Continuous Wavelet Transform

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Definition of the Continuous Wavelet Transform

Like the Fourier transform, the *continuous wavelet transform* (CWT) uses inner products to measure the similarity between a signal and an analyzing function. In the Fourier transform, the analyzing functions are complex exponentials, $e^{j\omega t}$. The resulting transform is a function of a single variable, ω. In the short-time Fourier transform, the analyzing functions are windowed complex exponentials, $w(t)e^{j\omega t}$, and the result in a function of two variables. The STFT coefficients, $F(\omega,\tau)$, represent the match between the signal and a sinusoid with angular frequency ω in an interval of a specified length centered at τ.

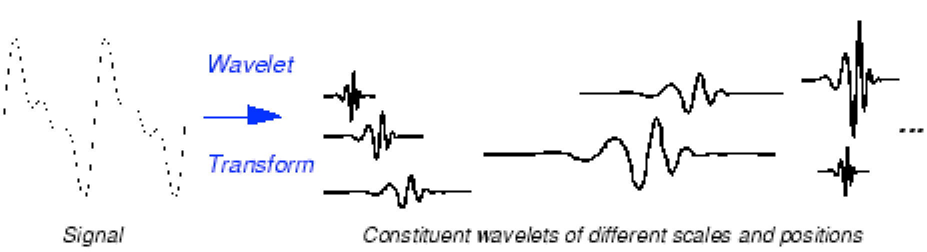
In the CWT, the analyzing function is a wavelet, ψ. The CWT compares the signal to shifted and compressed or stretched versions of a wavelet. Stretching or compressing a function is collectively referred to as *dilation* or *scaling* and corresponds to the physical notion of *scale*. By comparing the signal to the wavelet at various scales and positions, you obtain a function of two variables. The two-dimensional representation of a one-dimensional signal is redundant. If the wavelet is complex-valued, the CWT is a complex-valued function of scale and position. If the signal is real-valued, the CWT is a real-valued function of scale and position. For a scale parameter, a>0, and position, *b*, the CWT is:

$$C(a,b;f(t),\psi(t)) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{a}}\psi^*(\frac{t-b}{a})dt$$

where * denotes the complex conjugate. Not only do the values of scale and position affect the CWT coefficients, the choice of wavelet also affects the values of the coefficients.

By continuously varying the values of the scale parameter, *a*, and the position parameter, *b*, you obtain the *cwt coefficients C(a,b)*. Note that for convenience, the dependence of the CWT coefficients on the function and analyzing wavelet has been suppressed.

Multiplying each coefficient by the appropriately scaled and shifted wavelet yields the constituent wavelets of the original signal.



There are many different admissible wavelets that can be used in the CWT. While it may seem confusing that there are so many choices for the analyzing wavelet, it is actually a strength of wavelet analysis. Depending on what signal features you are trying to detect, you are free to select a wavelet that facilitates your detection of that feature. For example, if you are trying to detect abrupt discontinuities in your signal, you may choose one wavelet. On the other hand, if you are interesting in finding oscillations with smooth onsets and offsets, you are free to choose a wavelet that more closely matches that behavior.

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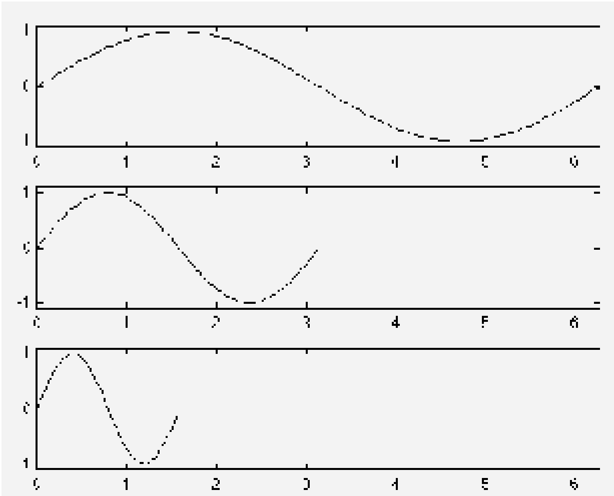
Scale

Like the concept of frequency, *scale* is another useful property of signals and images. For example, you can analyze temperature data for changes on different scales. You can look at year-to-year or decade-to-decade changes. Of course, you can examine finer (day-to-day), or coarser scale changes as well. Some processes reveal interesting changes on long time, or spatial scales that are not evident on small time or spatial scales. The opposite situation also happens. Some of our perceptual abilities exhibit *scale invariance*. You recognize people you know regardless of whether you look at a large portrait, or small photograph.

To go beyond colloquial descriptions such as "stretching" or "shrinking" we introduce the *scale factor*, often denoted by the letter *a*. The scale factor is a inherently positive quantity, *a*>0. For sinusoids, the effect of the scale factor is very easy to see.

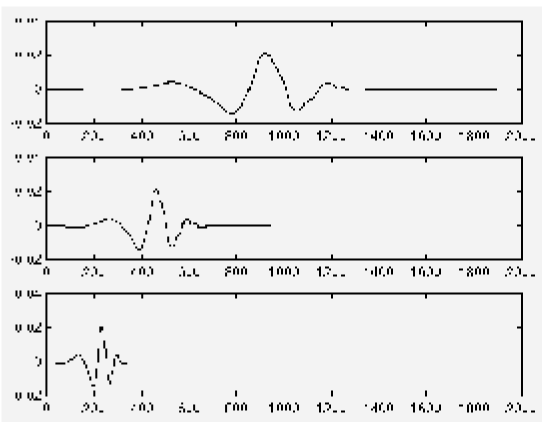


$f(t) = \sin(t)$; $a = 1$

$f(t) = \sin(2t)$; $a = \frac{1}{2}$

$f(t) = \sin(4t)$; $a = \frac{1}{4}$

The scale factor works exactly the same with wavelets. The smaller the scale factor, the more "compressed" the wavelet.



$f(t) = \psi(t)$; $a = 1$

$f(t) = \psi(2t)$; $a = \frac{1}{2}$

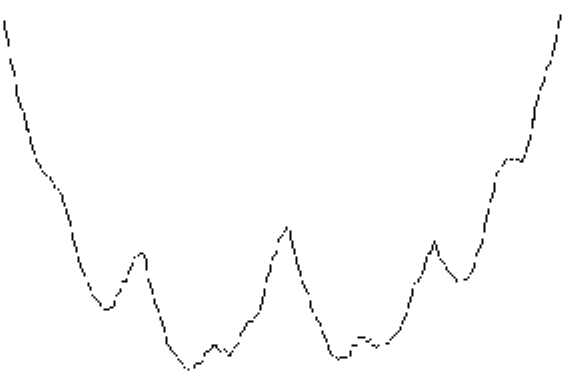
$f(t) = \psi(4t)$; $a = \frac{1}{4}$

It is clear from the diagrams that, for a sinusoid $\sin(\omega t)$, the scale factor a is related (inversely) to the radian frequency ω. This general inverse relationship between scale and frequency holds for signals in general. See CWT as a Filtering Technique and Scale and Frequency for more information on the relationship between scale and frequency.

Not only is a time-scale representation a different way to view data, it is a very natural way to view data derived from a great number of natural phenomena.

Consider a lunar landscape, whose ragged surface (simulated below) is a result of centuries of bombardment by meteorites whose sizes range from gigantic boulders to dust specks.

If we think of this surface in cross section as a one-dimensional signal, then it is reasonable to think of the signal as having components of different scales — large features carved by the impacts of large meteorites, and finer features created by small meteorites.

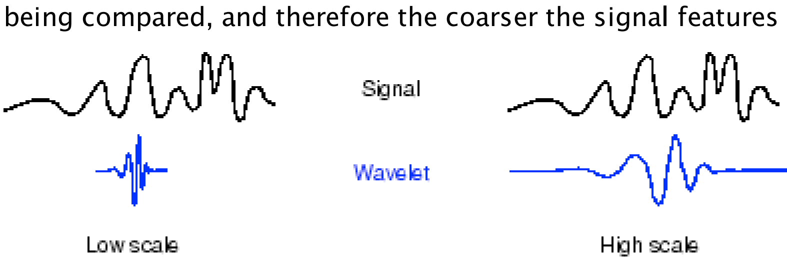


Here is a case where thinking in terms of scale makes much more sense than thinking in terms of frequency.

Even though this signal is artificial, many natural phenomena — from the intricate branching of blood vessels and trees, to the jagged surfaces of mountains and fractured metals — lend themselves to an analysis of scale.

Scale and Frequency

There is clearly a relationship between scale and frequency. Recall that higher scales correspond to the most "stretched" wavelets. The more stretched the wavelet, the longer the portion of the signal with which it is being compared, and therefore the coarser the signal features measured by the wavelet coefficients.



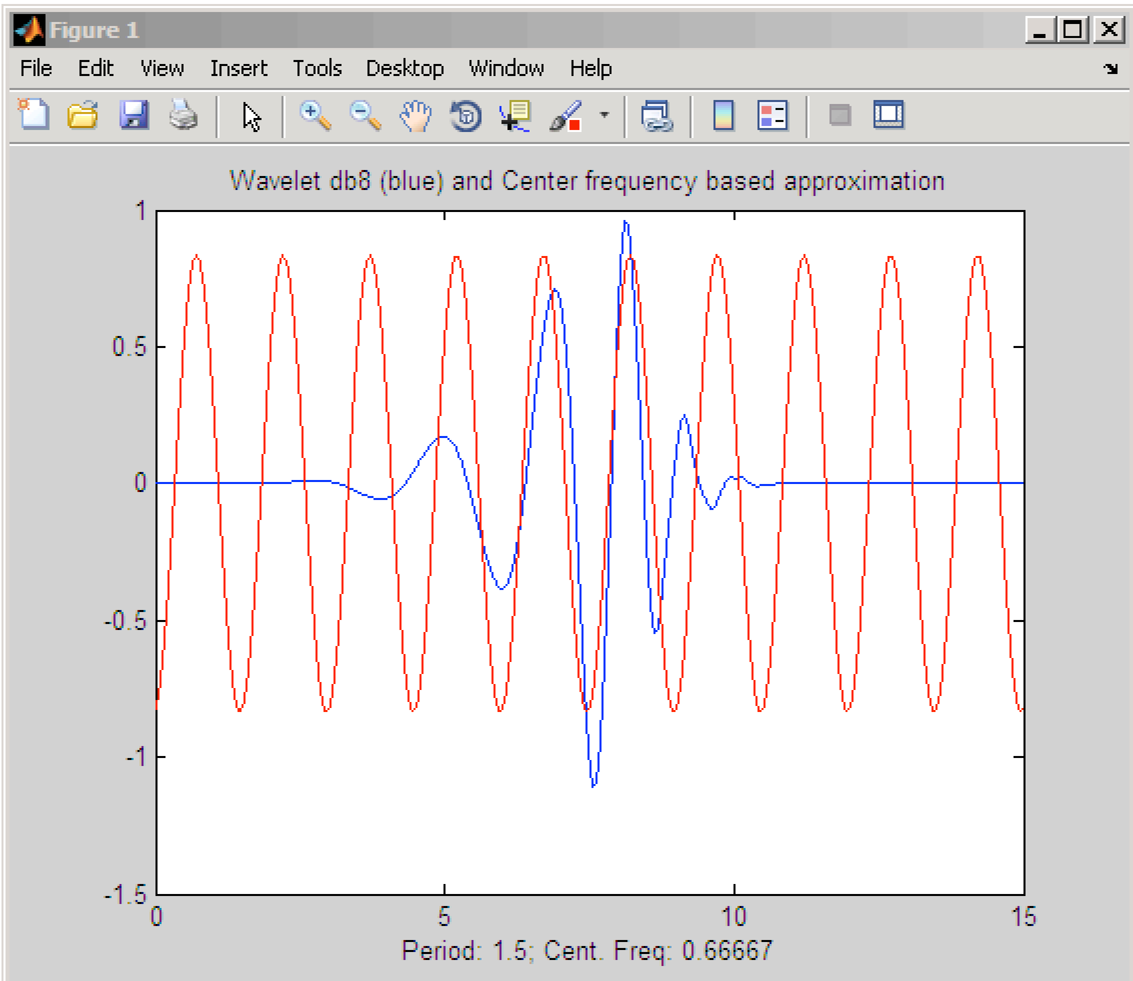
To summarize, the general correspondence between scale and frequency is:

- Low scale $a \Rightarrow$ Compressed wavelet $\Rightarrow$ Rapidly changing details $\Rightarrow$ High frequency ω.

- High scale a ⇒ Stretched wavelet ⇒ Slowly changing, coarse features ⇒ Low frequency ω.

While there is a general relationship between scale and frequency, no precise relationship exists. Users familiar with Fourier analysis often want to define a mapping between a wavelet at a given scale with a specified sampling period to a frequency in hertz. You can only do this in a general sense. Therefore, it is better to talk about the pseudo-frequency corresponding to a scale. The Wavelet Toolbox software provides two functions `centfrq` and `scal2frq`, which enable you to find these approximate scale-frequency relationships for specified wavelets and scales.

The basic approach identifies the peak power in the Fourier transform of the wavelet as its center frequency and divides that value by the product of the scale and the sampling interval. See `scal2frq` for details. The following example shows the match between the estimated center frequency of the `db8` wavelet and a sinusoid of the same frequency.



The relationship between scale and frequency in the CWT is also explored in CWT as a Filtering Technique.

