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# Parametric Analysis of Second Order Systems of Linear Differential Equations

***Summary***— **the general purpose of this report is to make an analysis of autonomous second order systems, starting from matrix "A2x2", called "characteristic matrix" of the system, composed by four parameters and then analyzing and providing all the possible cases for equilibrium behavior of a second order system with their respective restrictions, this analysis is given by representing the intervals where the system *Equilibrium* has a determined behavior. This analysis was also done by analyzing the characteristic polynomial of the system, this second perspective allows us to only focus on Eigen values and forget the parameters of the characteristic matrix "A". These two perspectives showed us that results obtained with these methods are the same and independent on the method used, because they are equivalent.**

***Key words— Equilibrium, characteristic matrix, characteristic polynomial*. Eigen values.**

## INTRODUCTION

Many physical systems are described by several differential equations. One of the most simple types of such systems is a system of two linear differential equations that on a general form can be written as:

$$\begin{cases} \dot{x}_1 = ax_1 + bx_2 \\ \dot{x}_2 = cx_1 + dx_2 \end{cases} \qquad (1.0)$$

here $X_1(t)$ and $X_2(t)$ are unknown functions of time t, and a, b, c, d are constants (parameters). System (1.0) by itself has many practically important applications, for example electrical circuits in physics, models in economics, etc. We could also represent system (1.0) on its matricidal form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (1.1)$$

This matrix A has two rows and three columns. We will call this a matrix of the size 2 × 2.
Let us introduce a definition:
**Definition 1:** A nonzero vector **v** and number **λ** are called an eigen vector and an eigen value of a square matrix A if they satisfy equation:

$$Av = \lambda v \qquad (1.2)$$

Eigen vectors are not unique, and it is easy to see that if we multiply it by an arbitrary constant **k** we get another eigen vector corresponding to the same eigen value. Indeed by multiplying (1.3) by k we get:

$$kAv = k\lambda v$$

therefore, we can say that **kv** is also an eigen vector of (1.2) corresponding to eigen value **λ**.
Eigen values can be calculated as following:

$$Det(A - I\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc) \qquad (1.3)$$

finally solving (1.3) for **λ** we have:

$$\lambda_{1,2} = \frac{(a+d)}{2} \pm \frac{\sqrt{((a+d)^2 - 4(ad-bc))}}{2} \qquad (1.4)$$

Let go back to system (1.0), if we solve this system by any method with some initial conditions $x_1(0) = a, x_2(0) = b$ we will find the explicit solution for the system for $x_1$ and $x_2$ . Let us now consider a two dimensional coordinate system with $Ox_1x_2$ the $x_1$ - axis for the variable $x_1$ and the - a $x_2$ xis for the variable . $x_2$ Such a coordinate system is called a **phase space**, if we plot the dynamic of $x_1$ vs $x_2$ we will get a unique trajectory according with **Existences and Uniqueness Theorem**, then if we plot several trajectories with several initial con conditions we will get a complete **phase portrait** of the system, that is a representative set of system solutions.
It is necessary to introduce another definition.
**definition 2:** *Equilibrium* are points where our system is stationary: placed at an equilibrium point the system will stay there forever. Mathematically *Equilibrium* points for second order systems it is required that at the equilibrium point both variables $\dot{x}_1$ and $\dot{x}_2$ do not change their values, i.e. both $\dot{x}_1 = 0 \wedge \dot{x}_2 = 0$ .
In order to analyze those *Equilibrium* points ans their behavior it is necessary to study eigen values of the matrix A.

now lets take system (1.0) and suppose we give an entry:

$$\begin{cases} \dot{x}_1 = ax_1 + bx_2 + eu \\ \dot{x}_2 = cx_1 + dx_2 + fu \end{cases} \qquad (1.5)$$

we also can write this system as a differential equation of order tow solving for $x_1$ with this we found:

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$$\frac{d^2 x_1}{dt^2} - (a+d)\frac{d x_1}{dt} + (ad - bc)x = e\frac{du}{dt} + (bf - ed)u$$
(1.6)

if we apply Laplace Transform to (1.6) we found:

$$X_1(s)(s^2 - (a+d)s + (ad - bc)) = U(s)(es + (bf - ed))$$
(1.7)

if we solve (1.7) for $X_1(s)/U(s)$ .
we found:

$$G(s) = \frac{X_1(s)}{U(s)} = \frac{(es + (bf - ed))}{(s^2 - (a+d)s + (ad - bc))} \quad (1.8)$$

that is called ***the transfer function of the system***.

