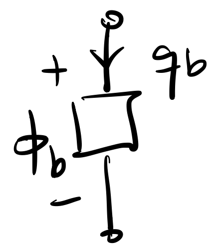
Fundamentals of circuit Quantisation.

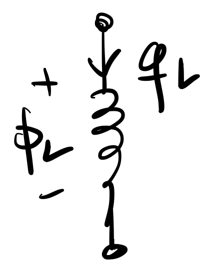
Lump element:



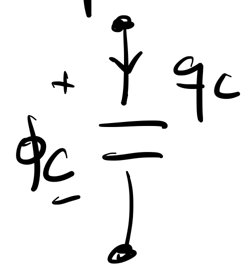
$$\phi_b = \int_0^t dz \, v(z)$$

$$q_b = \int_0^t dz \, i(z)$$

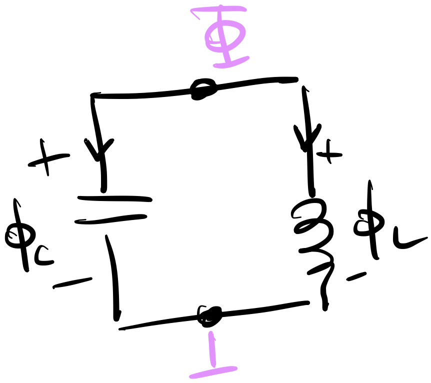
inductors



capacitors.



The node-flux method:



Only solve the KVL!

$$d\phi_C = d\phi_L = d\Phi$$

$$\ast = \frac{C\dot{\phi}_C^2}{2} - \frac{\phi_L^2}{2L}$$

(Integrate KVL constraints)

$$\phi_C = \Phi$$

$$\phi_L = \Phi + k$$

constant.

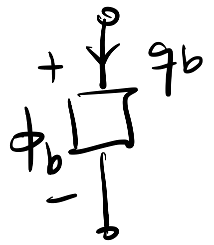
$$L = \frac{C\dot{\Phi}^2}{2} - \frac{(\Phi + k)^2}{2L}$$

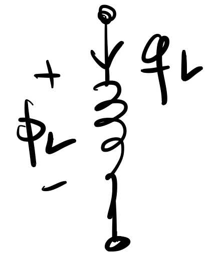
$$Q = C\dot{\Phi} \qquad \{\phi, Q\} = 1$$

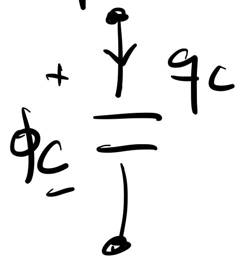
$$H = \frac{Q^2}{2C} + \frac{(\Phi + k)^2}{2L}$$

if constant, can be gauged away.

Lump element: inductors capacitors.







$\boxed{\text{Contributions to energy functions:}}$ $\frac{1}{2}\phi_L \dot{q}_L - \frac{\phi_L^2}{2L}$ $\frac{1}{2} q_C \dot{\phi}_C - \frac{q_C^2}{2C}$

Kirchhoff's laws:

(Constraints)

$$\begin{cases} KVL: & F_{loop}\, d\vec{\phi} = 0 \\ KCL: & F_{cutset}\, d\vec{q} = 0 \end{cases}$$

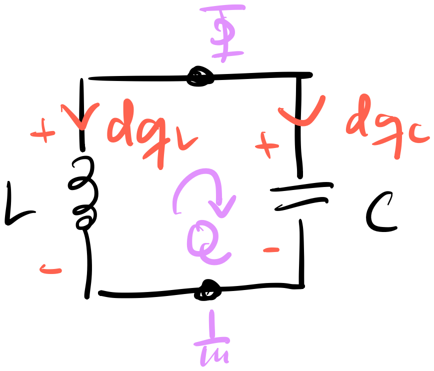
$$F d\vec{z} = \begin{pmatrix} F_{loop} & 0 \\ 0 & F_{cut} \end{pmatrix} \begin{pmatrix} d\bar{\phi} \\ d\bar{q} \end{pmatrix} = 0$$

$F\bar{k} = 0$ $\hookrightarrow$ Basis of fluxes and charges.

$K = K_{loop} \oplus K_{cut}$

Something more on the basis.