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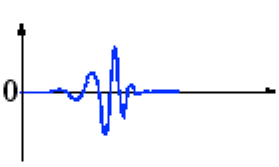
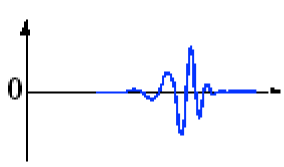
Shifting

Shifting a wavelet simply means delaying (or advancing) its onset. Mathematically, delaying a function f(t) by k is represented by f(t – k):



Wavelet function
$\psi(t)$



Shifted wavelet function
$\psi(t-k)$

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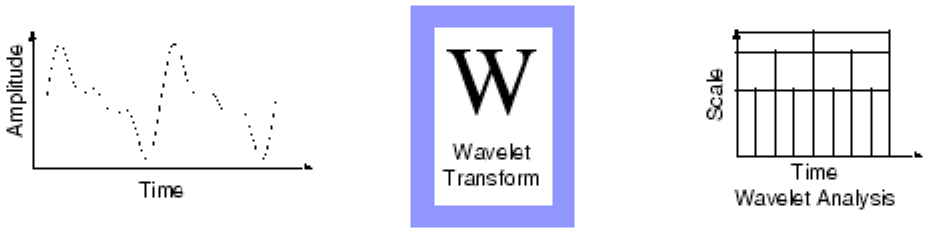
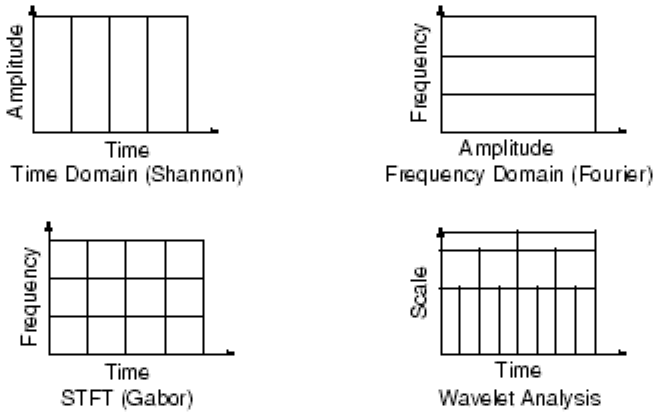
CWT as a Windowed Transform

In Short-Time Fourier Transform, the STFT is described as a windowing of the signal to create a local frequency analysis. A shortcoming of the STFT approach is that the window size is constant. There is a trade off in the choice of window size. A longer time window improves frequency resolution while resulting in poorer time resolution because the Fourier transform loses all time resolution over the duration of the window. Conversely, a shorter time window improves time localization while resulting in poorer frequency resolution.

Wavelet analysis represents the next logical step: a windowing technique with variable-sized regions. Wavelet analysis allows the use of long time intervals where you want more precise low-frequency information, and shorter regions where you want high-frequency information.



The following figure contrasts time, frequency, time-frequency, and time-scale representations of a signal.



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CWT as a Filtering Technique

The continuous wavelet transform (CWT) computes the inner product of a signal, $f(t)$, with translated and dilated versions of an analyzing wavelet, $\psi(t)$. The definition of the CWT is:

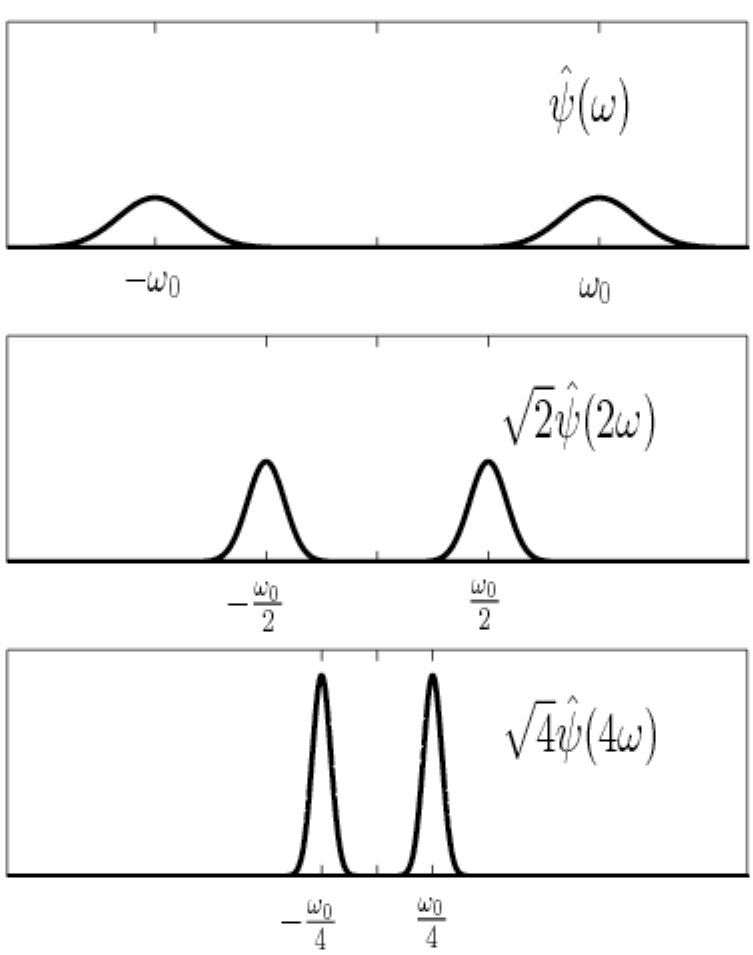
$$C(a,b;f(t),\psi(t)) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{a}}\psi^*\left(\frac{t-b}{a}\right)dt$$

You can also interpret the CWT as a frequency-based filtering of the signal by rewriting the CWT as an inverse Fourier transform.

$$C(a,b;f(t),\psi(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\sqrt{a}(\hat{\psi}(a\omega))^* e^{j\omega b}d\omega$$

where $\hat{f}(\omega)$ and $\hat{\psi}(\omega)$ are the Fourier transforms of the signal and the wavelet.

From the preceding equations, you can see that stretching a wavelet in time causes its support in the frequency domain to shrink. In addition to shrinking the frequency support, the center frequency of the wavelet shifts toward lower frequencies. The following figure demonstrates this effect for a hypothetical wavelet and scale (dilation) factors of 1,2, and 4.



This depicts the CWT as a bandpass filtering of the input signal. CWT coefficients at lower scales represent energy in the input signal at higher frequencies, while CWT coefficients at higher scales represent energy in the input signal at lower frequencies. However, unlike Fourier bandpass filtering, the width of the bandpass filter in the CWT is inversely proportional to scale. The width of the CWT *filters* decreases with increasing scale. This follows from the *uncertainty* relationships between the time and frequency support of a signal: the broader the support of a signal in time, the narrower its support in frequency. The converse relationship also holds.

In the wavelet transform, the scale, or dilation operation is defined to preserve energy. To preserve energy while shrinking the frequency support requires that the peak energy level increases. The *quality factor*, or *Q factor* of a filter is the ratio of its peak energy to bandwidth. Because shrinking or stretching the frequency support of a wavelet results in commensurate increases or decreases in its peak energy, wavelets are often referred to as constant-Q filters.

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Continuous Wavelet Transform via the Inverse Discrete Fourier Transform

The equation in the preceding section defined the CWT as the inverse Fourier transform of a product of Fourier transforms.

$$C(a,b;f(t),\psi(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\sqrt{a}\,\hat{\psi}^*(a\omega)e^{j\omega b}d\omega$$

The *time* variable in the inverse Fourier transform is the translation parameter, *b*.

This suggests that you can compute the CWT with the inverse Fourier transform. Because there are efficient algorithms for the computation of the discrete Fourier transform and its inverse, you can often achieve considerable savings by using `fft` and `ifft` when possible.

To obtain a picture of the CWT in the Fourier domain, start with the definition of the wavelet transform:

$$< f(t),\psi_{a,b}(t) >= \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} f(t)\psi^*(\tfrac{t-b}{a})dt$$

If you define:

$$\tilde{\psi}_a(t) = \frac{1}{\sqrt{a}}\psi^*(-t/a)$$

you can rewrite the wavelet transform as

$$(f*\tilde{\psi}_a)(b) = \int_{-\infty}^{\infty} f(t)\tilde{\psi}_a(b-t)dt$$

which explicitly expresses the CWT as a convolution.

To implement the discretized verion of the CWT, assume that the input sequence is a length N vector, *x[n]*. The discrete version of the preceding convolution is:

$$W_a[b] = \sum_{n=0}^{N-1} x[n]\tilde{\psi}_a[b-n]$$

To obtain the CWT, it appears you have to compute the convolution for each value of the shift parameter, *b*, and repeat this process for each scale, *a*.

However, if the two sequences are circularly-extended (periodized to length N), you can express the circular convolution as a product of discrete Fourier transforms. The CWT is the inverse Fourier transform of the product

$$W_a(b) = \frac{1}{N}\sqrt{\frac{2\pi a}{\Delta t}}\sum_{k=0}^{N-1}\hat{X}(2\pi k/N\Delta t)\hat{\psi}^*(a2\pi k/N\Delta t)e^{j2\pi kb/N}$$

where Δt is the sampling interval (period).

Expressing the CWT as an inverse Fourier transform enables you to use the computationally-efficient `fft` and `ifft` algorithms to reduce the cost of computing convolutions.

The `cwtft` function implements the CWT using an FFT-based algorithm. See `cwtftinfo` for information pertaining to the supported analyzing wavelets.

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Inverse Continuous Wavelet Transform

The `icwtft` function implements the inverse CWT. Using `icwtft` requires that you obtain the CWT from `cwtft`. The Wavelet Toolbox does not support the inverse CWT for a general CWT obtained using `cwt`.

Because the CWT is a redundant transform, there is not a unique way to define the inverse. The inverse CWT implemented in the Wavelet Toolbox utilizes a discrete version of the single integral formula due to Morlet.

The inverse CWT is classically presented in the double-integral form. Assume you have a wavelet with a

Fourier transform that satisfies the admissibility condition:

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$$

For wavelets satisfying the admissibility condition and finite-energy functions, *f(t)*, you can define the inverse CWT as:

$$f(t) = \frac{1}{C_\psi} \int_a \int_b < f(t), \psi_{a,b}(t) > \psi_{a,b}(t)\, db \frac{da}{a^2}$$

For analyzing wavelets and functions satisfying the following conditions, a single integral formula for the inverse CWT exists. These conditions are:

- The analyzed function, *f(t)*, is real-valued and the analyzing wavelet has a real-valued Fourier transform.
- The analyzed function, *f(t)*, is real-valued and the Fourier transform of the analyzing wavelet has support only on the set of nonnegative frequencies. This is referred to as an *analytic* wavelet. A function whose Fourier transform only has support on the set of nonnegative frequencies must be complex-valued.

The preceding conditions constrain the set of possible analyzing wavelets. If you inspect the list of wavelets supported by `cwtft`, each wavelet is either analytic or has a real-valued Fourier transform. Because the toolbox only supports the analysis of real-valued functions, the real-valued condition on the analyzed function is always satisfied.

To motivate the single integral formula, let ψ_1 and ψ_2 be two *wavelets* that satisfy the following two-*wavelet* admissibility condition:

$$\int \frac{|\hat{\psi}_1^*(\omega)|\,|\hat{\psi}_2(\omega)|}{|\omega|} d\omega < \infty$$

Define the constant:

$$C_{\psi_1,\psi_2} = \int \frac{\hat{\psi}_1^*(\omega)\hat{\psi}_2(\omega)}{|\omega|} d\omega$$

INote that the above constant may be complex-valued. Let *f(t)* and *g(t)* be two finite energy functions. If the two-*wavelet* admissibility condition is satisfied, the following equality holds:

$$C_{\psi_1,\psi_2} < f, g > = \int\int < f, \psi_1 > < g, \psi_2 >^* db \frac{da}{a^2}$$

where < , > denotes the inner product, * denotes the complex conjugate, and the dependence of ψ_1 and ψ_2 on scale and position has been suppressed for convenience.

The key to the single integral formula for the inverse CWT is to recognize that the two-*wavelet* admissibility condition can be satisfied even if one of the *wavelets* is not admissible. In other words, it is not necessary that both ψ_1 and ψ_2 be separately admissible. You can also relax the requirements further by allowing one of the *functions* and *wavelets* to be distributions. By first letting *g(t)* be the Dirac delta function (a distribution) and also allowing ψ_2 to be the Dirac delta function, you can derive the single integral formula for the inverse CWT

$$f(t) = 2\, Re\{\frac{1}{C_{\psi_1,\delta}} \int_0^{\infty} < f(t), \psi_1(t) > \frac{da}{a^{3/2}}\}$$

where *Re{ }* denotes the real part.

The preceding equation demonstrates that you can reconstruct the signal by summing the scaled CWT coefficients over all scales.

By summing the scaled CWT coefficients from select scales, you obtain an approximation to the original signal. This is useful in situations where your phenomenon of interest is localized in scale.

`icwtft` implements a discretized version of the above integral.

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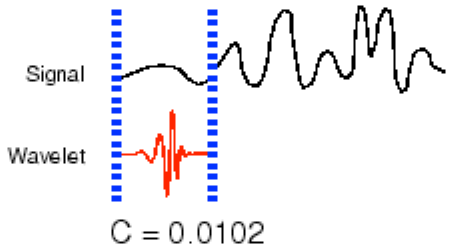
Five Easy Steps to a Continuous Wavelet Transform

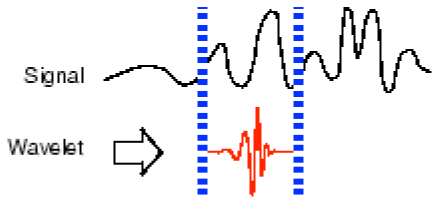
Here are the five steps of an easy recipe for creating a CWT:

1. Take a wavelet and compare it to a section at the start of the original signal.
2. Calculate a number, C, that represents how closely correlated the wavelet is with this section of the signal. The larger the number C is in absolute value, the more the similarity. This follows from the fact the CWT coefficients are calculated with an inner product. See Inner Products for more information on how inner products measure similarity. If the signal energy and the wavelet energy are equal to one, C may be interpreted as a correlation coefficient. Note that, in general, the signal energy does not equal one and the CWT coefficients are not directly interpretable as correlation coefficients.

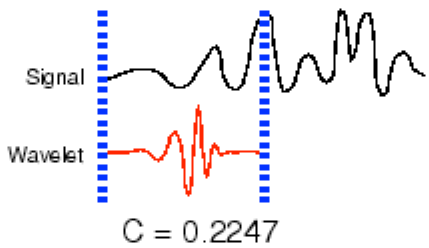
As described in Definition of the Continuous Wavelet Transform, the CWT coefficients explicitly depend on the analyzing wavelet. Therefore, the CWT coefficients are different when you compute the CWT for the same signal using different wavelets.