**definition 3:** The transfer function of a system described by a time-invariant linear differential equation is defined as the quotient between the Laplace transform of the output and the Laplace transform of the input (excitation function) under the assumption that all initial conditions are zero.

If we take the denominator polynomial of (1.8) and equals it to zero as follows:

$$s^2 - (a+d)s + (ad - bc) = 0 \quad (1.9)$$

we will find the poles of the system.

PARAMETRIC ANALYSIS OF DIFFERENT CASES OF EQUILIBRIUM BEHAVIOR

Given: (1.1) and (1.3), we can classify several *equilibrium behaviors*:

1.- **Spiral Sink** $(a+d)^2-4(ad-bc)<0 \wedge a+d<0$

*result:*

$$\beta=(a\geq 0\wedge((b<0\wedge c>-(a^2/b)\wedge a-2\sqrt{-bc}<d<-a)\vee(b>0\wedge c<-(a^2/b)\wedge a-2\sqrt{-bc}<d<-a)))$$
$$\alpha=(b>0\wedge((c\leq-(a^2/b)\wedge a-2\sqrt{-bc}<d<-a)\vee(-(a^2/b)<c<0\wedge a-2\sqrt{-bc}<d<a+2\sqrt{-bc})))$$
$$\delta=(b<0\wedge((0<c\leq-(a^2/b)\wedge a-2\sqrt{-bc}<d<a+2\sqrt{-bc})\vee(c>-(a^2/b)\wedge a-2\sqrt{-bc}<d<-a)))$$
$$\theta=a<0$$

*finally*:

$$(\theta\wedge(\delta\vee\alpha))\vee\beta$$

2.- **Nodal Sink** $(a+d)^2-4(ad-bc)>0 \wedge \lambda_1\lambda_2>0 \wedge \lambda_1+\lambda_2<0$

*result:*

$$\alpha=(a>0\wedge((b<0\wedge c>-(a^2/b)\wedge(b\,c)/a<d<a-2\sqrt{-bc})\vee(b>0\wedge c<-(a^2/b)\wedge(b\,c)/a<d<a-2\sqrt{-bc})))$$
$$\beta=(a=0\wedge((b<0\wedge c>0\wedge d<-2\sqrt{-bc})\vee(b>0\wedge c<0\wedge d<-2\sqrt{-bc})))$$
$$\delta=(c>0\wedge d<(b\,c)/a)$$
$$\theta=((c\leq-(a^2/b)\wedge d<a-2\sqrt{-bc})\vee(-(a^2/b)<c\leq 0\wedge(d<a-2\sqrt{-bc}\vee a+2\sqrt{-bc}<d<(b\,c)/a))\vee\delta)$$
$$\varepsilon=(b=0\wedge(d<a\vee a<d<0))\vee(b>0\wedge\theta)$$
$$\omega=(0\leq c<-(a^2/b)\wedge(d<a-2\sqrt{-bc}\vee a+2\sqrt{-bc}<d<(b\,c)/a))\vee(c\geq-(a^2/b)\wedge d<a-2\sqrt{-bc})$$
$$\rho=(a<0\wedge((b<0\wedge((c<0\wedge d<(b\,c)/a)\vee\omega))\vee\epsilon))$$

*finally:*

$$\rho\vee\beta\vee\alpha$$

3.- ***Spiral Source*** $(a+d)^2-4(ad-bc)<0\wedge(a+d)>0$

*result:*

$$\alpha=((c\leq-(a^2/b)\wedge-a<d<a+2\sqrt{-bc})\vee(-(a^2/b)<c<0\wedge a-2\sqrt{-bc}<d<a+2\sqrt{-bc}))$$
$$\beta=((b<0\wedge((0<c<-(a^2/b)\wedge a-2\sqrt{-bc}<d<a+2\sqrt{-bc})\vee(c\geq-(a^2/b)\wedge-a<d<a+2\sqrt{-bc})))\vee(b>0\wedge\alpha))$$
$$\theta=(a\leq 0\wedge((b<0\wedge c>-(a^2/b)\wedge-a<d<a+2\sqrt{-bc})\vee(b>0\wedge c<-(a^2/b)\wedge-a<d<a+2\sqrt{-bc})))$$
$$\varepsilon=(a>0\wedge\beta)$$

