Notes on Problem 1

As we have done multiple times in the past, we write the Taylor series expansions for all f_i $i \in [-2, 2]$ and get:

$$f_{j-2} = f_j - 2h f'_j + 2h^2 f''_j - \frac{4h^3}{3} f'''_j + \frac{2h^4}{3} f^{iv}_j \dots$$

$$f_{j-1} = f_j - h f'_j + \frac{h^2}{2} f''_j - \frac{h^3}{6} f'''_j + \frac{h^4}{24} f^{vi}_j \dots$$

$$f_{j+1} = f_j + h f'_j + \frac{h^2}{2} f''_j + \frac{h^3}{6} f'''_j + \frac{h^4}{24} f^{iv}_j \dots$$

$$f_{j+2} = f_j + 2h f'_j + 2h^2 f''_j + \frac{4h^3}{3} + \frac{2h^4}{3} f^{iv}_j \dots$$

A linear combination of the above that eliminates f', f''' and f^{iv}

we construct a taylor table by multiplying equations above by α_i $i \in [-2, 2]$

	f_j	f'_j	f''_j	f'''_j	f^{iv}_j
$\alpha_{-2} f_{j-2}$	α_{-2}	$\alpha_{-2}(-2h)$	$\alpha_{-2}\ 2h^2$	$\alpha_{-2}\left(-\frac{4h^3}{3}\right)$	$\alpha_{-2}\ \frac{2h^4}{3}$
$\alpha_{-1} f_{j-1}$	α_{-1}	$\alpha_{-1}(-h)$	$\alpha_{-1}\ \frac{h^2}{2}$	$\alpha_{-1}\left(-\frac{h^3}{6}\right)$	$\alpha_{-1}\ \frac{h^4}{24}$
$\alpha_0 f_j$	α_0	$\alpha_0\ 0$	$\alpha_0\ 0$	$\alpha_0\ 0$	$\alpha_0\ 0$
$\alpha_1 f_{j+1}$	α_1	$\alpha_1\ h$	$\alpha_1\ \frac{h^2}{2}$	$\alpha_1\ \frac{h^3}{6}$	$\alpha_1\ \frac{h^4}{24}$
$\alpha_2 f_{j+2}$	α_2	$\alpha_2\ 2h$	$\alpha_2\ 2h^2$	$\alpha_2\ \frac{4h^3}{3}$	$\alpha_2\ \frac{2h^4}{3}$
	0	0	1	0	0

After using Matlab to solve the system of equations obtained previously (see below) we get

$$\alpha_{-2} = \frac{-1}{12h^2}$$

$$\alpha_{-1} = \frac{4}{3h^2}$$

$$\alpha_0 = \frac{-5}{2h^2}$$

$$\alpha_1 = \frac{4}{3h^2}$$

$$\alpha_2 = \frac{-1}{12h^3}$$

This can be rationalized using a common denominator $12h^2$ to get

$$\frac{-f_{j-2} + 16f_{j-1} - 30f_j + 16f_{j+1} - f_{j+2}}{12h^2} = f''_j$$

Problem 1

```
syms h

Taylor_Table = [
    1 1 1 1 1
    -2*h -h 0 h 2*h
    2*h^2 h^2/2 0 h^2/2 2*h^2
    -4/3*h^3 -h^3/6 0 h^3/6 4/3*h^3
    2*h^4/3 h^4/24 0 h^4/24 2*h^4/3
    ];

Sol = [0, 0, 1, 0, 0];

Alph = linsolve(Taylor_Table, Sol')
```

```
Alph =

 -1/(12*h^2)
   4/(3*h^2)
  -5/(2*h^2)
   4/(3*h^2)
 -1/(12*h^2)
```

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Notes on Problem 2

$$L_{ni} = \frac{(x-x_0)}{(x_i-x_0)}\frac{(x-x_1)}{(x_i-x_1)} \cdots \frac{(x-x_n)}{(x_i-x_n)} \quad \text{skipping} \quad \frac{(x-x_i)}{(x_i-x_i)}$$

$L_{ni}(x_i) = 1$ and $L_{ni}(x_j) = 0$ $\quad j \neq i, \; j < n$

$$H_i(x) = (1 - 2L'_{ni}(x_i)(x-x_i))\, L^2_{ni}(x)$$

<u>Propositon</u> : $H_i(x_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

<u>Proof</u> :

We know $L_{ni} = \prod_{\substack{i=0 \\ j\neq i}}^{n} \frac{(x-x_i)}{(x_j-x_i)} \Rightarrow \frac{L'_{in}(x)}{L_{in}(x)} = \sum_{\substack{i=0 \\ j\neq i}}^{n} \frac{1}{x_j - x_i}$

$\therefore$ $L'_{in}(x) = L_{in}(x) \sum_{\substack{i=0 \\ j\neq i}}^{n} \frac{1}{x_j-x_i} \Rightarrow L'_{in}(x_i) = 0$

Also note that $L^2_{ni}(x) = \prod_{\substack{i=0 \\ j\neq i}}^{n} \frac{(x-x_i)^2}{(x_j-x_i)^2}$ mantains the characteristic that $\begin{cases} L^2_{ni}(x_i) = 1 \\ L^2_{nj}(x_i) = 0 \quad i \neq j \end{cases} \Rightarrow L^2_{ni}(x_j) = \delta_{ij}$

$\therefore$ $H_i(x_j) = [1 - 2(0)(x_j - x_i)]\,\delta_{ij} = \delta_{ij}$ $\quad\square$

<u>Corollary</u> $\hat{H}_i(x_j) = 0$

<u>Proof</u> $L^2_{ni}(x_j) = \delta_{ij} \Rightarrow L^2_{ni}(x_j)(x_j - x_i) \Rightarrow \delta_{ij}(x_j - x_i) = \hat{H}_i(x_j)$

$\delta_{ij}(x_j - x_i) = 0$ since $\begin{cases} (x_j - x_i) = 0 & \text{when } j = i \\ \delta_{ij} = 0 & \text{when } j \neq i \end{cases}$ $\quad\square$

Proposition $H_i'(x_j) = 0$

Proof

$$H_i'(x) = [1 - 2L_{in}'(x_i)(x - x_i)]' L_{ni}^2(x)$$
$$+ [1 - 2L_{in}'(x_i)(x - x_i)] L_{ni}^2(x)'$$

