Regular Grammars

Computation and Discrete Structures III

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Material adapted from Professor Oscar Bedoya

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Contents

Contenido

Languages, Machine Types and Grammars

Type	Languages	Machine Type	Grammar Rules
0	Recursively enumerable	Turing machine	Unrestricted
1	Context-sensitive	Linear bounded automaton	$\alpha \to \beta, \quad \lvert\alpha\rvert \leq \lvert\beta\rvert$
2	Context-free	Pushdown automaton	$A \to \gamma$
3	Regular	Finite automaton	$A \to aB$ $A \to a$

Regular Grammars

$$
\begin{aligned}
S &\rightarrow aA \mid bB \\
A &\rightarrow aA \mid a \\
B &\rightarrow bB \mid b
\end{aligned}
$$

- S, A and B are non-terminals, indicating they must be replaced according to the productions
- a and b are terminals belonging to an alphabet Σ

Regular Grammars

$$S \rightarrow aA \mid bB$$
$$A \rightarrow aA \mid a$$
$$B \rightarrow bB \mid b$$

Grammars generate strings

Regular Grammars

$$S \rightarrow aA \mid bB$$
$$A \rightarrow aA \mid a$$
$$B \rightarrow bB \mid b$$

Grammars generate strings

$S \rightarrow aA \rightarrow aaA \rightarrow aaa$

The string aaa is generated by the grammar

Regular Grammars

$$S \rightarrow aA \mid bB$$
$$A \rightarrow aA \mid a$$
$$B \rightarrow bB \mid b$$

$$S \rightarrow aA \rightarrow aa$$
$$S \rightarrow aA \rightarrow aaA \rightarrow aaaA \rightarrow aaaa$$
$$S \rightarrow bB \rightarrow bb$$

Regular Grammars

$$S \rightarrow aA \mid bB$$
$$A \rightarrow aA \mid a$$
$$B \rightarrow bB \mid b$$

Can the string *ab* be generated by the grammar?

Regular Grammars

$$S \rightarrow aAb \mid bBa$$
$$A \rightarrow aAb \mid \epsilon$$
$$B \rightarrow bBa \mid \epsilon$$

Indicate which of the following strings can be generated by the grammar:

- ab
- $aabb$
- $bbaa$
- abb
- $bbba$

Regular Grammars

$$S \rightarrow abS \mid \epsilon$$

Indicate which of the following strings can be generated by the grammar:

- ϵ
- $abab$
- $aaab$
- abb

Regular Grammars

$$S \rightarrow aE$$
$$E \rightarrow A \mid B$$
$$A \rightarrow aA \mid b$$
$$B \rightarrow aB \mid b$$

The string *aaab* can be generated as:
$S \rightarrow aE \rightarrow aA \rightarrow aaA \rightarrow aaaA \rightarrow aaab$
The notation $S \xrightarrow{*} w$ indicates that string w **can be generated** from S in 0 or more steps

Regular Grammars

Consider the regular language $a(a^* \cup b^*)b$. One way to express the strings accepted by the language is through the productions

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$$S \rightarrow aE$$

$$E \rightarrow A \mid B$$

$$A \rightarrow aA \mid b$$

$$B \rightarrow aB \mid b$$

Regular Grammars

A **regular grammar** is defined as a set of 4 elements, $G = (\Sigma, N, S, P)$ where:

- Σ is the alphabet
- N are the non-terminals
- S is the start symbol
- P is the collection of substitution rules or productions

$$S \rightarrow aE$$
$$E \rightarrow A \mid B$$
$$A \rightarrow aA \mid b$$
$$B \rightarrow aB \mid b$$

Regular Grammars

A **regular grammar** is defined as a set of 4 elements, $G = (\Sigma, N, S, P)$ where:

- Σ is the alphabet
- N are the non-terminals
- S is the start symbol
- P is the collection of substitution rules or productions of the form $A \to w$, where $A \in N$ and $w \in (\Sigma \cup N)^*$ that satisfies:
    1. w contains at most one non-terminal
    2. If w contains a non-terminal, it is the rightmost symbol in w

Regular Grammars

The following grammars are not regular:

$$
\begin{aligned}
S &\to AB \\
A &\to aA \mid a \\
B &\to bB \mid b
\end{aligned}
$$

$$
\begin{aligned}
S &\to aAb \\
A &\to cA \mid c
\end{aligned}
$$

$$
\begin{aligned}
aSb &\to aA \\
bAa &\to b \mid c
\end{aligned}
$$

Regular Grammars

Consider the following regular grammar:

- $\Sigma = \{a, b\}$
- $N = \{S, A\}$
- S is the start symbol
- P :

$$
\begin{aligned}
S &\rightarrow bA \\
A &\rightarrow aaA \mid b
\end{aligned}
$$

