Alphabets, strings, and languages

Computation and Discrete Structures III

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Material adapted from Professor Oscar Bedoya

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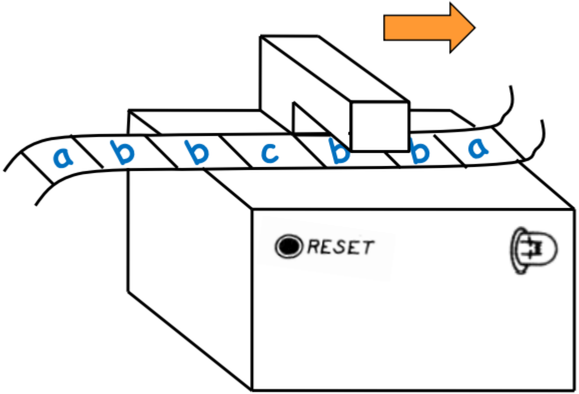
Contents

Contenido

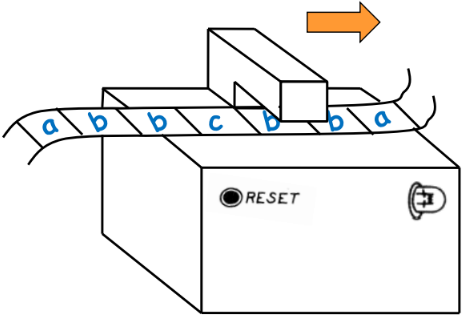
Languages, Machine Types and Grammars

Type	Languages	Machine Type	Grammar Rules
0	Recursively enumerable	Turing machine	Unrestricted
1	Context-sensitive	Linear bounded automaton	$\alpha \to \beta, \quad \lvert\alpha\rvert \leq \lvert\beta\rvert$
2	Context-free	Pushdown automaton	$A \to \gamma$
3	Regular	Finite automaton	$A \to aB$ $A \to a$

Theoretical machine (tape)



Theoretical machine (tape)



The **alphabet** is the **set of symbols** that may appear in the input of the machine

Alphabets

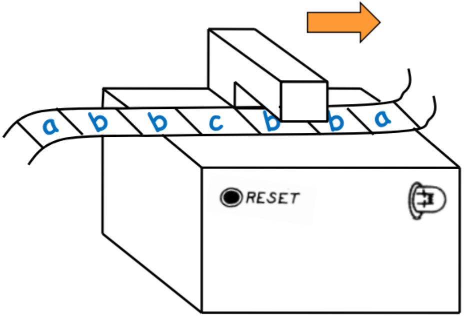
Alphabet

An alphabet is any non-empty set of symbols

- $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $\Sigma = \{a, b\}$

- Latin alphabet:
  $\Sigma = \{a,b,c,d,e,f,g,h,i,j,k,l,m,n,ñ,o,p,q,r,s,t,u,v,w,x,y,z\}$
- Greek alphabet:
  $\Sigma = \{\alpha, \beta, \gamma, \delta, \epsilon, ..., \Psi, \Omega\}$

Words or strings



Words or **strings** are finite sequences of symbols

Notion of word

Given the alphabet used in Spanish:
$\Sigma = \{a,b,c,d,e,f,g,h,i,j,k,l,m,n,ñ,o,p,q,r,s,t,u,v,w,x,y,z\}$
Words can be created:

- Colina
- Puente
- Dardo
- Fdkfjk

The notion of word has no associated semantics

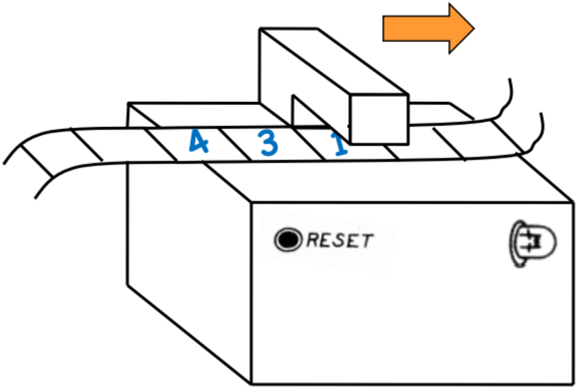
Strings or words

String or word

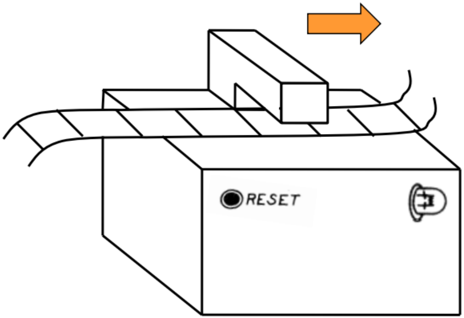
A word is a finite sequence of symbols from a given alphabet

- If $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then 431, 021, ϵ, are words over Σ
- If $\Sigma = \{a, b\}$ then ab, ba, aaab, ϵ, are words over Σ

Theoretical machine (tape) with word

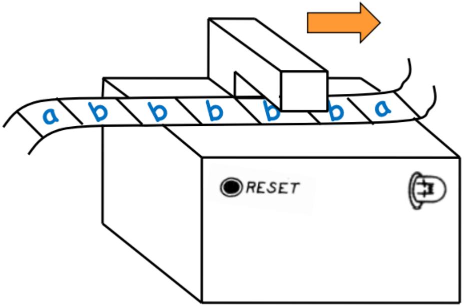


Machine (tape) with empty string



The **empty string** ϵ represents a word that has 0 symbols, that is, an empty tape

Machine (tape) that accepts strings



A machine **accepts** a specific set of words that can be generated from an alphabet

Language

Language

A language is a particular set of words

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- Show the following languages defined over $\Sigma = \{a,b\}$

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  - L1: set of words that have exactly 3 symbols

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- Show the following languages defined over $\Sigma = \{a,b\}$
  - L1: set of words that have exactly 3 symbols
  - L2: set of words that have at least one a
  - L3: set of words that have an even number of symbols
  - L4: set of all possible words

Universal language

Universal language over Σ

- It is denoted as Σ^* and is also known as the **closure**
- Σ^* is the language formed by all strings over the alphabet Σ

Universal language

Universal language over Σ

- It is denoted as Σ^* and is also known as the **closure**
- Σ^* is the language formed by all strings over the alphabet Σ

- Show the universal language Σ^* for the following alphabets:
  - $\Sigma = \{a,b,c\}$
  - $\Sigma = \{1\}$

Universal language

Universal language over Σ

- Σ^* is the language formed by all strings over the alphabet Σ

- For $\Sigma = \{a,b,c\}$, $\Sigma^* = \{\epsilon$, a, b, c, aa, ab, ac, ba, bb, bc, ... $\}$
- For $\Sigma = \{1\}$, $\Sigma^* = \{\epsilon$, 1, 11, 111, 1111, ...$\}$

Universal language

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Membership of the empty string and infiniteness of the universal language

- ϵ is always in Σ^* because the empty string can be obtained from any alphabet
- For any alphabet Σ, Σ^* is infinite because Σ cannot be empty

Language

Language

- A language L over an alphabet Σ is a subset of Σ^*, that is, $L \subseteq \Sigma^*$

Contenido

Concatenation of strings

Concatenation $\cdot$

Let x and y be two strings, their concatenation is the string $x \cdot y$

- $a \cdot aba = aaba$
- $ab \cdot ba = abba$

Power of a string

Power of a string

Let x be a string, the power of x is defined as:

$$x^n = \begin{cases} \epsilon & n = 0 \\ x \cdot x^{n-1} & n \geq 1 \end{cases}$$

$\cdot$ is the concatenation operator

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$\cdot$ is the concatenation operator

- $(aab)^3 = aab \cdot (aab)^2$
  $= aab \cdot aab \cdot aab^1$
  $= aab \cdot aab \cdot aab \cdot aab^0$
  $= aab \cdot aab \cdot aab \cdot \epsilon = aabaabaab$