Example 1 LC circuit.



$$\left.\begin{aligned} dq_C + dq_L &= 0 \\ d\phi_C - d\phi_L &= 0 \end{aligned}\right\} \Rightarrow \quad F_{cut} = \begin{pmatrix} 1 & 1 \end{pmatrix} \qquad F_{loop} = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

$$F_{cut} K_{cut} = 0 \qquad K_{cut} = \overbrace{\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}}^{1\ \text{dim}}$$

$$F_{loop} K_{loop} = 0 \qquad K_{loop} = \overbrace{\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}}^{1\ \text{dim}}$$

$$\begin{pmatrix} dq_C \\ dq_L \end{pmatrix} = \boxed{dQ} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} d\phi_C \\ d\phi_L \end{pmatrix} = \boxed{d\Phi} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We construct the energy function that will become a $\mathcal{L}$.

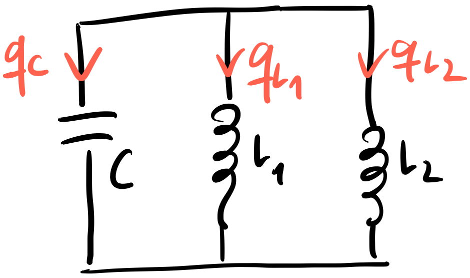
$$* = \frac{1}{2}\left(q_C \dot{\phi}_C + \phi_L \dot{q}_L\right) - \left(\frac{q_C^2}{2C} + \frac{\phi_L^2}{2L}\right)$$

(integrate kirchhoff's laws:

$$\mathcal{L} = \frac{1}{2}\underbrace{\left(Q\dot{\Phi} - \Phi\dot{Q}\right)}_{\frac{1}{2}\underbrace{\begin{pmatrix} Q \\ \Phi \end{pmatrix}^T}_{z^T}\underbrace{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}}_{W^{-1}}\dot{\begin{pmatrix} Q \\ \Phi \end{pmatrix}}} - \underbrace{\left(\frac{Q^2}{2C} + \frac{\Phi^2}{2L}\right)}_{H}$$

$W^{-1} = \{z, z^T\}$ ⇔ Poisson bracket → $\hat{C.R.}$

Example 2:



Kirchhoff's CONSTRAINTS

$$\begin{cases} dq_c + dq_{L_1} + dq_{L_2} = 0 & 1\text{ eq.} \to 2\ Q_s \\ d\phi_c = d\phi_{L_1} = d\phi_{L_2} & 2\text{ eqs.} \to 1\ \Phi \end{cases}$$

$$2 \times 3 \text{ variables} - 3 \text{ eqs} = \boxed{3}$$

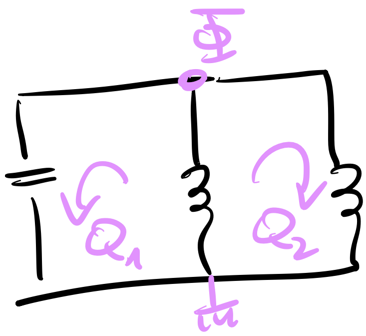
$$F_{cut} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \qquad F_{loop} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\dim \ker(F_{cut}) = 2 \qquad \dim \ker(F_{loop}) = 1$$

A basis for fluxes and charges:

$$dq_c = dQ_1$$
$$dq_{L_2} = dQ_2$$
$$dq_{L_1} = -d(Q_1 + Q_2)$$

$$d\phi_c = d\phi_{L_1} = d\phi_{L_2} = d\Phi$$



Energy function:

$$* = \frac{1}{2}\left(q_c\dot{\phi}_c + \phi_{L_1}\dot{q}_{L_2} + \phi_{L_2}\dot{q}_{L_2}\right) - \left(\frac{q_c^2}{2C} + \frac{\phi_{L_1}^2}{2L_1} + \frac{\phi_{L_2}^2}{2L_2}\right)$$

↓ Integrate constraints.

$$L = \frac{1}{2}\left(Q_1 \dot{\Phi} + \Phi\left(-\dot{Q}_1 - \cancel{\dot{Q}_2}\right) + \Phi\left(\cancel{\dot{Q}_2}\right)\right)$$

$$- \left(\frac{Q_1^2}{2C} + \frac{\Phi^2}{2}\left(\frac{1}{L_1} + \frac{1}{L_2}\right)\right)$$

⟶ Parallel summation rule.

$$= \frac{1}{2}\underbrace{\left(Q_1 \dot{\Phi} - \Phi \dot{Q}_1\right)}_{\tau W = (\xi, \zeta)^{-1}} - H$$

The first order term in the Lagrangian is associated with a 2-form.

$$L_W = \frac{1}{2}\left(q_c \dot{\phi}_c + \phi_L \dot{q}_L + \dots\right)$$

$$\Updownarrow$$

$$\omega = \frac{1}{2}\left(dq_c \wedge d\phi_c + d\phi_L \wedge dq_L + \dots\right)$$

$$d\alpha_1 \wedge d\alpha_2 = - d\alpha_2 \wedge d\alpha_1$$