*finally:*

$$\theta\vee\varepsilon$$

4.-**Nodal Source** $(a+d)^2-4(ad-bc)>0\wedge\lambda_1\lambda_2>0\wedge\lambda_1+\lambda_2>0$

*result:*

$\alpha=((c\leq-(a^2/b)\wedge d>a+2\sqrt{-bc})\vee(-(a^2/b)<c\leq 0\wedge((b\,c)/a<d<a-2\sqrt{-bc}\vee d>a+2\sqrt{-bc}))\vee(c>0\wedge d>(b\,c)/a))$

$\beta=((c<0\wedge d>(b\,c)/a)\vee(0\leq c<-(a^2/b)\wedge((b\,c)/a<d<a-2\sqrt{-bc}\vee d>a+2\sqrt{-bc}))\vee(c\geq-(a^2/b)\wedge d>a+2\sqrt{-bc}))$

$\delta=(a>0\wedge((b<0\wedge\beta)\vee(b=0\wedge(0<d<a\vee d>a))\vee(b>0\wedge\alpha)))$

$\theta=(a=0\wedge((b<0\wedge c>0\wedge d>2\sqrt{-bc})\vee(b>0\wedge c<0\wedge d>2\sqrt{-bc})))\vee\delta$

$\varepsilon=(a<0\wedge((b<0\wedge c>-(a^2/b)\wedge a+2\sqrt{-bc}<d<(b\,c)/a)\vee(b>0\wedge c<-(a^2/b)\wedge a+2\sqrt{-bc}<d<(b\,c)/a)))$

*finally*:

$\varepsilon\vee\theta$

5.- **Saddle** $(a+d)^2-4(ad-bc)>0\wedge(\lambda_1)(\lambda_2)<0$

*result:*

$\alpha=(a>0\wedge((b<0\wedge c>-(a^2/b)\wedge(b\,c)/a<d<a-2\sqrt{-bc})\vee(b>0\wedge c<-(a^2/b)\wedge(b\,c)/a<d<a-2\sqrt{-bc})))$

$\beta=(a=0\wedge((b<0\wedge c>0\wedge d<-2\sqrt{-bc})\vee(b>0\wedge c<0\wedge d<-2\sqrt{-bc})))$

$\delta=(c>0\wedge d<(b\,c)/a)$

*finally:*

$\rho\vee\beta\vee\alpha$

6.- ***Center*** $(a+d)^2-4(ad-bc)<0\wedge(a+d)=0$

*result:*

$\alpha=((b<0\wedge c>-(a^2/b))\vee(b>0\wedge c<-(a^2/b)))$

$\beta=(d=-a)$

*finally:*

$\beta\wedge\alpha$

7.-**Degenerate Nodal Source** $(a+d)^2-4(ad-bc)=0 \wedge \lambda_{12}=(a+d)/2 \wedge (a+d)>0$

*result:*

$\alpha=(b>0 \wedge ((c \leq -(a^2/b) \wedge d=a+2\sqrt{[-b\,c]}) \vee (-(a^2/b)<c<0 \wedge (d=a-2\sqrt{[-b\,c]} \vee d=a+2\sqrt{[-b\,c]})) \vee (c=0 \wedge d=a)))$

$\beta=((b<0 \wedge (\delta \vee (0<c<-(a^2/b) \wedge (d=a-2\sqrt{[-b\,c]} \vee d=a+2\sqrt{[-b\,c]})) \vee (c \geq -(a^2/b) \wedge d=a+2\sqrt{[-b\,c]}))) \vee (b=0 \wedge d=a) \vee \alpha)$

$\delta=(c=0 \wedge d=a)$

$\theta=(a \leq 0 \wedge ((b<0 \wedge c>-(a^2/b) \wedge d=a+2\sqrt{[-b\,c]}) \vee (b>0 \wedge c<-(a^2/b) \wedge d=a+2\sqrt{[-b\,c]})))$

$\varepsilon=(a>0 \wedge beta)$

*finally*:

