.500	0	0	0	1	1
mom_education	149	8.987	3.397	0	7	9	12	18
rural	239	0.540	0.499	0	0	1	1	1
city	239	0.661	0.474	0	0	1	1	1
z1	239	0.397	0.490	0	0	0	1	1
z2	239	0.669	0.471	0	0	1	1	1
IQscore	132	96.038	17.421	50	85	98.5	107	135
logWage	239	6.163	0.465	4.844	5.862	6.142	6.462	7.642
experienceSquare	239	128.577	98.151	1	49	100	196	529

Table 9: Missing Mother's Education

Statistic	N	Mean	St. Dev.	Min	Pctl(25)	Median	Pctl(75)	Max
X	128	1,765.523	870.253	53	996	1,796	2,572	3,009
wage	128	525.938	252.627	142	371.5	492.5	625	2,083
education	128	11.891	2.713	2	10	12	13	18
experience	128	10.898	4.742	0	7	10	15	21
age	128	28.789	3.513	24	26	28	32	34
raceColor	128	0.461	0.500	0	0	0	1	1
dad_education	38	9.184	3.840	2	6	9.5	12	16
rural	128	0.523	0.501	0	0	1	1	1
city	128	0.656	0.477	0	0	1	1	1
z1	128	0.414	0.494	0	0	0	1	1
z2	128	0.680	0.468	0	0	1	1	1
IQscore	71	93.352	15.900	60	80.5	96	104.5	124
logWage	128	6.175	0.418	4.956	5.918	6.199	6.438	7.642
experienceSquare	128	141.086	108.986	0	49	100	225	441

Table 10: Actual Observations

Statistic	N	Mean	St. Dev.	Min	Pctl(25)	Median	Pctl(75)	Max
X	723	1,399.028	853.489	5	680.5	1,314	2,125	2,998
wage	723	597.080	268.094	136	409	570	721	2,404
education	723	13.651	2.616	3	12	13	16	18
experience	723	8.145	4.005	0	5	8	10	21
age	723	27.797	3.066	24	25	27	30	34
raceColor	723	0.158	0.365	0	0	0	0	1
dad_education	723	10.234	3.739	0	8	11	12	18
mom_education	723	10.752	2.982	0	9	12	12	18
rural	723	0.346	0.476	0	0	0	1	1
city	723	0.730	0.444	0	0	1	1	1
z1	723	0.448	0.498	0	0	0	1	1
z2	723	0.685	0.465	0	0	1	1	1
IQscore	531	104.081	15.062	60	95	105	114	144
logWage	723	6.297	0.442	4.913	6.014	6.346	6.581	7.785
experienceSquare	723	82.367	78.067	0	25	64	100	441

```
                                                    na.rm=T))
model6 <- lm(logWage ~ education + experience + experienceSquare + raceColor +
               dad_education + mom_education + rural + city, data = data7)
```

Table 11: Regression summary

	Dependent variable:
	log(Wages)
Education	0.071***
	(0.007)
Experience	0.090***
	(0.011)
Expereince^2	−0.003***
	(0.001)
Race (White or Non-white)	−0.231***
	(0.031)
Father's Education	−0.00004
	(0.005)
Mother's Education	0.003
	(0.005)
Rural (Yes or No)	−0.095***
	(0.027)
City (Yes or No)	0.167***
	(0.026)
Constant	4.729***
	(0.128)
F Statistic	54.109***
df	8; 991
Observations	1,000
R^2	0.298
Adjusted R^2	0.292
Residual Std. Error	0.376

·p<0.1; *p<0.05; **p<0.01; ***p<0.001

4. **How about regress the variable(s) with missing values on education, experience, and raceColor, and use this regression(s) to predict (i.e., "impute") the missing values and then rerun the original regression using all of the observations?**

```
data8 <-data
model_dad<-lm(dad_education~education+experience+raceColor, data=data8)
model_mom<-lm(mom_education~education+experience+raceColor, data=data8)
coef_dad<-coef(model_dad)
coef_mom<-coef(model_mom)
# Impute values for missing observations
for (i in 1:nrow(data8)) {
  if (is.na(data8$dad_education[i])==TRUE) {
    data8$dad_education[i]= coef_dad[1]+coef_dad[2]*data8$education[i]+
      coef_dad[3]*data8$experience[i]+coef_dad[4]*data8$raceColor[i]
  }
  if (is.na(data8$mom_education[i])==TRUE) {
```

```
    data8$mom_education[i]= coef_mom[1]+coef_mom[2]*data8$education[i]+
      coef_mom[3]*data8$experience[i]+coef_mom[4]*data8$raceColor[i]
  }
}
```

Table 12: Regression summary

	Dependent variable:
	log(Wages)
Education	0.070^{***}
	(0.007)
Experience	0.089^{***}
	(0.011)
Expereince^2	-0.003^{***}
	(0.001)
Race (White or Non-white)	-0.227^{***}
	(0.031)
Father's Education	0.002
	(0.005)
Mother's Education	0.002
	(0.006)
Rural (Yes or No)	-0.095^{***}
	(0.027)
City (Yes or No)	0.167^{***}
	(0.026)
Constant	4.727^{***}
	(0.125)
F Statistic	54.147^{***}
df	8; 991
Observations	1,000
R^2	0.298
Adjusted R^2	0.293
Residual Std. Error	0.376

·p<0.1; *p<0.05; **p<0.01; ***p<0.001

5. **Compare the results of all of these regressions. Which one, if at all, would you prefer?**

Table 13:

	Dependent variable:		
	log(Wages)		
	(1)	(2)	(3)
Education	0.068***	0.071***	0.070***
	(0.008)	(0.007)	(0.007)
Experience	0.097***	0.090***	0.089***
	(0.013)	(0.011)	(0.011)
Expereince^2	−0.003***	−0.003***	−0.003***
	(0.001)	(0.001)	(0.001)
Race (White or Non-white)	−0.213***	−0.231***	−0.227***
	(0.041)	(0.031)	(0.031)
Father's Education	−0.001	−0.00004	0.002
	(0.006)	(0.005)	(0.005)
Mother's Education	0.011	0.003	0.002
	(0.007)	(0.005)	(0.006)
Rural (Yes or No)	−0.092**	−0.095***	−0.095***
	(0.032)	(0.027)	(0.027)
City (Yes or No)	0.178***	0.167***	0.167***
	(0.032)	(0.026)	(0.026)
Constant	4.642***	4.729***	4.727***
	(0.150)	(0.128)	(0.125)
F Statistic	32.592***	54.109***	54.147***
df	8; 714	8; 991	8; 991
Observations	723	1,000	1,000
R^2	0.275	0.298	0.298
Adjusted R^2	0.267	0.292	0.293
Residual Std. Error	0.379	0.376	0.376

·p<0.1; *p<0.05; **p<0.01; ***p<0.001

Comparing the regression results, it's worth noting that the magnitude and significance of the coefficients are broadly similar. However, given that the values that we imputed are for covariates that are neither statistically nor practically significant, I would prefer a model that used all of the observations and did not include the covariates for parent's education.

Question 4.6

1. **Consider using z_1 as the instrumental variable (IV) for education. What assumptions are needed on z_1 and the error term (call it, u)?**

Formally, we need to assume that the instrumental variable is uncorrelated with the error term. Ie.

$$Cov(z_{1j}, u) = 0 \ \forall j = 1, 2, \ldots, k$$

2. **Suppose z_1 is an indicator representing whether or not an individual lives in an area in which there was a recent policy change to promote the importance of education. Could z_1 be correlated with other unobservables captured in the error term?**

Yes. Living in an area with recent education policy changes could correlate to a number of variables that relate to the earnings in that area, such as a person's trust in educational and governmental institutions, or the willingness of people to pay taxes to support education spending. As with most instrumental variables, the assumption that it is truly uncorrelated with the error term can always be questioned, although it may represent an improvement in estimating coefficients over straightforward OLS.

3. **Using the same specification as that in Question 4.5, estimate the equation by 2SLS, using both z_1 and z_2 as instrument variables. Interpret the results. How does the coefficient estimate on education change?**

```
first_stage_a <- lm(education ~ z1, data = data )
first_stage_b <- lm(education ~ z2, data = data)
second_stage_a <- lm(data$logWage ~ first_stage_a$fitted + data$experience +
                       data$experienceSquare + data$raceColor +
                       data$dad_education + data$mom_education + data$rural +
                       data$city)
second_stage_b <- lm(data$logWage ~ first_stage_b$fitted + data$experience +
                       data$experienceSquare + data$raceColor +
                       data$dad_education + data$mom_education + data$rural +
                       data$city)
```

Table 14: Step One Regression Summary

	Dependent variable:	
	education	
	(1)	(2)
IV 1	0.242	
	(0.174)	
IV 2		1.006***
		(0.187)
Constant	13.113***	12.529***
	(0.114)	(0.158)
F Statistic	1.926	28.877***
df	1; 998	1; 998
Observations	1,000	1,000
R^2	0.002	0.029
Adjusted R^2	0.001	0.028
Residual Std. Error	2.728	2.690

·p<0.1; *p<0.05; **p<0.01; ***p<0.001

Note that z_1 is not correlated with education, and is thus not an appropriate instrumental variable to estimate the effect of education on wages.