- we know from first proof that the second term is zero

since $[1 - 2L_{in}'(x_i)(x_j - x_i)] = 1$ and $L_{ni}^2(x) = \delta_{ij}$

- we also know from proof one that $L_{ni}^2(x_j) = \delta_{ij}$

$$\therefore H_i'(x_j) = [1 - 2L_{in}'(x_i)(x - x_i)]' \delta_{ij} \, l_j$$

$$= -2(L_{in}'(x_i) + L_{in}''(x_i)\, x] \delta_{ij} \, l_j$$

Proof one shows $L_{in}'(x_i) = 0 \Rightarrow L_{in}''(x_i)\, x = 0$ $\square$

Corollary $\hat{H}_i'(x_j) = \delta_{ij}$

Proof

$$\hat{H}_i(x) = (x - x_i) L_{ni}^2(x)$$

$$\hat{H}_i'(x) = L_{ni}^2(x) + L_{ni}^2(x)'(x - x_i)$$

$$\hat{H}_i'(x_j) = \delta_{ij} + 0 \quad \square$$

In conclusion, our Polynomial defined as

$$P(x) = \sum_{0}^{n} H_i(x) f(x_i) + \sum_{0}^{\infty} \hat{H}_i(x) f'(x_i)$$

→ interpolates all $f(x_i)$ since $\hat{H}_i(x_j) = 0$ and

$H_i(x_j) = 1$ only when $i = j$

$$P(x_j) = \sum_{i=0}^{n} \delta_{ij} f(x_i) + 0 = 0 + 0 + \dots + f(x_j) + 0 + \dots$$

→ interpolates all $f'(x_j)$ since $H_i'(x_j) = 0$ and

$\hat{H}_i'(x_j) = 1$ only when $i = j$

$$P'(x_j) = 0 + \sum_{i=0}^{n} \delta_{ij} f'(x_i) = 0 + 0 + \dots + f'(x_j) + 0 + \dots$$

Also note that the degree of $P(x)$ is dictated by H and $\hat{H}$ which in term are dictated by the degree of the Lagrange basis polynomial (which is $\deg(L_{ni}) = n$)

$$H_i(x) = (1 - 2\underbrace{L'_{ni}(x_i)(x - x_i)}_{\text{degree } \leq 1})\underbrace{L^2_{ni}(x)}_{\text{degree } 2n} \quad \text{has degree } \leq 2n+1$$

$$\hat{H}_i(x) = \underbrace{(x - x_i)}_{\text{degree } 1}\underbrace{L^2_{ni}(x)}_{\text{degree } 2n} \quad \text{has degree } 2n+1$$

□

Notes on Problem 3

$$I = \int_a^b f(x)\,dx = \frac{b-a}{2}\left[f\left(\frac{b+a}{2} - \sqrt{\tfrac{1}{3}}\,\frac{b-a}{2} \right) + f\left(\frac{b+a}{2} + \sqrt{\tfrac{1}{3}}\,\frac{b-a}{2} \right) \right]$$

let $x = \frac{b-a}{2} t + \frac{b+a}{2}$ $\quad dx = \frac{b-a}{2} dt$

$t = 1 \Rightarrow x = \frac{b-a+b+a}{2} = b$

$t = -1 \Rightarrow x = \frac{a-b+b+a}{2} = a$

So the equality becomes

$$I = \int_{-1}^{1} f(t)\,\frac{(b-a)}{2}\,dt = f(-1/\sqrt{3}) + f(1/\sqrt{3}) \quad dx$$

Every polynomial f of degree 3 is a linear combination of $\{1, x, x^2, x^3\}$ and we wish to find w_1, w_2, x_1 and x_2 (hopefully $\pm\sqrt{1/3}$) which satisfy

$$\frac{(b-a)}{2}\int_{-1}^{1} 1 + x + x^2 + x^3 = w_1(1 + x + x^2 + x^3) + w_2(1 + x + x^2 + x^3)$$

Because the definite integral is linear, this yields 4 equations with 4 unknowns

$$w_1 + w_2 = \frac{b-a}{2}\int_{-1}^{1} dx = x \Big|_{-1}^{1} = 2$$

$$w_1 x_1 + w_2 x_2 = \frac{b-a}{2}\int_{-1}^{1} x\,dx = \frac{1}{2}x^2 \Big|_{-1}^{1} = 0$$

$$w_1 x_1^2 + w_2 x_2^2 = \frac{b-a}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{3}x^3 \Big|_{-1}^{1} = \frac{2}{3}$$

$$w_1 x_1^3 + w_2 x_2^2 = \frac{b-a}{2}\int_{-1}^{1} x^3\,dx = \frac{1}{4}x^4 \Big|_{-1}^{1} = 0$$

which has solutions (see below)

$$\begin{cases} w_1 = w_2 = \frac{b-a}{2} \\ x_1 = \sqrt{1/3} \\ x_2 = -\sqrt{1/3} \end{cases}$$

which are exactly the parameters we where looking for $\square$

→ To show this does not work for order higher than 3 polynomials, we can test for $f = x^4$ on $[-1, 1]$

$$\int_{-1}^{1} x^4\,dx = \frac{1}{5}x^5 \Big|_{-1}^{1} = 2/5 \neq$$

$$f(-\sqrt{1/3}) + f(\sqrt{1/3}) = \frac{2}{9} \quad \square$$

```
In[18]:= e1 = w1 + w2 - 2 ((b - a) / 2);
         e2 = w1 * x1 + w2 * x2;
         e3 = w1 * x1^2 + w2 * x2^2 - 2 / 3 ((b - a) / 2);
         e4 = w1 * x1^3 + w2 * x2^3;

In[22]:= Solve[{e1 == 0, e2 == 0, e3 == 0, e4 == 0}, {w1, w2, x1, x2}]
```

Out[22]=

$$\left\{\left\{w1 \to \frac{1}{2}\left(-a+b\right),\ w2 \to \frac{1}{2}\left(-a+b\right),\ x1 \to \frac{1}{\sqrt{3}},\ x2 \to -\frac{1}{\sqrt{3}}\right\},\right.$$
$$\left.\left\{w1 \to \frac{1}{2}\left(-a+b\right),\ w2 \to \frac{1}{2}\left(-a+b\right),\ x1 \to -\frac{1}{\sqrt{3}},\ x2 \to \frac{1}{\sqrt{3}}\right\}\right\}$$