$\varepsilon \vee \theta$

8.- **Degenerate Nodal Sink** $(a+d)^2-4(ad-bc)=0 \wedge \lambda_{12}=(a+d)/2 \wedge (a+d)<0$

*result:*

$\alpha=(a \geq 0 \wedge ((b<0 \wedge c>-(a^2/b) \wedge d=a-2\sqrt{[-b\,c]}) \vee (b>0 \wedge c<-(a^2/b) \wedge d=a-2\sqrt{[-b\,c]})))$

$\beta=(b>0 \wedge ((c \leq -(a^2/b) \wedge d=a-2\sqrt{[-b\,c]}) \vee (-(a^2/b)<c<0 \wedge (d=a-2\sqrt{[-b\,c]} \vee d=a+2\sqrt{[-b\,c]})) \vee (c=0 \wedge d=a)))$

$\delta=(b<0 \wedge ((c=0 \wedge d=a) \vee (0<c<-(a^2/b) \wedge (d=a-2\sqrt{[-b\,c]} \vee d=a+2\sqrt{[-b\,c]})) \vee (c \geq -(a^2/b) \wedge d=a-2\sqrt{[-b\,c]})))$

$\theta=(b=0 \wedge d=a)$

$\varepsilon=a<0$

*finally:*

$(\varepsilon \wedge (\delta \vee \theta \vee \beta)) \vee \alpha$

9.- ***Unstable Saddle Node*** $(a+d)^2-4(ad-bc)=0 \wedge (a+d) \neq 0, (a+d)>0, \lambda_1=0, \lambda_2=(a+d)$

*result:*

$\alpha=((a<0 \wedge ((b<0 \wedge c>-(a^2/b)) \vee (b>0 \wedge c<-(a^2/b)))) \vee (a>0 \wedge ((b<0 \wedge c<-(a^2/b)) \vee b=0 \vee (b>0 \wedge c>-(a^2/b))))) \wedge d=(b\,c)/a$

$\beta=(a=0 \wedge ((b<0 \wedge c=0 \wedge d>0) \vee (b=0 \wedge d>0) \vee (b>0 \wedge c=0 \wedge d>0)))$

*finally:*

$\beta \vee \alpha$

10.-***Stable Saddle Node*** $(a+d)^2-4(ad-bc)=0\wedge(a+d)\neq 0,(a+d)<0,\lambda_1=0,\lambda_2=(a+d)$

*result:*

$$\alpha=((a<0\wedge((b<0\wedge c<-(a^2/b))\vee b=0\vee(b>0\wedge c>-(a^2/b))))\vee(a>0\wedge((b<0\wedge c>-(a^2/b))\vee(b>0\wedge c<-(a^2/b)))))\wedge d=(bc)/a$$
$$\beta=(a=0\wedge((b<0\wedge c=0\wedge d<0)\vee(b=0\wedge d<0)\vee(b>0\wedge c=0\wedge d<0)))$$

*finally*:

$$\beta\vee\alpha$$

11.- **Both Eigen Values are 0** $\lambda_1=\lambda_2=0$

*result:*

$$\alpha=(a>0\wedge((b<0\wedge c=-(a^2/b)\wedge d=(bc)/a)\vee(b>0\wedge c=-(a^2/b)\wedge d=(bc)/a)))$$
$$\beta=(a=0\wedge((b<0\wedge c=0\wedge d=0)\vee(b=0\wedge d=0)\vee(b>0\wedge c=0\wedge d=0)))$$
$$\delta=(a<0\wedge((b<0\wedge c=-(a^2/b)\wedge d=(bc)/a)\vee(b>0\wedge c=-(a^2/b)\wedge d=(bc)/a)))$$