Table 15: Step Two Regression summary

	Dependent variable:
	log(Wages)
Education (z_2)	0.045
	(0.033)
Experience	0.072^{***}
	(0.013)
Expereince^2	-0.003^{***}
	(0.001)
Race (White or Non-white)	-0.245^{***}
	(0.045)
Father's Education	0.006
	(0.006)
Mother's Education	0.022^{**}
	(0.007)
Rural (Yes or No)	-0.096^{**}
	(0.034)
City (Yes or No)	0.191^{***}
	(0.034)
Constant	4.974^{***}
	(0.434)
F Statistic	21.634^{***}
df	8; 714
Observations	723
R^2	0.198
Adjusted R^2	0.189
Residual Std. Error	0.398

·p<0.1; *p<0.05; **p<0.01; ***p<0.001

The results of the regression show that the magnitude of the covariate coefficients is similar using the instrumental variable approach. The coefficient for education has dramatically decreased and is no longer statistically significant.

Question 5: CLM 2

The dataset, `wealthy candidates.csv`, contains candidate level electoral data from a developing country. Politically, each region (which is a subset of the country) is divided in to smaller electoral districts where the candidate with the most votes wins the seat. This dataset has data on the financial wealth and electoral performance (voteshare) of electoral candidates. We are interested in understanding whether or not wealth is an electoral advantage. In other words, do wealthy candidates fare better in elections than their less wealthy peers?

1. **Begin with a parsimonious, yet appropriate, specification. Why did you choose this model? Are your results statistically significant? Based on these results, how would you answer the research question? Is there a linear relationship between wealth and electoral performance?**

In order to decide on a parsimonious model and discover any potential issues with data quality or variable distributions, we will first explore summary statistics and conduct a univariate analysis.

Table 16: Summary Statistics for Voting Data

Statistic	X	urb	lit	voteshare	absolute_wealth	logWealth
N	2,497	2,497	2,497	2,497	2,497	2,497
Mean	1,249.930	0.187	0.451	0.288	5,034,105.000	11.961
St. Dev.	721.080	0.149	0.092	0.123	31,098,493.000	5.388
Min	1	0.028	0.242	0.006	2.000	0.693
Pctl(25)	626	0.084	0.385	0.200	187,500.000	12.142
Median	1,250	0.147	0.460	0.293	1,336,629.000	14.106
Pctl(75)	1,874	0.243	0.510	0.368	4,092,001.000	15.225
Max	2,498	0.802	0.652	0.693	1,216,399,232.000	20.919

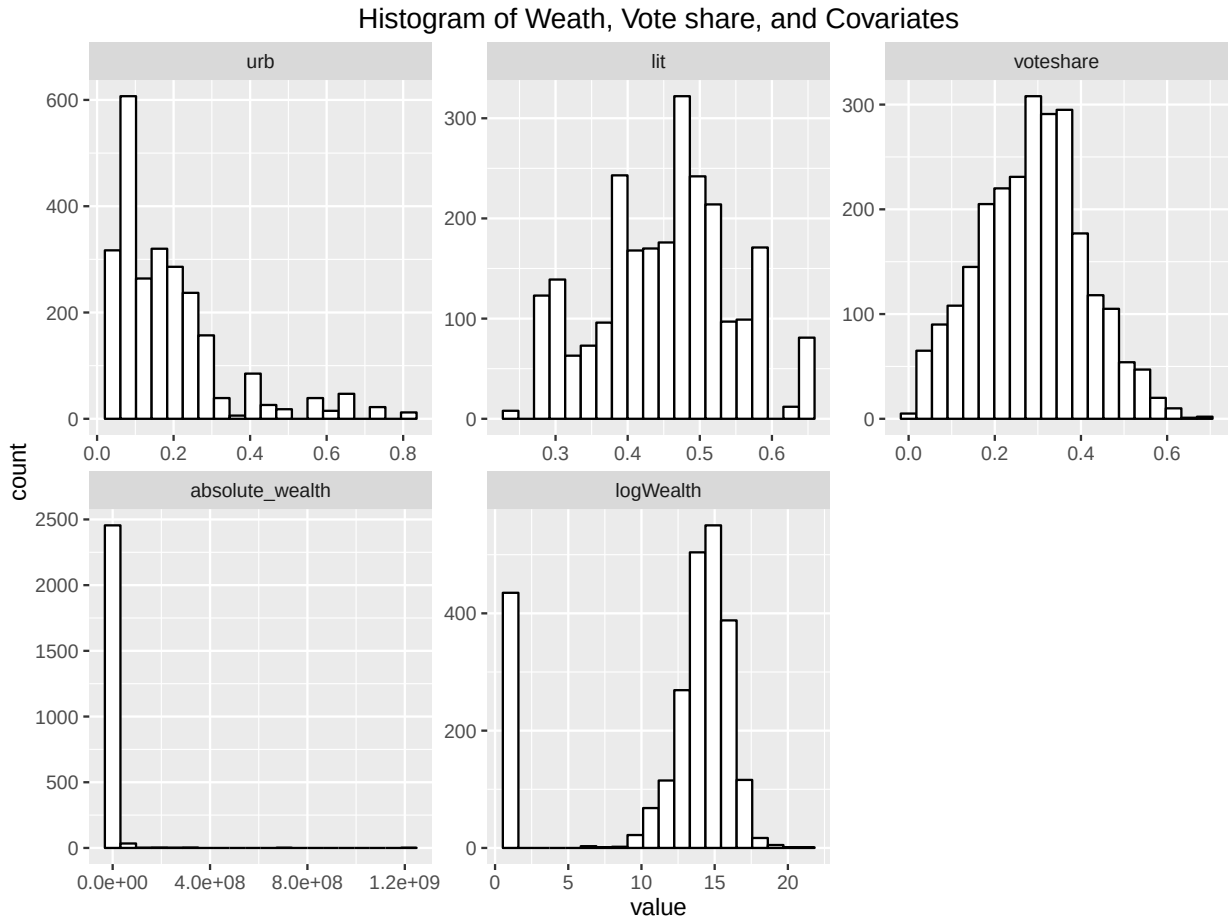


Figure 13: Histogram of Urban, Literacy, Voteshar, and Wealth Variables

Examining the absolute_wealth and log(absolute_wealth) variables reveals that a large number of observations have the same value. It seems likely that this value codes for having zero absolute wealth. Because we interested in the effect of wealth on voteshare, it seems reasonable to look at this effect among those with wealth greater than zero. By creating a factor variable representing wealth greater than zero, the equation for our parsimonious model will be

