CME 108/MATH 114
Introduction to Scientific Computing
Summer 2017

Problem Set 2

Due: Thursday, July 13 at 9:30 AM

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1. (10 points) Consider the following data

t	1	2	3	4
y	11	29	65	125

Construct a third-order interpolating polynomial of the form

$$P(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Use the interpolating points to form a matrix equation $Ax = b$ where x is the unknown vector $[a_0; a_1, a_2; a_3]$. Use Matlab `x = A\b` to solve for x. Interpolate for the value of $P(t = 3.3)$.

2. (25 points) Consider a form of the Runge's function below (not exactly the same as what was presented in class),

$$f(x) = \frac{1}{1 + x^2}.$$

on the interval $x \in [-5, 5]$. Do the following and plot your answers

(a) Use 11 equally spaced points and find the interpolating Lagrange polynomial. Plot the graph of the polynomial along with the original function. What do you observe?

(b) Now let's use unequally spaced points. Consider the points x_j given by

$$x_j = -5 \cos\left(\frac{j\pi}{n}\right), \quad j = 0, 1, 2, \ldots, n$$

With 11 points ($n = 10$), construct the Lagrange interpolating polynomial and plot. How does this interpolating polynomial behave compared to the one in the previous step?

(c) Now construct a cubic spline that passes through 11 equally spaced points in $x \in [-5, 5]$ and compare the result to the two polynomials you obtained earlier. Use Matlab `spline` for this part.

3. (25 points) (Problem from "Fundamentals of Engineering Numerical Analysis") The concentration of a certain toxin in a system of lakes downwind of an industrial area has been monitored very accurately at intervals from 1978 to 1992 as shown in the table below. It is believed that the concentration has varied smoothly between these data points.

Year	Toxin Concentration
1978	12.0
1980	12.7
1982	13.0
1984	15.2
1986	18.2
1988	19.8
1990	24.1
1992	28.1
1994	???

   (a) Interpolate the data with the Lagrange polynomial. Plot the polynomial and the data points. Use the polynomial to predict the condition of the lakes in 1994 (extrapolation). Discuss this prediction.

   (b) Use Matlab `spline` to find the toxin concentration in 1994 and compare with the Lagrange extrapolation.

   (c) Interpolation may also be used to fill holes in the data. Say the data from 1982 and 1984 disappeared. Predict these values using the Lagrange polynomial and Matlab `spline` fitted through the other known data points.

4. (25 points) In this problem you will explore a construction of a quadratic spline. The concept and development process is similar to that of a cubic spline, except the interpolating polynomial is piecewise quadratic (parabola). This means,

$$
\begin{aligned}
S_j'(x) &= \text{linear}, \quad j = 0, 1, \ldots, n-1 \\
S_j(x) &= \text{quadratic}, \quad j = 0, 1, \ldots, n-1
\end{aligned}
$$

The quadratic spline is of the form:

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2, \quad x_j \le x \le x_{j+1} \tag{1}$$

and $h_j = x_{j+1} - x_j$

Below are two additional conditions imposed on each spline:

$$S_j(x_j) = y_j \quad \text{must pass thru interpolating point}, x_j \tag{2}$$
$$S_j(x_{j+1}) = S_{j+1}(x_{j+1}) \quad \text{continuity of } S \tag{3}$$
$$S_j'(x_{j+1}) = S_{j+1}'(x_{j+1}) \quad \text{continuity of slope } S' \tag{4}$$

Follow the same procedure for cubic spline presented in class to construct the quadratic spline. Note the following.

(a) Combine Eq. (1) with the 3 conditions (2) through (4) to establish an equation for $b_j, j = 1, 2, 3, \ldots, n-1$. *Hint:* you will get a recursive relationship for b_j and not a linear tridiagonal system as in equation for c_j in cubic spline.

(b) Recall that, for the cubic spline, you need to select the end-conditions for $S_j''(x_0)$ and $S_j''(x_n)$ (e.g., free run-out, parabolic run-out, ...). Thus for the quadratic spline you also need to impose end-condition(s) for S'. How many end-conditions are needed? Propose your end-condition(s) for the problem and explain your choice (there is no unique answer for this).

5. (15 points) This problem is to be fully hand calculated (not Matlab). Make sure you show all you work, don't just guess.

   (a) Perform Gauss elimination on the following linear system

$$\begin{aligned} 2x_1 - 3x_2 + 3x_3 &= 2, \\ 3x_1 + 3x_2 + 9x_3 &= 15, \\ 3x_1 + 3x_2 + 5x_3 &= 11. \end{aligned}$$

   (b) Compute the LU factorization of the matrix in part (a) from your hand calculation. Then compare with the L and U matrices given by the Matlab command: `[L,U,P] = lu(A)`. Explain the difference.