3. Shift the wavelet to the right and repeat steps 1 and 2 until you've covered the whole signal.



4. Scale (stretch) the wavelet and repeat steps 1 through 3.



5. Repeat steps 1 through 4 for all scales.

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Interpreting the CWT Coefficients

Because the CWT is a redundant transform and the CWT coefficients depend on the wavelet, it can be challenging to interpret the results.

To help you in interpreting CWT coefficients, it is best to start with a simple signal to analyze and an analyzing wavelet with a simple structure.

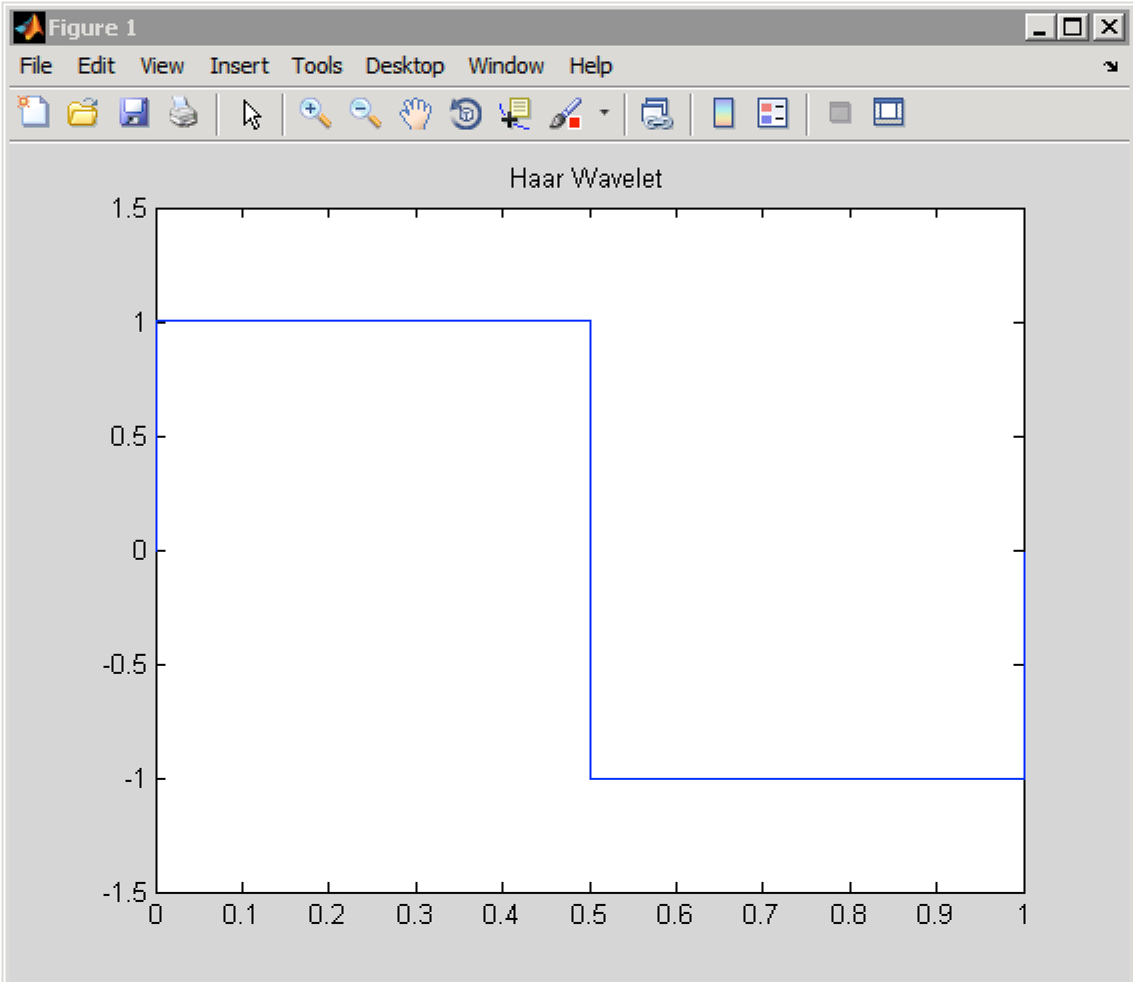
A signal feature that wavelets are very good at detecting is a discontinuity, or singularity. Abrupt transitions in signals result in wavelet coefficients with large absolute values.

For the signal create a shifted impulse. The impulse occurs at point 500.

```
x = zeros(1000,1);
x(500) = 1;
```

For the wavelet, pick the Haar wavelet.

```
[~,psi,xval] = wavefun('haar',10);
plot(xval,psi); axis([0 1 -1.5 1.5]);
title('Haar Wavelet');
```



To compute the CWT using the Haar wavelet at scales 1 to 128, enter:

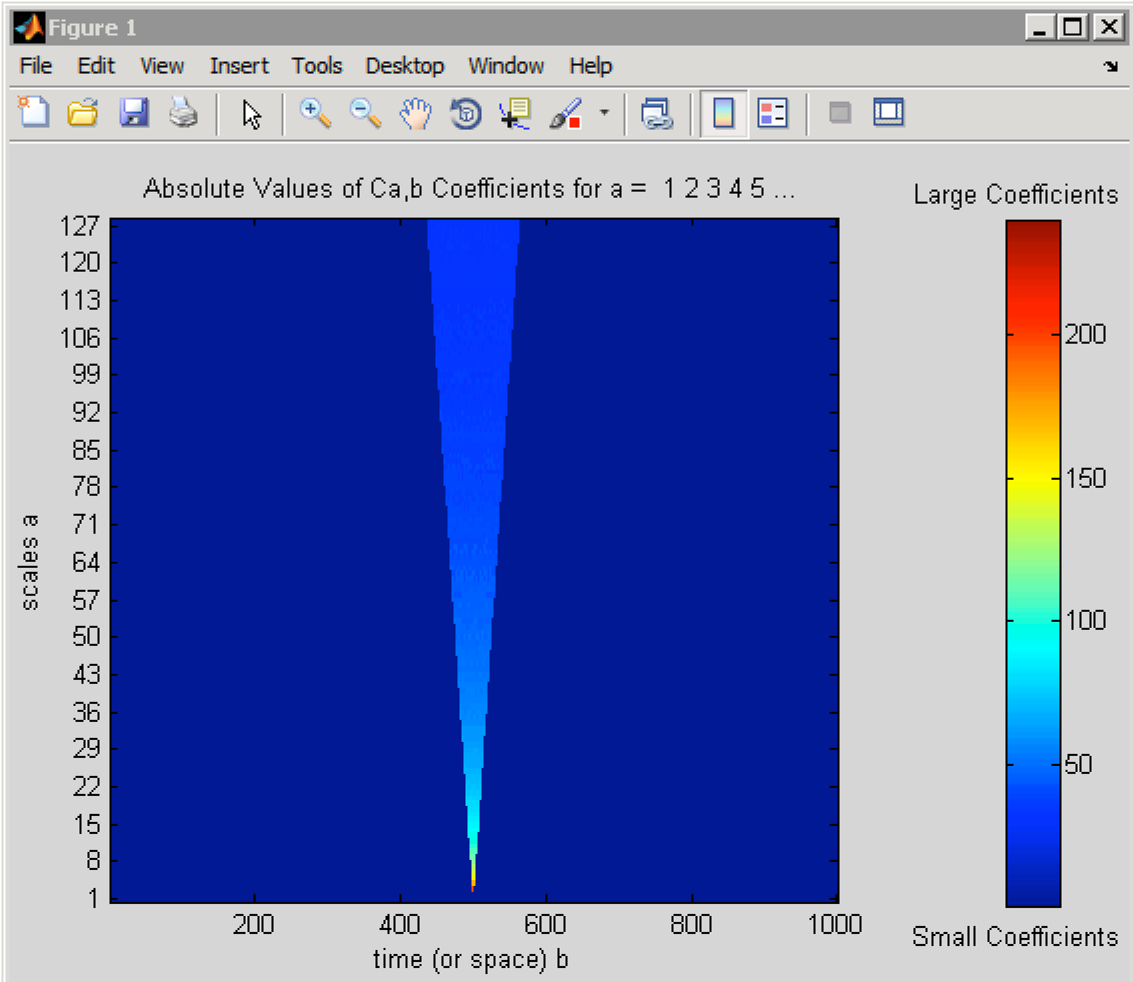
```
CWTcoeffs = cwt(x,1:128,'haar');
```

`CWTcoeffs` is a 128-by-1000 matrix. Each row of the matrix contains the CWT coefficients for one scale. There are 128 rows because the `SCALES` input to `cwt` is `1:128`. The column dimension of the matrix matches the length of the input signal.

Recall that the CWT of a 1D signal is a function of the scale and position parameters. To produce a plot of the CWT coefficients, plot position along the *x*-axis, scale along the *y*-axis, and encode the magnitude, or size of the CWT coefficients as color at each point in the *x*-*y*, or time-scale plane.

You can produce this plot using `cwt` with the optional input argument `'plot'`.

```
cwt(x,1:128,'haar','plot');
colormap jet; colorbar;
```



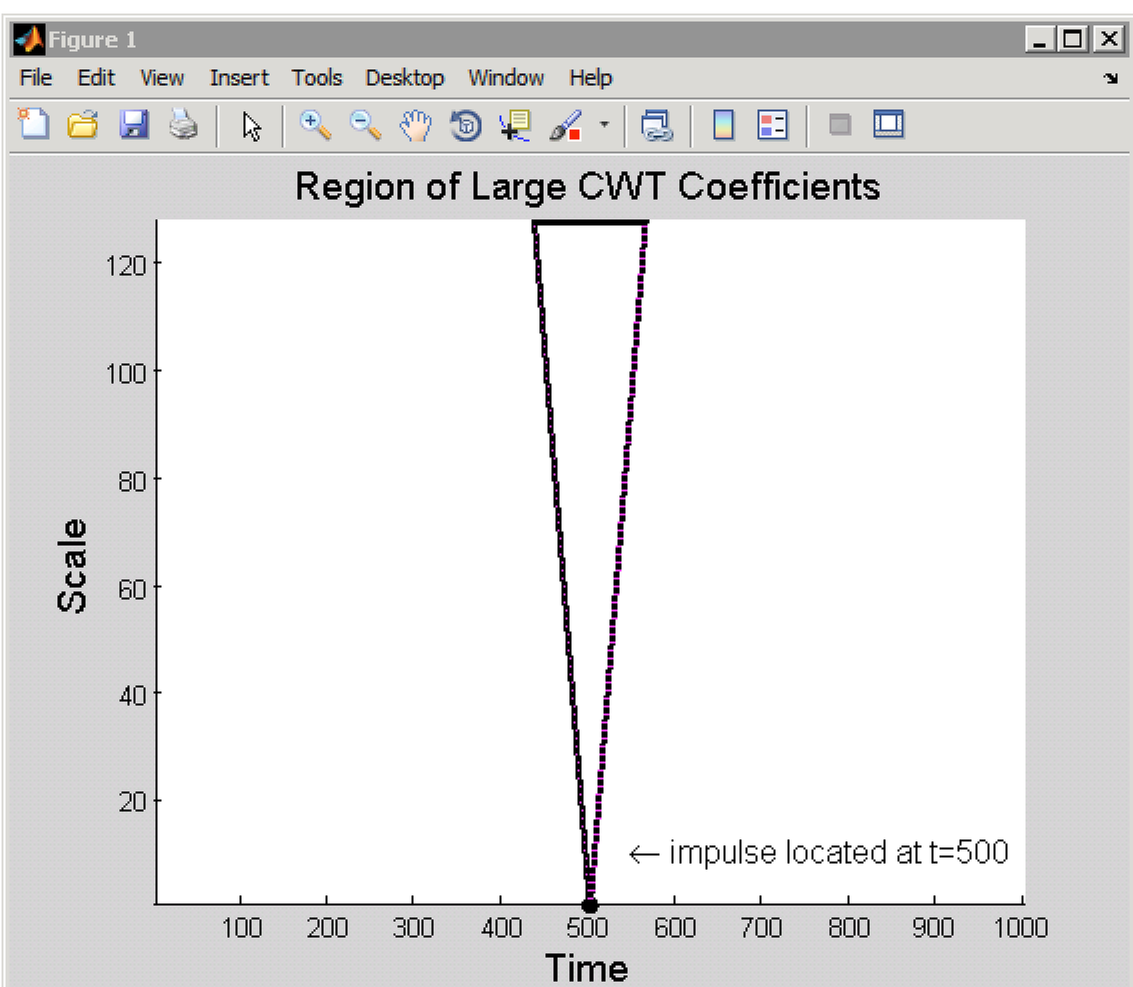
The preceding figure was modified with text labels to explicitly show which colors indicate large and small CWT coefficients.

You can also plot the size of the CWT coefficients in 3D with

```
cwt(x,1:64,'haar','3Dplot'); colormap jet;
```

where the number of scales has been reduced to aid in visualization.

Examining the CWT of the shifted impulse signal, you can see that the set of large CWT coefficients is concentrated in a narrow region in the time-scale plane at small scales centered around point 500. As the scale increases, the set of large CWT coefficients becomes wider, but remains centered around point 500. If you trace the border of this region, it resembles the following figure.



