A Practical Comparison Between Hopfield Networks and Restricted Boltzmann Machines as Content-Addressable Autoassociative Memories

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Abstract

In this paper we analyze the performance of Hopfield Networks and Restricted Boltzmann Machines when they are used as autoassociative memories for several kinds of addressable binary content. The aim of this work is to recommend to the reader a model that offers the best results for a specific given pattern data set, based on the results obtained after testing several parameterized models against different random pattern data sets that satisfy the same statistical properties.

Keywords: Artificial Neural Networks, Recurrent Neural Networks, Hopfield Networks, Restricted Boltzmann Machine, Autoassociative Memory, Addressable Content

1. Problem statement and goals

Attractor networks such as Hopfield Networks [Hopfield (1982)] and Restricted Boltzmann Machines [Smolensky (1987)] are widely used as binary content-addressable memory systems, [DAI (1998)][SCH (1995)][Krizhevsky and Hinton] and the theory behind them has been deeply studied in the past years. [S. V. B. Aiyer and Fallside (1990)][Zhuang and Huang (1993)][Nagatani and Hagiwara (2014)] Both models have been compared on specific problems [R. Sammouda and Nishitani (1996)] and mathematical relations of equivalence have been demonstrated. [Bar (2012)][Agl (2013)]

In this paper we analyze the behaviour of both models after being trained with several random pattern data sets that satisfy certain statistical properties. These properties can be extracted from any kind of binary data set and, in conjunction with the result tables presented in this paper, the reader can have an approximate idea about what model offers the best results for his specific problem. In addition, if the reader has a sample of expected input for the system, it is possible to extract other statistical properties from the sample and compare them with the statistical properties of the tested random input pattern data sets.

What we expected:

- A higher ability to memorize patterns/capacity of Hopfield Networks trained with the Storkey rule vs Hebbian rule.
- A worse performance of Hopfield networks when:
  - Incrementing the number of patterns to memorize.
  - Incrementing the correlation between the patterns to memorize.
- Nothing special about RBMs (perhaps the possibility to adapt to the data by tuning its parameters? Able to memorize whatever thanks to overfitting?).

2. Analyzed Models

Hopfield Networks are well-known content-addressable memories for which the Hebbian learning rule has been the traditional training approach. [Hopfield (1982)][Hebb (1950)] This paper analyzes the Storkey learning rule [Storkey (1997)] as well as the previously mentioned Hebbian. Storkey is being considered because it has been proved to provide a great increase in the capacity of the network, that is, it is able to recall more patterns.

Additionally, units in the Hopfield Network may be updated either synchronously or asynchronously. This paper only contains an analysis of the asynchronous method, since the synchronous is considered less realistic based on the absence of observed global clock influencing analogous biological or physical systems of interest. [MacKay (2003)]

Boltzmann Machines are usually considered a stochastic analogue of the Hopfield Networks. [Ack (1985)] Despite their theoretical usefulness, it is well known that in practice they cannot learn properly for large enough problems. Thats why we are considering its constrained version, the Restricted Boltzmann Machine, that permits an efficient training using the Contrastive Divergence algorithm. [Carreira-Perpinan and Hinton (2005)]

Learning in a Restricted Boltzmann Machine is dependent on several parameters that have to be tuned appropriately. Our experiments try to follow Hintons recommendations [Hinton (2012)] on the following issues, although some details have been implemented in a different way:

3. Previous work

Hopfield Networks and Restricted Boltzman Machines had been compared several times for associative memory, even though both have divers applications, this is the most known one, at least for Hopfield Networks. This comparison has been made by applying both Neural Networks to several databases, which raises a new problem, how to generate this databases. For this task several algorithm had been proposed, genetic algorithms between them, however the solution was a simple randomized setting to generate patterns with $0s$ and $1s$ of length N. The databases are classified by certain property in relation with the Hamming distance between vectors. For this we try to assure that the distance between vectors in this datasets has an standard deviation of σ and a mean of μ.

This properties along with the size of the vectors L and the total number of vectors N will characterize the datasets for training and testing both algorithms.

4. The CI methods

Do not repeat well-known theory or formulas. Just mention which methods you use and why you choose them, and provide relevant citations.

5. Results and Discussion

The main part of the document.

6. Strengths and weaknesses

Be critic with your work ...