$$voteshare = \beta_0 + \beta_1 log(wealth) + \beta_2 hasWealth$$

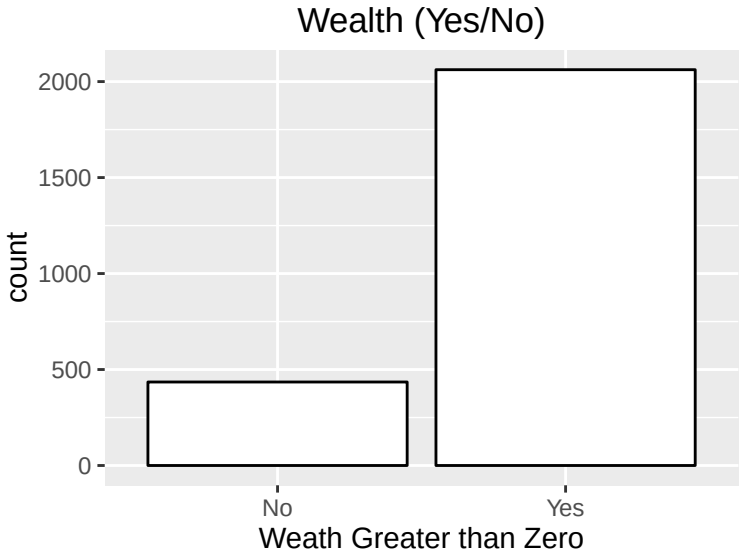


Figure 14: Bar Plot of Wealth and No Wealth

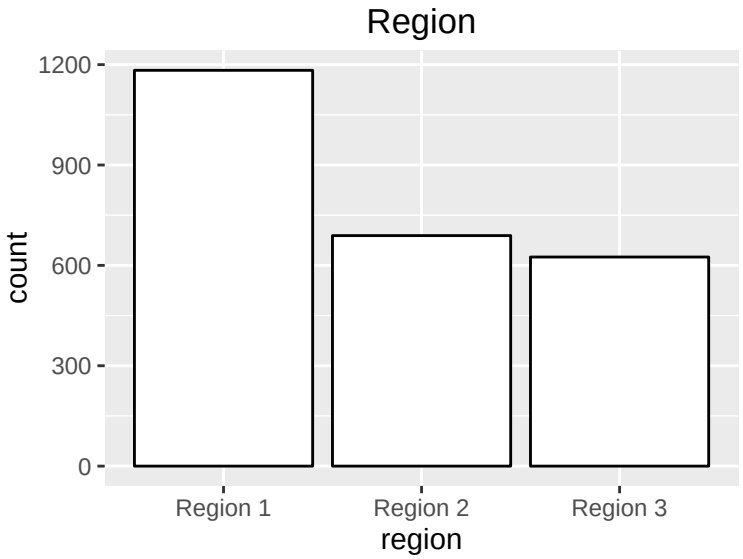


Figure 15: Bar Plot of Region

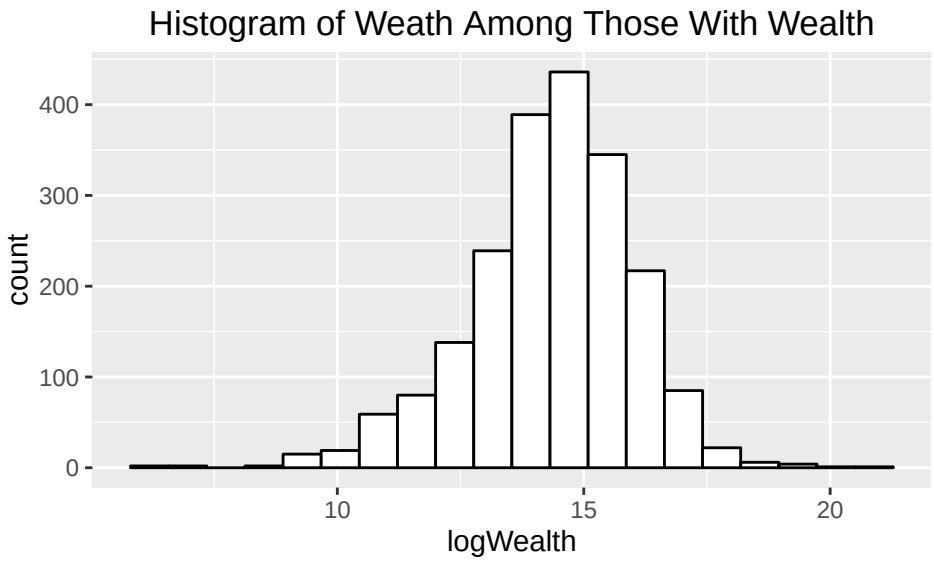


Figure 16: Histogram of log(Wealth) Among Those With Wealth

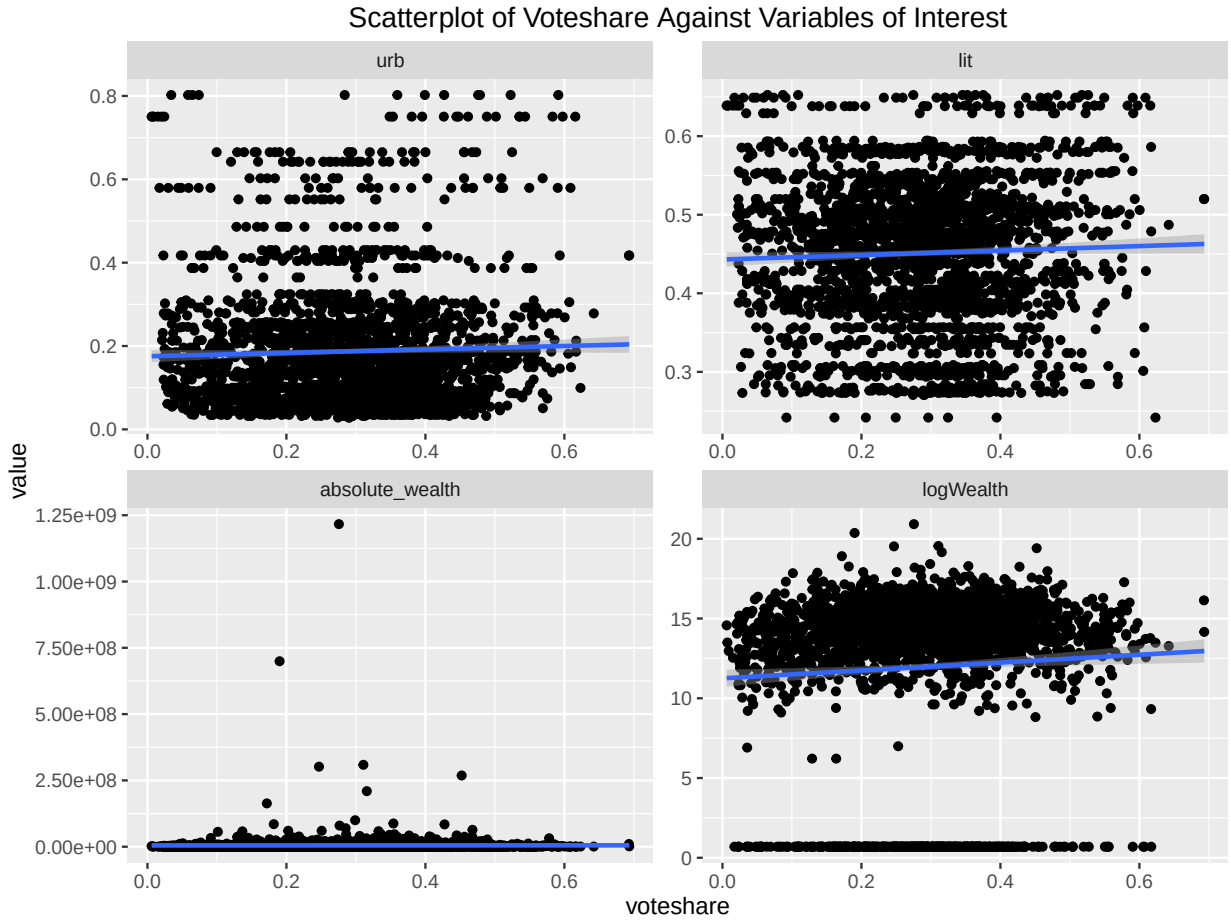


Figure 17: Scatterplots of Voteshare against Urban, Literacy, Absolute Wealth, and log(Absolute Wealth)

Table 17: Regression summary - Parsimonious Model

	Dependent variable:
	Voteshare
log(Wealth)	0.005**
	(0.002)
Has Wealth = Yes	−0.057*
	(0.026)
Constant	0.273***
	(0.006)
F Statistic	7.189***
df	2; 2494
Observations	2,497
R^2	0.006
Adjusted R^2	0.005
Residual Std. Error	0.123

·p<0.1; *p<0.05; **p<0.01; ***p<0.001

The parsimonious model reveals a very weak positive effect of wealth on voteshare. The effect is not of practical significance and suggests that there may not be a linear relationship between wealth and voteshare.

2. **A team-member suggests adding a quadratic term to your regression. Based on your prior model, is such an addition warranted? Add this term and interpret the results. Do wealthier candidates fare better in elections?**

Theoretically, the addition of a quadratic seems questionable because it suggests there is a point at which the returns to wealth stop increasing. Even if the quadratic model resulted in an improvement in the R^2 value, it may be difficult to justify the inclusion of a quadratic term without a theoretical reason.

Table 18: Regression summary - Quadratic Model

	Dependent variable:
	Voteshare
log(Wealth)	0.005**
	(0.002)
$\log(\text{Wealth})^2$	−0.057*
	(0.026)
Has Wealth = Yes	0.273***
	(0.006)
F Statistic	7.189***
df	2; 2494
Observations	2,497
R^2	0.006
Adjusted R^2	0.005
Residual Std. Error	0.123

·p<0.1; *p<0.05; **p<0.01; ***p<0.001

3. **Another team member suggests that it is important to take into account the fact that different regions have different electoral contexts. In particular, the relationship between candidate wealth and electoral performance might be different across states. Modify your model and report your results. Test the hypothesis that this addition is not needed.**

First, we will look at how the variables of interest are distributed across regions.