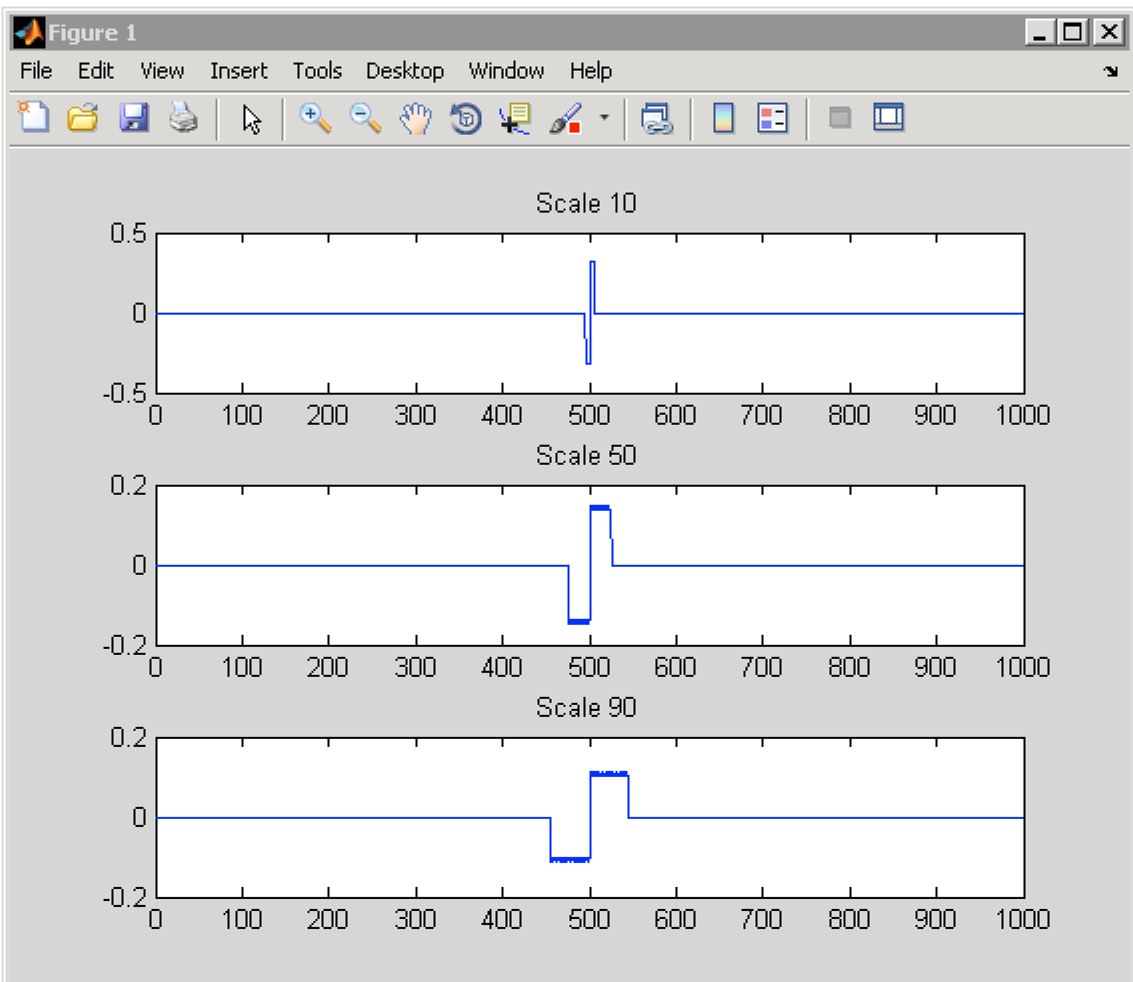
This region is referred to as the *cone of influence* of the point *t*=500 for the Haar wavelet. For a given point, the cone of influence shows you which CWT coefficients are affected by the signal value at that point.

To understand the cone of influence, assume that you have a wavelet supported on [−*C*, *C*]. Shifting the wavelet by *b* and scaling by *a* results in a wavelet supported on [−*Ca*+*b*, *Ca*+*b*]. For the simple case of a shifted impulse, $\delta(t-\tau)$, the CWT coefficients are only nonzero in an interval around τ equal to the support of the wavelet at each scale. You can see this by considering the formal expression of the CWT of the shifted impulse.

$$C(a,b;\delta(t-\tau),\psi(t)) = \int_{-\infty}^{\infty} \delta(t-\tau)\frac{1}{\sqrt{a}}\psi^*(\tfrac{t-b}{a})dt = \frac{1}{\sqrt{a}}\psi^*(\tfrac{\tau-b}{a})$$

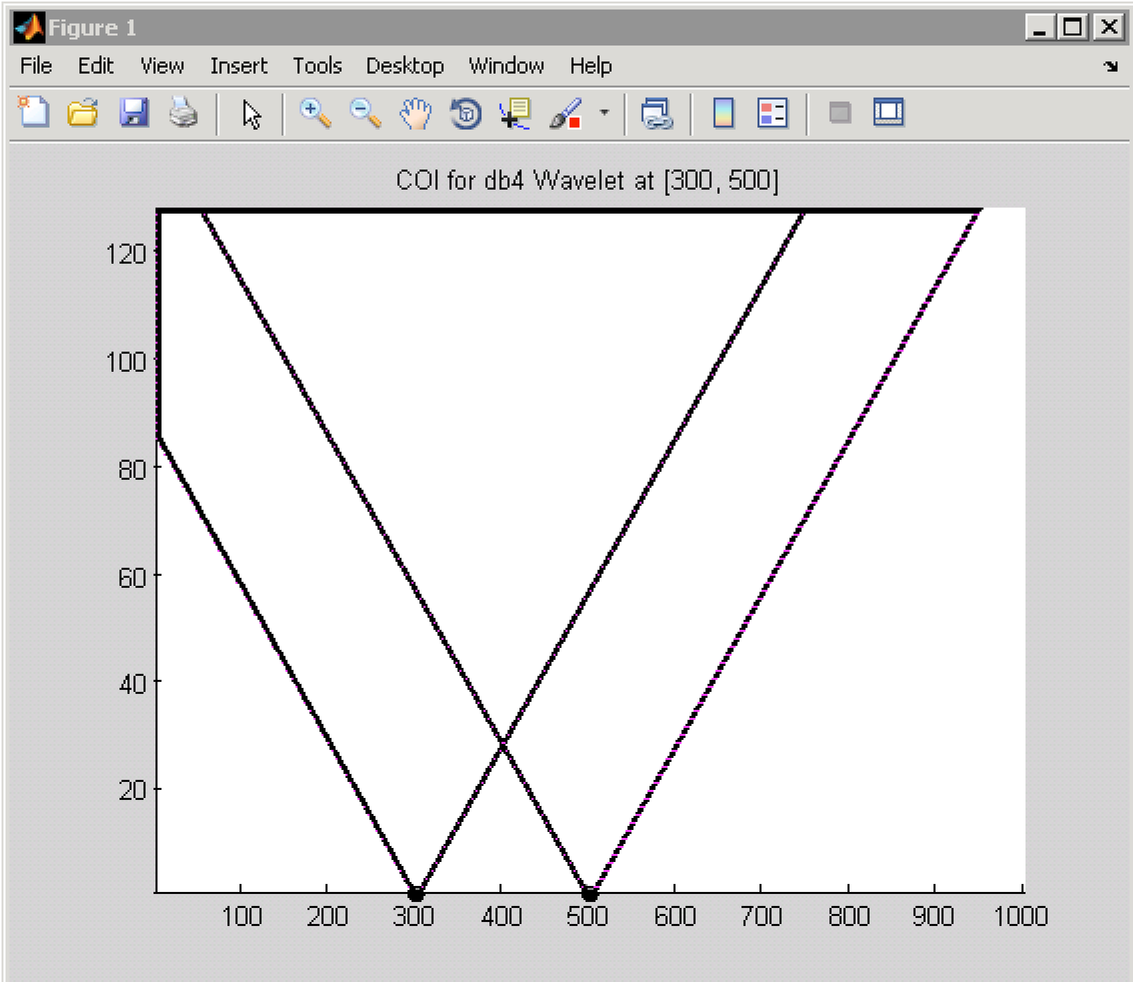
For the impulse, the CWT coefficients are equal to the conjugated, time-reversed, and scaled wavelet as a function of the shift parameter, *b*. You can see this by plotting the CWT coefficients for a select few scales.

```
subplot(311)
plot(CWTcoeffs(10,:)); title('Scale 10');
subplot(312)
plot(CWTcoeffs(50,:)); title('Scale 50');
subplot(313)
plot(CWTcoeffs(90,:)); title('Scale 90');
```



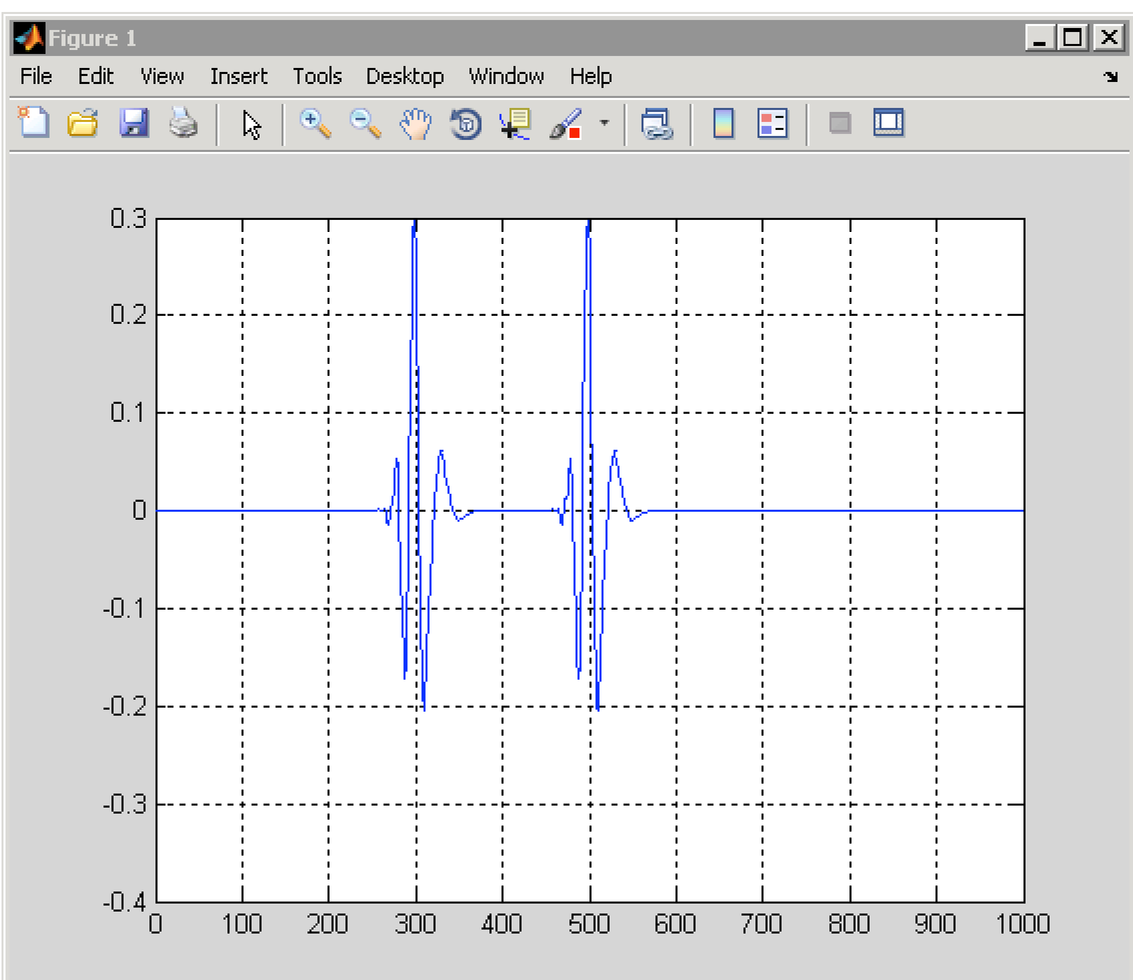
The cone of influence depends on the wavelet. You can find and plot the cone of influence for a specific wavelet with conofinf.

The next example features the superposition of two shifted impulses, $\delta(t-300)+\delta(t-500)$. In this case, use the Daubechies' extremal phase wavelet with four vanishing moments, `db4`. The following figure shows the cone of influence for the points 300 and 500 using the `db4` wavelet.



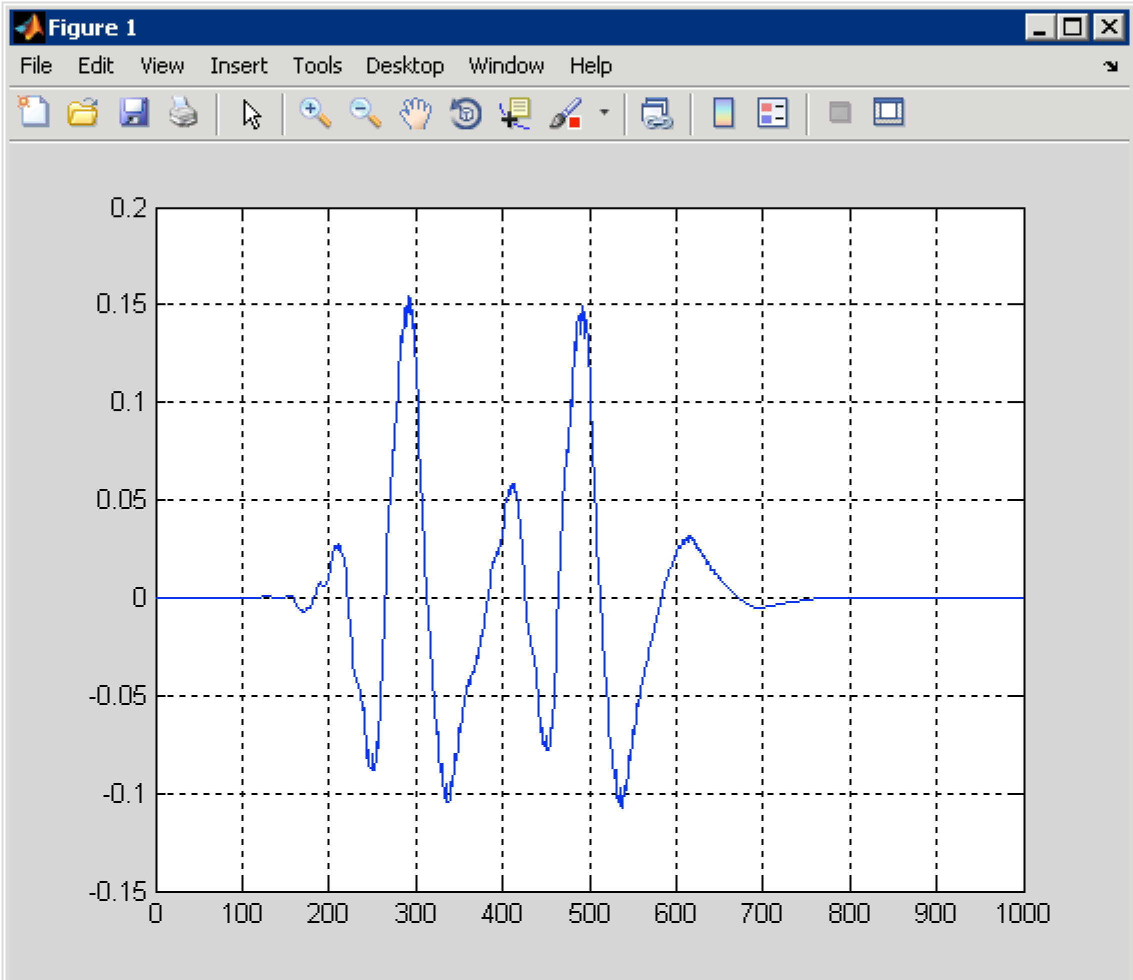
Look at point 400 for scale 20. At that scale, you can see that neither cone of influence overlaps the point 400. Therefore, you can expect that the CWT coefficient will be zero at that point and scale. The signal is only nonzero at two values, 300 and 500, and neither cone of influence for those values includes the point 400 at scale 20. You can confirm this by entering:

```
x = zeros(1000,1);
x([300 500]) = 1;
CWTcoeffs = cwt(x,1:128,'db4');
plot(CWTcoeffs(20,:)); grid on;
```