7. Conclusions and future work

The conclusions are not a mere repetition of the abstract. Basically, you should describe "what you know now that you did *not* before doing the work". In addition, mention what would be natural follow-up lines of work.

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Appendix A. Annalyzed Pattern Data Sets

	P_1	P_2	P_3	P_4	P_5	P_6	P_7
$\|P_i\|$	5	10	15	20	25	30	35
n	100	100	100	100	100	100	100
$\frac{\mu_{P_i}}{n}$	0.20	0.20	0.20	0.20	0.20	0.20	0.20
$\frac{\sigma_{P_i}}{n}$	0.02	0.02	0.02	0.02	0.02	0.02	0.02

Table 1: Analyzed pattern data sets (1 of 4)

	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
$\|P_i\|$	5	10	15	20	25	30	35
n	100	100	100	100	100	100	100
$\frac{\mu_{P_i}}{n}$	0.30	0.30	0.30	0.30	0.30	0.30	0.30
$\frac{\sigma_{P_i}}{n}$	0.03	0.03	0.03	0.03	0.03	0.03	0.03

Table 2: Analyzed pattern data sets (2 of 4)

	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}
$\|P_i\|$	5	10	15	20	25	30	35
n	100	100	100	100	100	100	100
$\frac{\mu_{P_i}}{n}$	0.40	0.40	0.40	0.40	0.40	0.40	0.40
$\frac{\sigma_{P_i}}{n}$	0.04	0.04	0.04	0.04	0.04	0.04	0.04

Table 3: Analyzed pattern data sets (3 of 4)

	P_{22}	P_{23}	P_{24}	P_{25}	P_{26}	P_{27}	P_{28}
$\|P_i\|$	5	10	15	20	25	30	35
n	100	100	100	100	100	100	100
$\frac{\mu_{P_i}}{n}$	0.50	0.50	0.50	0.50	0.50	0.50	0.50
$\frac{\sigma_{P_i}}{n}$	0.05	0.05	0.05	0.05	0.05	0.05	0.05

Table 4: Analyzed pattern data sets (4 of 4)

Appendix B. Analyzed Models

	M_1	M_2	M_3	M_4
(1)	Hopfield	Hopfield	RBM	RBM
(2)	Hebbian	Storkey	CD	CD
(3)	n/a	n/a	50	100
(4)	n/a	n/a	1	1

Table 5: Analyzed models

(1) Model

(2) Learning rule.

(3) Number of hidden neurons.

(4) Patterns per batch

Appendix C. Tested Input Data Sets

	I_1	I_2	I_3	I_4	I_5
(1)	10	10	10	10	10
$\frac{\mu_{I_i}}{\mu_{P_j}}$	0.1 ± 0.0	0.2 ± 0.0	0.3 ± 0.0	0.4 ± 0.0	0.5 ± 0.0

Table 6: Randomly generated and tested input data sets

(1) Number of generated inputs per each pattern of the pattern data set, $\frac{|I_i|}{|P_j|}$

Appendix D. Training and Validation Results

	M_1	M_2	M_3	M_4
P_1	n/a	n/a	18124 ± 7069	14541 ± 2420
P_2	n/a	n/a	29057 ± 8731	11242 ± 2781
P_3	n/a	n/a	33583 ± 11031	12570 ± 3410
P_4	n/a	n/a	28940 ± 16282	12377 ± 5766
P_5	n/a	n/a	35530 ± 14682	9834 ± 2972
P_6	n/a	n/a	25162 ± 9013	12035 ± 2854
P_7	n/a	n/a	22319 ± 8792	12006 ± 3489
P_8	n/a	n/a	31690 ± 8718	21964 ± 4678
P_9	n/a	n/a	35074 ± 8181	18915 ± 3348
P_{10}	n/a	n/a	42173 ± 21376	23287 ± 9620
P_{11}	n/a	n/a	42339 ± 25028	26494 ± 15153
P_{12}	n/a	n/a	64959 ± 39081	36511 ± 14325
P_{13}	n/a	n/a	38354 ± 22668	24570 ± 11618
P_{14}	n/a	n/a	36352 ± 23636	30642 ± 14164
P_{15}	n/a	n/a	44274 ± 14970	44308 ± 13092
P_{16}	n/a	n/a	60494 ± 12280	37955 ± 7551
P_{17}	n/a	n/a	86497 ± 46676	50112 ± 22391
P_{18}	n/a	n/a	64698 ± 33553	37774 ± 12024
P_{19}	n/a	n/a	95758 ± 71621	60481 ± 30881
P_{20}	n/a	n/a	76866 ± 60918	98700 ± 16541
P_{21}	n/a	n/a	58147 ± 41255	74267 ± 29266
P_{22}	n/a	n/a	74411 ± 38141	59317 ± 11266
P_{23}	n/a	n/a	69858 ± 33073	66048 ± 10678
P_{24}	n/a	n/a	132268 ± 57657	82277 ± 23651
P_{25}	n/a	n/a	217777 ± 271439	93176 ± 38089
P_{26}	n/a	n/a	53481 ± 55669	114696 ± 31592
P_{27}	n/a	n/a	43289 ± 35583	131611 ± 65733
P_{28}	n/a	n/a	56949 ± 69490	151782 ± 41599