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Let x be a string, the power of x is defined as:

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Show:

- $a^3 \cdot (aba)^2$
- $(ab)^2 \cdot (ba)^3$

Examples of languages

- $\Sigma = \{a\}$,
  $L = \{a, aa, aaa, aaaa, ...\}$

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- $\Sigma = \{0, 1\}$,
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Examples of languages

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Length of a string

Length of a string

Let x be a string that belongs to a language L, its length is denoted by $|x|$ and defined as:

$$|x| = \begin{cases} 0 & x = \epsilon \\ n & x = a_1 a_2 ... a_n \end{cases}$$

Length of a string

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Let x be a string that belongs to a language L, its length is denoted by $|x|$ and defined as:

$$|x| = \begin{cases} 0 & x = \epsilon \\ n & x = a_1a_2...a_n \end{cases}$$

- $|\epsilon| = 0$
- $|ababaa| = 6$

Concatenation of languages

Concatenation of languages

Let A and B be two languages defined over Σ, the concatenation $A \cdot B$ is defined as:

$$A \cdot B = \{u \cdot v \mid u \in A \land v \in B\}$$

- $A = \{a, ab, ac\}, B = \{b, b^2\}$
- $A \cdot B =$

Concatenation of languages

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- $A = \{a, ab, ac\}, B = \{b, b^2\}$
- $A \cdot B = \{ab, abb, acb, ab^2, abb^2, acb^2\} = \{ab, ab^2, acb, ab^2, ab^3, acb^2\} = \{ab, acb, ab^2, ab^3, acb^2\}$

Power of a language

Power of a language

Given a language A over Σ the power is defined as:

$$A^n = \begin{cases} \{\epsilon\} & n = 0 \\ A \cdot A^{n-1} & n \geq 1 \end{cases}$$

Compute A^3 for $A = \{ab, b\}$

Power of a language

Power of a language

Given a language A over Σ the power is defined as:

$$A^n = \begin{cases} \{\epsilon\} & n = 0 \\ A \cdot A^{n-1} & n \geq 1 \end{cases}$$

Compute A^3 for $A = \{ab, b\}$,
$A^3 = A \cdot A \cdot A \cdot A^0 = \{ab, b\}\{ab, b\}\{ab, b\}\{\epsilon\} = \{ab, b\}\{abab, bab, abb, bb\} = \{ababab, abbab, ababb, abbb, babab, bbab, babb, bbb\} \cdot \{\epsilon\}$

Exercises: power of languages

Given $A = \{ab, ca, ad\}$,

- Is $abcaab \in A^3$?
- Is $adca \in A^2$?
- Is $caba \in A^2$?
- Is $abcaaa \in A^3$?
- Is $adcaab \in A^3$?

Exercises: power of languages

Given $A = \{ab, ca, ad\}$,

- Is $abcaab \in A^3$? Yes
- Is $adca \in A^2$? Yes
- Is $caba \in A^2$? No
- Is $abcaaa \in A^3$? No
- Is $adcaab \in A^3$? Yes

Exercises: power of languages

Given $A = \{ab, c, ac\}$,

- Is $accab \in A^3$?
- Is $abacca \in A^3$?
- Is $abcc \in A^3$?
- Is $abcba \in A^3$?

Exercises: power of languages

Given $A = \{ab, c, ac\}$,

- Is $accab \in A^3$? Yes
- Is $abacca \in A^3$? No
- Is $abcc \in A^3$? Yes
- Is $abcba \in A^3$? No

Kleene closure

Kleene closure

The Kleene closure of a language A is the union of its powers, denoted by A^*

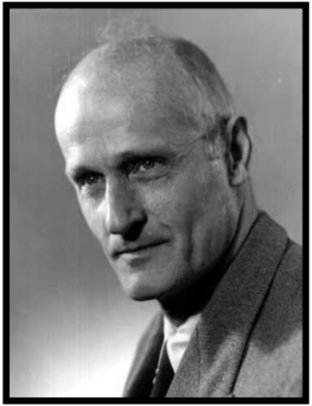
$$A^* = A^0 \cup A^1 \cup A^2 \cup \ldots$$