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$$

$bb, baab, baaaab, baaaaaab, \ldots$

Regular Grammars

Consider the following regular grammar:

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- $N = \{S, A\}$
- S is the start symbol
- P :

$$
\begin{aligned}
S &\rightarrow bA \\
A &\rightarrow aaA \mid b
\end{aligned}
$$

The language accepted by the grammar, $L(G)$, contains strings of the form $b(aa)^*b$

Regular Grammars

Indicate the regular expression associated with the following grammar:

- $\Sigma = \{a, b\}$
- $N = \{S\}$
- S is the start symbol
- P :

$$S \to aS \mid b$$

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Regular Grammars

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- $N = \{S, B\}$
- S is the start symbol
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The language accepted by the grammar, $L(G)$, contains strings of the form a^*b^*

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\begin{aligned}
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- $\Sigma = \{a, b\}$
- $N = \{S, A\}$
- S is the start symbol
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$$\begin{aligned} S &\rightarrow abS \mid A \\ A &\rightarrow a \mid b \end{aligned}$$

The language accepted by the grammar, $L(G)$, contains strings of the form $(ab)^*(a \cup b)$

Regular Grammars

Design a regular grammar that recognizes $(ab)^+$

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- $\Sigma = \{a, b\}$
- $N = \{S\}$
- S is the start symbol
- P :

$$S \rightarrow abS \mid ab$$

Regular Grammars

Design a regular grammar that recognizes the language given by $(a \cup b)a^*(a \cup b)$

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- $\Sigma = \{a, b\}$
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- S is the start symbol
- P :

$$
\begin{aligned}
S &\rightarrow aA \mid bA \\
A &\rightarrow aA \mid a \mid b
\end{aligned}
$$

Regular Grammars

Design a regular grammar that recognizes the language given by $(a \cup b)^* a (a \cup b)^*$

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- $\Sigma = \{a, b\}$
- $N = \{S, A\}$
- S is the start symbol
- P :

$$\begin{aligned} S &\rightarrow aS \mid bS \mid aA \\ A &\rightarrow aA \mid bA \mid \epsilon \end{aligned}$$

Regular Grammars

Design a regular grammar that recognizes the language given by $(ab)^{+}(a \cup b)^{*}$

Regular Grammars

Design a regular grammar that recognizes the language given by $(ab)^+(a \cup b)^*$

- $\Sigma = \{a, b\}$
- $N = \{S, A\}$
- S is the start symbol
- P :

$$\begin{aligned} S &\rightarrow abS \mid abA \\ A &\rightarrow aA \mid bA \mid \epsilon \end{aligned}$$

Regular Grammars

Design a regular grammar that recognizes the language given by $a^*b^*c^*$

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Design a regular grammar that recognizes the language given by $a^*b^*c^*$

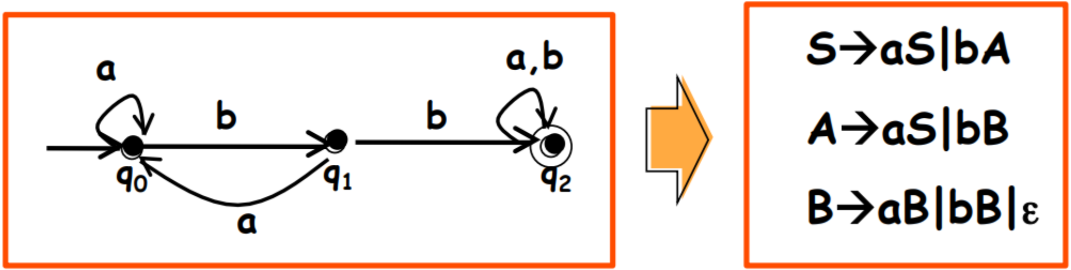
- $\Sigma = \{a, b, c\}$
- $N = \{S, B, C\}$
- S is the start symbol
- P :

$$
\begin{aligned}
S &\rightarrow aS \mid bB \mid cC \mid \epsilon \\
B &\rightarrow bB \mid cC \mid \epsilon \\
C &\rightarrow cC \mid \epsilon
\end{aligned}
$$

Regular Grammars

Theorem

Given an automaton M, there exists a grammar G such that $L(M) = L(G)$



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Given an automaton M, there exists a grammar G such that $L(M) = L(G)$

The productions are obtained by taking the automaton states as non-terminals and alphabet symbols as terminals

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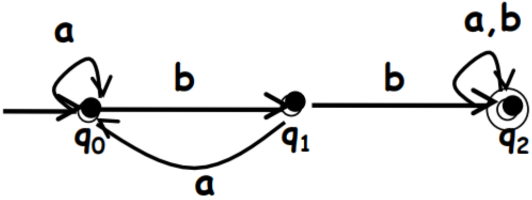