Notes on Problem 4

Problem has the form
$$\begin{cases} f(x,y,z)=0 \\ g(x,y,z)=0 \\ h(x,y,z)=0 \end{cases}$$

$$[J^{(0)}]\,\delta^{(1)} = -\phi^{(0)}$$

$$\begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix}_0 \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - z_0 \end{bmatrix} = -\begin{bmatrix} f \\ g \\ h \end{bmatrix}_0$$

In general for step k

$$[J^{(k)}]\,\delta^{(k+1)} = -\phi^{(k)} \quad \text{and} \quad x_j^{(k+1)} = x_j^{(k)} + \delta_j^{(k+1)}$$

solve for δ

use δ to find next points.

test $\phi^{(k)}$ for convergence $|\phi^{(k)}| < \varepsilon$

The Jacobian in this case is

$$J = \begin{bmatrix} 16x & 2y & -1 \\ 2 & 6y & 1 \\ 8x-4 & 18y & 2z \end{bmatrix}$$

We can observe in the matlab script below that the solution converges after 50 steps to $[x=0 \quad y=0 \quad z=1]$

In[48]:=
```
a = 8 * x^2 + y^2 - z + 1 == 0;
b = 2 * x + 3 * y^2 + z - 1 == 0;
c = 4 * x^2 + 9 * y^2 + z^2 - 4 * x - 1 == 0;
```

In[42]:=
```
J = {{D[a, x], D[a, y], D[a, z]},
    {D[b, x], D[b, y], D[b, z]},
    {D[c, x], D[c, y], D[c, z]}} // MatrixForm
```

Out[42]//MatrixForm=

$$\begin{pmatrix} (16x) & (2y) & (-1) \\ (2) & (6y) & (1) \\ (-4+8x) & (18y) & (2z) \end{pmatrix}$$

In[53]:=
```
Solve[{a, b, c}, {x, y, z}][[1]]
```

Out[53]= $\{x \to 0, y \to 0, z \to 1\}$

Problem 4

```
clear
x = [1;1;1]; % Initial guess
residual = 1;
k = 0;
fprintf('  0   x           y           z           abs(f) \n')
fprintf(' %2i %10.6f %10.6f %10.6f %10.6f \n',k,x,residual);
while residual > 1e-6
f = [8*x(1)*x(1)+x(2)*x(2)-x(3)+1; % compute the f system
    2*x(1)+3*x(2)*x(2)+x(3)-1;
    4*x(1)*x(1)+9*x(2)*x(2)+x(3)*x(3)-4*x(1)-1];


residual = norm(f); % compute the residual
J = [16*x(1) 2*x(2) -1; % compute the Jacobian
    2 6*x(2) 1;
    8*x(1) 18*x(2) 2*x(3)];
x = x-(J\f); % solve system and update
k = k+1;
fprintf(' %2i %10.6f %10.6f %10.6f %10.6f\n',k,x,residual)
end
```

```
  0   x           y           z           abs(f)
  0   1.000000    1.000000    1.000000    1.000000
  1  -0.210526    1.973684   -7.421053   13.674794
  2   1.414431    1.131882   -3.546442   91.080829
  3   0.539444    0.866098   -2.117338   33.600632
  4  -0.009084    0.782975   -0.800256   11.127212
  5  -0.979117    0.681502    1.595790    5.728068
  6  -0.084993    1.534414   -3.710917   15.442516
  7   2.339432    0.649921   -2.599072   35.128667
  8   1.054710    0.747195   -2.755935   52.714617
  9   0.336874    0.798635   -1.579263   17.749298
 10  -0.169602    0.737587   -0.281719    7.564519
 11   0.276902    0.441280    0.125402    5.193402
 12  -0.079207    0.538673    0.316364    1.703527
 13  -0.027037    0.285580    1.001574    2.294850
 14   0.020316    0.131412    0.978865    0.873946
 15  -0.014502    0.100919    1.001239    0.089323
 16   0.010732    0.029041    0.991505    0.153382
 17  -0.007723    0.090677    1.002176    0.055030
 18   0.005666    0.031339    0.996284    0.110272
 19  -0.004097    0.050859    1.001577    0.024125
 20   0.002993    0.011528    0.998255    0.042923
 21  -0.002170    0.054728    1.000953    0.015091
 22   0.001582    0.020349    0.999140    0.038035
 23  -0.001149    0.024588    1.000537    0.005730
 24   0.000836    0.003921    0.999562    0.011116
 25  -0.000608    0.041145    1.000294    0.004300
 26   0.000442    0.017907    0.999773    0.018775
 27  -0.000322    0.013469    1.000158    0.001835
 28   0.000234    0.002412    0.999882    0.003236
```

```
  29  -0.000170   0.018891   1.000084   0.001185
  30   0.000124   0.007812   0.999938   0.004151
  31  -0.000090   0.006790   1.000045   0.000395
  32   0.000065   0.000989   0.999967   0.000864
  33  -0.000048   0.012538   1.000024   0.000336
  34   0.000035   0.005579   0.999983   0.001706
  35  -0.000025   0.003918   1.000013   0.000187
  36   0.000018   0.000791   0.999991   0.000264
  37  -0.000013   0.004602   1.000007   0.000091
  38   0.000010   0.001775   0.999995   0.000261
  39  -0.000007   0.001879   1.000004   0.000032
  40   0.000005   0.000259   0.999997   0.000067
  41  -0.000004   0.003731   1.000002   0.000026
  42   0.000003   0.001684   0.999999   0.000149
  43  -0.000002   0.001134   1.000001   0.000018
  44   0.000001   0.000252   0.999999   0.000021
  45  -0.000001   0.001161   1.000001   0.000007
  46   0.000001   0.000418   1.000000   0.000018
  47  -0.000001   0.000539   1.000000   0.000003
  48   0.000000   0.000083   1.000000   0.000005
  49  -0.000000   0.000916   1.000000   0.000002
  50   0.000000   0.000400   1.000000   0.000009
  51  -0.000000   0.000296   1.000000   0.000001
```

