Mathematical Reflections
Author: Juan José Granier .
February 10, 2025

0690

P1. Let $\mathcal{P}$ the set of all prime numbers. We show the result by an inductive construction. First, for $k = 1$, let $p_1 = 2$, we have that $p_1 \in \mathcal{P}$, then p_1 is not a perfect power of any positive integer $b \geq 2$. For $k = 2$, notice that for $p_2 = 3 \in \mathcal{P}$, we have for the 3 non-empty subsets of $S_2 = \{p_1, p_2\} = \{2, 3\}$, defined by $A_1, A_2, A_3 = \{2\}, \{3\}, \{2, 3\}$ that the sum equals $2, 3, 5$ respectively, so for $p_2 = 3$ it is enough for the construction in the step $k = 2$. Notice that for $k = 2$, by Bertrand's Postulate, we have $3 \in (2, 4)$, with $4 = 2 \times 2$ Suppose we have the construction up to k, so for $3 \leq t \leq k$ we have $S_t = \{p_1, p_2, \cdots, p_{t-1}, p_t$ such that for every $A \subseteq S_t$, $S_A = \sum_{p_i \in A} p_i \neq b^j, \forall b, j \geq 2$. For the case $t = k + 1$, notice that if we select a candidate $p_{k+1} \in (p_k, 2p_k)$, we notice that for the set of restrictions

$$\mathcal{S}_{\mathcal{R},S_k} = \{b^n - S_A | b^n > S_A, n \geq 2, A \subseteq S_k\}$$

we have if $n \geq 2$, then $b \geq \left\lceil (S_A + 1)^{\frac{1}{n}} \right\rceil$ and by the election of the candidate, $b^n \leq S_A + 2p_k \Rightarrow b \leq \left\lfloor (S_A + 2p_k)^{\frac{1}{n}} \right\rfloor$. If we use the approximation $p_k \approx k \log k, \forall k \geq 1$, with p_k the $k-$th prime number, we notice that $p_1 = 2$ is the first prime number, $p_2 = 3$ is the second prime number, and by construction $p_3, \cdots, p_k$ are the $s_3, \cdots, s_k$-th prime numbers, with $s_3, \cdots, s_k \in \mathbb{N}$. So, we can approximate $2p_k \approx 2s_k \log s_k$ and notice that $S_{S_k} \leq kp_{s_k} \approx ks_k \log s_k \leq \frac{s_k^2}{2} \log s_k$. This 2 bounds and the fact $a^{\frac{1}{n}}$ with $a \geq 1$ is decreasing for $n \geq 1$ implies that

$$\left\lceil \sqrt{S_A + 1} \right\rceil \leq b \leq \left\lfloor \sqrt{S_A + 2p_{s_k}} \right\rfloor \leq \left\lfloor \sqrt{\frac{s_k \log s_k}{2} (4 + s_k)} \right\rfloor$$
$$2 \leq n \leq \lfloor \log_2 (S_A + 2p_k) \rfloor \leq \left\lfloor \log_2 \left(\frac{s_k \log s_k}{2} (4 + s_k) \right) \right\rfloor$$

and then $|\mathcal{S}_{\mathcal{R},\mathcal{S}_\|}| < +\infty$. Furthermore, we notice that

$$|\mathcal{P}(S_k)| = \mathcal{O}(2^k)$$
$$b = \mathcal{O}\left(s_k \sqrt{\log s_k}\right) = \mathcal{O}\left(k \sqrt{\log k}\right)$$
$$n = \mathcal{O}(\log k)$$

with the second and third asymptotic equalities derived from the fact that again by the approximation $p_k \approx k \log k$ and the inequality $s_k \log s_k < p_{s_{k+1}} < 2s_k \log s_k$, we notice that by the Prime Number Theorem $s = \pi(p_s) \approx \frac{p_s}{\log p_s} \Rightarrow s_{k+1} \approx \frac{p_{s_{k+1}}}{\log p_{s_{k+1}}}$ and then

$$\frac{s_k \log s_k}{\log (2s_k \log s_k)} < s_{k+1} < \frac{2s_k \log s_k}{\log (2s_k \log s_k)}$$

and then using the approximation $\log(s_k \log s_k) \approx \log s_k$, we have

$$s_{k+1} > \frac{s_k \log s_k}{\log s_k} = s_k \Rightarrow s_{k+1} = \Omega(s_k)$$
$$s_{k+1} < \frac{2s_k \log s_k}{\log s_k} = 2s_k \Rightarrow s_{k+1} = \mathcal{O}(s_k)$$