Table 7: Number of epochs needed to train model M_i with pattern data set P_j

	M_1	M_2	M_3	M_4
P_1	0.00 ± 0.00	0.42 ± 0.12	1.00 ± 0.00	1.00 ± 0.00
P_2	0.00 ± 0.00	0.17 ± 0.04	1.00 ± 0.00	1.00 ± 0.00
P_3	0.00 ± 0.00	0.07 ± 0.02	1.00 ± 0.00	1.00 ± 0.00
P_4	0.00 ± 0.00	0.05 ± 0.00	0.87 ± 0.33	1.00 ± 0.00
P_5	0.00 ± 0.00	0.04 ± 0.00	0.95 ± 0.09	1.00 ± 0.01
P_6	0.00 ± 0.00	0.04 ± 0.01	0.99 ± 0.01	0.99 ± 0.01
P_7	0.00 ± 0.00	0.03 ± 0.01	0.80 ± 0.33	1.00 ± 0.01
P_8	0.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00
P_9	0.00 ± 0.00	0.47 ± 0.10	1.00 ± 0.00	1.00 ± 0.00
P_{10}	0.00 ± 0.00	0.30 ± 0.06	1.00 ± 0.00	1.00 ± 0.00
P_{11}	0.00 ± 0.00	0.17 ± 0.04	1.00 ± 0.00	1.00 ± 0.00
P_{12}	0.00 ± 0.00	0.10 ± 0.03	1.00 ± 0.00	1.00 ± 0.00
P_{13}	0.00 ± 0.00	0.10 ± 0.02	0.98 ± 0.04	1.00 ± 0.00
P_{14}	0.00 ± 0.00	0.05 ± 0.02	1.00 ± 0.01	1.00 ± 0.00
P_{15}	0.82 ± 0.16	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00
P_{16}	0.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00
P_{17}	0.00 ± 0.00	0.93 ± 0.03	1.00 ± 0.00	1.00 ± 0.00
P_{18}	0.00 ± 0.00	0.81 ± 0.09	1.00 ± 0.00	1.00 ± 0.00
P_{19}	0.00 ± 0.00	0.61 ± 0.06	1.00 ± 0.00	1.00 ± 0.00
P_{20}	0.00 ± 0.00	0.44 ± 0.07	1.00 ± 0.00	1.00 ± 0.00
P_{21}	0.00 ± 0.00	0.25 ± 0.04	1.00 ± 0.00	1.00 ± 0.00
P_{22}	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00
P_{23}	0.99 ± 0.03	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00
P_{24}	0.72 ± 0.12	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00
P_{25}	0.42 ± 0.09	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00
P_{26}	0.08 ± 0.03	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00
P_{27}	0.03 ± 0.03	0.99 ± 0.02	1.00 ± 0.00	1.00 ± 0.00
P_{28}	0.01 ± 0.02	0.96 ± 0.03	1.00 ± 0.00	1.00 ± 0.00

Table 8: Number of stored patterns, proportional to $|P_i|$

Appendix E. Testing Results

E.1. Testing Results for Pattern Data Set P_1

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.370 ± 0.066	0.375 ± 0.071	0.305 ± 0.046	0.273 ± 0.042	0.258 ± 0.045
M_3	1.000 ± 0.000	0.998 ± 0.007	0.995 ± 0.009	0.948 ± 0.041	0.855 ± 0.060
M_4	1.000 ± 0.000	1.000 ± 0.000	0.998 ± 0.007	0.972 ± 0.026	0.840 ± 0.036

Table 9: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_1, proportional to $|I_i|$

