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Parametric Analysis of Second Order Systems on Closed Loop of Linear Differential Equations

***Summary*— the general purpose of this report is to make an analysis of autonomous second order systems on closed loop, starting from matrix "A_{2x2}", called "characteristic matrix" of the system, composed by four parameters and the entry of the system "μ_{1x2}" multiplied by a vector of constants "B_{2x1}", them analyzing and providing all the possible cases for equilibrium behavior of a second order system on closed loop with their respective restrictions, this analysis is given by representing the intervals where the system *Equilibrium* has a determined behavior.**

Key words*— *Equilibrium, characteristic matrix.

Introduction

Control theory in control systems engineering is a sub-field of mathematics that deals with the control of continuously operating dynamical systems in engineered processes and machines. The objective is to develop a control model for controlling such systems using a control action in an optimum manner without *delay or overshoot* and ensuring control stability. One of the techniques used to control the system behavior is to close the loop of the system, a system on closed loop looks as follows:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}h\\ k\end{bmatrix}\begin{bmatrix}x_1 & x_2\end{bmatrix} \quad (1.0)$$

In general the form of (1.0):

$$\dot{X}=AX+B\mu \quad (1.0.1)$$

were **μ** is called the entry of the system.
Closing the loop, we will be able to control the response of the system.
In these case we will analyze following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}h\\ k\end{bmatrix}\begin{bmatrix}x_1 & 0\end{bmatrix} \quad (1.1)$$

here $X_1(t)$ and $X_2(t)$ are unknown functions of time t, and a, b, c, d, h and k are constants (parameters). System (1.1) by itself has many practically important applications. We could also represent system (1.1) on this form:

$$\begin{cases}\dot{x}_1=(a+h)\,x_1+bx_2\\ \dot{x}_2=(c+k)\,x_1+dx_2\end{cases} \quad (1.2)$$

Let us introduce a definition:
Definition 1: A nonzero vector **v** and number **λ** are called an eigen vector and an eigen value of a square matrix A if they satisfy equation:

$$Av=\lambda v \quad (1.3)$$

Eigen vectors are not unique, and it is easy to see that if we multiply it by an arbitrary constant **k** we get another eigen vector corresponding to the same eigen value. Indeed by multiplying (1.3) by k we get:

$$kAv=k\lambda v$$

therefore, we can say that **kv** is also an eigen vector of (1.3) corresponding to eigen value **λ**.
Eigen values can be calculated as following:

$$Det(A-I\lambda)=\lambda^2-((a+h)+d)\lambda+((a+h)d-b(c+k))$$
(1.4)

finally solving (1.4) for **λ** we have:

$$\lambda_{1,2}=\frac{((a+h)+d)}{2}\pm\frac{\sqrt{(((a+h)+d)^2-4((a+h)d-b(c+k)))}}{2}$$
(1.5)

In order to analyze *Equilibrium* points and their behavior it is necessary to study eigen values of the matrix A.

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Parametric Analysis of different cases of equilibrium behavior with a system on closed loop

Given: (1.1) and (1.5), we can classify several *equilibrium behaviors*:

1.- **Spiral Sink** $((a+h)+d)^2-4((a+h)d-b(c+k))<0\wedge(a+h)+d<0$

result:

$$\beta=(h\geq-a\wedge k<(-a^2-bc-2ah-h^2)/b\wedge a+h-2\sqrt{[-bc-bk]}<d<-a-h)$$
$$\alpha=((-a^2-bc-2ah-h^2)/b<k<-c\wedge a+h-2\sqrt{[-bc-bk]}<d<a+h+2\sqrt{[-bc-bk]})$$
$$\delta=(k\leq(-a^2-bc-2ah-h^2)/b\wedge a+h-2\sqrt{[-bc-bk]}<d<-a-h)$$

finally:

$$(b>0\wedge((h<-a\wedge(\delta\vee\alpha))\vee\beta))ta$$

2.- **Nodal Sink** $((a+h)+d)^2-4((a+h)d-b(c+k))>0\wedge\lambda_1\lambda_2>0\wedge\lambda_1+\lambda_2<0$