and this implies $s_k \leq s_{k+1} \leq 2s_k$, so $s_{k+1} = \Theta(s_k)$ and then $s_{k+1} = \mathcal{O}(s_k)$. So, we have that $s_k \approx k$ is a good approximation for the second asymptotic equality. For the third asymptotic equality we use an analogous approach, first notice

$$n \leq \log_2 \left(\frac{k^2 \log k}{2} \right)$$

and using the property $\log_2(ab) = \log_2 a + \log_2 b$, we have

$$n \leq \log_2 k^2 + \log_2(\log k) - 1$$

and asymptotically for big values of k, the term $\log_2(\log k)$ is small compared to $2\log_2 k$ and then we can bound $n = \mathcal{O}(\log k)$

This implies

$$|\mathcal{S}_{\mathcal{R},S_k}| = \mathcal{O}\left(2^k k \log^{\frac{3}{2}} k\right)$$

and then the size of the restrictions is big but finite, and the increase in size is bounded. Notice that in the interval $(p_k, 2p_{s_k})$ the growth of perfect power numbers has a law of $\mathcal{O}\left((2p_{s_k})^{\frac{1}{2}}\right)$, that implies that the number of perfect power numbers in the interval is considerably smaller than the total number of numbers in the interval, so the fraction of numbers affected by the restrictions is significantly smaller. More generally, given an interval $[1, x], x \in \mathbb{N}$, the number of perfect power numbers increases at order $\mathcal{O}\left(x^{\frac{1}{2}}\right)$. Thus, combining with the bound we obtained before, we know that the number of subsets of restrictions is considerably smaller than we estimated before. The factor 2^k changes to $2^{\frac{k}{2}}$, and then

$$|\mathcal{S}_{\mathcal{R},S_k}| = \mathcal{O}\left(2^{\frac{k}{2}} k \log^{\frac{3}{2}} k\right)$$

Notice that the bound in size is the same for each candidate $p_{s_{k+1}} > p_{s_k}$. Also, we know by the algorithm that for $l \geq 2$ candidates of $p_{s_{k+1}}$, let's say $p_{s_{k_l}} > p_{s_{k_{l-1}}} > \cdots > p_{s_{k_1}} = p_{s_{k+1}} > p_{s_k}$, then we notice that the second and third asymptotic equalities apply, because in the worst case we have for $l \geq 1$ that $2^{l-1}p_{s_{k_{l-1}}} < p_{s_{k_l}} < 2^l p_{s_{k_{l-1}}}$ and l are constant parameters with respect to $k \geq 1$, then we have a bound even for the candidates at each step.

It's easy to notice that the sums $S_A, A \subseteq S_k$ increase at a linear rate, because $p_{s_k} \approx s_k \log s_k \approx k \log k$. Also, some additional facts:

- Notice that we want to construct an infinite set $S \subset \mathcal{P}$ such that no perfect power number equals the sum of elements of $A, \forall A \subseteq S$. Let $\mathcal{S}$ be the set of finite sums of finite subsets of the Prime Numbers. We want to argue that S can be constructed inductively in finite steps for each $k \geq 1$ by using $\mathcal{S}$
- It's easy to see that finite combinations of the first $k \geq 1$ prime numbers $p_1, p_2, \cdots, p_k$ generate $\mathcal{S}$ and that $\mathcal{S}$ is countable: just noticing that

$$\mathcal{S} = \bigcup_{k=1}^{+\infty} \{ \sum_{p_i \in A} p_i | A \subseteq S_k \}$$

  with $S_k = \{p_1, p_2, \cdots, p_k\}$ the first $k \geq 1$ prime numbers. Then, we have the union over all finite sums of elements of S_k and each one of these sets is finite, so the union over all $k \geq 1$ of these finite sets is countable.

- The set $\mathcal{S}$ is discrete in $\mathbb{N}$ with a minimum gap $\delta = 2$: To see this, just notice that given A, B finite subsets of prime numbers, assume without loss of generalization that

$$S_A = \sum_{p_i \in A} p_i < S_B = \sum_{p_i \in B} p_i$$

  Since the primes are strictly increasing, the smallest possible difference occurs when

$$B = A \cup \{p^*\}, p^* \in \mathcal{P} \Rightarrow S_B - S_A = p^*$$

  The smallest such p^* is 2 and is the only even prime number, then $\delta = 2$ and since the set is bounded below by 2 and every change in sum increases by at least 2 or another prime, no accumulation points exist. Thus, $\mathcal{S}$ is discrete in $\mathbb{N}$

- Given $b \geq 2$, we notice that

$$b^{j+1} - b^j = b^j (b-1) \geq 2^j (b-1) \geq 2^j, \forall j \geq 2$$

  Then, the gap between powers of a given number increases at an exponential rate.