E.2. Testing Results for Pattern Data Set P_2

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.164 ± 0.040	0.149 ± 0.042	0.140 ± 0.035	0.128 ± 0.025	0.115 ± 0.022
M_3	0.991 ± 0.014	0.938 ± 0.055	0.752 ± 0.101	0.490 ± 0.104	0.254 ± 0.064
M_4	0.998 ± 0.007	0.966 ± 0.017	0.801 ± 0.065	0.528 ± 0.049	0.230 ± 0.040

Table 10: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_2, proportional to $|I_i|$

E.3. Testing Results for Pattern Data Set P_3

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.068 ± 0.004	0.068 ± 0.002	0.067 ± 0.000	0.067 ± 0.000	0.067 ± 0.003
M_3	0.930 ± 0.053	0.700 ± 0.091	0.383 ± 0.085	0.162 ± 0.036	0.063 ± 0.022
M_4	0.981 ± 0.008	0.833 ± 0.067	0.494 ± 0.061	0.195 ± 0.049	0.054 ± 0.018

Table 11: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_3, proportional to $|I_i|$

E.4. Testing Results for Pattern Data Set P_4

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.050 ± 0.000	0.050 ± 0.000	0.050 ± 0.000	0.050 ± 0.000	0.050 ± 0.000
M_3	0.746 ± 0.288	0.443 ± 0.185	0.179 ± 0.077	0.053 ± 0.024	0.007 ± 0.006
M_4	0.907 ± 0.044	0.586 ± 0.065	0.219 ± 0.047	0.048 ± 0.020	0.008 ± 0.004

Table 12: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_4, proportional to $|I_i|$

E.5. Testing Results for Pattern Data Set P_5

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.040 ± 0.000	0.040 ± 0.000	0.040 ± 0.000	0.040 ± 0.001	0.040 ± 0.001
M_3	0.662 ± 0.103	0.283 ± 0.059	0.075 ± 0.017	0.018 ± 0.008	0.003 ± 0.003
M_4	0.829 ± 0.049	0.378 ± 0.074	0.087 ± 0.024	0.011 ± 0.011	0.002 ± 0.002

Table 13: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_5, proportional to $|I_i|$

E.6. Testing Results for Pattern Data Set P_6

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.033 ± 0.000	0.033 ± 0.002	0.033 ± 0.000	0.032 ± 0.002	0.032 ± 0.003
M_3	0.609 ± 0.064	0.175 ± 0.042	0.042 ± 0.013	0.006 ± 0.005	0.001 ± 0.002
M_4	0.746 ± 0.086	0.248 ± 0.056	0.035 ± 0.011	0.004 ± 0.003	0.000 ± 0.000

Table 14: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_6, proportional to $|I_i|$

E.7. Testing Results for Pattern Data Set P_7

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.032 ± 0.009	0.031 ± 0.006	0.029 ± 0.001	0.029 ± 0.000	0.026 ± 0.003
M_3	0.388 ± 0.162	0.112 ± 0.049	0.020 ± 0.011	0.003 ± 0.004	0.000 ± 0.000
M_4	0.672 ± 0.097	0.185 ± 0.041	0.018 ± 0.007	0.000 ± 0.001	0.000 ± 0.000

Table 15: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_7, proportional to $|I_i|$

E.8. Testing Results for Pattern Data Set P_8

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.998 ± 0.007	0.968 ± 0.042	0.890 ± 0.087	0.843 ± 0.087	0.735 ± 0.086
M_3	1.000 ± 0.000	1.000 ± 0.000	1.000 ± 0.000	0.985 ± 0.017	0.902 ± 0.064
M_4	1.000 ± 0.000	1.000 ± 0.000	1.000 ± 0.000	0.995 ± 0.009	0.950 ± 0.035

Table 16: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_8, proportional to $|I_i|$

E.9. Testing Results for Pattern Data Set P_9

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.440 ± 0.077	0.395 ± 0.072	0.345 ± 0.034	0.295 ± 0.036	0.273 ± 0.038
M_3	1.000 ± 0.000	0.996 ± 0.005	0.921 ± 0.033	0.736 ± 0.074	0.471 ± 0.073
M_4	1.000 ± 0.000	1.000 ± 0.000	0.966 ± 0.015	0.820 ± 0.039	0.474 ± 0.037

Table 17: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_9, proportional to $|I_i|$