- It is also known as **Kleene star / Star closure**
- A^* is the set of possible concatenations over A

Stephen Kleene

Stephen Kleene

- Creator of regular expressions
- Introduced Kleene closure, A^*



(1909 - 1994)

Kleene closure

Compute A^* for $A = \{a, ab\}$

Kleene closure

$A = \{a, ab\}$

- $A^0 = \{\epsilon\}$
- $A^1 = \{a, ab\}$
- $A^2 = \{aa, aab, aba, abab\}$
- ...

$A^* = \{\epsilon, a, ab, aa, aab, aba, abab, \ldots\}$

Kleene closure

$A = \{a, ab\}$

- $A^0 = \{\epsilon\}$
- $A^1 = \{a, ab\}$
- $A^2 = \{aa, aab, aba, abab\}$
- $\ldots$

$A^* = \{\epsilon, a, ab, aa, aab, aba, abab, \ldots\}$

- Is $ababab \in A^*$?
- Is $abbbbb \in A^*$?
- Is $abaaaaaa \in A^*$?

Kleene closure

$A = \{a, ab\}$

- $A^0 = \{\epsilon\}$
- $A^1 = \{a, ab\}$
- $A^2 = \{aa, aab, aba, abab\}$
- $\ldots$

$A^* = \{\epsilon, a, ab, aa, aab, aba, abab, \ldots\}$

- Is $ababab \in A^*$? Yes
- Is $abbbbb \in A^*$? No
- Is $abaaaaaa \in A^*$? Yes

Kleene closure

$A = \{a, ab\}$

- $A^0 = \{\epsilon\}$
- $A^1 = \{a, ab\}$
- $A^2 = \{aa, aab, aba, abab\}$
- ...

$A^* = \{\epsilon, a, ab, aa, aab, aba, abab, ...\}$

Relationship between A^* and Σ^*

What relationship does A^* have with Σ^*?
Compute Σ^* over $\Sigma = \{a, b\}$

Kleene closure

$A = \{a, ab\}$

- $A^0 = \{\epsilon\}$
- $A^1 = \{a, ab\}$
- $A^2 = \{aa, aab, aba, abab\}$
- ...

$A^* = \{\epsilon, a, ab, aa, aab, aba, abab, ...\}$
$\Sigma^* = \{\epsilon, a, b, aa, ab, ba, bb, aaa, aab, ...\}$

Kleene closure A^* and Σ^* closure

Closures

- Σ^* is defined over the alphabet and corresponds to all strings that can be created over an alphabet Σ
- A^* is defined over a language A and consists of all possible concatenations

Kleene closure A^* and Σ^* closure

Closures

- Σ^* is defined over the alphabet and corresponds to all strings that can be created over an alphabet Σ
- A^* is defined over a language A and consists of all possible concatenations

$A = \{a, ab\}$ is defined over $\Sigma = \{a, b\}$

- $A^* = \{\epsilon, a, ab, aa, aab, aba, abab, ...\}$
- $\Sigma^* = \{\epsilon, a, b, aa, bb, ab, ba, aaa, aab, aba, ...\}$
- **In general, $A^* \subseteq \Sigma^*$ holds**

Kleene positive closure

Kleene positive closure A^+

The Kleene positive closure of a language A is the union of the powers excluding $A^0 = \{\epsilon\}$,

$$A^+ = A^1 \cup A^2 \cup A^3 \cup \ldots$$

Positive closure Σ^+

Positive closure Σ^+

It is the set of words that can be formed over Σ excluding the empty string ϵ

Exercises – Closures

- Let $A = \{a, b, ab\}$, show A^* and A^+. Indicate if $abba \in A^*$, $bbaa \in A^*$
- Let $A = \{a, aa, ac\}$ and $B = \{b, ba\}$, show $A \cdot B$, $B \cdot A$ and B^*
- Let $A = \{a\}$, show A^*

