Calculus of parametric equations *

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1 Introduction

2 Derivatives of parametric equations

Definition 2.1. *If $x = x\left(t\right)$ and $y = y\left(t\right)$ both have derivatives that are continuous at $t = c$, then the **chain rule** is given in (1).*

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \tag{1}$$

Derivation 2.1. *The derivation of the first derivative of parametric equations, $\left(x, y\right) = \left[f\left(t\right), g\left(t\right)\right]$, considers a section of the parametric-defined curve, termed $F\left(x\right)$ on some interval $a \le t \le b$. Since $y = g\left(t\right)$ and $y = F\left(x\right)$ for this interval, we have that $g\left(t\right) = F\left(f\left(t\right)\right)$. We now take the derivative of both sides with respect to t in (2-4).*

$$\frac{dy}{dt} = \frac{d}{dt}\left[F\left(f\left(t\right)\right)\right] \text{ note that } F\left(f\left(t\right)\right) = F\left(x\right) \tag{2}$$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \text{ (chain rule)} \tag{3}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{4}$$

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Definition 2.2. *If the denominator in (1) is not zero, then the **derivative of a parametric equation** is given in (5).*

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{5}$$

Definition 2.3. *The **second order derivative of a parametric equation** is given in (6).*

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{d}{dx}\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \tag{6}$$

Derivation 2.2. *The derivation is given in (7-11).*

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right] \tag{7}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}}{\frac{dx}{dt}}\left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right] \text{ multiply by } \frac{\frac{1}{dt}}{\frac{1}{dt}} \tag{8}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{dy}{dt}}{\frac{dx}{dt}\frac{dx}{dt}} \tag{9}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{dy}{dt}\frac{dt}{dx}}{\frac{dx}{dt}} \text{ rewrite } \frac{dx}{dt} \text{ in denominator to numerator} \tag{10}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} \text{ simplify} \tag{11}$$

Theorem 2.1. *Let $x'(t)$ and $y'(t)$ be continuous functions. Then, for the curve defined by the parametric equations $x = x(t)$ and $y = y(t)$, we have the following:*

3 Horizontal and vertical tangent lines

For the parametric equations shown in (5), we have a horizontal tangent line when (12) holds.

$$\frac{dy}{dt} = 0 \text{ whenever } \frac{dx}{dt} \neq 0 \tag{12}$$

For the parametric equations shown in (5), we have a vertical tangent line when (13) holds.

$$\frac{dx}{dt} = 0 \text{ whenever } \frac{dy}{dt} \neq 0 \tag{13}$$

4 Area under a parametric curve

A function, $y = f(x)$, that is continuous on an interval $[a, b]$, has an area under the curve for the interval as given in (14).

$$A = \int_a^b f(x)\, dx = \int_a^b y\, dx \tag{14}$$

Suppose that a parameter, t, indicates clockwise traversal of a parametric curve on the interval $c \leq t \leq d$, and that on this interval the curve does not intersect itself (except perhaps that the starting and end-point coincide, $x(c) = x(d)$ and $y(c)) = y(d)$), then the enclosed area is given in (15-17).

Derivation 4.1.

$$x = x(t), y = y(t) \tag{15}$$

$$\frac{dx}{dt} = x'(t) \rightarrow dx = x'(t)\, dx \tag{16}$$

$$\int_a^b y dt = \int_c^d y(t)\, x'(t)\, dt \tag{17}$$

If the curve is traversed counterclockwise then the derivative is as given in (18).

$$\int_c^d y'(t)\, x(t)\ dt = -\int_c^d y(t)\, x'(t)\ dt \tag{18}$$