E.10. Testing Results for Pattern Data Set P_{10}

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.264 ± 0.041	0.213 ± 0.043	0.172 ± 0.027	0.132 ± 0.026	0.109 ± 0.017
M_3	0.994 ± 0.006	0.928 ± 0.027	0.709 ± 0.059	0.377 ± 0.079	0.133 ± 0.042
M_4	0.999 ± 0.002	0.978 ± 0.014	0.822 ± 0.038	0.470 ± 0.076	0.152 ± 0.033

Table 18: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{10}, proportional to $|I_i|$

E.11. Testing Results for Pattern Data Set P_{11}

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.153 ± 0.030	0.127 ± 0.028	0.108 ± 0.026	0.089 ± 0.025	0.078 ± 0.023
M_3	0.981 ± 0.013	0.814 ± 0.049	0.473 ± 0.045	0.166 ± 0.024	0.037 ± 0.009
M_4	0.992 ± 0.018	0.918 ± 0.049	0.587 ± 0.084	0.210 ± 0.043	0.031 ± 0.012

Table 19: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{11}, proportional to $|I_i|$

E.12. Testing Results for Pattern Data Set P_{12}

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.083 ± 0.026	0.074 ± 0.023	0.060 ± 0.017	0.049 ± 0.009	0.042 ± 0.010
M_3	0.948 ± 0.029	0.611 ± 0.071	0.232 ± 0.027	0.054 ± 0.017	0.007 ± 0.004
M_4	0.988 ± 0.008	0.820 ± 0.046	0.365 ± 0.041	0.079 ± 0.017	0.006 ± 0.002

Table 20: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{12}, proportional to $|I_i|$

E.13. Testing Results for Pattern Data Set P_{13}

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.082 ± 0.013	0.074 ± 0.008	0.051 ± 0.009	0.041 ± 0.007	0.028 ± 0.009
M_3	0.823 ± 0.063	0.436 ± 0.080	0.096 ± 0.024	0.015 ± 0.010	0.003 ± 0.003
M_4	0.969 ± 0.015	0.650 ± 0.064	0.185 ± 0.028	0.023 ± 0.010	0.001 ± 0.001

Table 21: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{13}, proportional to $|I_i|$

E.14. Testing Results for Pattern Data Set P_{14}

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.048 ± 0.012	0.042 ± 0.011	0.036 ± 0.009	0.028 ± 0.004	0.023 ± 0.004
M_3	0.754 ± 0.053	0.285 ± 0.052	0.051 ± 0.013	0.006 ± 0.005	0.000 ± 0.001
M_4	0.934 ± 0.032	0.479 ± 0.078	0.098 ± 0.019	0.005 ± 0.003	0.000 ± 0.001

Table 22: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{14}, proportional to $|I_i|$

E.15. Testing Results for Pattern Data Set P_{15}

	I_1	I_2	I_3	I_4	I_5
M_1	0.723 ± 0.103	0.583 ± 0.053	0.425 ± 0.060	0.272 ± 0.055	0.170 ± 0.070
M_2	1.000 ± 0.000	1.000 ± 0.000	1.000 ± 0.000	0.998 ± 0.007	0.990 ± 0.014
M_3	1.000 ± 0.000	1.000 ± 0.000	0.998 ± 0.007	0.970 ± 0.032	0.912 ± 0.077
M_4	1.000 ± 0.000	1.000 ± 0.000	1.000 ± 0.000	0.995 ± 0.009	0.960 ± 0.032

Table 23: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{15}, proportional to $|I_i|$

E.16. Testing Results for Pattern Data Set P_{16}

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.988 ± 0.029	0.981 ± 0.033	0.970 ± 0.033	0.925 ± 0.046	0.847 ± 0.033
M_3	1.000 ± 0.000	0.994 ± 0.013	0.954 ± 0.033	0.810 ± 0.108	0.529 ± 0.110
M_4	1.000 ± 0.000	1.000 ± 0.000	0.993 ± 0.008	0.883 ± 0.061	0.631 ± 0.105

Table 24: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{16}, proportional to $|I_i|$

E.17. Testing Results for Pattern Data Set P_{17}

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.912 ± 0.033	0.880 ± 0.040	0.802 ± 0.047	0.696 ± 0.058	0.537 ± 0.054
M_3	0.997 ± 0.003	0.955 ± 0.024	0.785 ± 0.084	0.466 ± 0.085	0.214 ± 0.045
M_4	1.000 ± 0.000	0.998 ± 0.005	0.943 ± 0.021	0.645 ± 0.048	0.287 ± 0.028