- Let

$$\mathcal{N} = \{b^j, b, j \geq 2\}$$

  the set of Perfect Power Numbers. As we saw before, S_k of the algorithm increases its sums at a linear rate, and the same applies for the set $\mathcal{S}$. Also, we saw before that $\mathcal{N}$ increases its size at an exponential rate, and therefore $\mathcal{N}$ is not dense in $\mathbb{N}$. Then, the difference between a number in $\mathcal{S}$ and a number in $\mathcal{N}$ tends to increase as we advance in the construction. Formally speaking, define the minimum gap as

$$\Delta = \text{mín}\{|s - n| : s \in \mathcal{S}, n \in \mathcal{N}, s \neq n\}$$

  Given that $\mathcal{S}$ is discrete and $\mathcal{N}$ grows exponentially, there exists a positive constant $\varepsilon > 0$ such that $\Delta \geq \varepsilon$. That guarantees that there is a gap between $\mathcal{S}$ and $\mathcal{N}$, because $\mathcal{S}$ is not dense in $\mathcal{N}$.

- As before, we have that the number of perfect power numbers in an interval $[1,x], x \in \mathbb{N}$ is $\mathcal{N}(x) = \mathcal{O}\left(x^{\frac{1}{2}}\right)$, then if we define $\mathbb{N}(x)$ as the number of elements of $\mathbb{N}$ in the interval $[1,x]$, then $\mathbb{N}(x) = x$, and we have

$$\lim_{x\to+\infty} \frac{\mathcal{N}(x)}{\mathbb{N}(x)} = \lim_{x\to+\infty} \frac{\mathcal{O}\left(x^{\frac{1}{2}}\right)}{x} = \lim_{x\to+\infty} \mathcal{O}\left(x^{-\frac{1}{2}}\right) = 0$$

  Then, we deduce that $\mathcal{N}$ is not dense in $\mathbb{N}$

- Let Δ the minimum gap between $\mathcal{S}$ and $\mathcal{N}$. At each step of the search for $p_{s_{k+1}}$, the length of the interval we are looking duplicates, but by the Prime Number Theorem the density in terms of relative proportion tends to 0 and given that $|\mathcal{P}| = +\infty$, it always exists a prime $p_{s_{k_{t_k}}}$ before the difference between $\mathcal{N}$ and $S_{k+1} = S_k \cup \{p_{s_{k_{t_l}}} | l \geq 1\}$ is less than ε. Let $\mathcal{I}_{k,n}$ be the Bertrand's Intervals as we have defined before, i.e $\left(2^{n-1}p_{s_k}, 2^n p_{s_n}\right)$ with $n \geq 1$, let . Let

$$t_{k,n} = \inf_{l\geq 1} \left\{ \begin{array}{l} p_{s_{k_{l,n}}} \\ p_{s_{k_{1,n}}} < p_{s_{k_{2,n}}} < \cdots < p_{s_{k_{l,n}}} \\ p_{s_{k_r},n} \in \mathcal{I}_{k,n}, \forall 1 \leq r \leq l \\ p_{s_{k_r,n}} \in \mathcal{P}, \forall 1 \leq r \leq l \\ \displaystyle\sum_{p_i \in A} p_i \neq b^j, \quad \forall b, j \geq 2, \forall A \subseteq S_{k+1} = S_k \cup \{p_{s_{k_{l,n}}}\} \end{array} \right\}$$

  with $p_{s_{k_{1,0}}} = p_{s_k}$, considering:

  - By the Prime Number Theorem, the number of primes in the interval $\left(2^{r-1}p_{s_k}, 2^r p_{s_k}\right)$ is

$$\pi\left(2^r p_{s_k}\right) - \pi\left(2^{r-1}p_{s_k}\right) \approx \frac{2^r p_{s_k}}{\log\left(2^r p_{s_k}\right)} - \frac{2^{r-1}p_{s_k}}{\log\left(2^{r-1}p_{s_k}\right)}$$

    It's not hard to notice that

$$\frac{2^r p_{s_k}}{\log p_{s_k} + r\log 2} \approx \frac{2^r p_{s_k}}{\log p_{s_k}}\left(1 - \frac{r\log 2}{\log p_{s_k}}\right),$$
$$\frac{2^{r-1}p_{s_k}}{\log p_{s_k} + (r-1)\log 2} \approx \frac{2^{r-1}p_{s_k}}{\log p_{s_k}}\left(1 - \frac{(r-1)\log 2}{\log p_{s_k}}\right).$$