Next, look at the point 400 at scale 80. At scale 80, the cones of influence for both points 300 and 500 include the point 400. Even though the signal is zero at point 400, you obtain a nonzero CWT coefficient at that scale. The CWT coefficient is nonzero because the support of the wavelet has become sufficiently large at that scale to allow signal values 100 points above and below to affect the CWT coefficient. You can confirm this by entering:

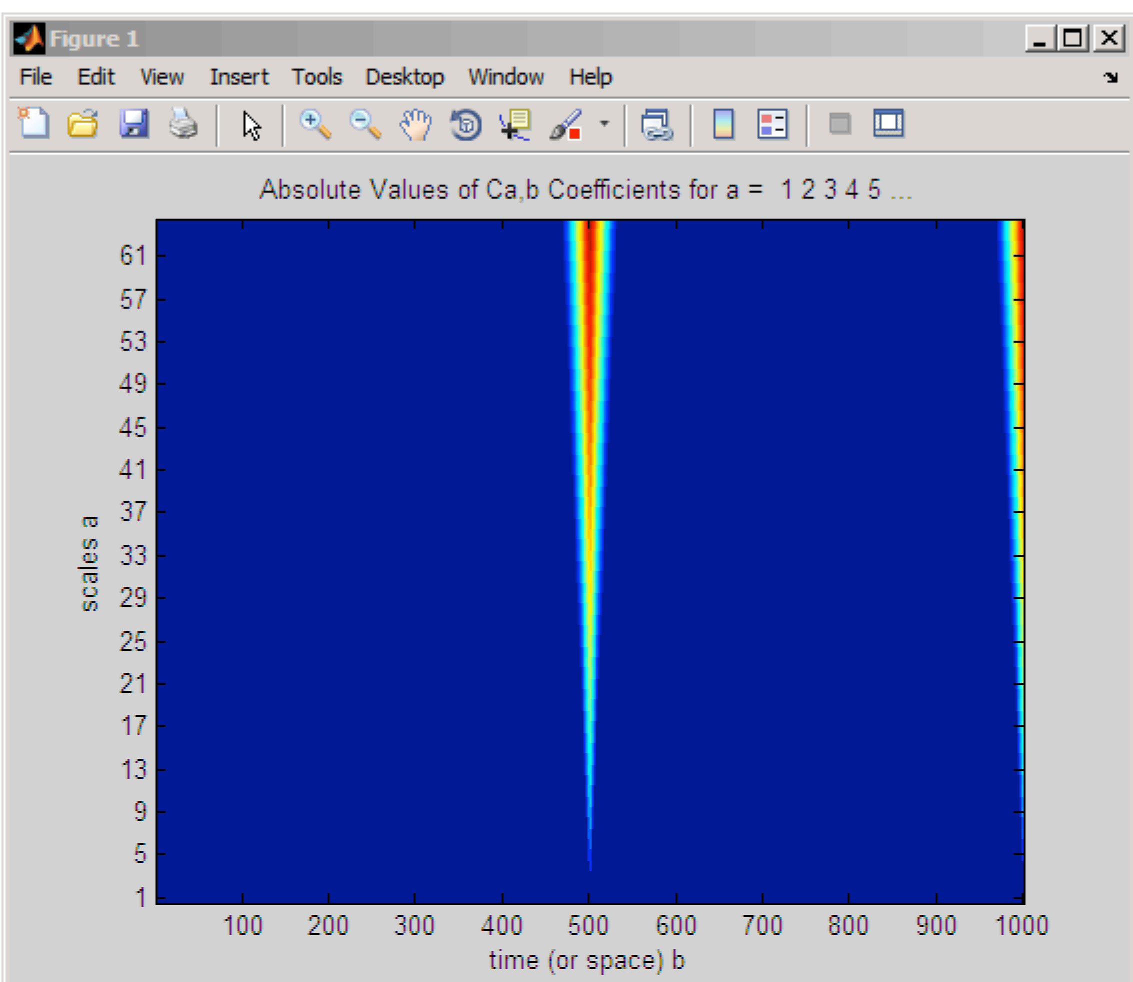
```
plot(CWTcoeffs(80,:));
grid on;
```



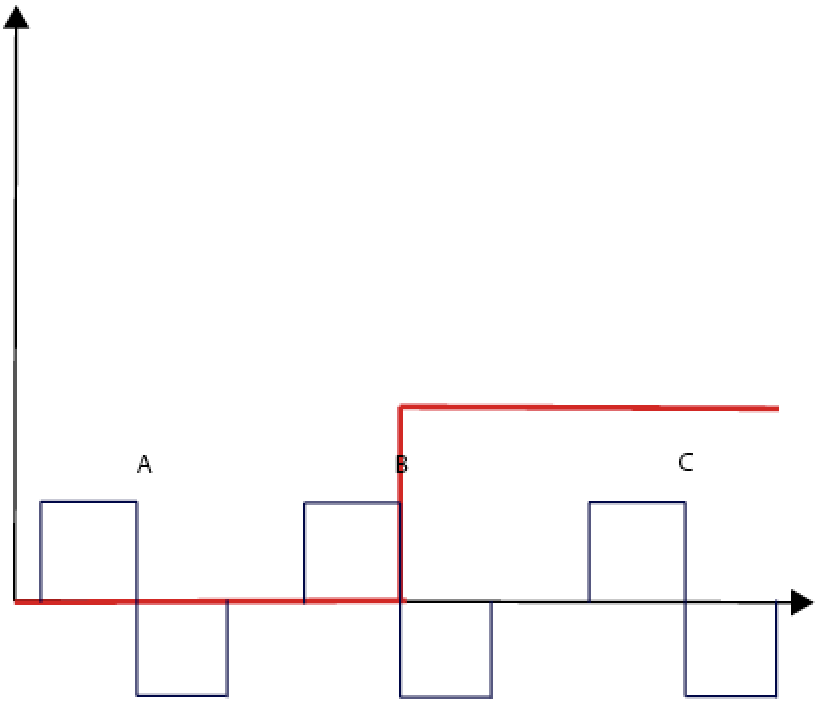
In the preceding example, the CWT coefficients became large in the vicinity of an abrupt change in the signal. This ability to detect discontinuities is a strength of the wavelet transform. The preceding example also demonstrated that the CWT coefficients localize the discontinuity best at small scales. At small scales, the small support of the wavelet ensures that the singularity only affects a small set of wavelet coefficients.

To demonstrate why the wavelet transform is so adept at detecting abrupt changes in the signal, consider a shifted Heaviside, or unit step signal.

```
x = [zeros(500,1); ones(500,1)];
CWTcoeffs = cwt(x,1:64,'haar','plot'); colormap jet;
```



Similar to the shifted impulse example, the abrupt transition in the shifted step function results in large CWT coefficients at the discontinuity. The following figure illustrates why this occurs.



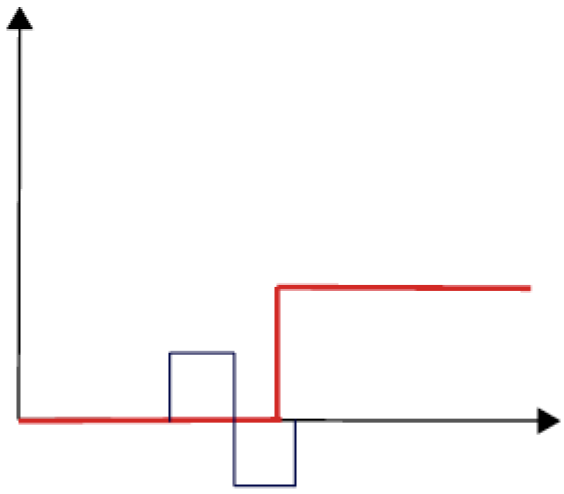
In the preceding figure, the red function is the shifted unit step function. The black functions labeled A, B, and C depict Haar wavelets at the same scale but different positions. You can see that the CWT coefficients around position A are zero. The signal is zero in that neighborhood and therefore the wavelet transform is also zero because any wavelet integrates to zero.

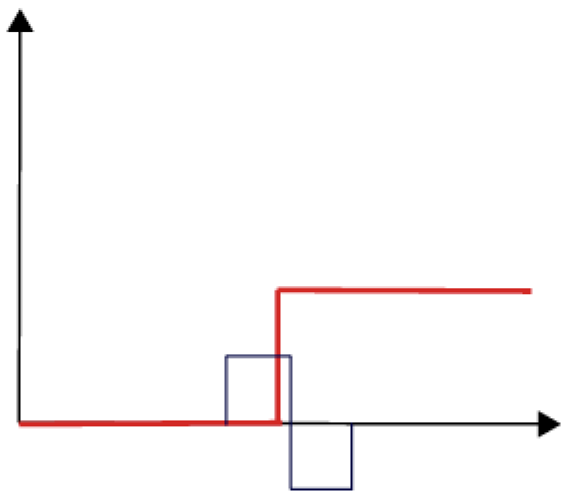
Note the Haar wavelet centered around position B. The negative part of the Haar wavelet overlaps with a region of the step function that is equal to 1. The CWT coefficients are negative because the product of the Haar wavelet and the unit step is a negative constant. Integrating over that area yields a negative number.

Note the Haar wavelet centered around position C. Here the CWT coefficients are zero. The step function is equal to one. The product of the wavelet with the step function is equal to the wavelet. Integrating any wavelet over its support is zero. This is the zero moment property of wavelets.

At position B, the Haar wavelet has already shifted into the nonzero portion of the step function by 1/2 of its support. As soon as the support of the wavelet intersects with the unity portion of the step function, the CWT coefficients are nonzero. In fact, the situation illustrated in the previous figure coincides with the CWT coefficients achieving their largest absolute value. This is because the entire negative deflection of the wavelet oscillation overlaps with the unity portion of the unit step while none of the positive deflection of the wavelet does. Once the wavelet shifts to the point that the positive deflection overlaps with the unit step, there will be some positive contribution to the integral. The wavelet coefficients are still negative (the negative portion of the integral is larger in area), but they are smaller in absolute value than those obtained at position B.

The following figure illustrates two other positions where the wavelet intersects the unity portion of the unit step.



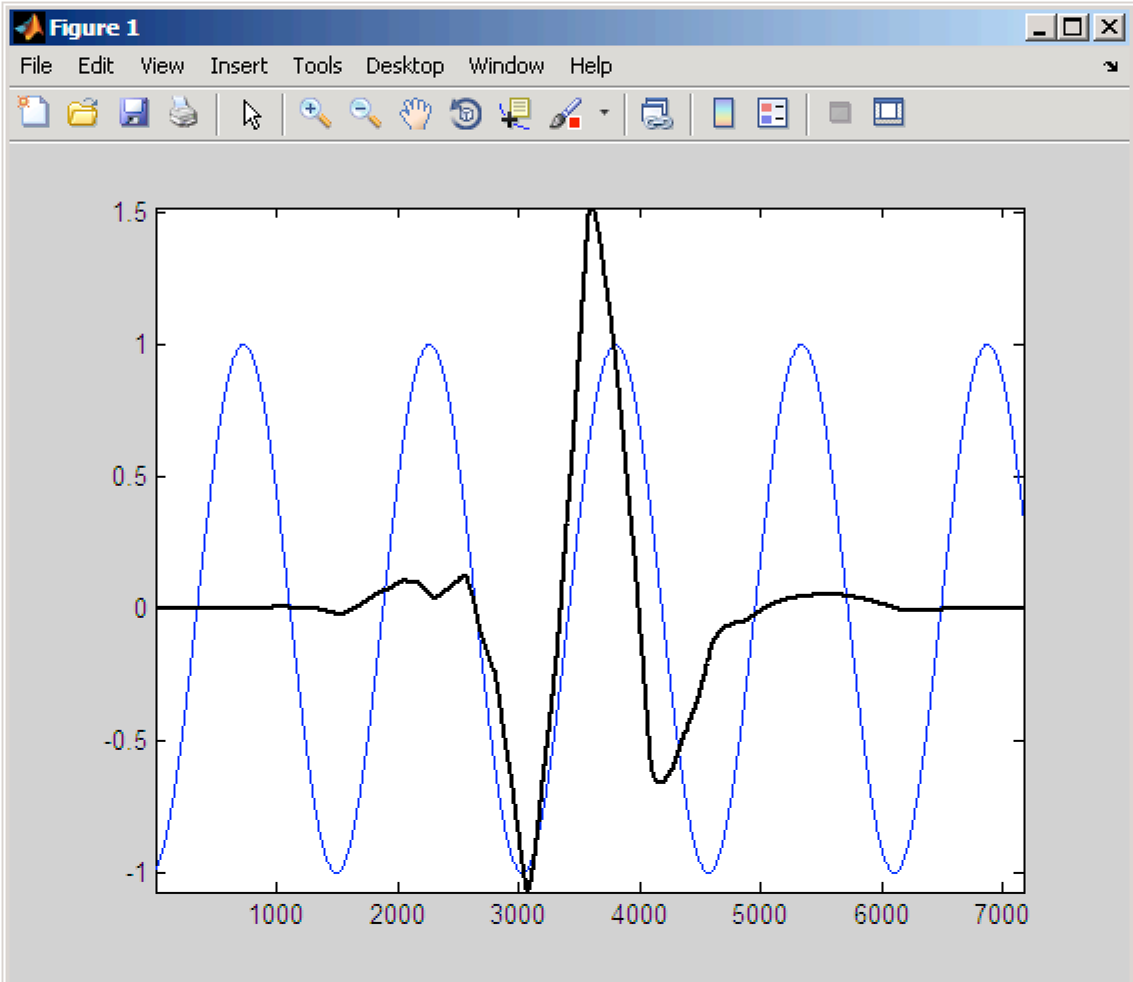


In the top figure, the wavelet has just begun to overlap with the unity portion of the unit step. In this case, the CWT coefficients are negative, but not as large in absolute value as those obtained at position B. In the bottom figure, the wavelet has shifted past position B and the positive deflection of the wavelet begins to contribute to the integral. The CWT coefficients are still negative, but not as large in absolute value as those obtained at position B.