Table 25: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{17}, proportional to $|I_i|$

E.18. Testing Results for Pattern Data Set P_{18}

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.748 ± 0.074	0.689 ± 0.068	0.571 ± 0.057	0.445 ± 0.037	0.292 ± 0.034
M_3	0.989 ± 0.009	0.868 ± 0.067	0.577 ± 0.070	0.251 ± 0.046	0.067 ± 0.025
M_4	0.999 ± 0.002	0.976 ± 0.011	0.790 ± 0.046	0.372 ± 0.038	0.090 ± 0.020

Table 26: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{18}, proportional to $|I_i|$

E.19. Testing Results for Pattern Data Set P_{19}

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.518 ± 0.049	0.420 ± 0.044	0.325 ± 0.030	0.205 ± 0.039	0.112 ± 0.022
M_3	0.969 ± 0.014	0.728 ± 0.036	0.344 ± 0.054	0.099 ± 0.030	0.014 ± 0.010
M_4	0.998 ± 0.003	0.933 ± 0.026	0.607 ± 0.046	0.201 ± 0.030	0.023 ± 0.013

Table 27: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{19}, proportional to $|I_i|$

E.20. Testing Results for Pattern Data Set P_{20}

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.326 ± 0.045	0.253 ± 0.027	0.173 ± 0.029	0.104 ± 0.025	0.048 ± 0.013
M_3	0.929 ± 0.028	0.559 ± 0.063	0.188 ± 0.033	0.041 ± 0.009	0.005 ± 0.004
M_4	0.996 ± 0.006	0.877 ± 0.028	0.420 ± 0.027	0.079 ± 0.016	0.005 ± 0.003

Table 28: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{20}, proportional to $|I_i|$

E.21. Testing Results for Pattern Data Set P_{21}

	I_1	I_2	I_3	I_4	I_5
M_1	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000	0.000 ± 0.000
M_2	0.204 ± 0.039	0.156 ± 0.028	0.095 ± 0.020	0.055 ± 0.015	0.023 ± 0.010
M_3	0.876 ± 0.037	0.437 ± 0.060	0.101 ± 0.029	0.011 ± 0.007	0.001 ± 0.002
M_4	0.987 ± 0.008	0.742 ± 0.038	0.258 ± 0.034	0.029 ± 0.008	0.002 ± 0.002

Table 29: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{21}, proportional to $|I_i|$

E.22. Testing Results for Pattern Data Set P_{22}

	I_1	I_2	I_3	I_4	I_5
M_1	1.000 ± 0.000	1.000 ± 0.000	1.000 ± 0.000	1.000 ± 0.000	0.988 ± 0.010
M_2	1.000 ± 0.000	1.000 ± 0.000	1.000 ± 0.000	1.000 ± 0.000	0.995 ± 0.009
M_3	1.000 ± 0.000	1.000 ± 0.000	1.000 ± 0.000	0.998 ± 0.007	0.950 ± 0.020
M_4	1.000 ± 0.000	1.000 ± 0.000	1.000 ± 0.000	1.000 ± 0.000	0.970 ± 0.026

Table 30: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{22}, proportional to $|I_i|$

E.23. Testing Results for Pattern Data Set P_{23}

	I_1	I_2	I_3	I_4	I_5
M_1	0.968 ± 0.045	0.953 ± 0.065	0.926 ± 0.088	0.884 ± 0.089	0.802 ± 0.129
M_2	1.000 ± 0.000	1.000 ± 0.000	1.000 ± 0.000	1.000 ± 0.000	0.994 ± 0.007
M_3	1.000 ± 0.000	0.998 ± 0.004	0.981 ± 0.013	0.881 ± 0.061	0.635 ± 0.056
M_4	1.000 ± 0.000	1.000 ± 0.000	0.996 ± 0.005	0.924 ± 0.044	0.726 ± 0.067

Table 31: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{23}, proportional to $|I_i|$