Notes on Problem 5

→ The minimization problem becomes finding the set $\{x, y\}$ where all first partial derivatives are 0

$$f(x,y) = 2x^2 - 2x + 4y^2 - 6y + 2xy + 7$$

$$\frac{\partial f}{\partial x} = 4x + 2y - 2 = 0 \quad (a)$$

$$\frac{\partial f}{\partial y} = 8y + 2x - 6 = 0 \quad (b)$$

$\frac{1}{2}(a) - (b)$ gives $\quad -7y + 5 = 0 \Rightarrow y = 5/7, \quad x = 1/7$

→ Now do this by finding Hessian

$$x^{(k+1)} = x^{(k)} - (H^{-1})^{(k)} \nabla f^{(k)}$$

cofactors

$$H = \begin{bmatrix} 4 & 2 \\ 2 & 8 \end{bmatrix} \qquad H^{-1} = \frac{1}{28} \begin{bmatrix} 8 & -2 \\ -2 & 4 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$

1/det

$$\therefore H^{-1} \nabla f = \frac{1}{14} \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4x + 2y - 2 \\ 8y + 2x - 6 \end{bmatrix} =$$

$$= \frac{1}{14} \begin{bmatrix} 16x + 8y - 8 - 8y - 2x + 6 \\ -4x - 2y + 2 + 16y + 4x - 12 \end{bmatrix} =$$

$$= \frac{1}{14} \begin{bmatrix} 14x - 2 \\ 14y - 10 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 7x - 1 \\ 7y - 5 \end{bmatrix}$$

$$\therefore \quad [H^{-1}]^{(0)} \nabla f^{(0)} = \frac{1}{7}\begin{bmatrix} 7x_0 - 1 \\ 7y_0 - 5 \end{bmatrix}$$

$$x^{(0)} - [H^{-1}]^{(0)} \nabla f^{(0)} = \frac{1}{7}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{matrix} 1/7 \\ 5/7 \end{matrix} = x^{(1)}$$

we now test for convergence $|\nabla f^{(1)}| < \varepsilon$

$$\nabla f^{(1)} = \begin{bmatrix} 4/7 + 10/7 - 2 \\ 40/7 + 2/7 - 6 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 4 + 10 - 14 \\ 40 + 2 - 42 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$|\nabla f^{(1)}| = 0 < \varepsilon \quad \forall \varepsilon \in \mathbb{R} \quad \square$$

Problem 5

```
clear

x(1) = 1; %This could be anything as shown above
x(2) = -2;

tol = 1.e-6;
dim = length(x);
[f,df,ddf] = func(x); % calculate f,gradien(f),hessian(f)
iter = 0;

fprintf(' iter   x          y           f         \n')
format = '%5i %10.6f %10.6f %10.6f\n';

fprintf(format,iter,x,f);

while norm(df) > tol && iter <= 5000
    x = x-df/ddf;
    [f,df,ddf] = func(x);
    iter = iter+1;
    fprintf(format,iter,x,f);
end


function [phi,dphi,hphi] = func(x)
phi = 2*x(1)^2-2*x(1)+4*x(2)^2-6*x(2)+2*x(1)*x(2);
% gradient
dphi(1) = 4*x(1)+2*x(2)-2;
dphi(2) = 8*x(2)+2*x(1)-6;
% Hessian
hphi(1,1) = 4;
hphi(1,2) = 2;
hphi(2,1) = 2;
hphi(2,2) = 8;
end
```

```
 iter   x          y           f
    0   1.000000  -2.000000  24.000000
    1   0.142857   0.714286  -2.285714
```

Notes on Problem 6

Consider the boundary value problem

$$\begin{cases} (x-1)y'' - xy' + y + (1-x)^2 = 0 & \\ y(0)=0 \quad , \quad y(2) + \frac{1}{2}y'(2) + \frac{1}{4} = 0 & x \in [0,2] \end{cases}$$

First discretize the ODE with finite dif representation of the derivative, y'' and y' as follows

$$(x_j - 1)\left[\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}\right] - x_j\left[\frac{y_{j+1} - y_{j-1}}{2h}\right] + y_j + (1-x_j)^2 = 0$$

using $N = 41$ node points over the interval $[x_L, x_R] = [0,2]$

valid for $j = 2, 3 \dots, 40$, $h = \frac{2}{40} = 0.05$

. we know $x_j = (j-1)h$, by multiplying by h^2 get.

$$[(j-1)h - 1](y_{j+1} - 2y_j + y_{j-1}) - \frac{1}{2}(j-1)h\left[(y_{j+1} - y_{j-1})h\right] +$$
$$+ \; h^2\left[y_j + [1-(j-1)h]^2\right] = 0$$

Now group the y_{j-1} , y_j and y_{j+1} terms

$$\begin{cases} y_{j+1}\left[(j-1)h - 1 - \frac{1}{2}(j-1)h^2\right] + \\ y_j\left[-2(j-1)h + 2 + h^2\right] + \\ y_{j-1}\left[(j-1)h - 1 + \frac{1}{2}(j-1)h^2\right] = \\ = -h^2\left[1-(j-1)h\right]^2 \end{cases}$$

applicable to all interior points

We now look at the boundary conditions $y(0)=0$

$\therefore$ when $j=2 \Rightarrow y_{j-1}=0$ and we get the condition

$$y_3\left[h-1-\frac{1}{2}h^2\right] + y_2\left[-2h+2+h^2\right] = -h^2(1-h)^2$$

The condition at the right boundary is given by the expression $\frac{1}{2}y'(2) + y(2) = -\frac{1}{4}$, obtained at $j=41$

$$y_j + \frac{1}{2}y_j' = -\frac{1}{4} \qquad \text{where} \quad y_j' = \frac{y_{j+1}-y_{j-1}}{2h}$$

by the central diff formula. Where y_{j+1} is a phantom node since its outside the grid

$$\therefore \quad y_{j+1} = 2h\,y_j' + y_{j-1} \qquad \text{at } j=41=N$$

This way we can introduce y_j' into the original equation and hopefully find the boundary condition reflected.

$$2h\,y_j'\left[40h-1-20h^2\right] + y_j\left[-80h+2+h^2\right] + y_{j-1}\left[40h-1-\cancel{20h^2}+40h-1+\cancel{20h^2}\right] = -h^2(1-40h)^2$$

$$\leftarrow [80h-2]$$

We can now use the boundary condition to write y_j' in terms of y_j as follows: $y_j' = -\frac{1}{2} - 2y_j$

$$y_j \left[-4h\,(40h - 1 - 20h^2) + (-80h + 2 + h^2) \right] +$$

$$+\ y_{j-1} \left[80h - 2 \right] = \left[-h^2 (1 - 40h)^2 + h\,(40h - 1 - 20h^2) \right]$$