    Subtracting both terms, we have

$$\pi(2^r p_{s_k}) - \pi(2^{r-1}p_{s_k}) \approx \frac{2^r p_{s_k}}{\log p_{s_k}}\left(1 - \frac{r\log 2}{\log p_{s_k}}\right) - \frac{2^{r-1}p_{s_k}}{\log p_{s_k}}\left(1 - \frac{(r-1)\log 2}{\log p_{s_k}}\right).$$

    Extracting a common factor, we arrive to

$$\pi(2^r p_{s_k}) - \pi(2^{r-1}p_{s_k}) \approx 2^{r-1}p_{s_k}\left(\frac{2\log p_{s_k} + r\log 2 - 1}{\log p_{s_k} + (r-1)\log 2}\right).$$

    Then even for big values of $p_{s_{k_r}}$, it's still positive and finite

  - The restrictions grow at a sub-exponential rate and for each failed candidate, we are left with infinite number of primes, such that we can look over the same Bertrand interval, if there are primes left, or the next one.

  If we define

$$n_k = \inf\{n \geq 1 | t_{k,n} < +\infty\}$$

Using these facts, we have a first alternative to prove that n_k and $\hat{t}_k = t_{k,n_k}$ are finite, because we have the infinite cardinality of the Prime Numbers and the fact that $\mathcal{S}$ is not dense in $\mathcal{N}$ and $\mathcal{N}$ is not dense in $\mathbb{N}$ implies that $\mathcal{S}$ is not dense in $\mathbb{N}$, and joint with the fact the restriction sets grow at a sub-exponential rate, we have that there exist gaps between $\mathcal{N}$ and $\mathcal{S}$. For a second alternative, next there is a detailed proof using contradiction by supposing $\hat{t}_k = +\infty$ and getting a contradiction with the algorithm procedure. By the definition of $\hat{t}_k$, it's enough to prove that $n_k < +\infty$.

So, putting all together, if we prove that last fact we have that in a finite number of intervals, say n_k, we have $\hat{t}_k$ candidates, with $\hat{t}_k$ finite as well, we can find a suitable $p_{s_{k+1}} = p_{s_{k_{n_{\hat{t}_k}},n_k}}$ such that $S_{k+1} = S_k \cup \{p_{s_{k+1}}\}$ satisfies all the conditions, because there exist gaps between $\mathcal{S}$ and $\mathcal{N}$ and those gaps define S, because we are adding in the construction the primes such that every subset sum is not equal to a perfect power number. Notice that for $C > 1$ the same reasoning applies for the selection of $p_k < p_{k+1} < Cp_k, k \geq 1$. Notice that another strategy is to define $\mathcal{J}_{k,n} = \left(p_{s_{k_{1,n-1}}}, 2p_{s_{k_{1,n-1}}}\right)$, with $n \geq 1$, with $p_{s_{k_{1,0}}} = p_{s_k}$.

If we define $\hat{\tau}_k$ as the $s_{k+1} = s_{k_{n_{\hat{t}_k}},n_k}$ position, notice that we have n_k intervals, so if $\tau_{k,n}$ is the number of primes in the Bertrand Intervals, then we have the expression

$$\hat{\tau}_k = t_{k,n_k} + \sum_{n=1}^{n_k-1} \tau_{k,n} \Rightarrow s_{k+1} = \hat{\tau}_k + s_k$$

Then, we showed an explicit expression for s_{k+1}, in terms of s_k and the previous definition.

First, notice that this is an inductive algorithm to construct S. Given the arguments, we see that we can start with S_1, S_2 already constructed. By first inspection, we find $S_3 = \{2, 3, 17\}$ when we see the Bertrand intervals.

$$\begin{gathered}
\mathcal{I}_{2,1} = (3, 6)\,, \quad t_{2,1} = +\infty \\
\mathcal{J}_{2,1} = (5, 10)\,, \; w_{2,1} = +\infty \\
\mathcal{J}_{2,2} = (7, 14)\,, \; w_{2,2} = +\infty \\
\mathcal{J}_{2,3} = (11, 22)\,, \; w_{2,3} = 2 \\
\mathcal{I}_{2,2} = (6, 12)\,, \quad t_{2,2} = +\infty \\
\mathcal{I}_{2,3} = (12, 18)\,, \quad t_{2,3} = 2
\end{gathered}$$