E.24. Testing Results for Pattern Data Set P_{24}

	I_1	I_2	I_3	I_4	I_5
M_1	0.648 ± 0.102	0.603 ± 0.122	0.537 ± 0.110	0.439 ± 0.117	0.307 ± 0.087
M_2	1.000 ± 0.000	1.000 ± 0.000	1.000 ± 0.000	0.991 ± 0.007	0.924 ± 0.018
M_3	1.000 ± 0.000	0.947 ± 0.027	0.791 ± 0.062	0.481 ± 0.095	0.220 ± 0.049
M_4	1.000 ± 0.000	0.997 ± 0.005	0.939 ± 0.031	0.708 ± 0.058	0.338 ± 0.053

Table 32: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{24}, proportional to $|I_i|$

E.25. Testing Results for Pattern Data Set P_{25}

	I_1	I_2	I_3	I_4	I_5
M_1	0.339 ± 0.083	0.268 ± 0.070	0.192 ± 0.058	0.134 ± 0.035	0.064 ± 0.024
M_2	1.000 ± 0.000	0.999 ± 0.002	0.994 ± 0.008	0.926 ± 0.013	0.761 ± 0.036
M_3	0.995 ± 0.004	0.902 ± 0.053	0.639 ± 0.097	0.297 ± 0.055	0.089 ± 0.030
M_4	0.999 ± 0.002	0.984 ± 0.012	0.839 ± 0.057	0.480 ± 0.072	0.161 ± 0.027

Table 33: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{25}, proportional to $|I_i|$

E.26. Testing Results for Pattern Data Set P_{26}

	I_1	I_2	I_3	I_4	I_5
M_1	0.042 ± 0.025	0.025 ± 0.018	0.009 ± 0.008	0.008 ± 0.006	0.000 ± 0.000
M_2	0.998 ± 0.003	0.988 ± 0.011	0.933 ± 0.011	0.748 ± 0.031	0.488 ± 0.026
M_3	0.972 ± 0.025	0.759 ± 0.070	0.389 ± 0.079	0.123 ± 0.049	0.024 ± 0.010
M_4	0.998 ± 0.004	0.952 ± 0.026	0.675 ± 0.093	0.285 ± 0.063	0.055 ± 0.022

Table 34: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{26}, proportional to $|I_i|$

E.27. Testing Results for Pattern Data Set P_{27}

	I_1	I_2	I_3	I_4	I_5
M_1	0.011 ± 0.013	0.007 ± 0.010	0.003 ± 0.004	0.001 ± 0.002	0.000 ± 0.000
M_2	0.970 ± 0.020	0.912 ± 0.023	0.772 ± 0.027	0.519 ± 0.038	0.275 ± 0.040
M_3	0.925 ± 0.022	0.611 ± 0.071	0.223 ± 0.049	0.053 ± 0.015	0.006 ± 0.004
M_4	0.999 ± 0.002	0.890 ± 0.034	0.477 ± 0.056	0.127 ± 0.020	0.014 ± 0.006

Table 35: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{27}, proportional to $|I_i|$

E.28. Testing Results for Pattern Data Set P_{28}

	I_1	I_2	I_3	I_4	I_5
M_1	0.005 ± 0.007	0.001 ± 0.001	0.001 ± 0.002	0.000 ± 0.000	0.000 ± 0.000
M_2	0.914 ± 0.031	0.798 ± 0.016	0.572 ± 0.014	0.310 ± 0.032	0.124 ± 0.022
M_3	0.874 ± 0.048	0.475 ± 0.060	0.152 ± 0.023	0.020 ± 0.006	0.003 ± 0.003
M_4	0.991 ± 0.006	0.789 ± 0.052	0.336 ± 0.055	0.050 ± 0.012	0.005 ± 0.003

Table 36: Number of successful recalls when input data set I_i is given to model M_j, trained with pattern data set P_{28}, proportional to $|I_i|$

Appendix F. Reproducibility

The source code of this project is public on github[1]. The default configured random seed produces the same exact results as the ones published in this paper. Requirements

- $Python \ \rightarrow \ 2.7.0$
- $NumPy \ \rightarrow \ 1.12.0$
- $UNIX - like\ system\ shell$

Execution Just execute the following command in the root of the project to generate data sets and test the models:

$$\$\ sh\ generate\_results.sh$$

This will create several JSON files in *out/results/*. When it finishes, execute the following command:

$$\$\ sh\ generate\_latex.sh\ >\ out/tables.tex$$

This will create a TeX file in *out/tables.tex* containing the result tables used in this paper.

1. https://github.com/juanlao7/bblr-hopfield-boltzmann