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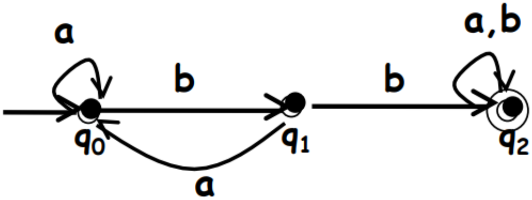
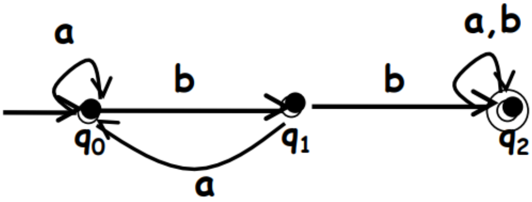




The automaton *M* induces the regular grammar:

$q_0 \to aq_0 \mid bq_1$

$q_1 \to aq_0 \mid bq_2$

$q_2 \to aq_2 \mid bq_2 \mid \varepsilon$

$\Rightarrow$

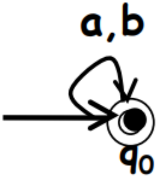
$S \to aS \mid bA$

$A \to aS \mid bB$

$B \to aB \mid bB \mid \varepsilon$

Regular Grammars

Automaton that recognizes $(a \cup b)^*$



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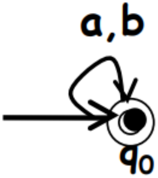
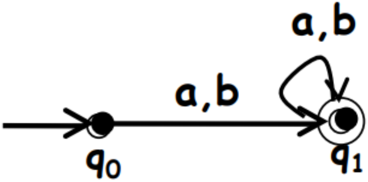


The automaton M induces the regular grammar:

$$q_0 \to aq_0 \,|\, bq_0 \,|\, \varepsilon \quad \Rightarrow \quad S \to aS \,|\, bS \,|\, \varepsilon$$

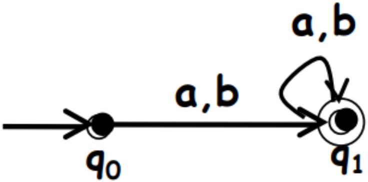
Regular Grammars

Automaton that recognizes $(a \cup b)^+$



Regular Grammars

Automaton that recognizes $(a \cup b)^+$

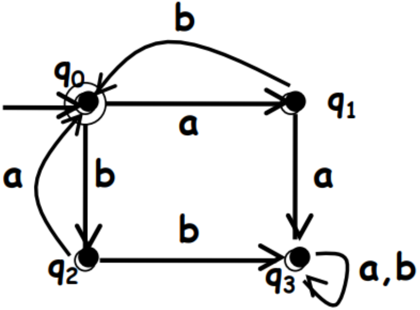


The automaton M induces the regular grammar:

$$\begin{aligned} q_0 &\to aq_1 \mid bq_1 \\ q_1 &\to aq_1 \mid bq_1 \mid \varepsilon \end{aligned} \quad \Rightarrow \quad \begin{aligned} S &\to aA \mid bA \\ A &\to aA \mid bA \mid \varepsilon \end{aligned}$$

Regular Grammars

Show the regular grammar for the following automaton that recognizes $(ab \cup ba)^*$



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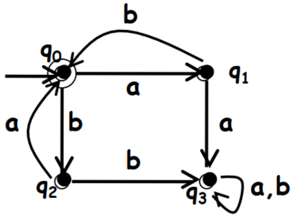


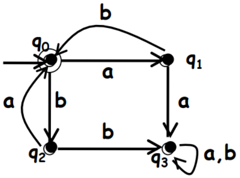
The automaton M induces the regular grammar:

$$\begin{aligned} q_0 &\rightarrow aq_1 \mid bq_2 \mid \varepsilon \\ q_1 &\rightarrow bq_0 \mid aq_3 \\ q_2 &\rightarrow aq_0 \mid bq_3 \\ q_3 &\rightarrow aq_3 \mid bq_3 \end{aligned}$$

$$\begin{aligned} S &\rightarrow aA \mid bB \mid \varepsilon \\ A &\rightarrow bS \mid aC \\ B &\rightarrow aS \mid bC \\ C &\rightarrow aC \mid bC \end{aligned}$$

Regular Grammars

Show the regular grammar for the following automaton that recognizes $(ab \cup ba)^*$



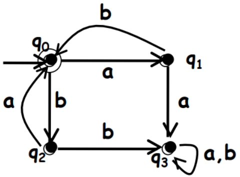
The automaton M induces the regular grammar:

$q_0 \rightarrow aq_1 \mid bq_2 \mid \varepsilon$
$q_1 \rightarrow bq_0 \mid aq_3$
$q_2 \rightarrow aq_0 \mid bq_3$
$q_3 \rightarrow aq_3 \mid bq_3$