*finally:*

$$\delta\vee\beta\vee\alpha$$

## System response analysis with function transfer poles for equilibrium behavior

With (1.9) and:

$$T=(a+d)$$
$$D=(ad+bc)$$

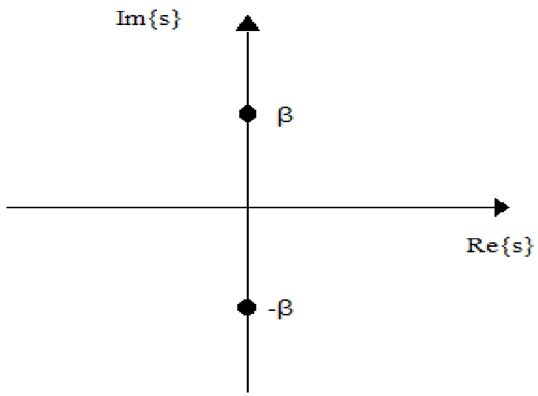
now:

$$s^2-Ts+D=0$$

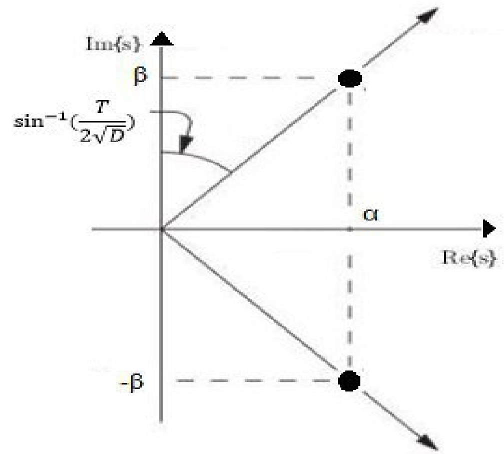
we can classify the same equilibrium behaviors starting from different cases as follows:

*1.-* $s_1 \wedge s_2$ *are imaginary* $T^2-4D<0; s_{1,2}=\alpha\pm j\beta; \alpha=T/2, \beta=\sqrt{[4D-T^2]}/2$