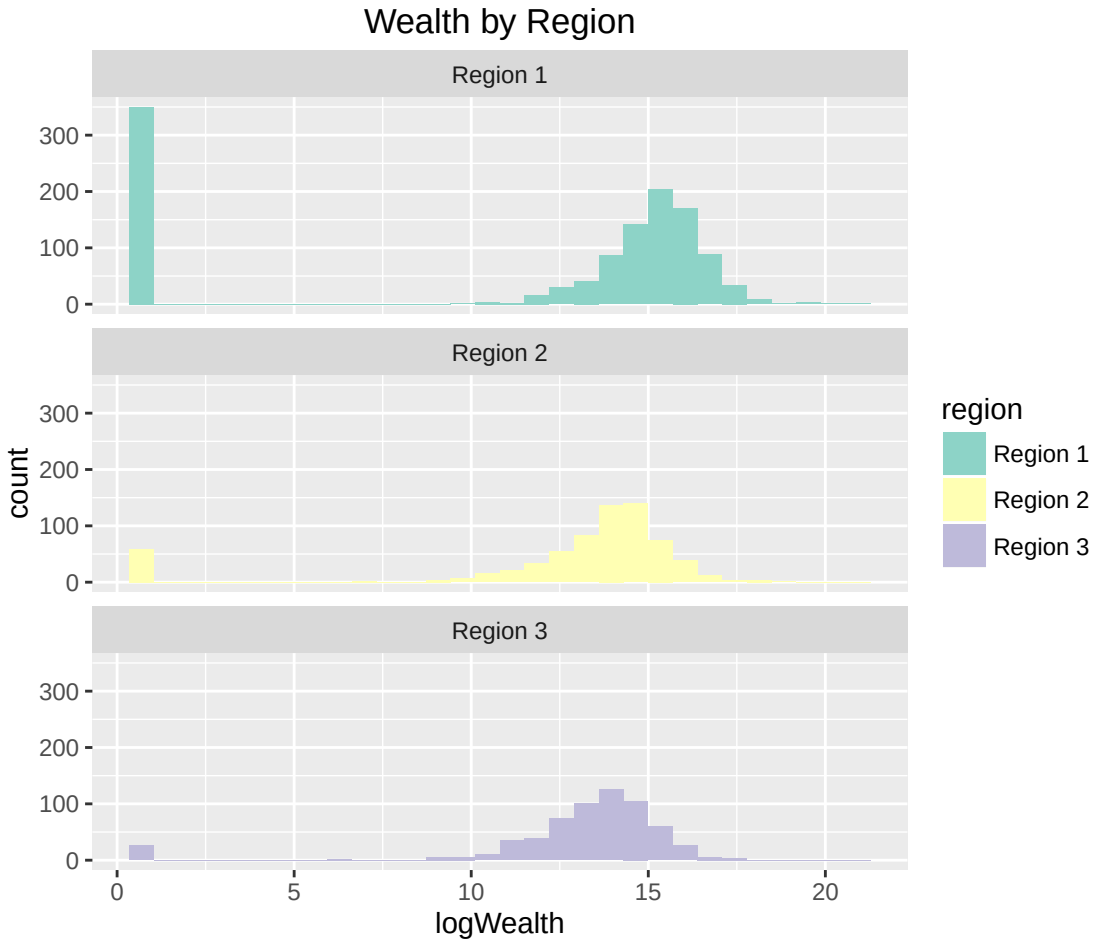


Figure 18: Histograms of Wealth By Region

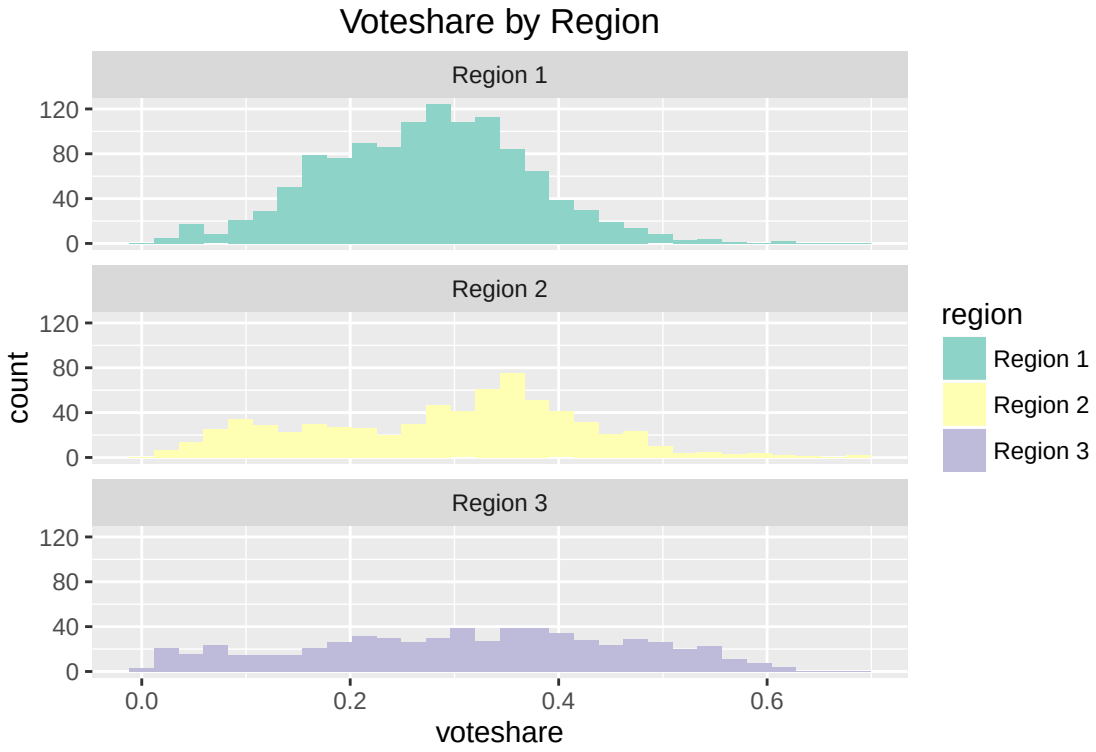


Figure 19: Histograms of Voteshare By Region

The histograms of wealth by region and voteshare by region do seem to reveal some genuine structural differences between the regions. The first region has much greater wealth inequality, with a large share of the observations having no wealth and greater average wealth among the wealthy than regions 2 and 3. The voteshare histogram for region one also suggests a difference in election structure, as voteshare percentages tend to be clustered between 15% and 40%, while regions two and three have relatively uniform voteshare distributions between 5% and 60%.

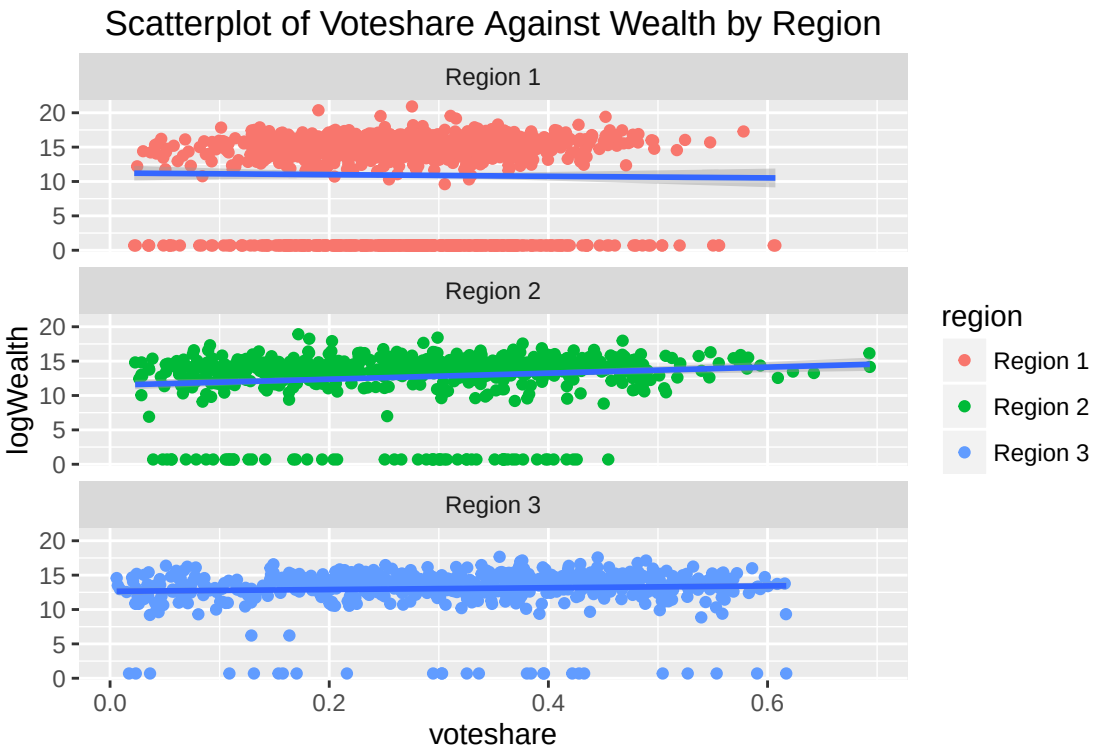


Figure 20: Scatterplots of Voteshare Against Wealth By Region

The scatter plots of voteshare against the wealth variables reveal that is all regions, there does not seem to be a strong linear relationship between wealth and voteshare. There seems to be a level of $log(wealth)$ around 10, below which there are few observations, but once that floor is cleared, there doesn't seem to be a strong relationship between additional wealth and additional voteshare.

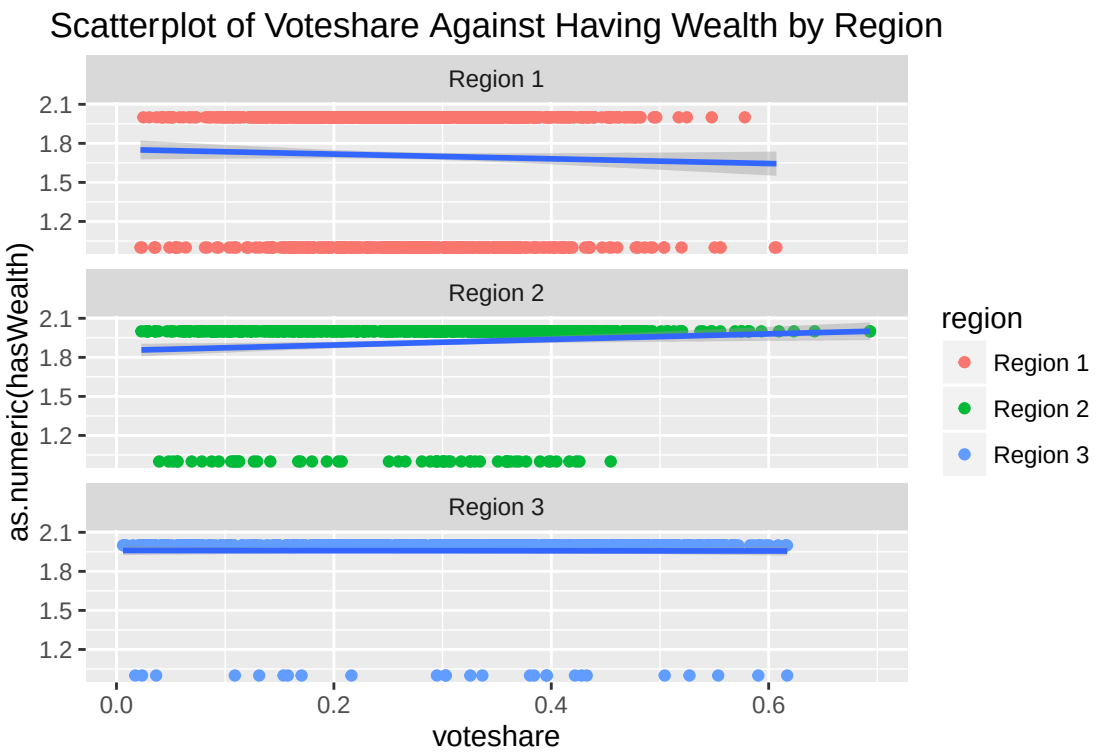


Figure 21: Scatterplots of Voteshare Against Having Wealth By Region

Table 19: Regression summary - Region Model

	Dependent variable:
	Voteshare
log(Wealth)	0.011***
	(0.002)
Has Wealth = Yes	−0.156***
	(0.029)
Region = 2	0.031***
	(0.006)
Region = 3	0.055***
	(0.007)
Constant	0.262***
	(0.006)
F Statistic	19.367***
df	4; 2492
Observations	2,497
R^2	0.032
Adjusted R^2	0.030
Residual Std. Error	0.121

·p<0.1; *p<0.05; **p<0.01; ***p<0.001

The regression output shows that the model including regions accounts for a greater share of the overall variation than the parsimonious model, with the difference between the two models being practically and

Table 20: Model Comparison

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
Parsimonious Model	2,494	37.736				
Region Model	2,492	36.741	2	0.995	33.755	0

statistically significant (F(2, 2492) = 33.755, p < 0.001). This supports the notion that the structural differences in voteshare and wealth between the regions are important to take into account when estimating the effect of wealth on voteshare.