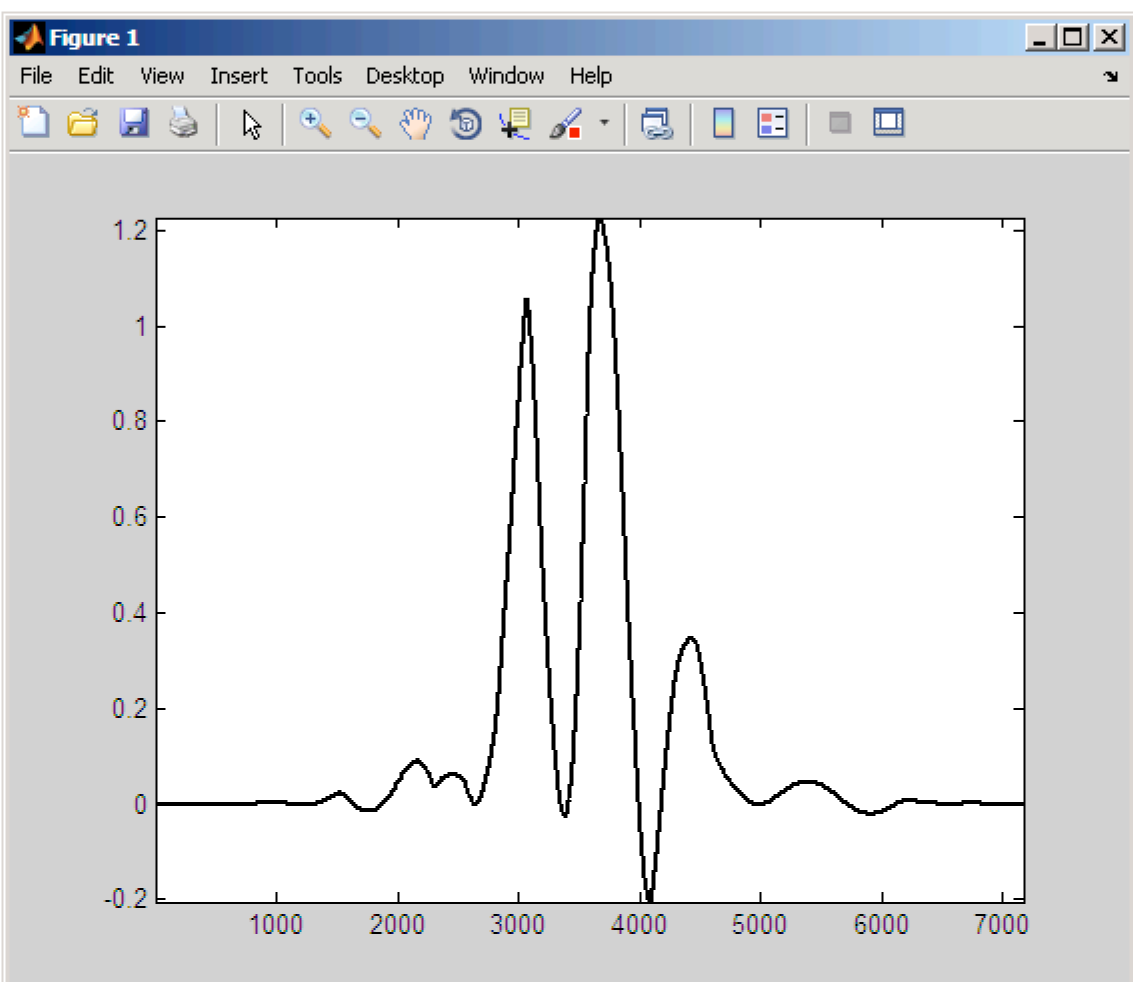
You can now visualize how the wavelet transform is able to detect discontinuities. You can also visualize in this simple example exactly why the CWT coefficients are negative in the CWT of the shifted unit step using the Haar wavelet. Note that this behavior differs for other wavelets.

```
x = [zeros(500,1); ones(500,1)];
CWTcoeffs = cwt(x,1:64,'haar','plot'); colormap jet;
% plot a few scales for visualization
subplot(311);
plot(CWTcoeffs(5,:)); title('Scale 5');
subplot(312);
plot(CWTcoeffs(10,:)); title('Scale 10');
subplot(313);
plot(CWTcoeffs(50,:)); title('Scale 50');
```

Next consider how the CWT represents smooth signals. Because sinusoidal oscillations are a common phenomenon, this section examines how sinusoidal oscillations in the signal affect the CWT coefficients. To begin, consider the `sym4` wavelet at a specific scale superimposed on a sine wave.

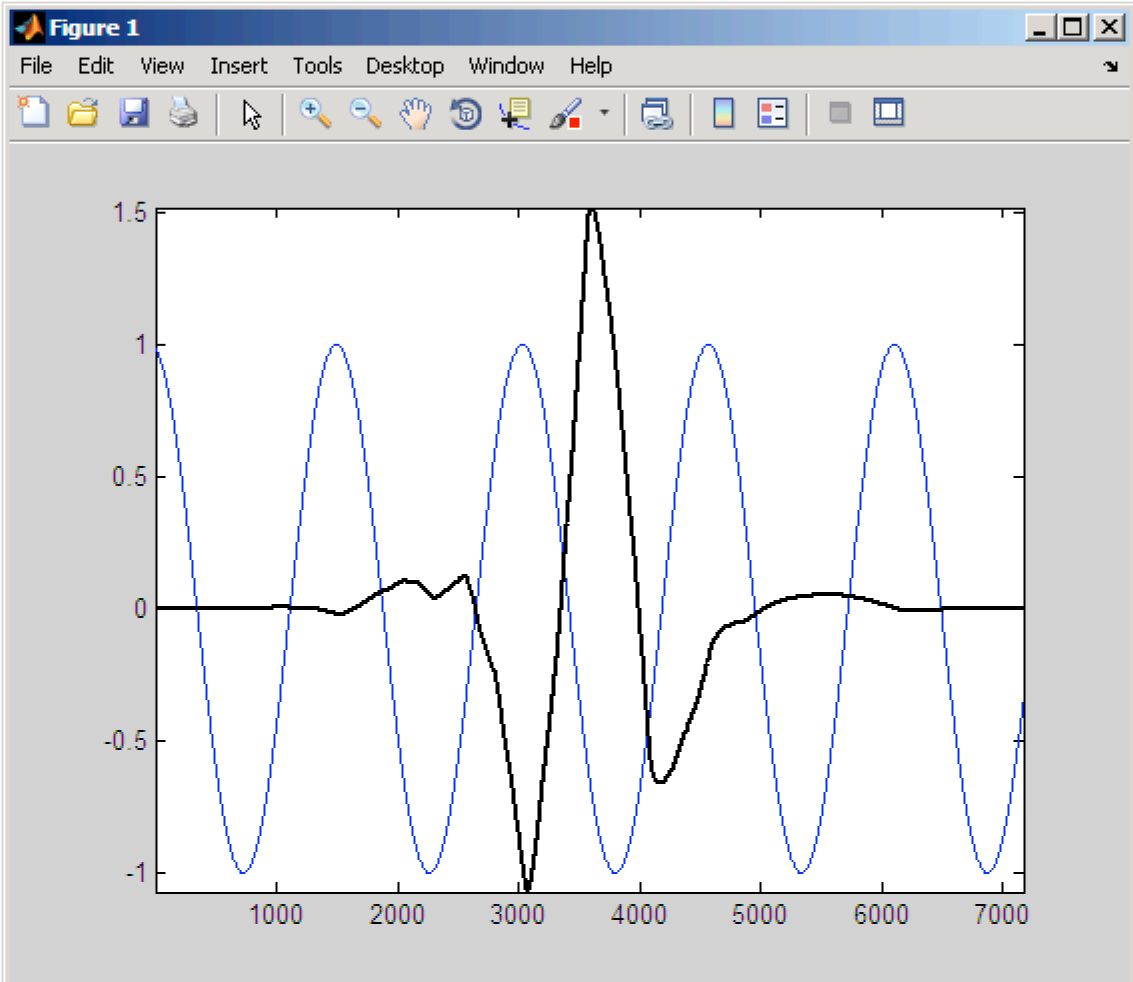


Recall that the CWT coefficients are obtained by computing the product of the signal with the shifted and scaled analyzing wavelet and integrating the result. The following figure shows the product of the wavelet and the sinusoid from the preceding figure.

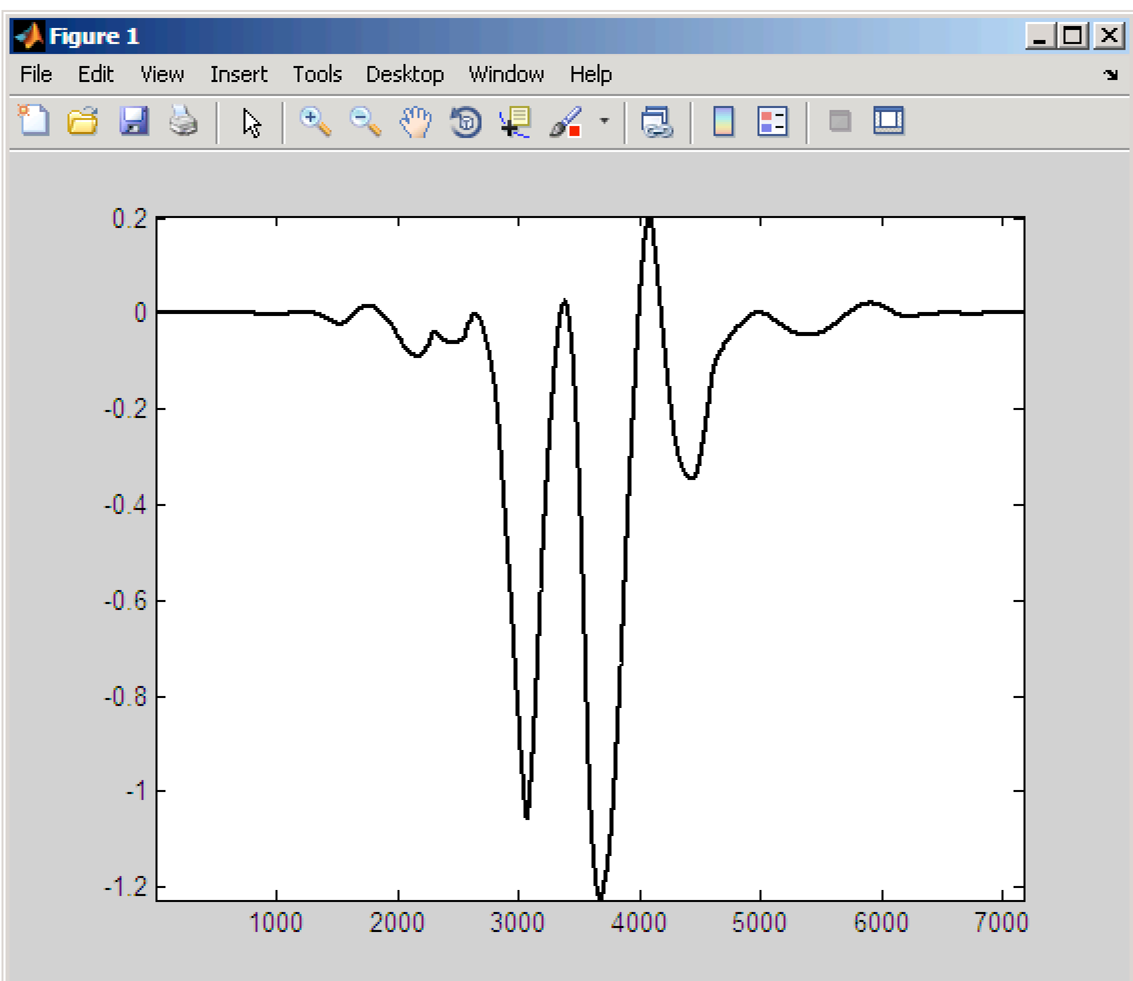


You can see that integrating over this product produces a positive CWT coefficient. That results because the oscillation in the wavelet approximately matches a period of the sine wave. The wavelet is *in phase* with the sine wave. The negative deflections of the wavelet approximately match the negative deflections of the sine wave. The same is true of the positive deflections of both the wavelet and sinusoid.

The following figure shifts the wavelet 1/2 of the period of the sine wave.