We can now write the vectors $\{a\}$, $\{b\}$, $\{c\}$ and $\{f\}$

$c_j = \left[(j-1)\left(h - \frac{1}{2}h^2\right) - 1 \right]$ $\quad j = 2, 3, \ldots 40$

$b_j = \left[h^2 - 2(j-1)\,h + 2 \right]$ $\quad j = 2, 3, \ldots, 41$

$a_j = \left[(j-1)\left(h + \frac{h^2}{2}\right) - 1 \right]$ $\quad j = 2, 3, \ldots, 40$

$f_j = -h^2 \left[1 - (j-1)h \right]^2$ $\quad j = 3, 4, \ldots, 40$

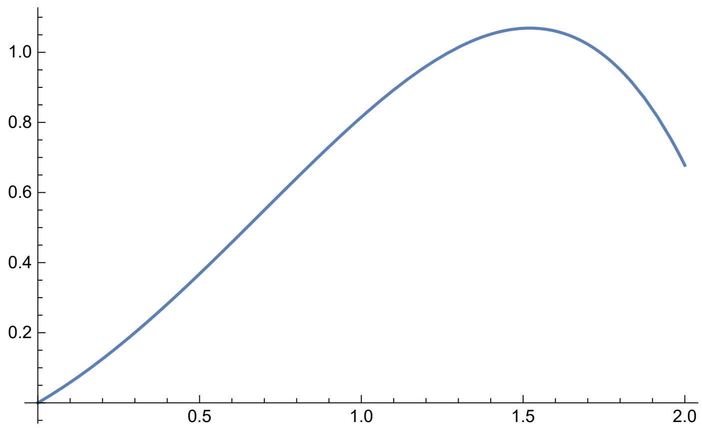
$f_2 = -h^2 (1-h)^2$

$f_{40} = \left[-h^2 (1 - 40h)^2 +, h\,(40h - 1 - 20h^2) \right]$

```
sol = DSolve[
  {(x - 1) * y''[x] - x * y'[x] + y[x] == -(1 - x)^2, y[0] == 0, y[2] + 1/2 y'[2] == -1/4}, y, {x, 0, 2}]
```

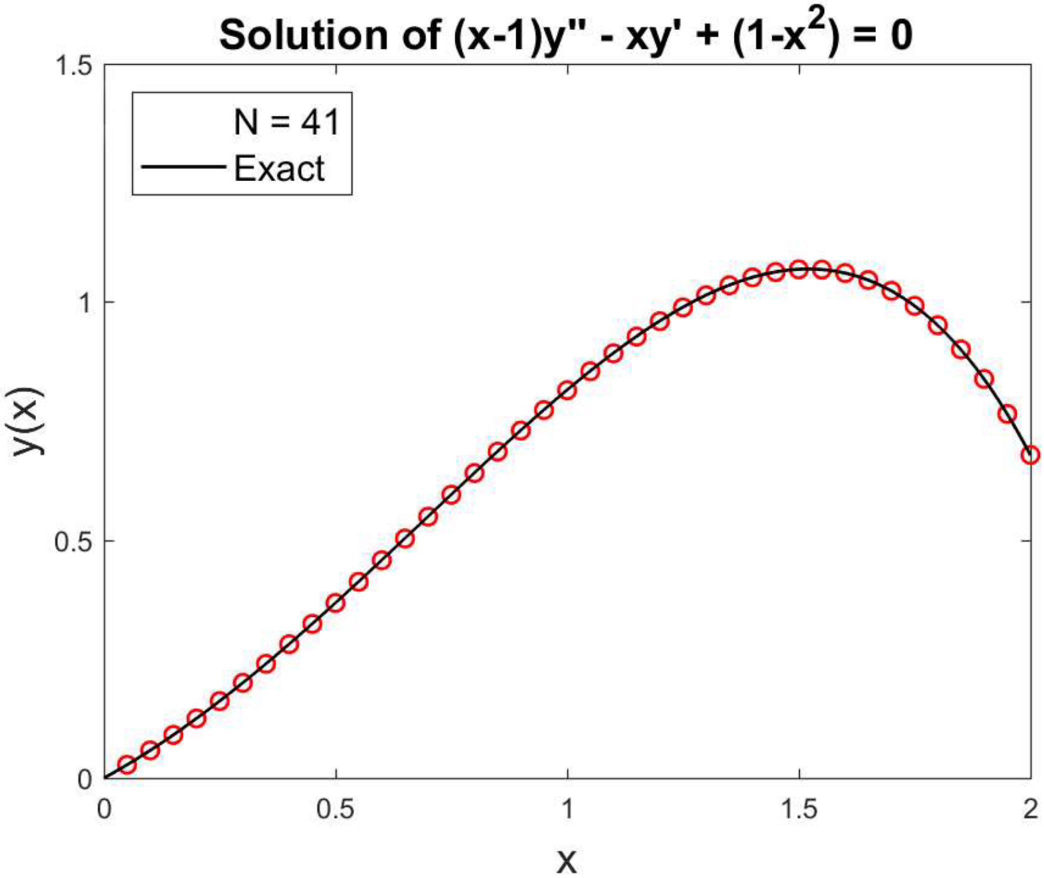
$$\left\{\left\{y \to \text{Function}\left[\{x\},\ 1 - e^{x} - \frac{29\,x}{10} + \frac{3\,e^{2}\,x}{5} + x^{2}\right]\right\}\right\}$$

```
Plot[First[y[x] /. sol], {x, 0, 2}]
```