4. **Return to your parsimonious model. Do you think you have found a causal and unbiased estimate? Please state the conditions under which you would have an unbiased and causal estimates. Do these conditions hold?**

In order for OLS to make unbiased estimates of the OLS parameters, 4 conditions must hold:

1. The parameters have a linear relationship
2. Observations are observed through random sampling
3. There is no perfect collinearity among the independent variables
4. There is zero conditional mean of the error term and zero correlation of the error with any of the independent variables

And in order for our coefficients to represent causal effects, changes in the independent variable must not change the expected value of the error term. Ie.

$$\frac{\partial u}{\partial x} = 0$$

Two primary issues that lead me to conclude that we have not found a causal and unbiased estimate. The first is the lack of apparent linearity in the relationship between voteshare and wealth. This is supported by the low explanatory power of our model, as well as the scatter plot of wealth and voteshare that seems to imply non linear factors may be more important than linear. The second issue is that wealth is probably not exogenous, but are in fact correlated with a host of measures that are contained in the error term. For example, having more money may be correlated with being a well-known public figure, which may influence voteshare.

5. **Someone proposes a difference in difference design. Please write the equation for such a model. Under what circumstances would this design yield a causal effect?**

Supposing we had data collected at two different time periods, with λ_t being an indicator for the fixed time effect, a_i representing the time-fixed effect of being individual i, and u_{it} representing the time-variable portion of the error term. Then our population model would look like:

$$voteshare_{it} = \beta_0 + \lambda_t + \beta_1 log(absolute\_wealth)_{it} + \beta_2 hasWealth_{it} + a_i + u_{it}$$

The first difference model would be obtained by subtracting the model at t_2 from t_1 and assuming parallel trends, ie that trends would have continued in the same way at t_2 as in t_1 in the absence of treatment.

$$\Delta voteshare_i = \lambda_t + \beta_1 \Delta log(absolute\_wealth)_i + \beta_2 \Delta hasWealth_i + \Delta u_i$$

In order for the first difference model to be causal, changes in the difference in the independent variables must not change the expectation of changes in the error term.

$$\frac{\partial \Delta u}{\partial \Delta x} = 0$$

Question 6: CLM 3

Your analytics team has been tasked with analyzing aggregate revenue, cost and sales data, which have been provided to you in the R workspace/data frame `retailSales.Rdata`.

Your task is two fold. First, your team is to develop a model for predicting (forecasting) revenues. Part of the model development documentation is a backtesting exercise where you train your model using data from the first two years and evaluate the model's forecasts using the last two years of data.

Second, management is equally interested in understanding variables that might affect revenues in support of management adjustments to operations and revenue forecasts. You are also to identify factors that affect revenues, and discuss how useful management's planned revenue is for forecasting revenues.

Your analysis should address the following:

- **Exploratory Data Analysis: focus on bivariate and multivariate relationships.**
- **Be sure to assess conditions and identify unusual observations.**

First we explore the whole dataset.

```
load("retailSales.Rdata")
data <- retailSales; rm(retailSales)
summary(data)
```

```
##       Year                          Product.line
##  Min.   :2004   Camping Equipment       :24108
##  1st Qu.:2005   Golf Equipment          : 8820
##  Median :2006   Mountaineering Equipment:12348
##  Mean   :2006   Outdoor Protection      : 8820
##  3rd Qu.:2006   Personal Accessories    :30576
##  Max.   :2007
##
##                Product.type                  Product
##  Eyewear              : 9408   Aloe Relief       :  588
##  Watches              : 7644   Astro Pilot       :  588
##  Lanterns             : 7056   Auto Pilot        :  588
##  Cooking Gear         : 5880   Bear Edge         :  588
##  Navigation           : 5880   Bear Survival Edge:  588
##  Climbing Accessories: 4116   Bella             :  588
##  (Other)              :44688   (Other)           :81144
##    Order.method.type  Retailer.country    Revenue
##  E-mail     :12096   Australia: 4032   Min.   :       0
##  Fax        :12096   Austria  : 4032   1st Qu.:   18579
##  Mail       :12096   Belgium  : 4032   Median :   59867
##  Sales visit:12096   Brazil   : 4032   Mean   :  189418
##  Special    :12096   Canada   : 4032   3rd Qu.:  190193
##  Telephone  :12096   China    : 4032   Max.   :10054289
##  Web        :12096   (Other)  :60480   NA's   :59929
##  Planned.revenue     Product.cost        Quantity        Unit.cost
##  Min.   :      16   Min.   :      6   Min.   :     1   Min.   :  0.85
##  1st Qu.:   19557   1st Qu.:   9432   1st Qu.:   328   1st Qu.: 11.43
```

```
##  Median :   63907   Median :  32784   Median :  1043   Median : 36.83
##  Mean   :  198818   Mean   : 111625   Mean   :  3607   Mean   : 84.89
##  3rd Qu.:  203996   3rd Qu.: 111371   3rd Qu.:  3288   3rd Qu.: 80.00
##  Max.   :10054289   Max.   :6756853   Max.   :313628   Max.   :690.00
##  NA's   :59929      NA's   :59929     NA's   :59929    NA's   :59929
##    Unit.price       Gross.profit     Unit.sale.price
##  Min.   :   2.06   Min.   : -18160   Min.   :   0.00
##  1st Qu.:  23.00   1st Qu.:   8333   1st Qu.:  20.15
##  Median :  66.77   Median :  25794   Median :  62.65
##  Mean   : 155.99   Mean   :  77793   Mean   : 147.23
##  3rd Qu.: 148.30   3rd Qu.:  78254   3rd Qu.: 140.96
##  Max.   :1359.72   Max.   :3521098   Max.   :1307.80
##  NA's   :59929     NA's   :59929     NA's   :59929
```

The dataset contains 84,672 observations of 14 variables. 5 of them are categorical (`Product.line`, `Product.type`, `Product`, `Order.method.type`, `Retailer.country`), and `Year` should also be considered as categorical, since there are data from only 4 years (from 2004 to 2007).

```
data <- data %>%
  mutate(Year = as.factor(Year))
```

We also notice (from the output of `summary`) that some of the variables (all of them numerical) has a high number of `NA`s, the same in all cases (59929, i.e., 70.78% of the total number of observations). Do the `NA`s appear in the same observations for all those variables? Yes, they do.

```
# data_isNA <- as.data.frame(sapply(data, is.na))
data_isNA <- data %>% mutate_each(funs(is.na(.)))
head(data_isNA)
```

```
##    Year Product.line Product.type Product Order.method.type
## 1 FALSE        FALSE        FALSE   FALSE             FALSE
## 2 FALSE        FALSE        FALSE   FALSE             FALSE
## 3 FALSE        FALSE        FALSE   FALSE             FALSE
## 4 FALSE        FALSE        FALSE   FALSE             FALSE
## 5 FALSE        FALSE        FALSE   FALSE             FALSE
## 6 FALSE        FALSE        FALSE   FALSE             FALSE
##   Retailer.country Revenue Planned.revenue Product.cost Quantity Unit.cost
## 1            FALSE   FALSE           FALSE        FALSE    FALSE     FALSE
## 2            FALSE   FALSE           FALSE        FALSE    FALSE     FALSE
## 3            FALSE    TRUE            TRUE         TRUE     TRUE      TRUE
## 4            FALSE    TRUE            TRUE         TRUE     TRUE      TRUE
## 5            FALSE   FALSE           FALSE        FALSE    FALSE     FALSE
## 6            FALSE    TRUE            TRUE         TRUE     TRUE      TRUE
##   Unit.price Gross.profit Unit.sale.price
## 1      FALSE        FALSE           FALSE
## 2      FALSE        FALSE           FALSE
## 3       TRUE         TRUE            TRUE
## 4       TRUE         TRUE            TRUE
## 5      FALSE        FALSE           FALSE
## 6       TRUE         TRUE            TRUE
```

```
# vars_with_NAs <- apply(data_isNA, 2, sum)
vars_with_NAs <- data_isNA %>% summarise_each(funs(sum))
(vars_with_NAs <- names(vars_with_NAs)[vars_with_NAs>0])
```

```
## [1] "Revenue"         "Planned.revenue" "Product.cost"    "Quantity"
## [5] "Unit.cost"       "Unit.price"      "Gross.profit"    "Unit.sale.price"
```

```
sapply(data_isNA[, vars_with_NAs[-1]], identical,
       as.vector(data_isNA[, vars_with_NAs[1]]))
```

```
## Planned.revenue    Product.cost        Quantity       Unit.cost
##            TRUE            TRUE            TRUE            TRUE
##      Unit.price    Gross.profit Unit.sale.price
##            TRUE            TRUE            TRUE
```