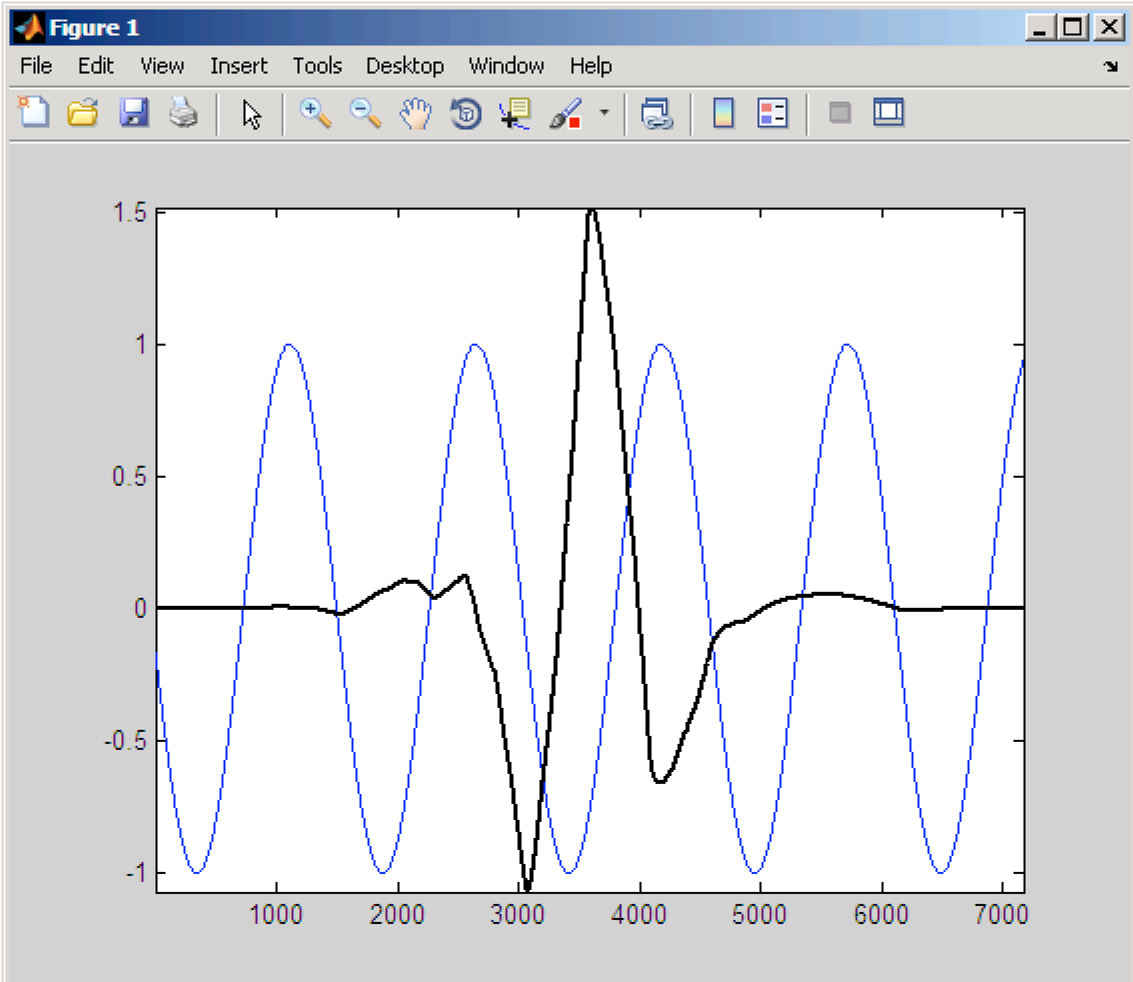


Examine the product of the shifted wavelet and the sinusoid.

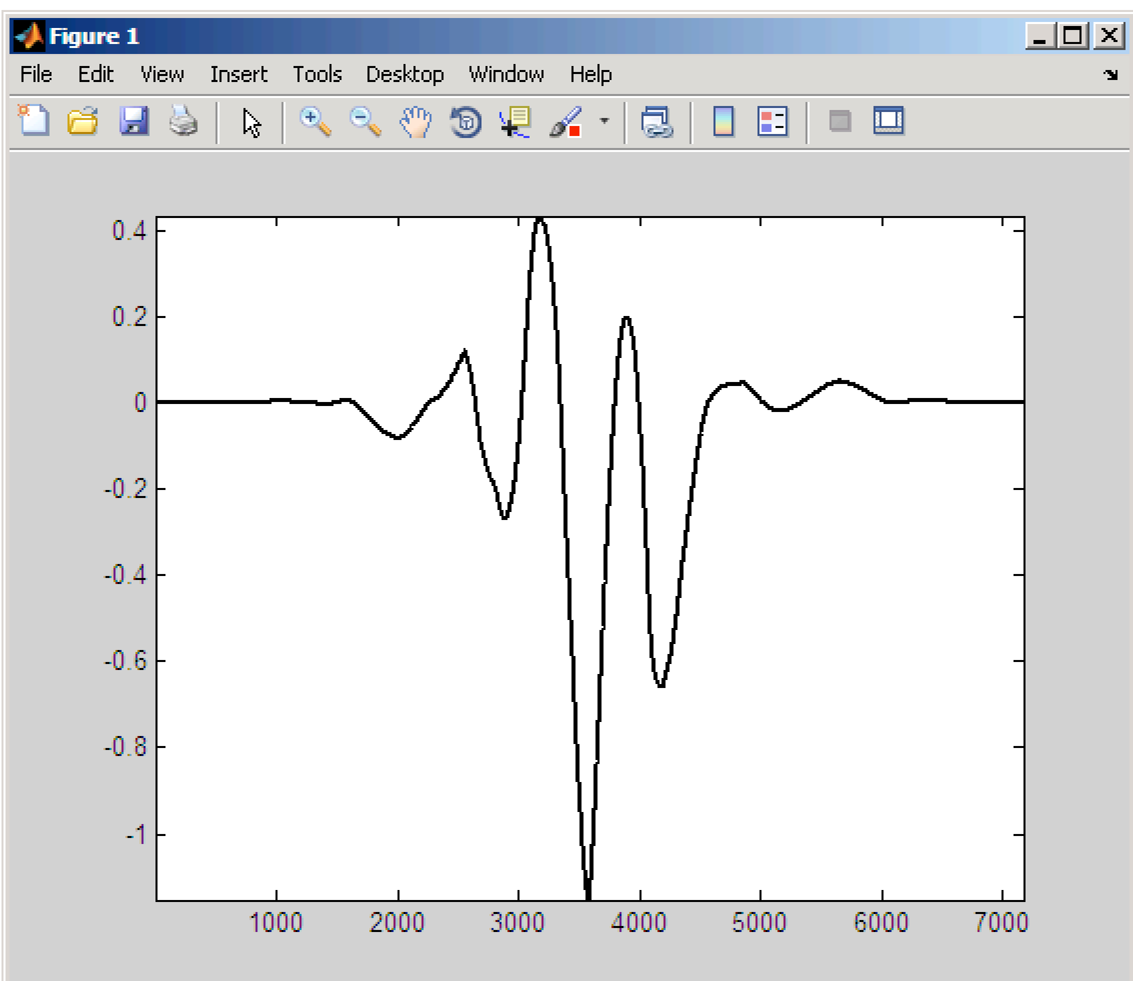


You can see that integrating over this product produces a negative CWT coefficient. That results because the wavelet is 1/2 cycle out of phase with the sine wave. The negative deflections of the wavelet approximately match the positive deflections of the sine wave. The positive deflections of the wavelet approximately match the negative deflections of the sinusoid.

Finally, shift the wavelet approximately one quarter cycle of the sine wave.

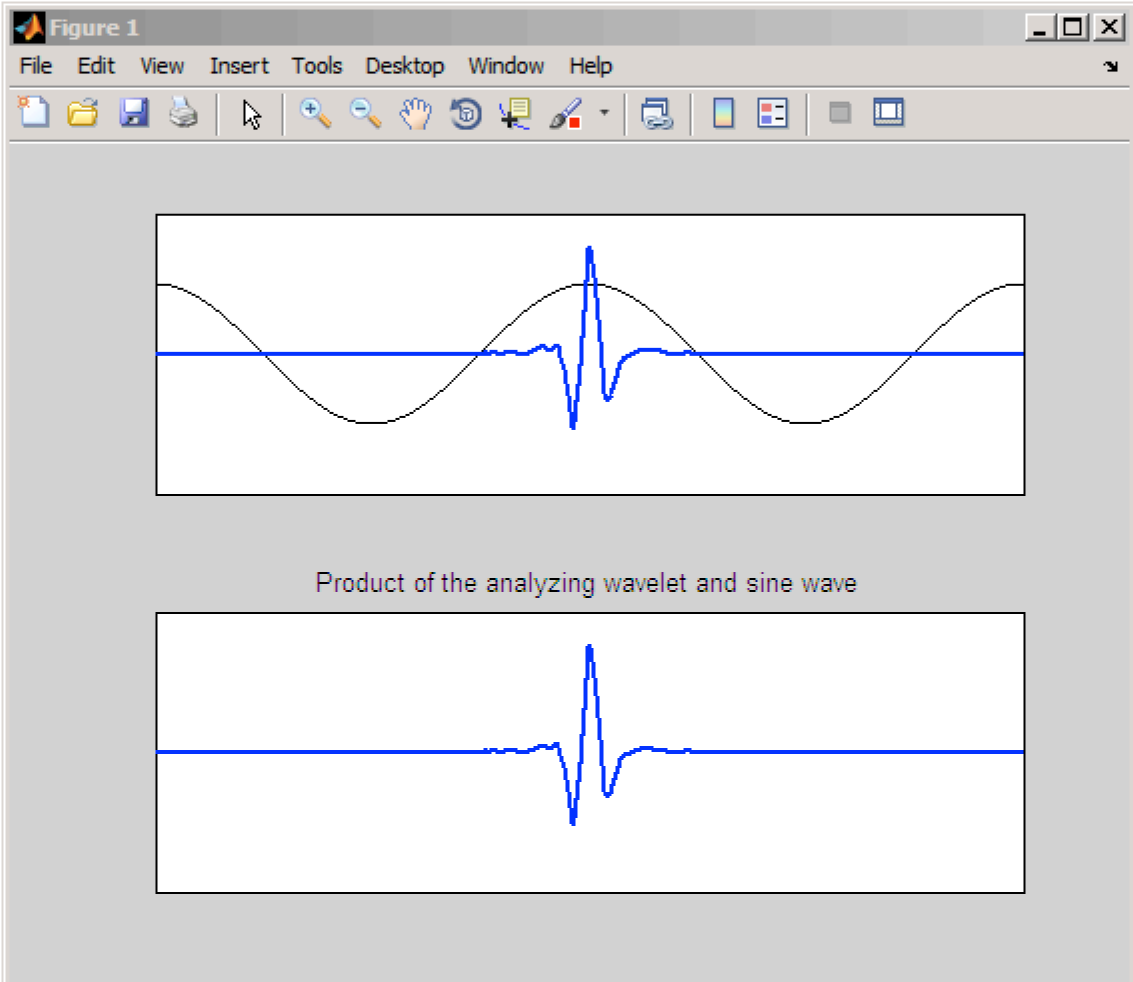


The following figure shows the product of the shifted wavelet and the sinusoid.



Integrating over this product produces a CWT coefficient much smaller in absolute value than either of the two previous examples. That results because the negative deflection of the wavelet approximately aligns with a positive deflection of the sine wave. Also, the main positive deflection of the wavelet approximately aligns with a positive deflection of the sine wave. The resulting product looks much more like a wavelet than the other two products. If it looked exactly like a wavelet, the integral would be zero.

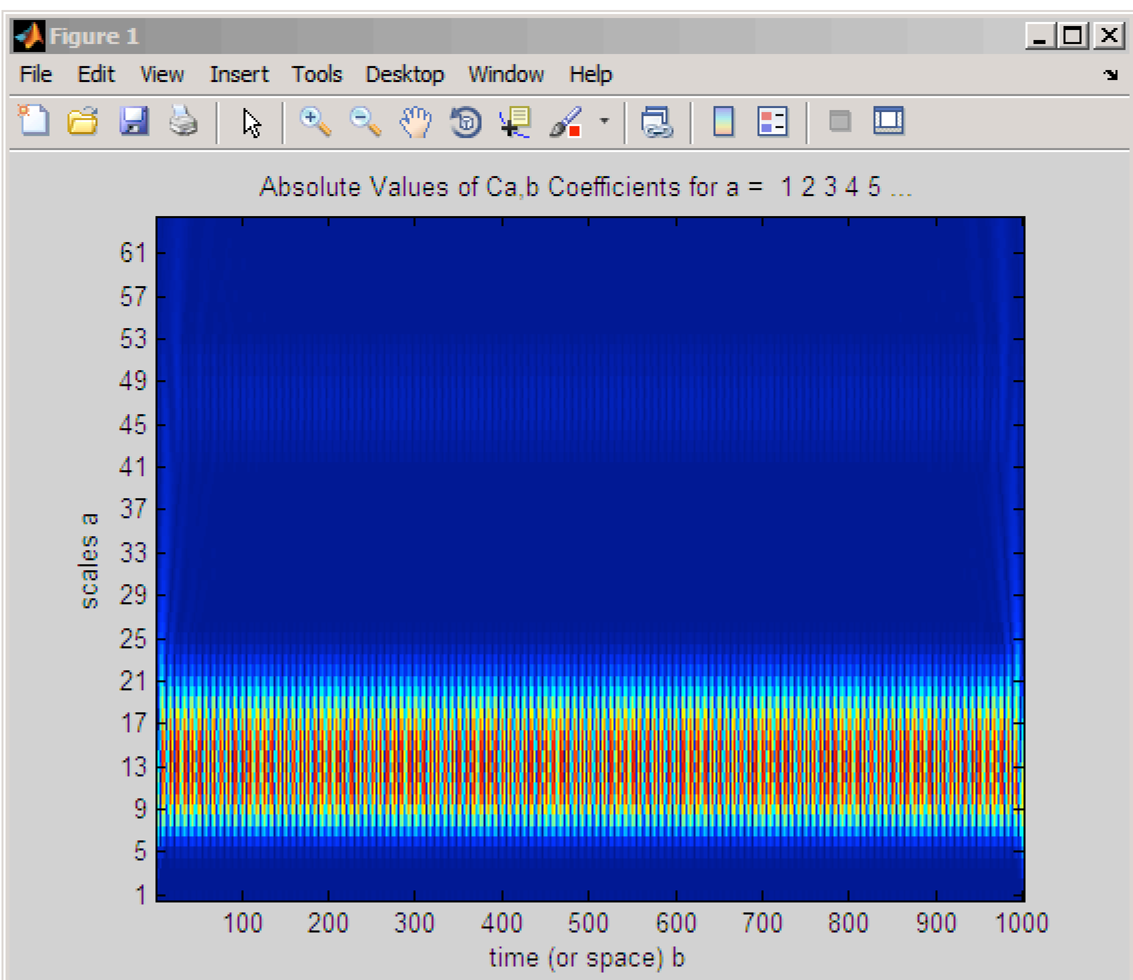
At scales where the oscillation in the wavelet occurs on either a much larger or smaller scale than the period of the sine wave, you obtain CWT coefficients near zero. The following figure illustrates the case where the wavelet oscillates on a much smaller scale than the sinusoid.