Problem 6

```
close all

N = 41;

xL = 0; % left boundary
xR = 2; % right boundary
h = (xR-xL)/(N-1); % grid spacing
x = linspace(xL,xR,N); % x(j) for plots

a(1:N-2) = x(2:N-1) - 1 + 1/2.*x(2:N-1).*h;
a(N-1) = 80*h-2;
a(1) = 0; %not used

b(1:N-2) = - 2*x(2:N-1) + 2 + h^2;
b(N-1) = -4*h*(40*h-1-20*h^2)+(-80*h+2+h^2);

c(1:N-2) = x(2:N-1) - 1 - 1/2.*x(2:N-1).*h;
c(N-1) = 0; %not used

f(1) = -h^2*(1-h)^2;
f(2:N-2) = -h^2*(1-x(3:N-1)).^2;
f(N-1) = -h^2*(1-40*h)^2 + h*(40*h-1-20*h^2);

y = tridiag(N-1, a, b, c, f);
y = [yL y yR];


x1 = linspace(xL,xR,1025);
yexact = 1-exp(x1)-29*x1./10 + 3.*exp(2)*x1./5 + x1.^2;
plot(x(1:N),y(1:N),'or',x1,yexact,'k','linewidth',1.15)
%ylim([-0.2 1.6])
h_leg = legend(['N = ',num2str(N)],'Exact','Location','NorthWest');
set(h_leg,'fontsize',13)
xlabel('x','fontsize',15)
ylabel('y(x)','fontsize',15)
title('Solution of (x-1)y'''' - xy'' + (1-x^2) = 0','fontsize',15)
text(0.35,3.75,'Dirichlet BC, y N = 1.43','fontsize',13)

function x = tridiag(n,a,b,c,f)
    for j = 2:n
        b(j) = b(j)-a(j)/b(j-1)*c(j-1);
        f(j) = f(j)-a(j)/b(j-1)*f(j-1);
    end

    x(n) = f(n)/b(n);

    for j=n-1:-1:1
        x(j) = (f(j)-c(j)*x(j+1))/b(j);
    end
end
```


Solution of (x-1)y'' - xy' + (1-x^2) = 0
N = 41
Exact
y(x)
x
0
0.5
1
1.5
2

Notes on Problem 7

$$\nabla^2 \phi = 0$$

$$\frac{\partial \phi}{\partial x}(x,0) = \frac{\partial \phi}{\partial x}(x,1) = 0$$

$$\phi(0,y) = 0 \qquad \phi(2,y) = y$$

The discretized Laplace equation at (i,j) using second order finite difference approximation of the 2nd derivs

$$\frac{\phi_{i+1\,j} - 2\phi_{ij} + \phi_{i-1\,j}}{\Delta^2} + \frac{\phi_{i\,j+1} - 2\phi_{ij} + \phi_{i\,j-1}}{\Delta^2} = 0$$

$$\therefore \frac{1}{4}\left[\phi_{i-1\,j} + \phi_{i+1\,j} + \phi_{i\,j+1} + \phi_{i\,j-1}\right] = \phi_{ij}$$

valid for all interior points, the residual is therefore

$$d_{ij}^{(k+1)} = \frac{1}{4}\left[\phi_{i-1\,j}^{(k)} + \phi_{i+1\,j}^{(k)} + \phi_{i\,j-1}^{(k)} + \phi_{i\,j+1}^{(k)}\right] - \phi_{ij}^{(k)}$$

use this equation to calculate all points except for boundary for which we use

$$\left.\begin{aligned} \frac{\phi_{1j} - \phi_{2j}}{2} = 0 \quad &\Rightarrow \quad \phi_{1j} = \phi_{2j} \\ \frac{\phi_{mj} - \phi_{m-1\,j}}{2} = 0 \quad &\Rightarrow \quad \phi_{mj} = \phi_{m-1\,j} \end{aligned}\right\} \text{ insulated at } i=1 \text{ and } i=m$$

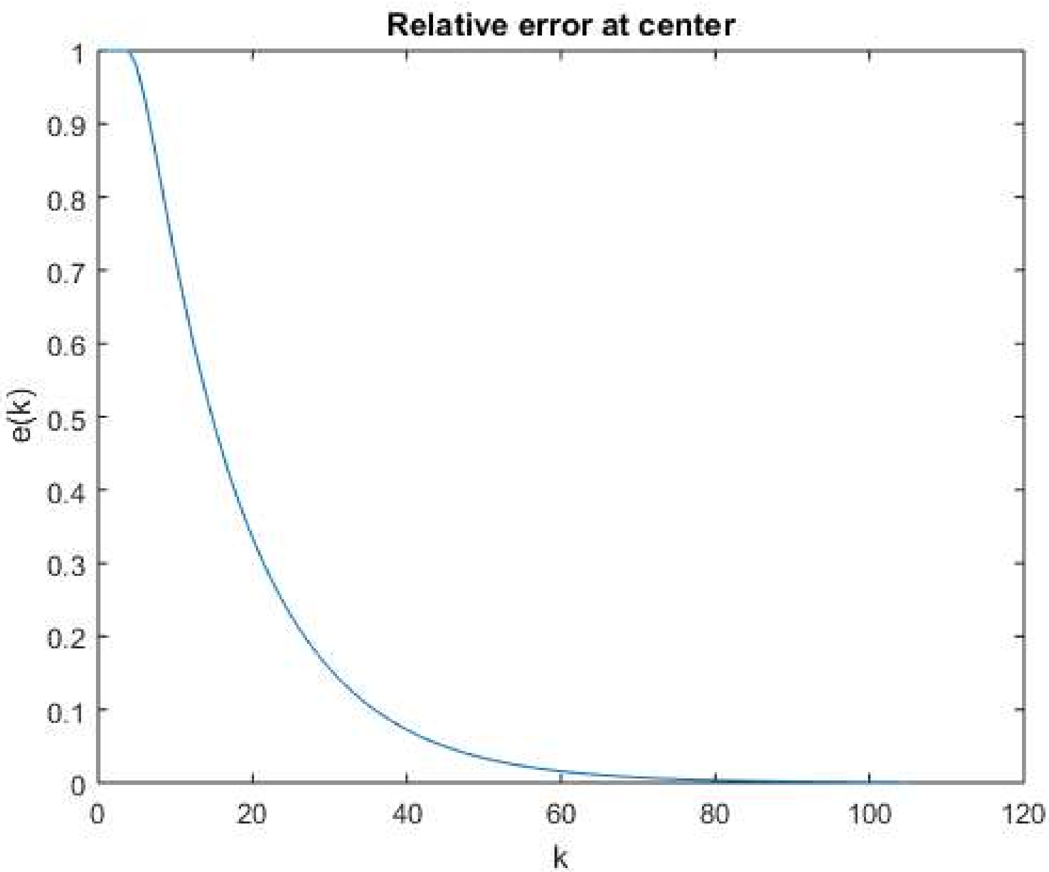
$$\left.\phi_{i1} = 0, \quad \phi_{in} = i \right\} \text{ left side kept at } 0° \text{ and right side at } y°$$