And the amount of NAs per category is roughly the same for all categorical values (or at least there are non-missing data for all categories; below we just show the percentage per category for three of the numerical variables).

```
data_categorical <- data %>%
  select(which(names(data) %in% names(data)[sapply(data, is.factor)])) %>%
  mutate_each(funs(as.character(.))) %>% mutate(Revenue = data$Revenue)
data_categorical %>%
  select(Revenue, Year) %>%
  group_by(Year) %>%
  summarise_each(funs(100*mean(is.na(.)))) %>%
  rename("% of NAs in numerical variables" = Revenue)
```

```
## Source: local data frame [4 x 2]
##
##    Year % of NAs in numerical variables
##   (chr)                           (dbl)
## 1  2004                        67.95163
## 2  2005                        65.49981
## 3  2006                        71.70257
## 4  2007                        77.95729
```

```
data_categorical %>%
  select(Revenue, Product.line) %>%
  group_by(Product.line) %>%
  summarise_each(funs(100*mean(is.na(.)))) %>%
  rename("% of NAs in numerical variables" = Revenue)
```

```
## Source: local data frame [5 x 2]
##
##               Product.line % of NAs in numerical variables
##                      (chr)                           (dbl)
## 1        Camping Equipment                        65.26049
## 2           Golf Equipment                        68.67347
## 3 Mountaineering Equipment                        76.13379
## 4       Outdoor Protection                        66.62132
## 5     Personal Accessories                        74.77106
```

```
## Source: local data frame [21 x 2]
##
##    Retailer.country % of NAs in numerical variables
##                (chr)                           (dbl)
## 1          Australia                        77.15774
## 2            Austria                        72.44544
## 3            Belgium                        75.99206
## 4             Brazil                        81.49802
## 5             Canada                        57.66369
## 6              China                        77.33135
## 7            Denmark                        80.28274
## 8            Finland                        79.46429
## 9             France                        60.49107
## 10           Germany                        59.37500
## 11             Italy                        69.07242
## 12             Japan                        58.60615
## 13             Korea                        74.47917
## 14            Mexico                        73.36310
## 15       Netherlands                        70.03968
## 16         Singapore                        70.70933
## 17             Spain                        71.55258
## 18            Sweden                        74.25595
## 19       Switzerland                        80.03472
## 20    United Kingdom                        70.23810
## 21     United States                        52.28175
```

So we can ommit all those missing observations (reducing our sample size to 24743), and continue with a further analysis of the numerical variables:

```
data <- data %>% na.omit()
data_categorical <- data %>%
  select(which(names(data) %in% names(data)[sapply(data, is.factor)]))
data_non_categorical <- data %>%
  select(which(names(data) %in% names(data)[!sapply(data, is.factor)]))
round(stat.desc(data_non_categorical, desc = TRUE, basic = TRUE), 2)
```

```
##                    Revenue Planned.revenue Product.cost    Quantity
## nbr.val       2.474300e+04    2.474300e+04 2.474300e+04    24743.00
## nbr.null      7.600000e+01    0.000000e+00 0.000000e+00        0.00
## nbr.na        0.000000e+00    0.000000e+00 0.000000e+00        0.00
## min           0.000000e+00    1.569000e+01 5.760000e+00        1.00
## max           1.005429e+07    1.005429e+07 6.756853e+06   313628.00
## range         1.005429e+07    1.005427e+07 6.756847e+06   313627.00
## sum           4.686776e+09    4.919342e+09 2.761941e+09 89237091.00
## median        5.986727e+04    6.390684e+04 3.278372e+04     1043.00
## mean          1.894182e+05    1.988175e+05 1.116251e+05     3606.56
## SE.mean       2.484130e+03    2.559050e+03 1.515680e+03       55.80
## CI.mean.0.95  4.869040e+03    5.015880e+03 2.970830e+03      109.38
## var           1.526863e+11    1.620349e+11 5.684198e+10 77048387.56
## std.dev       3.907509e+05    4.025355e+05 2.384156e+05     8777.72
## coef.var      2.060000e+00    2.020000e+00 2.140000e+00        2.43
##                Unit.cost Unit.price  Gross.profit Unit.sale.price
## nbr.val         24743.00   24743.00  2.474300e+04        24743.00
```

```
## nbr.null           0.00       0.00  0.000000e+00          76.00
## nbr.na             0.00       0.00  0.000000e+00           0.00
## min                0.85       2.06 -1.815960e+04           0.00
## max              690.00    1359.72  3.521098e+06        1307.80
## range            689.15    1357.66  3.539257e+06        1307.80
## sum          2100344.99 3859701.42  1.924835e+09     3642909.71
## median            36.83      66.77  2.579376e+04          62.65
## mean              84.89     155.99  7.779311e+04         147.23
## SE.mean            0.83       1.57  1.005230e+03           1.48
## CI.mean.0.95       1.63       3.08  1.970320e+03           2.89
## var            17190.71   60912.60  2.500267e+10       53846.65
## std.dev          131.11     246.80  1.581223e+05         232.05
## coef.var           1.54       1.58  2.030000e+00           1.58
```

All numerical variables are right-skewed, with long right tails (i.e., with several observations more than 2 standard deviations far from the mean), especially the ones corresponding to aggregate—non-unit—results.

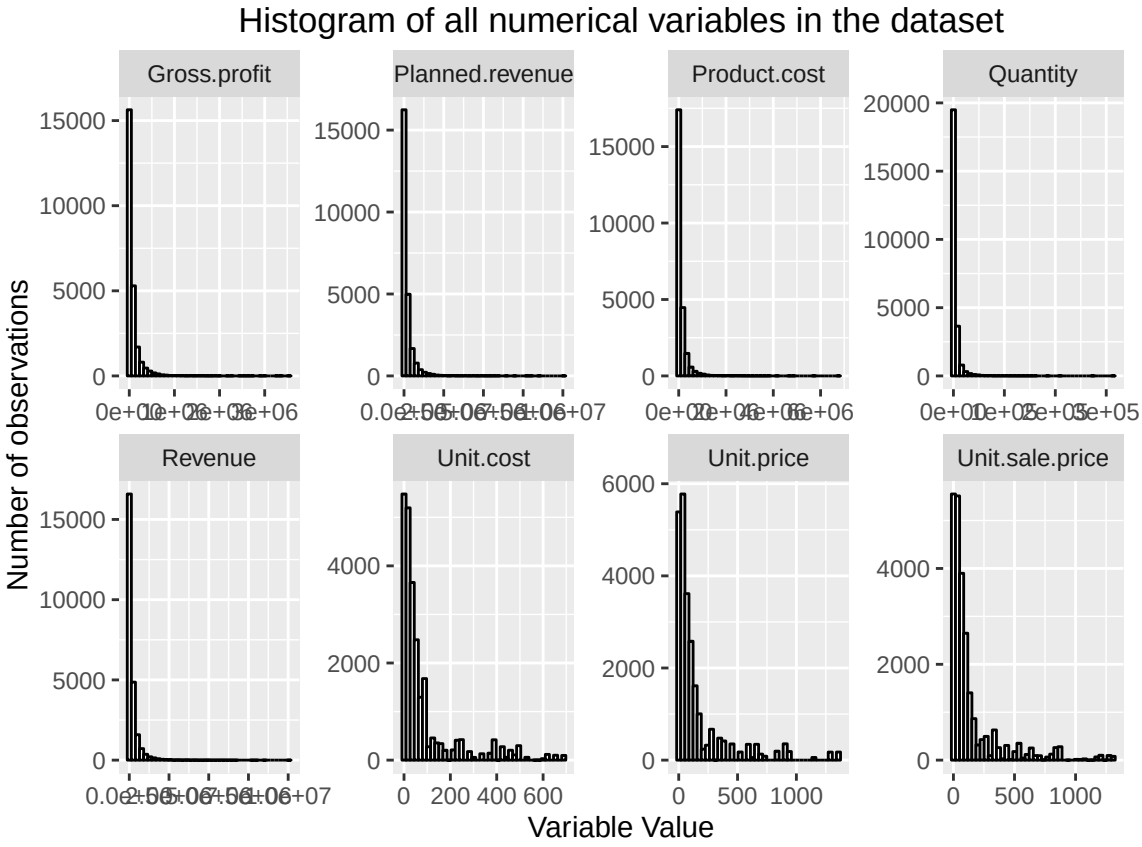


Figure 22: Histogram of all non-categorical variables in the dataset

Below we show the correlation matrix of the numerical variables, as well as two different representations of the scatterplot matrix (where we've used a sample of the data of size 500 because the plotting functions consume a lot of resources; that's why the correlations shown in the second Figure, only approximate, differ from the ones shown right below). As we might have expected, the correlations between `Revenue`,

`Planned.revenue`, `Product.cost`, and `Gross.profit` (i.e., the aggregate values), as well as those between `Unit.cost`, `Unit.price`, and `Unit.sale.price` (i.e., the values per unit), are positive and very high. `Quantity` is negatively correlated with the unitary variables (but that correlation is negligible in absolute value), and is moderately correlated ($\rho \simeq 0.5$) with the aggregate values.

```
cor(data_non_categorical)
```

```
##                   Revenue Planned.revenue Product.cost   Quantity
## Revenue         1.0000000       0.9990586    0.9903575  0.5055979
## Planned.revenue 0.9990586       1.0000000    0.9895792  0.4994770
## Product.cost    0.9903575       0.9895792    1.0000000  0.5061298
## Quantity        0.5055979       0.4994770    0.5061298  1.0000000
## Unit.cost       0.2463441       0.2550054    0.2415089 -0.1687497
## Unit.price      0.2332806       0.2421026    0.2194407 -0.1677662
## Gross.profit    0.9779407       0.9767878    0.9395732  0.4862920
## Unit.sale.price 0.2360448       0.2444078    0.2220105 -0.1674531
##                  Unit.cost Unit.price Gross.profit Unit.sale.price
## Revenue          0.2463441  0.2332806    0.9779407       0.2360448
## Planned.revenue  0.2550054  0.2421026    0.9767878       0.2444078
## Product.cost     0.2415089  0.2194407    0.9395732       0.2220105
## Quantity        -0.1687497 -0.1677662    0.4862920      -0.1674531
## Unit.cost        1.0000000  0.9886870    0.2446187       0.9889263
## Unit.price       0.9886870  1.0000000    0.2456107       0.9992750
## Gross.profit     0.2446187  0.2456107    1.0000000       0.2485667
## Unit.sale.price  0.9889263  0.9992750    0.2485667       1.0000000
```