Then in the case $k = 3$, we have $n_2 = 3, m_2 = 3$ and then we need 3 intervals of the Bertrand form. And in the third interval we need 2 steps, i.e. 2 consecutive prime candidates to reach our solutions, with the previous one being the last failed candidate: in this way, we are using in each step the failed candidate as the initial point or the same last prime in S_2 just duplicated by another power of 2 to find the candidate to construct. In total, we use in both cases we explore the prime numbers $\{5, 7, 11, 13\}$, i.e. 4 failed candidates before the candidate $\{17\}$ that accomplishes our goal for $k = 3$, so the number of candidate steps used to construct S_3 from S_2 is 5 and the number of Bertrand Intervals steps used is 3, with

$$w_{2.m_2} = w_{2.3} = \hat{w}_2 = t_{2,n_2} = t_{2,3} = \hat{t}_2 = 3$$

We define our inductive algorithm, using the $\mathcal{I}_{k,n}$ Bertrand's Intervals; the other case is a similar argument.

Algorithm 1: Inductive Construction of a Prime Set Avoiding Perfect Powers

Input: An initial sets of primes $S_1 = \{2\}$.
Output: An infinite set S where no subset sum is a perfect power.

Initialize $S_1 = \{2\}$;
for $k = 1, 2, \dots$ **do**
  Set $n = 1$, $\mathcal{I}_{k,n} = (2^{n-1} p_{s_k}, 2^n p_{s_k})$;
  repeat
    Select a candidate prime p_{k+1} using:
    - **Bertrand's Postulate:** Choose $p_{k+1} \in \mathcal{I}_{k,n}$.

    Compute all subset sums $S_A = \sum_{p_i \in A} p_i, \quad A \subseteq S_k \cup \{p_{k+1}\}$;
    Generate the restricted set:

$$\mathcal{S}_{\mathcal{R},S_k} = \mathcal{S}_{\mathcal{R},S_k}\,(p_{k+1}) = \bigcup_{A \subseteq S_k} \{b^n - S_A \mid 2 \le b \le \lfloor \sqrt{S_A + p_{k+1}} \rfloor, \quad 2 \le n \le \log_2(S_A + p_{k+1})\}$$

    if $p_{k+1} \notin \mathcal{S}_{\mathcal{R},S_k}$ **then**
      Append p_{k+1} to $S_k : S_k \cup \{p_{k+1}\} \leftarrow S_{k+1}$;
    else
      if *all primes in* $\mathcal{I}_{k,n}$ *have been checked* **then**
        Increment n, update $\mathcal{I}_{k,n} = (2^{n-1} p_{s_k}, 2^n p_{s_k})$;
      end
    end
  until *a valid prime* p_{k+1} *is found*;
end
return S;

To finish the proof, we prove by contradiction that $n_k < +\infty$. Before that, we notice:

- Given $p \in \mathcal{P} \setminus \{2\}$, if we define S^p as the infinite set generated by primes, starting with p and using the same algorithm as before, if we prove $S = S^2$ is infinite, then we can prove that S^p is infinite as well, $\forall p \in \mathcal{P} \setminus \{2\}$. That says the definition of S is not unique and is possible using the same algorithm $\forall p \in \mathcal{P}$ as a starting point in S_1^p

- By construction, notice that

$$S_1^p \neq S_1^q, \forall p, q \in \mathcal{P}, p \neq q$$
$$\Rightarrow S_k^p \neq S_k^q, \forall p, q \in \mathcal{P}, p \neq q, \forall k \geq 1$$

  Because in the first step, $\{p\} \neq \{q\}$. So, given k steps, we infer by induction that the set S_k^p built using the algorithm depends completely on the choice of p, then by the algorithm, we notice that the sets constructed are not completely equal at finite steps. Hence, the sets constructed differ in the primes selected by the algorithm at least in the order they were selected, uniquely defined by the restrictions generated by the candidates.

- Let $p_k^{(p)}$ be the $k-$th prime selected by the algorithm, starting with the prime p. The function

$$f : \mathbb{N} \times \mathcal{P} \longrightarrow \mathcal{P}$$
$$f(k, p) = p_k^{(p)}$$
$$f(1, p) = p, \forall p \in \mathcal{P}$$

  is strictly increasing with fixed $p \in \mathcal{P}$, let $g_k(p) = f(k, p)$. If $p \neq q$ and $S^p = S^q$, we would have $g_{k_0}(p) = g_{k_0}(q)$ for some $k_0 \geq 1$. However, the construction is deterministic and in each step we selected a unique prime based on the restriction set, then for $1 \leq j \leq k_0, g_j(p) = g_j(q)$, but then $g_1(p) = p = g_1(q) = q$, a contradiction. Then, if we prove that $|S^p| = +\infty, \forall p \in \mathcal{P}$, then we have proved that if $p \neq q \in \mathcal{P} \Rightarrow S^p \neq S^q$.