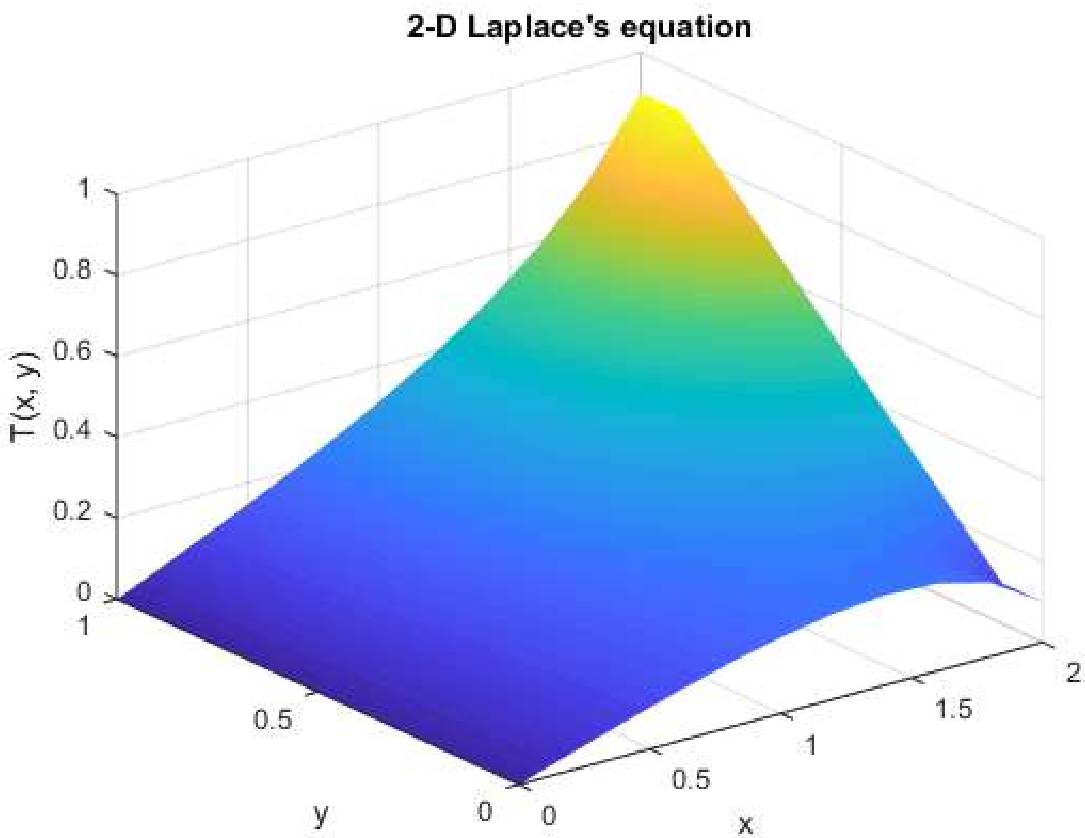
Problem 7

```
clear

N=11;
M=11;
dx=2/(N-1);
dy=1/(M-1);
x = 0:dx:2;
y = 0:dy:1;

tol = 1e-5;

p = zeros(M,N);
res = zeros(M,N);


p(:,1)=0;                         %left side at 0
p(:,N)=y;                         %Right side at y
p(1,:)=p(2,:);                    %Derivative at upper boundary is 0
p(M,:)=p(M-1,:);                  %Derivative at lower boundary is 0

j=2:N-1;
i=2:M-1;

error = 10;
iter_count = 0;
sor = 1.25;

while error > tol
    error = 0;

    for j = 2:N-1
        for i = 2:M-1
            res(i,j) = (p(i-1,j)+p(i+1,j)+p(i,j+1)+p(i,j-1))/4-p(i,j);
            p(i,j) = p(i,j) + sor*res(i,j);
            error = max(error, abs(res(i,j)));
        end
    end

    % Reinforce the boundary contitions
    p(:,1)=0;
    p(:,N)=y;
    p(1,:)=p(2,:);
    p(M,:)=p(M-1,:);

    iter_count = iter_count+1;
    ek(iter_count) = abs((0.25-p(6,6))/0.25);
end

fprintf('SOR = %1.2f gives %i iterations\n',sor,iter_count);
plot(ek)
title('Relative error at center')
ylabel('e(k)')
```

```
xlabel('k')
figure

surf(x,y,p,'EdgeColor','none');
shading interp
title('2-D Laplace''s equation')
xlabel('x')
ylabel('y')
zlabel('T(x, y)')
```

```
SOR = 1.25 gives 105 iterations
```