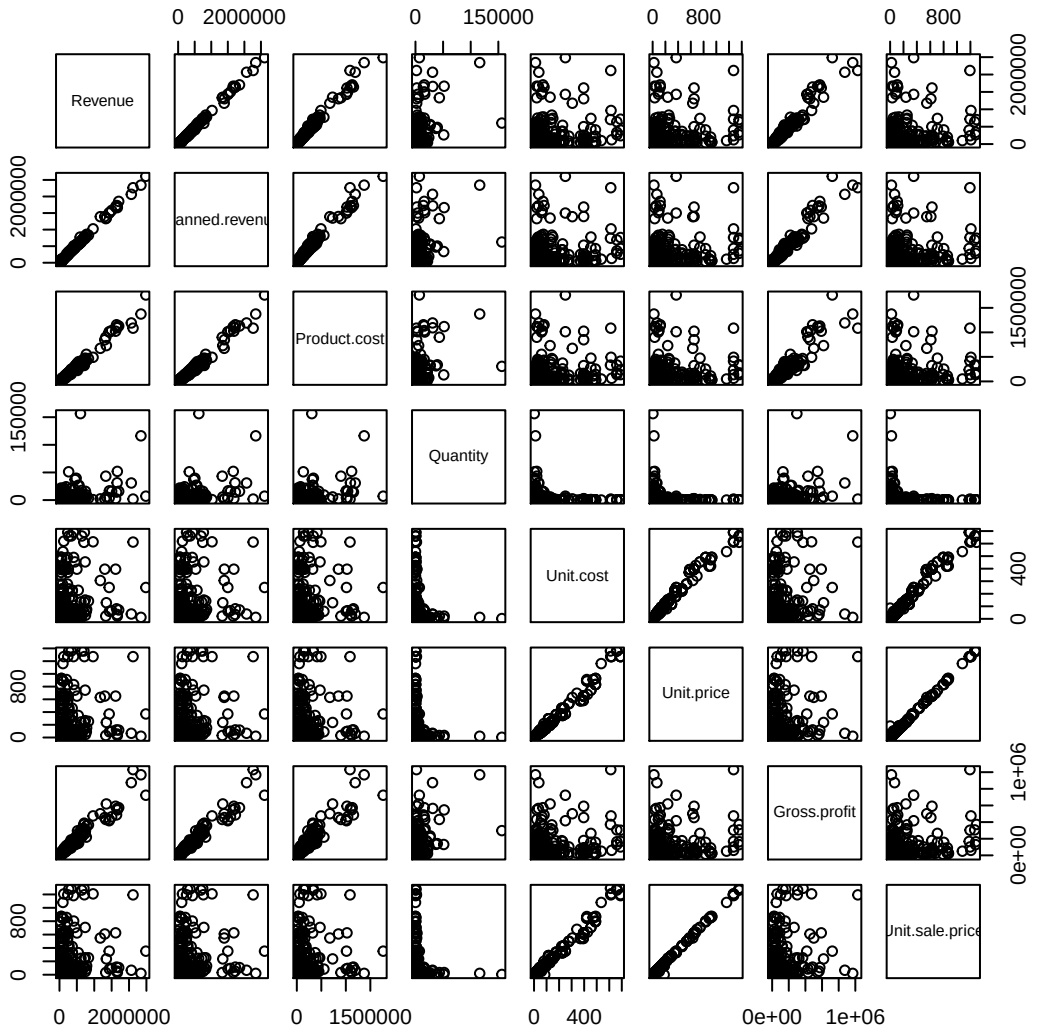


Figure 23: Scatterplot matrix of a sample of the dataset

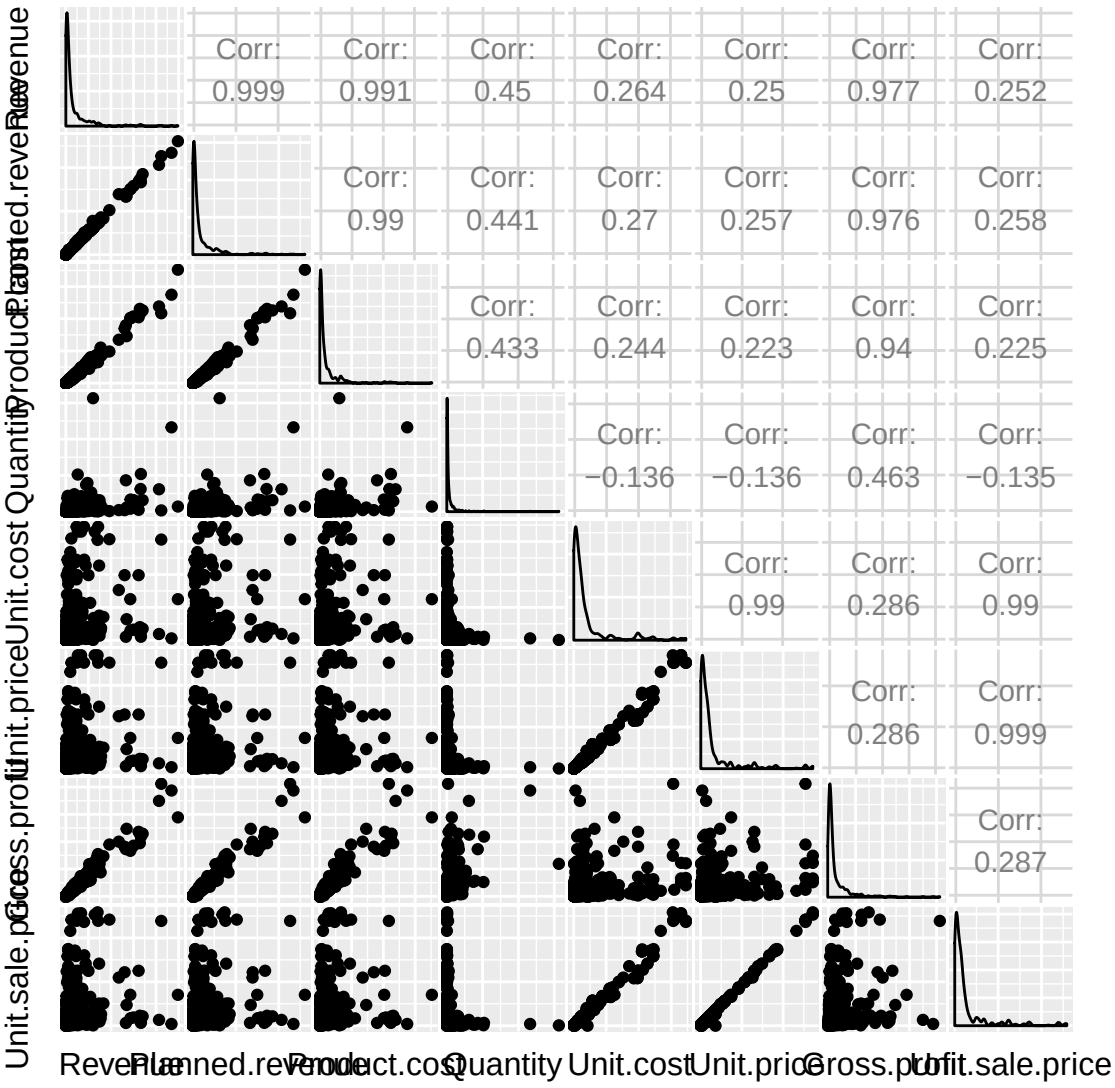


Figure 24: Scatterplot matrix of a sample of the dataset (with correlations)

After our EDA, we can divide the dataset into two separate ones, to train and evaluate the model. Now we can convert `Year` back to a numerical variable (subtracting 2004 so the baseline is 0; that will make the intercept more intuitive when including `Year` in the regression model).

```
# Year back to integer (factor only useful for vizs)
data <- data %>% mutate(Year = as.numeric(levels(Year))[Year] - 2004)
# One dataset per couple of years
# data_200405 <- data %>% filter(Year <= 2005)
# data_200607 <- data %>% filter(Year > 2005)
data_200405 <- data %>% filter(Year <= 1)
data_200607 <- data %>% filter(Year > 1)
```

Not all products appear in both periods so some re-factoring is needed:

```
# Re-factor Product (since the levels differ by period)
products_200405 <- data.frame(Product = levels(droplevels(data_200405$Product)))
products_200607 <- data.frame(Product = levels(droplevels(data_200607$Product)))
continuing_products <- intersect(products_200405, products_200607)
(new_or_discontinuted_products <- union(products_200405, products_200607) %>%
  setdiff(continuing_products))
```

```
##        Product
## 1 Trail Master
## 2   Trail Star
## 3   Auto Pilot
```

```
# Products present in one period and not the other are labelled as "Other"
data_200405 <- data_200405 %>%
  mutate(Product = ifelse(Product %in% new_or_discontinuted_products$Product,
                          "Other", as.character(Product))) %>%
  mutate(Product = factor(Product))
data_200607 <- data_200607 %>%
  mutate(Product = ifelse(Product %in% new_or_discontinuted_products$Product,
                          "Other", as.character(Product))) %>%
  mutate(Product = factor(Product))
```