- We have that if $p < q \in \mathcal{P}$, then $p = S_1^p < q = S_1^q$, then by induction we prove that $p_k^{(p)} \leq p_k^{(q)}, \forall k \geq 1$

- The restrictions are unique: Suppose we have $(b_1, n_1, A_1), (b_2, n_2, A_2$ for the same m in the restriction set, with $b_1^{n_1} > b_2^{n_2}$. Then $S_{A_1} > S_{A_2}$ and we have:

$$m = b_1^{n_1} - S_{A_1} = b_2^{n_2} - S_{A_2}$$
$$\Rightarrow b_1^{n_1} - b_2^{n_2} = S_{A_1} - S_{A_2} \geq 2$$

  As $b_1^{n_1} - b_2^{n_2} \geq b_2^{n_2}(b_1^{n_1-n_2} - 1) \geq b_2^{n_2}(b_1 - 1)$, we have $S_{A_1} - S_{A_2} \geq b_2^{n_2}$, but then we have

$$S_{A_1} \geq b_2^{n_2} + S_{A_2} \geq b_2^{n_2}$$

  Then, $m \geq b_2^{n_2}$, but

$$b_1^{n_1} - S_{A_1} \geq b_2^{n_2}$$
$$\Rightarrow b_1^{n_1} - b_2^{n_2} = S_{A_1} - S_{A_2} \geq S_{A_1}$$

  A clear contradiction, because $S_{A_2} > 0$.

- Henceforth, we define S^2 as S in the original algorithm.

Let's prove by contradiction. If $n_k = +\infty$, then S^2 is finite and there exists $k_2 \geq 2$ such that $S^2 = S_k^2$, w.l.o.g by the induction suppose that $k_2 = k \geq 2$. Then, we have S_k^2 is a maximal set and is unique by the algorithm for $p = 2$. If not, then we could have selected in $2 \leq j \leq k$, another prime $p'_j \neq p_j$ using the same definitions of the algorithm, but as we saw before p_j is uniquely determined, a contradiction. As a consequence, we can assume that the set is unique; then by the maximal condition.

$$p \in \mathcal{S}_{\mathcal{R}, S_k}(p), \forall p > p_k$$

Suppose even that all sets $(S^p)_{p \in \mathcal{P} \setminus \{2\}}$ are finite, then $\exists k_p \geq 2 : S^p = S_{k_p}^p, \forall p \in \mathcal{P}$, with $k_2 = k$. Then, there are 3 cases we can analyze:

- If there exists $p \neq 2$ such that $k_p > k$, then the process of building related to S^p stops after the process related to S^2. As $p > 2$ and $k_p \geq k + 1$, there exists at least one prime in $S_p^k \cap \overline{S_k^2}$. Consequently, let $\hat{p}^{(p)}$ be a candidate of primes that are in $S_{k_p}^p$, but not in S_k^2. Then as the algorithm defines uniquely the set of restrictions, we have that:

– If $\hat{p}^{(p)} \notin \mathcal{S}_{\mathcal{R},S_k^2}\left(\hat{p}^{(p)}\right)$, then we can form a set $S_{k+1}^2 = S_k^2 \cup \{\hat{p}^{(p)}\}$, but this contradicts that the S_k^2 is maximal.

– If $\hat{p}^{(p)} \in \mathcal{S}_{\mathcal{R},S_k^2}\left(\hat{p}^{(p)}\right)$, then by the uniqueness of the restrictions, we have that:

$$\exists! \left(\hat{b}, \hat{n}, \hat{A}\right) \in \mathcal{S}_{\mathcal{R},S_k^2}\left(\hat{p}^{(p)}\right) : \hat{b}^{\hat{n}} - S_{\hat{A}} = \hat{p}^{(p)}$$

Then, letting $\hat{p}^{(2)} \in S_k^2$ such that $\hat{p}^{(2)} \in \hat{A}$, if we define

$$\hat{S_k^2} = \left(S_k^2 \setminus \{\hat{p}^{(2)}\}\right) \bigcup \{\hat{p}^{(p)}\}$$

The swap of $\hat{p}^{(2)}$ with $\hat{p}^{(p)}$

Then, as $\hat{S}_2^k$ defines a new maximal set by the algorithm for S^2, we have a contradiction with the uniqueness of S_k^2.