result:

$$\alpha=(h>-a\wedge k<(-a^2-bc-2ah-h^2)/b\wedge(bc+bk)/(a+h)<d<a+h-2\sqrt{[-bc-bk]})$$
$$\beta=(h=-a\wedge k<-c\wedge d<a+h-2\sqrt{[-bc-bk]})\vee\alpha$$
$$\delta=(k=-c\wedge(d<a+h+2\sqrt{[-bc-bk]}\vee a+h+2\sqrt{[-bc-bk]}<d<(bc+bk)/(a+h)))$$
$$\theta=((-a^2-bc-2ah-h^2)/b<k<-c\wedge(d<a+h-2\sqrt{[-bc-bk]}\vee a+h+2\sqrt{[-bc-bk]}<d<(bc+bk)/(a+h)))$$
$$\varepsilon=(b>0\wedge((h<-a\wedge((k\leq(-a^2-bc-2ah-h^2)/b\wedge d<a+h-2\sqrt{[-bc-bk]})\vee\theta\vee\delta\vee(k>-c\wedge d<(bc+bk)/(a+h))))\vee\beta))$$
$$\omega=(h>-a\wedge k>(-a^2-bc-2ah-h^2)/b\wedge(bc+bk)/(a+h)<d<a+h-2\sqrt{[-bc-bk]})$$
$$\rho=(k\geq(-a^2-bc-2ah-h^2)/b\wedge d<a+h-2\sqrt{[-bc-bk]})$$
$$\iota=(d<a+h-2\sqrt{[-bc-bk]}\vee a+h+2\sqrt{[-bc-bk]}<d<(bc+bk)/(a+h))$$
$$\gamma=(k=-c\wedge(d<a+h+2\sqrt{[-bc-bk]}\vee a+h+2\sqrt{[-bc-bk]}<d<(bc+bk)/(a+h)))$$
$$\psi=(h<-a\wedge((k<-c\wedge d<(bc+bk)/(a+h))\vee\gamma\vee(-c<k<(-a^2-bc-2ah-h^2)/b\wedge\iota)\vee\rho))$$

finally:

$$(b<0\wedge(\psi\vee(h=-a\wedge k>-c\wedge d<a+h-2\sqrt{[-bc-bk]})\vee\omega))\vee(b=0\wedge h<-a\wedge(d<a+h\vee a+h<d<0))\vee\varepsilon$$

3.- ***Spiral Source*** $((a+h)+d)^2-4((a+h)d-b(c+k))<0\wedge((a+h)+d)>0$

result:

$$\alpha=((-a^2-bc-2ah-h^2)/b<k<-c\wedge a+h-2\sqrt{[-bc-bk]}<d<a+h+2\sqrt{[-bc-bk]})$$
$$\beta=(h>-a\wedge((k\leq(-a^2-bc-2ah-h^2)/b\wedge-a-h<d<a+h+2\sqrt{[-bc-bk]})\vee\alpha))$$
$$\theta=(b>0\wedge((h\leq-a\wedge k<(-a^2-bc-2ah-h^2)/b\wedge-a-h<d<a+h+2\sqrt{[-bc-bk]})\vee\beta))$$
$$\varepsilon=(-c<k<(-a^2-bc-2ah-h^2)/b\wedge a+h-2\sqrt{[-bc-bk]}<d<a+h+2\sqrt{[-bc-bk]})$$
$$\nu=(\varepsilon\vee(k\geq(-a^2-bc-2ah-h^2)/b\wedge-a-h<d<a+h+2\sqrt{[-bc-bk]}))$$

finally:

$$(b<0\wedge((h\leq-a\wedge k>(-a^2-bc-2ah-h^2)/b\wedge-a-h<d<a+h+2\sqrt{[-bc-bk]})\vee(h>-a\wedge\nu)))\vee\theta$$