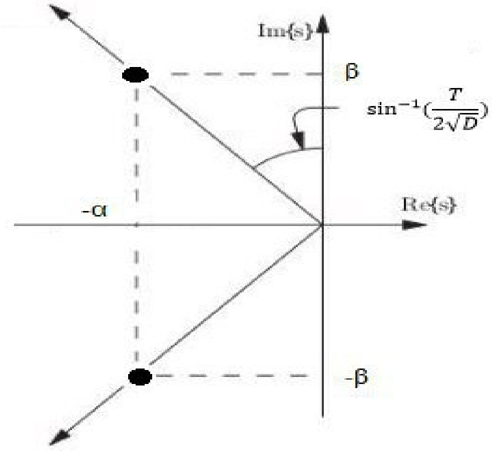
***Center*** $\alpha=0$




***Spiral Source*** $\alpha>0$

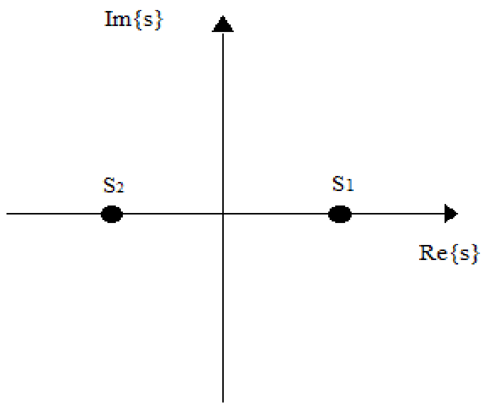



***Spiral Sink*** $\alpha<0$

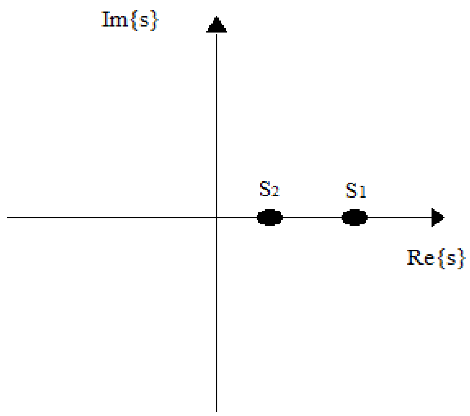



*2.-* $s_1 \wedge s_2$ *are real* $T^2-4D>0; s_{1,2}=T/2\pm\sqrt{[T^2-4D]}/2$

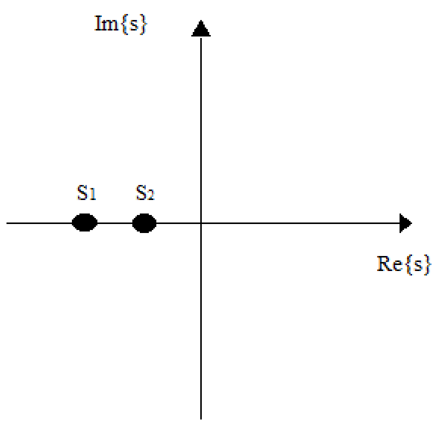
***Saddle*** $s_1>0>s_2$



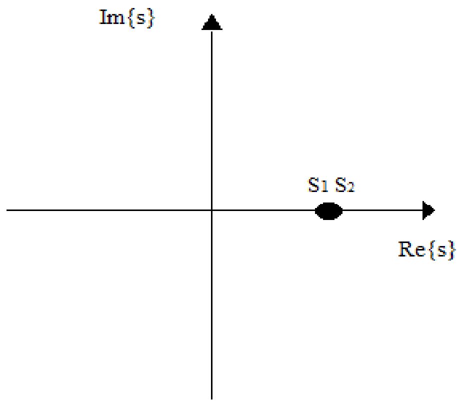
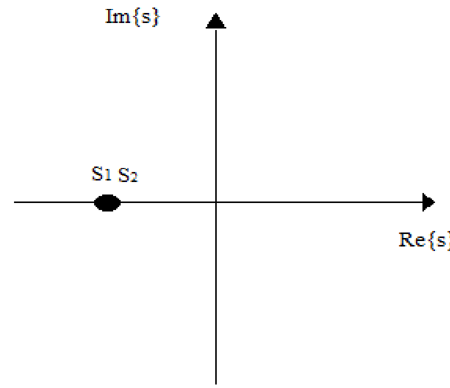

***Nodal Source*** $s_1>s_2>0$



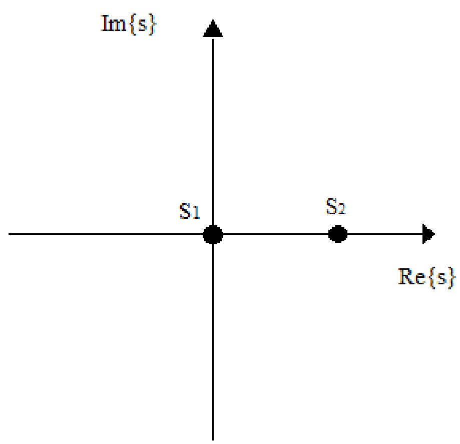
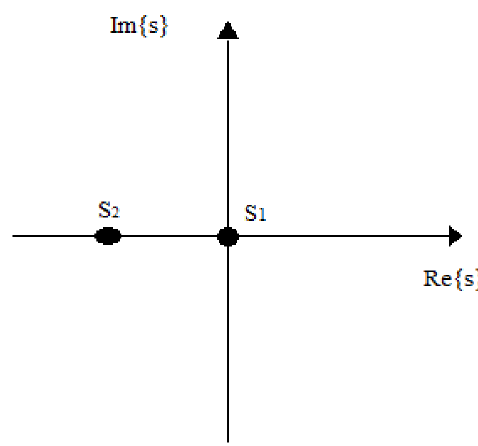

***Nodal Sink*** $s_1<s_2<0$



*3.-* $s_1 \wedge s_2$ *are real and equal* $T^2-4D=0; s_{1,2}=T/2$

| **Degenerate Nodal Source** $T>0$ | **Degenerate Nodal Sink** $T<0$ |
| --- | --- |
|   |   |

*4.-* $s_1 \wedge s_2$ *are real and one is zero* $s_1=0; s_2=T/2$

| **Unstable Saddle-Node** $T>0$ | **Stable Saddle-Node** $0<T$ |
| --- | --- |
|   |   |

## CONCLUSIONS

We have found all possible types of equilibrium which can occur in 2D systems: saddle, non-stable node, stable node, center, non-stable spiral and stable spiral. By analyzing tow different methods for this analysis, this proves that this tow methods are equivalent and that we can start any study of second order system equilibrium by any of them depending on resources we have for this study.

In spite of this tow method are equivalent for equilibrium behavior analysis, the parametric analysis could give us more information about the system, but, both give the same information about system equilibrium, this is because analyzing the characteristic polynomial of the system involve studying the transfer function, and as we know from the theory we can have same transfer function for many systems.

These results are applicable to any linear system which can be expressed in the forms showed above.

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