Calculus of polar coordinates *

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1 Derivatives of polar coordinate functions

Definition 1.1. *The derivative of a polar equation in rectangular coordinates is given in (1).*

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{d\theta}{dt}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \tag{1}$$

Derivation 1.1. *In polar equations r is a function of θ, seen in (2). The familiar conversions from rectangular coordinates to polar coordinates are shown in (3) and (4).*

$$r = f(\theta) \tag{2}$$
$$x = r\cos\theta = f(\theta)\cos\theta \tag{3}$$
$$y = r\sin\theta = f(\theta)\sin\theta \tag{4}$$

The derivatives of y and x are shown in (5) and (6), where the power rule of differentiation is applied.

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta \tag{5}$$
$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \tag{6}$$

This gives rise to equation (1).

*A course in vector calculus

2 Integrals of polar coordinate functions

Definition 2.1. *The integral of a polar coordinate curve, $r = f(\theta)$, on the interval $\theta = [a, b]$, given that r is continuous and positive on this interval, is given in (7)*

$$A = \int_a^b \frac{1}{2}(f(\theta))^2 \, d\theta \tag{7}$$

Derivation 2.1. *The fraction of a circle with radius r at angle θ (counterclockwise from the positive x-axis) is given in (8).*

$$A = \pi r^2 \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta \tag{8}$$

Dividing the interval $\theta = a$ to $\theta = b$ up into n equal angled sections results in (9).

$$\Delta\theta = \frac{b - a}{n} = \theta_i - \theta_{i-1}, n \in \mathbb{N} \tag{9}$$

The approximate area of one such sectional area, A_i, is given in (10).

$$A_i \approx \frac{1}{2}(f(\theta_i))^2 \Delta\theta \tag{10}$$

The addition of all the sectional areas, A_i is given in (11).

$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{1}{2}(f(\theta_i))^2 \Delta\theta \tag{11}$$

Taking the limit as $n \to \infty$ results in the integral given in (12).

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2}(f(\theta_i))^2 \Delta\theta = \int_a^b \frac{1}{2}(f(\theta))^2 \, d\theta \tag{12}$$