4.-**Nodal Source** $((a+h)+d)^2-4((a+h)d-b(c+k))>0\wedge\lambda_1\lambda_2>0\wedge\lambda_1+\lambda_2>0$

result:

$$\alpha=((c\leq-(a^2/b)\wedge d>a+2\sqrt{-bc})\vee(-(a^2/b)<c\leq 0\wedge((bc)/a<d<a-2\sqrt{-bc}\vee d>a+2\sqrt{-bc}))\vee(c>0\wedge d>(bc)/a))$$
$$\beta=((c<0\wedge d>(bc)/a)\vee(0\leq c<-(a^2/b)\wedge((bc)/a<d<a-2\sqrt{-bc}\vee d>a+2\sqrt{-bc}))\vee(c\geq-(a^2/b)\wedge d>a+2\sqrt{-bc}))$$
$$\delta=(a>0\wedge((b<0\wedge\beta)\vee(b=0\wedge(0<d<a\vee d>a))\vee(b>0\wedge\alpha)))$$
$$\theta=(a=0\wedge((b<0\wedge c>0\wedge d>2\sqrt{-bc})\vee(b>0\wedge c<0\wedge d>2\sqrt{-bc})))\vee\delta$$
$$\varepsilon=(a<0\wedge((b<0\wedge c>-(a^2/b)\wedge a+2\sqrt{-bc}<d<(bc)/a)\vee(b>0\wedge c<-(a^2/b)\wedge a+2\sqrt{-bc}<d<(bc)/a)))$$

finally:

$$\varepsilon\vee\theta$$

5.- **Saddle** $((a+h)+d)^2-4((a+h)d-b(c+k))>0\wedge(\lambda_1)(\lambda_2)<0$

result:

$$\alpha=(a>0\wedge((b<0\wedge c>-(a^2/b)\wedge(bc)/a<d<a-2\sqrt{-bc})\vee(b>0\wedge c<-(a^2/b)\wedge(bc)/a<d<a-2\sqrt{-bc})))$$
$$\beta=(a=0\wedge((b<0\wedge c>0\wedge d<-2\sqrt{-bc})\vee(b>0\wedge c<0\wedge d<-2\sqrt{-bc})))$$
$$\delta=(c>0\wedge d<(bc)/a)$$

finally:

$$\rho\vee\beta\vee\alpha$$

6.- ***Center*** $((a+h)+d)^2-4((a+h)d-b(c+k))<0\wedge((a+h)+d)=0$

result:

$$\alpha=((b<0\wedge c>-(a^2/b))\vee(b>0\wedge c<-(a^2/b)))$$
$$\beta=(d=-a)$$

finally:

$$\beta\wedge\alpha$$

7.-**Degenerate Nodal Source** $((a+h)+d)^2-4((a+h)d-b(c+k))=0 \wedge \lambda_{12}=((a+h)+d)/2 \wedge ((a+h)+d)>0$

result:

$\alpha=(b>0 \wedge ((c \leq -(a^2/b) \wedge d=a+2\sqrt{[-b\,c]}) \vee (-(a^2/b)<c<0 \wedge (d=a-2\sqrt{[-b\,c]} \vee d=a+2\sqrt{[-b\,c]})) \vee (c=0 \wedge d=a)))$

$\beta=((b<0 \wedge (\delta \vee (0<c<-(a^2/b) \wedge (d=a-2\sqrt{[-b\,c]} \vee d=a+2\sqrt{[-b\,c]})) \vee (c \geq -(a^2/b) \wedge d=a+2\sqrt{[-b\,c]}))) \vee (b=0 \wedge d=a) \vee \alpha)$

$\delta=(c=0 \wedge d=a)$

$\theta=(a \leq 0 \wedge ((b<0 \wedge c>-(a^2/b) \wedge d=a+2\sqrt{[-b\,c]}) \vee (b>0 \wedge c<-(a^2/b) \wedge d=a+2\sqrt{[-b\,c]})))$

$\varepsilon=(a>0 \wedge beta)$

finally:

$\varepsilon \vee \theta$

8.- **Degenerate Nodal Sink** $((a+h)+d)^2-4((a+h)d-b(c+k))=0 \wedge \lambda_{12}=((a+h)+d)/2 \wedge ((a+h)+d)<0$

result:

$\alpha=(a \geq 0 \wedge ((b<0 \wedge c>-(a^2/b) \wedge d=a-2\sqrt{[-b\,c]}) \vee (b>0 \wedge c<-(a^2/b) \wedge d=a-2\sqrt{[-b\,c]})))$

$\beta=(b>0 \wedge ((c \leq -(a^2/b) \wedge d=a-2\sqrt{[-b\,c]}) \vee (-(a^2/b)<c<0 \wedge (d=a-2\sqrt{[-b\,c]} \vee d=a+2\sqrt{[-b\,c]})) \vee (c=0 \wedge d=a)))$

$\delta=(b<0 \wedge ((c=0 \wedge d=a) \vee (0<c<-(a^2/b) \wedge (d=a-2\sqrt{[-b\,c]} \vee d=a+2\sqrt{[-b\,c]})) \vee (c \geq -(a^2/b) \wedge d=a-2\sqrt{[-b\,c]})))$

$\theta=(b=0 \wedge d=a)$

$\varepsilon=a<0$

finally:

$(\varepsilon \wedge (\delta \vee \theta \vee \beta)) \vee \alpha$

9.- ***Unstable Saddle Node*** $((a+h)d-b(c+k))=0 \wedge ((a+h)+d) \neq 0, ((a+h)+d)>0, \lambda_1=0, \lambda_2=((a+h)+d)$

result:

$\alpha=((a<0 \wedge ((b<0 \wedge c>-(a^2/b)) \vee (b>0 \wedge c<-(a^2/b)))) \vee (a>0 \wedge ((b<0 \wedge c<-(a^2/b)) \vee b=0 \vee (b>0 \wedge c>-(a^2/b))))) \wedge d=(b\,c)/a$

$\beta=(a=0 \wedge ((b<0 \wedge c=0 \wedge d>0) \vee (b=0 \wedge d>0) \vee (b>0 \wedge c=0 \wedge d>0)))$

finally:

$\beta \vee \alpha$

10.-***Stable Saddle Node*** $((a+h)d-b(c+k))=0\wedge((a+h)+d)\neq 0,((a+h)+d)<0,\lambda_1=0,\lambda_2=((a+h)+d)$

result:

$\alpha=((a<0\wedge((b<0\wedge c<-(a^2/b))\vee b=0\vee(b>0\wedge c>-(a^2/b))))\vee(a>0\wedge((b<0\wedge c>-(a^2/b))\vee(b>0\wedge c<-(a^2/b)))))\wedge d=(b\,c)/a$

$\beta=(a=0\wedge((b<0\wedge c=0\wedge d<0)\vee(b=0\wedge d<0)\vee(b>0\wedge c=0\wedge d<0)))$

finally:

$\beta\vee\alpha$

11.- **Both Eigen Values are 0** $\lambda_1=\lambda_2=0$

result:

$\alpha=(a>0\wedge((b<0\wedge c=-(a^2/b)\wedge d=(b\,c)/a)\vee(b>0\wedge c=-(a^2/b)\wedge d=(b\,c)/a)))$

$\beta=(a=0\wedge((b<0\wedge c=0\wedge d=0)\vee(b=0\wedge d=0)\vee(b>0\wedge c=0\wedge d=0)))$

$\delta=(a<0\wedge((b<0\wedge c=-(a^2/b)\wedge d=(b\,c)/a)\vee(b>0\wedge c=-(a^2/b)\wedge d=(b\,c)/a)))$

finally:

$\delta\vee\beta\vee\alpha$

CONCLUSIONS

We have found all possible types of equilibrium which can occur in 2D systems: saddle, non-stable node, stable node, center, non-stable spiral and stable spiral. By analyzing eigen values of the selected system.

With the ranges exposed above we are able to determine the behavior of the system once we close the loop on a second order system with the entry specified above, with this and according with the control theory we could modify systems to give a certain wanted behavior. For example closing the loop will allow us to make a system that is not stable, stable by changing the equilibrium behavior from a source to a sink, this is very useful on a lot of topics.

These results are applicable to any linear system which can be expressed in the form of system (1.1).

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