- If there exists $p \neq 2$ such that $k_p \leq k-1$, then there exists a prime $\hat{p}^{(2)}$ such that $\hat{p}^{(2)} \in \overline{S_p^k} \cap S_k^2$. As in the previous case where $k_p \geq k+1$, there are two cases:

  – If $\hat{p}^{(2)} \notin \mathcal{S}_{\mathcal{R},S_k^p}\left(\hat{p}^{(2)}\right)$, then we can form a set $S_{k+1}^p = S_k^p \cup \{\hat{p}^{(2)}\}$, but this contradicts that S_k^p is maximal.

  – If $\hat{p}^{(2)} \in \mathcal{S}_{\mathcal{R},S_k^p}\left(\hat{p}^{(2)}\right)$, then

$$\exists! \left(\hat{b}, \hat{n}, \hat{A}\right) \in \mathcal{S}_{\mathcal{R},S_k^p}\left(\hat{p}^{(2)}\right) : \hat{b}^{\hat{n}} - S_{\hat{A}} = \hat{p}^{(2)}$$

  Let $\hat{p}^{(2)} \in \hat{A}$, if we define as in the case $k_p \geq k+1$ the swap of $\hat{p}^{(p)}$ with $\hat{p}^{(2)}$, then we have another maximal set $\hat{S}_{k_p}^p$ for the algorithm, a clear contradiction with the uniqueness of $S_{k_p}^p$

- If not, then $k = k_p, \forall p \in \mathcal{P}$, then since the algorithm is deterministic, if $p_i^{(2)} = p_i^{(p)}, \forall 1 \leq i \leq k$, then $S_2^2 = S_2^p$, contradicting the fact that they are not equal in the selection by the algorithm, as we proved before. Then, there exists $1 \leq i \leq k$ such that $p_i^{(2)} \neq p_i^{(p)}$. We can suppose that $2 \leq i \leq k$, because $k > 2$. Let

$$\hat{i} = \inf\{2 \leq i \leq k | p_i^{(2)} \neq p_i^{(p)}\}$$

  We have three cases:

  – If $\hat{i} = k$, then we can swap $p_k^{(2)}$ with $p_k^{(p)}$ and we define a new maximal set $\hat{S}_k^2$, a contradiction.

  – If $\hat{i} < k$, we have two cases here

    ∗ If $p_{\hat{i}}^{(p)} \notin S_k^2$, then we can swap the $\hat{i}-$th element in S_k^2 with the $\hat{i}-$th element of S_k^p, having other maximal set, a contradiction.

    ∗ Else, as $p_i^{(2)} \leq p_i^{(p)}, \forall 2 \leq i \leq k$, we know that

$$p_i^{(2)} = p_i^{(p)}, \forall 2 \leq i \leq \hat{i} - 1$$
$$p_{\hat{i}}^{(2)} < p_{\hat{i}}^{(p)}$$

    Then, there exists $\hat{i} + 1 \leq \hat{j}_1 \leq k$ such that

$$p_{\hat{j}_1}^{(2)} = p_{\hat{i}}^{(p)}$$

    if $p_{\hat{i}+1}^{(p)} \notin S_k^2$, we can swap $p_{\hat{j}_1}^{(2)}$ with $p_{\hat{i}+1}^{(p)}$, giving s contradiction with the uniqueness of the maximal set generated by the algorithm. If not, then iteratively by the argument, we can find $j_2, j_3, \cdots j_{r_{2,p}}$ such that

$$
\begin{gathered}
j_l < j_{l+1}, \forall l \leq i \leq r_{2,p} - 1 \\
j_l \geq \hat{i} + l, \forall 1 \leq l \leq r_{2,p} \\
p_{j_l} = p_{\hat{i}+l}, \forall 1 \leq l \leq r_{2,p} \\
\hat{i} + 1 \leq j_{r_{2,p}} \leq k - 1 \\
p^{(p)}_{j_{2_{r,p}+1}} \notin S^2_k
\end{gathered}
$$

because with $\hat{i}$ we create a gap between the available elements in S^2_k which could be equal to elements of S^p_k, then at least one element in S^k_p could be swaped with one element in S^2_k, giving the contradiction with the uniqueness of the maximal set generated by the algorithm.

- If there exists $p \neq 2$ such that S^p is infinite, then as S^p is infinite, $\exists \hat{p}^{(p)}_{k+1}, \hat{p}^{(p)}_{k+2}, \cdots$ a sequence of primes in S^p such that $\hat{p}^{(p)}_{k+j} \in S^p \cap \overline{S^k_2}, \forall j \geq 1$. Then, we can swap elements of S^2_k with elements of the sequence, contradicting the uniqueness of the set generated by the algorithm, starting from $q = 2$.