$S \rightarrow aA \mid bB \mid \varepsilon$
$A \rightarrow bS \mid aC$
$B \rightarrow aS \mid bC$
$C \rightarrow aC \mid bC$

Evaluate the string *aab*

Show the regular grammar for the following automaton that recognizes $(ab \cup ba)^*$



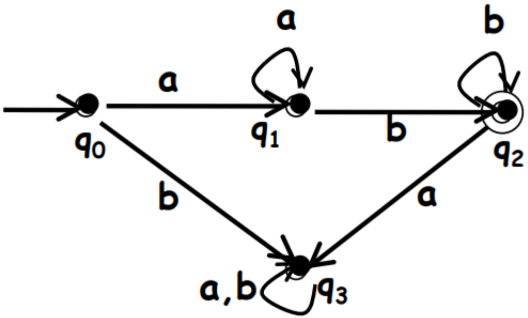
The automaton M induces the regular grammar:

$$q_0 \to aq_1 \mid bq_2 \mid \varepsilon \qquad S \to aA \mid bB \mid \varepsilon$$
$$q_1 \to bq_0 \mid aq_3 \qquad A \to bS \mid aC$$
$$q_2 \to aq_0 \mid bq_3 \qquad B \to aS \mid bC$$
$$q_3 \to aq_3 \mid bq_3 \qquad C \to aC \mid bC$$

The string *aab* is not generated by the grammar

Regular Grammars

Show the regular grammar for the following automaton that recognizes a^+b^+



Regular Grammars

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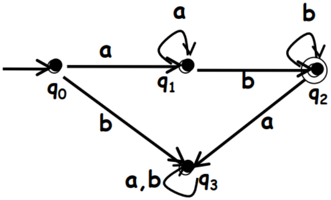

The automaton M induces the regular grammar:

$$q_0 \rightarrow aq_1 \mid bq_3 \qquad S \rightarrow aA \mid bC$$

$$q_1 \rightarrow aq_1 \mid bq_2 \qquad A \rightarrow aA \mid bB$$

$$q_2 \rightarrow bq_2 \mid \varepsilon \mid aq_3 \qquad B \rightarrow bB \mid \varepsilon \mid aC$$

$$q_3 \rightarrow aq_3 \mid bq_3 \qquad C \rightarrow aC \mid bC$$

Regular Grammars

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Given a regular grammar G, there exists an automaton M such that $L(M) = L(G)$

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The automaton construction uses the productions, considering non-terminals as states and terminals as transitions

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Given a regular grammar G, there exists an automaton M such that $L(M) = L(G)$

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$$S \rightarrow aS \mid bA$$
$$A \rightarrow aB \mid bB \mid \epsilon$$
$$B \rightarrow aB \mid bB$$

Regular Grammars

Design the automaton for the following grammar:

$$S \rightarrow aS \mid bA$$

$$A \rightarrow aB \mid bB \mid \epsilon$$

$$B \rightarrow aB \mid bB$$

Regular Grammars

Design the automaton for the following grammar:

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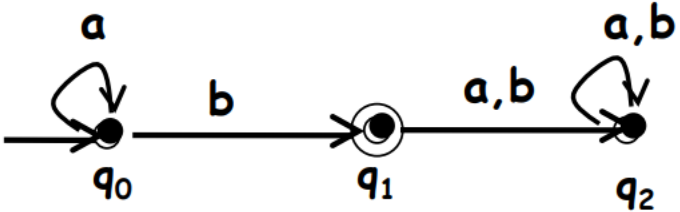

Regular Grammars

Design the automaton for the following grammar:

$$S \to aB \mid bA \mid \epsilon$$

$$A \to abaS$$

$$B \to babS$$

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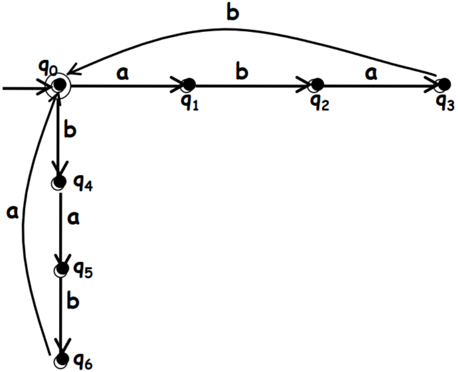
$$A \rightarrow abaS$$
$$B \rightarrow babS$$



Regular Grammars

Design the automaton for the following grammar:

$$S \rightarrow aA \mid \epsilon$$

$$A \rightarrow abA \mid baB \mid \epsilon$$

$$B \rightarrow aB \mid bA$$

References

Kozen, D. C. (2007)
Automata and computability
Springer Science & Business Media.