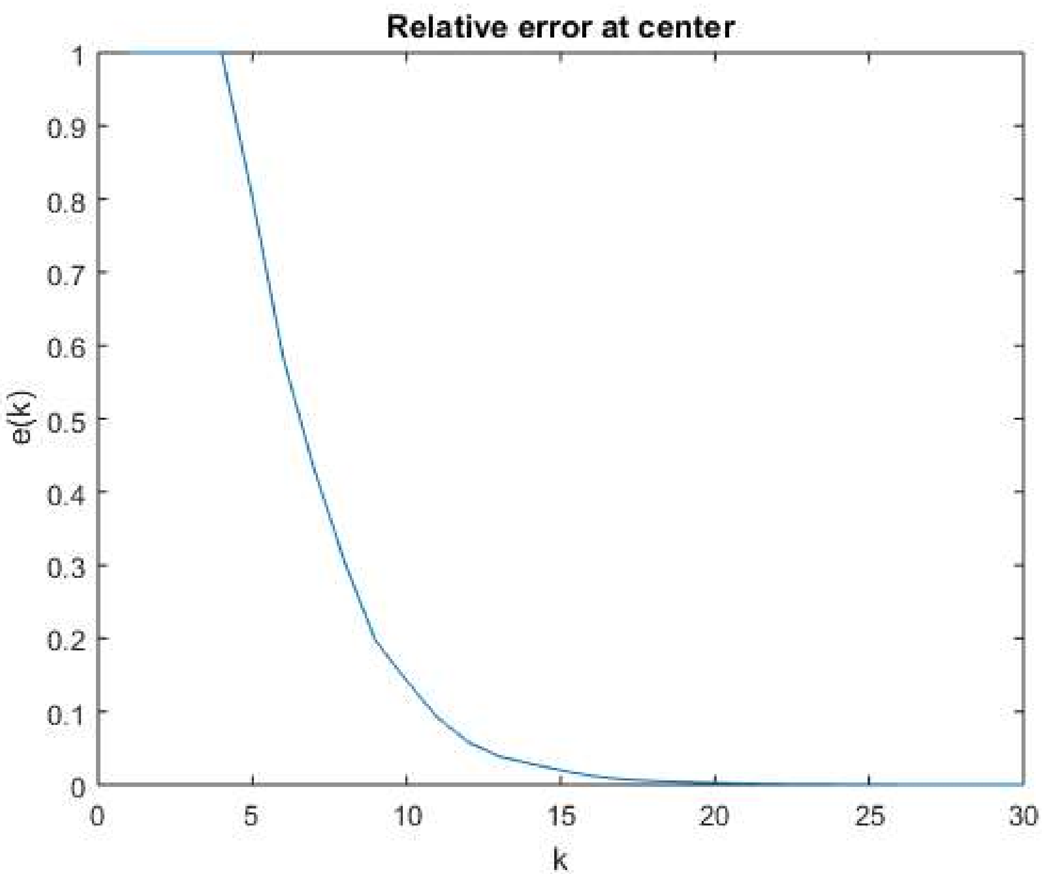

2-D Laplace's equation
T(x, y)
1
0.8
0.6
0.4
0.2
0
1
0.5
0
y
0
0.5
1
1.5
2
x

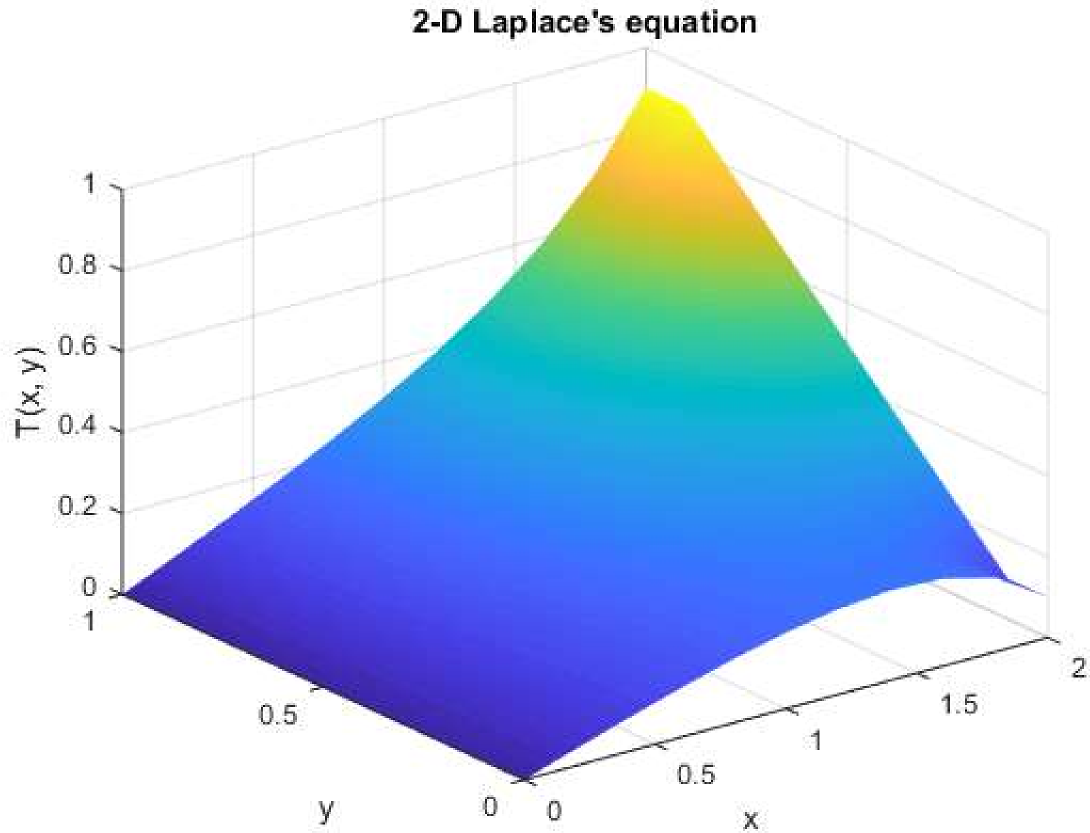
Problem 7

```
clear

N=11;
M=11;
dx=2/(N-1);
dy=1/(M-1);
x = 0:dx:2;
y = 0:dy:1;

tol = 1e-5;

p = zeros(M,N);
res = zeros(M,N);


p(:,1)=0;                        % Left side at 0
p(:,N)=y;                        % Right side at y
p(1,:)=p(2,:);                   % Derivative at upper boundary is 0
p(M,:)=p(M-1,:);                 % Derivative at lower boundary is 0

j=2:N-1;
i=2:M-1;

error = 10;
iter_count = 0;
sor = 1.75;

while error > tol
    error = 0;

    for j = 2:N-1
        for i = 2:M-1
            res(i,j) = (p(i-1,j)+p(i+1,j)+p(i,j+1)+p(i,j-1))/4-p(i,j);
            p(i,j) = p(i,j) + sor*res(i,j);
            error = max(error, abs(res(i,j)));
        end
    end

    % Reinforce the boundary contitions
    p(:,1)=0;
    p(:,N)=y;
    p(1,:)=p(2,:);
    p(M,:)=p(M-1,:);

    iter_count = iter_count+1;
    ek(iter_count) = abs((0.25-p(6,6))/0.25);
end

fprintf('SOR = %1.2f gives %i iterations\n',sor,iter_count);
plot(ek)
title('Relative error at center')
ylabel('e(k)')
```

```
xlabel('k')
figure

surf(x,y,p,'EdgeColor','none');
shading interp
title('2-D Laplace''s equation')
xlabel('x')
ylabel('y')
zlabel('T(x, y)')
```

```
SOR = 1.75 gives 30 iterations
```




2-D Laplace's equation
T(x, y)
1
0.8
0.6
0.4
0.2
0
1
0.5
0
y
0
0.5
1
1.5
2
x