With this, we have proved that there exists a prime $p_{k+1} > p_k$, with $p_{k+1} \notin \mathcal{S}_{\mathcal{R},S_k}(p_{k+1})$, and that implies that $\hat{t}_k < +\infty$. To prove with the previous result that S^p is an infinite set for every $p \in \mathcal{P} \setminus \{2\}$, we argue in the same way we did with S^2. By contradiction, we assume that $S^p = S^p_k$ for some $k \geq 2$, then using that S^2 is an infinite set, we have the same procedure as before with the case: S^p being an infinite set and S^2 being a finite set. The conclusion follows and therefore the proof is finished. Notice that with the proof finished, the algorithm applies replacing $S_1 = \{2\}$ with $S_1 = \{p\}$, with $p \neq 2$ a prime.

Maybe, what is interesting is that if we define the same forbidden set, i.e $\mathcal{N}$, the same base set, i.e the Prime Numbers, and for example we define the infinite sets.

$$S = \lim_{k \nearrow +\infty} S_k, S_k = \{p_1, p_2, \cdots, p_{k+1} \in \mathcal{P} : \sum_{p_i \in A} p_i^n \neq b^j, \forall b, j \geq 2, \forall A \subseteq S_k, |A| \geq 2, p_1 < p_2 \cdots < p_{k+1}\}, k \geq 1, n \geq 2$$
$$S = \lim_{k \nearrow +\infty} S_k, S_k = \{p_1, p_2, \cdots, p_{k+1} \in \mathcal{P} : \sum_{p_i \in A} p_i^{k+1} \neq b^j, \forall b, j \geq 2, \forall A \subseteq S_k, |A| \geq 2, p_1 < p_2 \cdots < p_{k+1}\}, k \geq 1$$

Then, by a similar procedure, it can be proved that those sets are valid, i.e., are well defined and there exist inductive algorithms for their constructions. The proof is left to the reader; however, essentially the proof relies on proving the finiteness and decreasing density of the restriction sets that allow new candidates of prime numbers, joined with the infinite cardinality of the Prime Numbers. Using analog techniques, with some modifications, we can prove these results. Notice that the second definition of S, maybe for S_1 we can have the same definition as in the initial question, and for $S_2, S_3, \cdots$ define with the additional condition $|A| \geq 2$ and the exponent $k+1$ replaced by k, with $k \geq 2$, then we have the same base condition to define the S^p, with $p \in \mathcal{P}$. In the new definitions of S, notice that the base case needs two primes, so it's well defined, because we need subsets of cardinality at least 2 to have sums not equal to a perfect power number.

Let $T \subseteq \mathbb{N}$ an infinite set. Let $x \in \mathbb{N}$ and define $T(x)$ as the number of elements of T in $[1, x]$.

Then, maybe a more interesting question is if the following is true:

Question : Let $B \subset \mathbb{N}$ be an **allowed Set**, i.e., an infinite set in $\mathbb{N}$. Let $F \subset \mathbb{N}$ a **forbidden set**, i.e. an infinite set $F \neq B$ such that for $x \in \mathbb{N}, F(x) = o(B(x))$. Then, does there exist an algorithmic construction that allows us to find an infinite set $S \subseteq B$ such that no sum of subsets of size at least $z_{B,F}^k$ of S belongs to F at the k-th step of the algorithm, with

$$1 \leq z_{B,F}^k \leq z_{B,F}, \forall k \geq 1$$
$$z_{B,F} = \sup_{k \geq 1} z_{B,F}^k < +\infty?$$

Notice that in the original case $z_{\mathcal{B},\mathcal{N}} = z_{B,\mathcal{N}}^k = 1, \forall k \geq 1$, with $\mathcal{B} = \mathcal{P}$ and in the other cases $z_{\mathcal{B}_{1,n},\mathcal{N}} = z_{\mathcal{B}_2,\mathcal{N}} = 2$ and $z_{B_{1,n},\mathcal{N}}^k = 2, z_{\mathcal{B}_2,\mathcal{N}}^k = 2, \forall k \geq 1$, with $\mathcal{B}_{1,n} = \{p^n : p \in \mathcal{P}\}, n \geq 2$ and $B_2 = \bigcup_{n \in \mathbb{N}} \mathcal{B}_{1,n}$. Notice that with our variation for the second problem, $z_{\mathcal{B}_2.\mathcal{N}}^1 = 1$.