Exercises – Closures

- **Let** $A = \{a, b, ab\}$**, show** A^* **and** A^+**. Indicate if** $abba \in A^*$**,** $bbaa \in A^*$
  - $A^* = A^0 \cup A^1 \cup A^2 \cup \ldots$
    $= \{\epsilon\} \cup \{a, b, ab\} \cup \{aa, ab, aab, ba, bb, bab, aba, abb, abab\} \cup \ldots$
    $= \{\epsilon, a, b, ab, aa, ab, aab, ba, bb, bab, aba, abb, abab, \ldots\}$
  - $A^+ = A^1 \cup A^2 \cup \ldots$
    $= \{a, b, ab\} \cup \{aa, ab, aab, ba, bb, bab, aba, abb, abab\} \cup \ldots$
    $= \{a, b, ab, aa, ab, aab, ba, bb, bab, aba, abb, abab, \ldots\}$
- **Let** $A = \{a, aa, ac\}$ **and** $B = \{b, ba\}$**, show** $A \cdot B$**,** $B \cdot A$ **and** B^*
  - $A \cdot B = \{ab, aba, aab, aaba, acb, acba\}$
  - $B \cdot A = \{ba, baa, bac, baa, baaa, baac\}$
  - $B^* = \{\epsilon, b, ba, bba, bab, bbba, babb, \ldots\}$
- **Let** $A = \{a\}$**, show** A^*
  - $A^* = \{\epsilon, a, aa, aaa, aaaa, aaaaa, \ldots\}$

Exercises – Languages

Show strings that belong to the following languages. Indicate if the empty string ϵ belongs to the languages and express in general terms (in words) the type of strings that belong to each one.

- $L_1 = \{w_1 c w_2 \mid |w_1| = |w_2|$ where $w_1, w_2 \in \Sigma^*$ with $\Sigma = \{a, b\}\}$
- $L_2 = \{a^n b^m \mid n \neq m, n, m \geq 0\}$
- $L_3 = \{a^n b^{2n} c^n \mid n \geq 0\}$

Exercises – Languages

$L_1 = \{w_1 c w_2 \mid |w_1| = |w_2|$ **where** $w_1, w_2 \in \Sigma^*$ **with** $\Sigma = \{a, b\}\}$
$aca, acb, bca, abbbabcaaaaaa$
In general, strings that have a c in the middle, such that the substrings on either side have the same length. $\epsilon \notin L_1$

Exercises – Languages

$L_2 = \{a^n b^m \mid n \neq m, n, m \geq 0\}$
$abb, aab, aabbb, aaabb$
In general, strings that have a different number of a’s than b’s where the a’s are to the left of the b’s. $\epsilon \notin L_2$

Exercises – Languages

$L_3 = \{a^n b^{2n} c^n \mid n \geq 0\}$
$abbc, aabbbbcc, aaabbbbbbccc$
In general, strings that have twice as many b's as a's and c's where a's, b's, and then c's appear from left to right. $\epsilon \in L_3$

Exercises – Languages

Express formally the following languages:

- L_1 is the set of strings from the universal language of $\Sigma = \{a, b, c\}$ that start with a and end with a
- L_2 is the set of strings with even length defined over the universal language of $\Sigma = \{a, b\}$

Exercises – Languages

- L_1 **is the set of strings from the universal language of** $\Sigma = \{a, b, c\}$ **that start with a and end with a**
  - $L_1 = \{aw_1a \mid w_1 \in \Sigma^* \text{ with } \Sigma = \{a, b, c\}\}$
- L_2 **is the set of strings with even length defined over the universal language of** $\Sigma = \{a, b\}$
  - $L_2 = \{w_i \mid |w_i| = 2k, \text{ where there exists } k \geq 1, w_i \in \Sigma^* \text{ with } \Sigma = \{a, b\}\}$

References

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