Calculus of parametric equations *

Dr Juan H Klopper

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1 Introduction

2 Arc length

We consider an arc as a segment of a parametric curve, $(x, y) = (x(t), y(t))$, of the interval of the parameter $a \le t \le b$. The length of this arc can be calculated if it, as well as $x'(t)$ and $y'(t)$, are continuous on the interval and the arc does not intersect itself, other than perhaps, at a finite number of points.

The arc is divided into n equal length sections. For two consecutive points on the arc $((x(t_{i-1}), y(t_{i-1})), (x(t_i), y(t_i)))$ for $i = \{1, 2, 3, \ldots, n\}$. On the arc we then have n subintervals $a = t_0 < t_1 < t_2 < \ldots < t_n = b$ where each subinterval has length Δt and (1) holds.

$$t_i - t_{i-1} = \Delta t \frac{b-a}{n} \tag{1}$$

The length, s_i, of the subinterval, i, is shown in (2, 3).

$$s_i \approx d\left[\left(\left(x(t_{i-1}), y(t_{i-1})\right), \left(x(t_i), y(t_i)\right)\right)\right] \tag{2}$$

$$s_i \approx \sqrt{\left[x(t_i) - x(t_{i-1})\right]^2 + \left[y(t_i) - y(t_{i-1})\right]^2} \tag{3}$$

From the mean value theorem, there will be a point on the interval $c = (a, b)$ where (4, 5) hold.

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$$\frac{dx}{dt}\left(c_i\right) = \frac{x\left(t_i\right) - x\left(t_{i-1}\right)}{t_i - t_{i-1}} \to x\left(t_i\right) - x\left(t_{i-1}\right) = \frac{dx\left(c_i\right)}{dt}\Delta t \tag{4}$$

$$\frac{dy}{dt}\left(d_i\right) = \frac{y\left(t_i\right) - y\left(t_{i-1}\right)}{t_i - t_{i-1}} \to y\left(t_i\right) - y\left(t_{i-1}\right) = \frac{dy\left(d_i\right)}{dt}\Delta t \tag{5}$$

Here, c_i and d_i are two points in the interval (t_{i-1}, t_i). By substitution the subinterval length in (3) is then as shown in (6, 7).

$$s_i \approx \sqrt{\left[\frac{dx}{dt}\left(c_i\right)\Delta t\right]^2 + \left[\frac{dy}{dt}\left(d_i\right)\right]^2} \tag{6}$$

$$s_i \approx \sqrt{\left[\frac{dx}{dt}\left(c_i\right)\right]^2 + \left[\frac{dy}{dt}\left(d_i\right)\right]^2}\,\Delta t \tag{7}$$

The total arc length, s, is then approximated in (8).

$$s \approx \sum_{i-1}^{n} \sqrt{\left[\frac{dx}{dt}\left(c_i\right)\right]^2 + \left[\frac{dy}{dt}\left(d_i\right)\right]^2}\,\Delta t \tag{8}$$

The arc length, s, is then shown in (9) as $n \to \infty$.

$$s = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt \tag{9}$$