There are some variables that are calculated from `Revenue` (or vice versa) so including them in any regression model would lead to a perfect fit. In particular, `Gross.profit` = `Revenue` - `Product.cost`. And `Revenue` should be equal to `Unit.sale.price` times `Quantity`, though this is not always the case, and there are differences in many cases (53.4% of the total number of observations).

```
head(data %>% select(Revenue, Product.cost, Gross.profit) %>%
       mutate(Revenue2 = Product.cost + Gross.profit))
```

```
##     Revenue Product.cost Gross.profit  Revenue2
## 1 315044.33    158371.76    156672.57 315044.33
## 2  13444.68      6298.80      7145.88  13444.68
## 3 181120.24     89413.06     91707.18 181120.24
## 4  69608.15     35326.25     34281.90  69608.15
## 5  30940.35     16370.97     14569.38  30940.35
## 6  74321.18     36531.63     37789.55  74321.18
```

```
all(round(data$Revenue - data$Product.cost, 2) == round(data$Gross.profit, 2))
```

```
## [1] TRUE
```

```
head(data %>% select(Revenue, Unit.sale.price, Quantity) %>%
       mutate(Revenue2 = Unit.sale.price * Quantity))
```

```
##     Revenue Unit.sale.price Quantity  Revenue2
## 1 315044.33        5.195714    66385 344917.49
## 2  13444.68        6.190000     2172  13444.68
## 3 181120.24        5.488000    35696 195899.65
## 4  69608.15        5.040000    15205  76633.20
## 5  30940.35        3.950000     7833  30940.35
## 6  74321.18        5.585000    14328  80021.88
```

So `Revenue` and `Product.cost` should definitely not be included in the regression model, but `Unit.sale.price` and `Quantity` might.

Let's start with the simplest model:

```
# Simplest model
params = c("Planned.revenue")
model1 <- lm(as.formula(paste("Revenue", paste(params, sep = "",
                                               collapse = " + "),
                              sep = " ~ ")), data_200405)
coeftest(model1, vcov = vcovHC)
```

```
##
## t test of coefficients:
##
##                    Estimate  Std. Error t value  Pr(>|t|)
## (Intercept)     -3.2766e+03  3.5760e+02  -9.163 < 2.2e-16 ***
## Planned.revenue  9.6938e-01  2.7204e-03 356.333 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
new_data <- data.frame(data_200607[, params])
```

We'll use the RMSE to compare different models:

```
(RMSE <- sqrt(sum((model1_predictions[, 1] - data_200607$Revenue)^2) /
                dim(data_200607)[1]))
```

```
## [1] 19593.67
```

- **Is the change in the average revenue different from 95 cents when the planned revenue increases by $1?**

As shown below, the change in the average revenue is significantly different from \$0.95 when the revenue increases by \$1 (while the F statistic of the exact value, which is quite close to \$0.95, has a p value equal to 1):

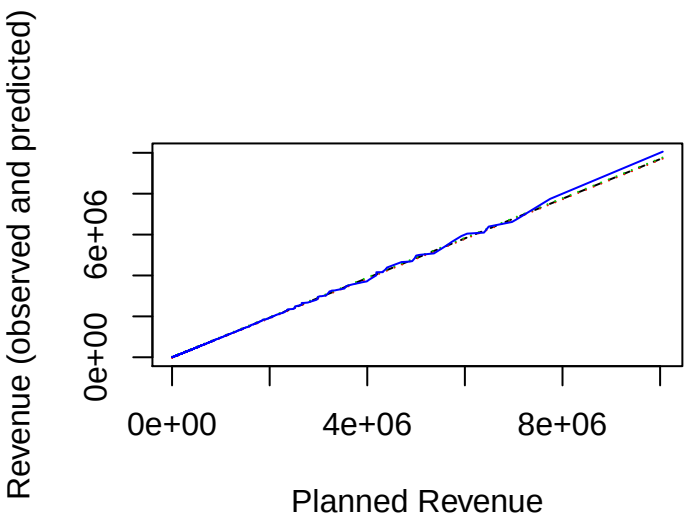


Figure 25: Planned Revenue vs. Revenue (observed and predicted) in 2006 and 2007

```
model1_full <- lm(as.formula(paste("Revenue", paste(params, sep = "",
                                                    collapse = " + "),
                                   sep = " ~ ")), data)
coeftest(model1_full, vcov = vcovHC)
```

```
##
## t test of coefficients:
##
##                    Estimate  Std. Error t value  Pr(>|t|)
## (Intercept)     -3.3970e+03  3.0377e+02 -11.183 < 2.2e-16 ***
## Planned.revenue  9.6981e-01  1.8151e-03 534.297 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
linearHypothesis(model1_full, "Planned.revenue = 0.95", vcov = vcovHC)
```

```
## Linear hypothesis test
##
## Hypothesis:
## Planned.revenue = 0.95
##
## Model 1: restricted model
## Model 2: Revenue ~ Planned.revenue
##
## Note: Coefficient covariance matrix supplied.
##
##   Res.Df Df      F    Pr(>F)
## 1  24742
## 2  24741  1 119.12 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

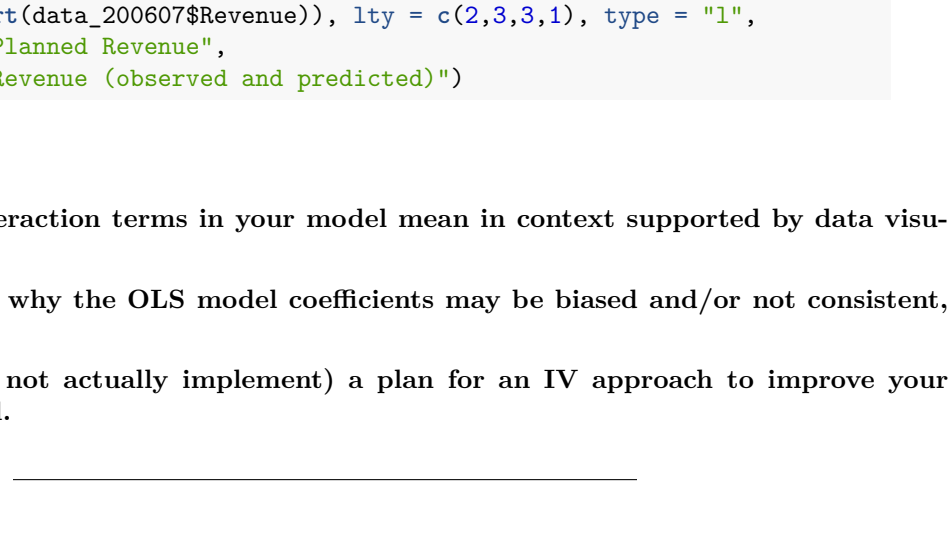
```
linearHypothesis(model1_full, paste("Planned.revenue =",
                                    coeftest(model1, vcov = vcovHC)[2, 1]),
                 vcov = vcovHC)
```

```
## Linear hypothesis test
##
## Hypothesis:
## Planned.revenue = 0.969384229459145
##
## Model 1: restricted model
## Model 2: Revenue ~ Planned.revenue
##
## Note: Coefficient covariance matrix supplied.
##
##   Res.Df Df      F Pr(>F)
## 1  24742
## 2  24741  1 0.0551 0.8145
```

```
params = c("Year", "Planned.revenue")
model2 <- lm(as.formula(paste("Revenue", paste(params, sep = "",
                                               collapse = " + "),
                              sep = " ~ ")), data_200405)
coeftest(model2, vcov = vcovHC)
```

```
##
## t test of coefficients:
##
##                     Estimate  Std. Error  t value  Pr(>|t|)
## (Intercept)     -3.6017e+03  3.6855e+02  -9.7728 < 2.2e-16 ***
## Year             6.3868e+02  2.4457e+02   2.6115  0.009025 **
## Planned.revenue  9.6935e-01  2.7230e-03 355.9794 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
model2_predictions <- predict(model2, data_200607[, params],
                              interval = "prediction")
matplot(data_200607[order(data_200607$Planned.revenue) , c("Planned.revenue")],
        cbind(model2_predictions[order(data_200607$Planned.revenue), ],
              sort(data_200607$Revenue)), lty = c(2,3,3,1), type = "l",
        xlab = "Planned Revenue",
        ylab = "Revenue (observed and predicted)")
```



```
## [1] 19649.12
```

- **Explain what interaction terms in your model mean in context supported by data visualizations.**
- **Give two reasons why the OLS model coefficients may be biased and/or not consistent, be specific.**
- **Propose (but do not actually implement) a plan for an IV approach to improve your forecasting model.**

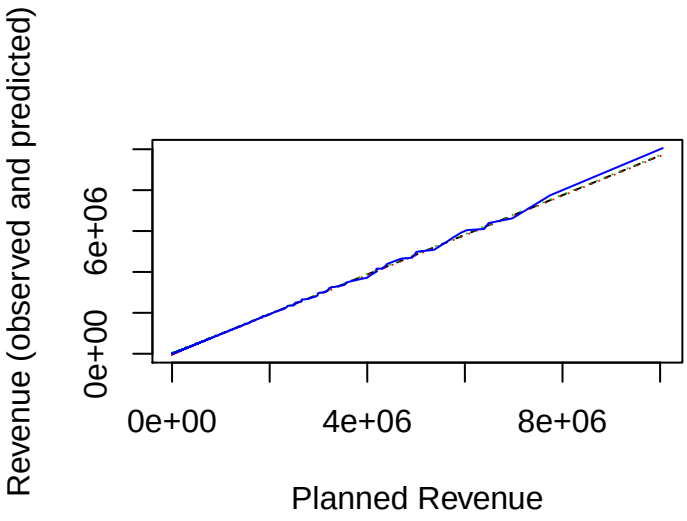


Figure 26: Planned Revenue vs. Revenue (observed and predicted) in 2006 and 2007