The product shown in the bottom pane closely resembles the analyzing wavelet. Integrating this product results in a CWT coefficient near zero.

The following example constructs a 60-Hz sine wave and obtains the CWT using the `sym8` wavelet.

```
t = linspace(0,1,1000);
x = cos(2*pi*60*t);
CWTcoeffs = cwt(x,1:64,'sym8','plot'); colormap jet;
```

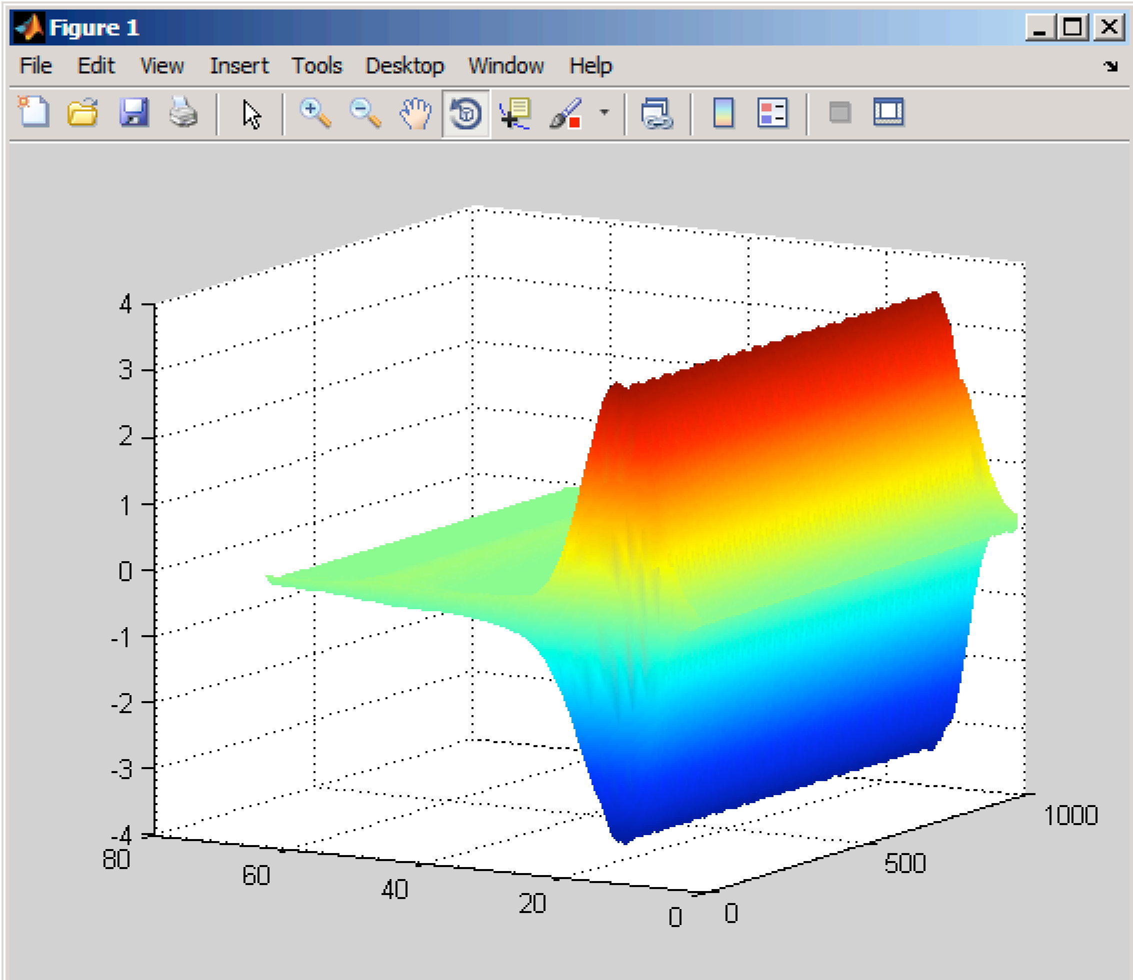


Note that the CWT coefficients are large in absolute value around scales 9 to 21. You can find the pseudo-frequencies corresponding to these scales using the command:

```
freq = scal2frq(9:21,'sym8',1/1000);
```

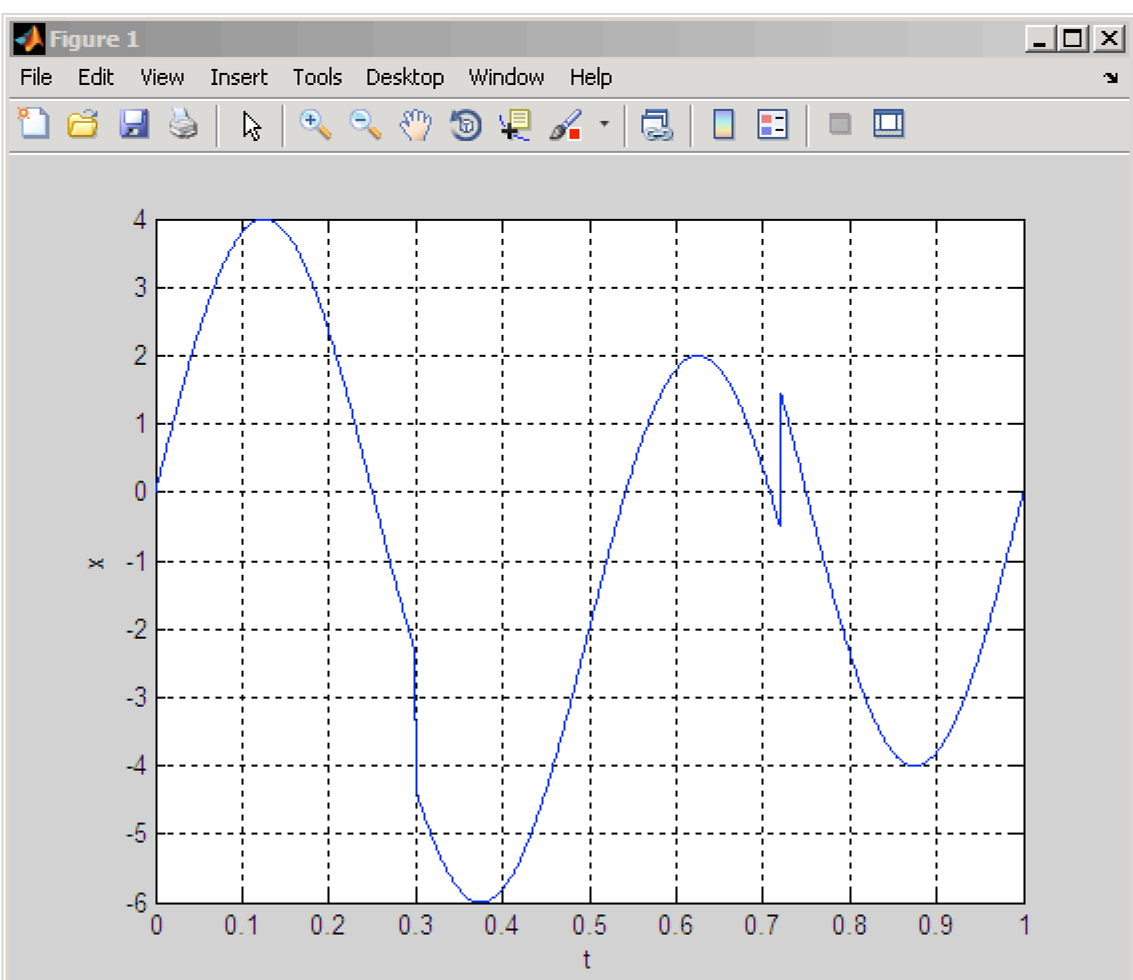
Note that the CWT coefficients are large at scales near the frequency of the sine wave. You can clearly see the sinusoidal pattern in the CWT coefficients at these scales with the following code.

```
surf(CWTcoeffs); colormap jet;
shading('interp'); view(-60,12);
```



The final example constructs a signal consisting of both abrupt transitions and smooth oscillations. The signal is a 4-Hz sinusoid with two introduced discontinuities.

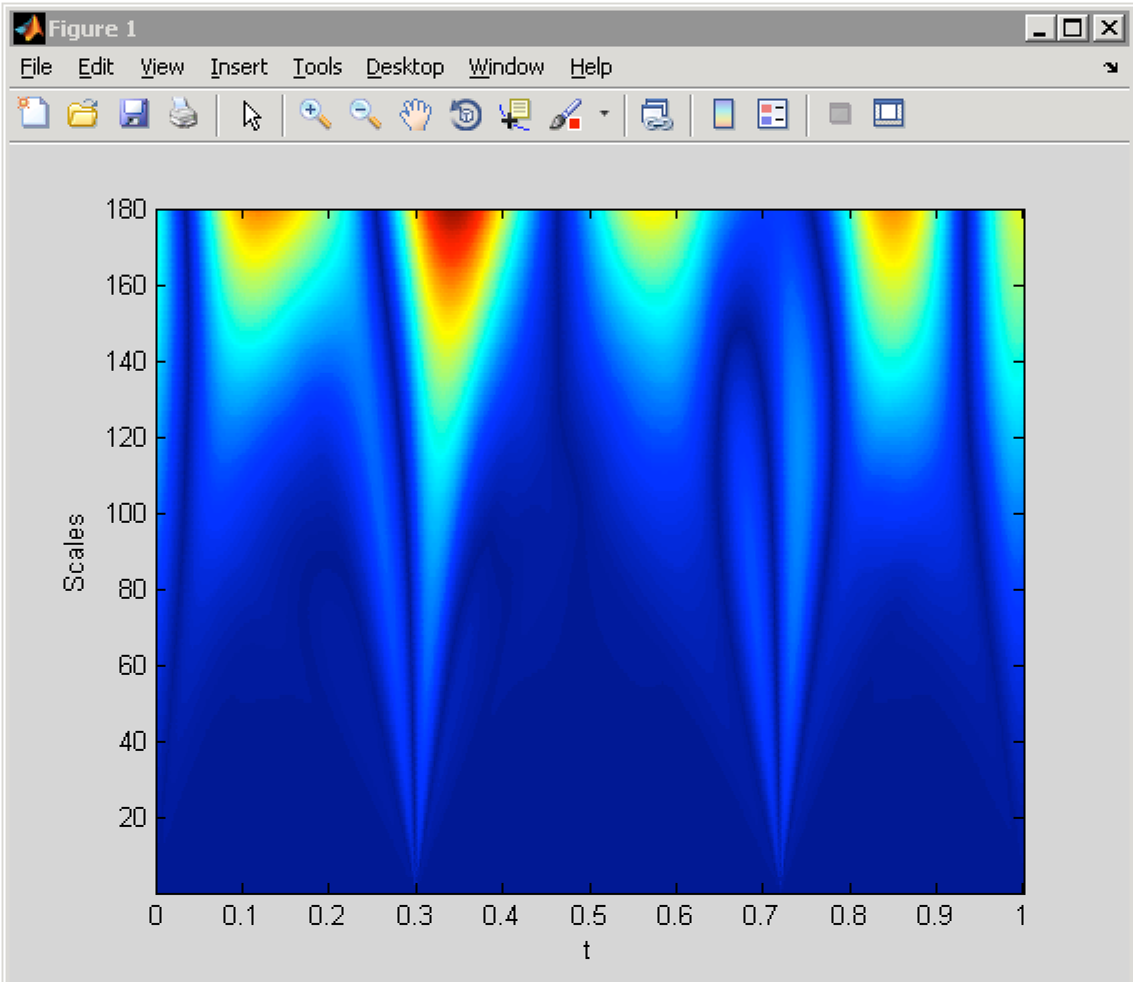
```
N = 1024;
t = linspace(0,1,1024);
x = 4*sin(4*pi*t);
x = x - sign(t - .3) - sign(.72 - t);
plot(t,x); xlabel('t'); ylabel('x');
grid on;
```



Note the discontinuities near t=0.3 and t=0.7.

Obtain and plot the CWT using the `sym4` wavelet.

```
CWTcoeffs = cwt(x,1:180,'sym4');
imagesc(t,1:180,abs(CWTcoeffs));
colormap jet; axis xy;
xlabel('t'); ylabel('Scales');
```



Note that the CWT detects both the abrupt transitions and oscillations in the signal. The abrupt transitions affect the CWT coefficients at all scales and clearly separate themselves from smoother signal features at small scales. On the other hand, the maxima and minima of the 2-Hz sinusoid are evident in the CWT coefficients at large scales and not apparent at small scales.

The following general principles are important to keep in mind when interpreting CWT coefficients.

- **Cone of influence**— Depending on the scale, the CWT coefficient at a point can be affected by signal values at points far removed. You have to take into account the support of the wavelet at specific scales. Use `conofinf` to determine the cone of influence. Not all wavelets are equal in their support. For example, the Haar wavelet has smaller support at all scales than the `sym4` wavelet.
- **Detecting abrupt transitions**— Wavelets are very useful for detecting abrupt changes in a signal. Abrupt changes in a signal produce relatively large wavelet coefficients (in absolute value) centered around the discontinuity at all scales. Because of the support of the wavelet, the set of CWT coefficients affected by the singularity increases with increasing scale. Recall this is the definition of the cone of influence. The most precise localization of the discontinuity based on the CWT coefficients is obtained at the smallest scales.
- **Detecting smooth signal features**— Smooth signal features produce relatively large wavelet coefficients at scales where the oscillation in the wavelet correlates best with the signal feature. For sinusoidal oscillations, the CWT coefficients display an oscillatory pattern at scales where the oscillation in the wavelet approximates the period of the sine wave.

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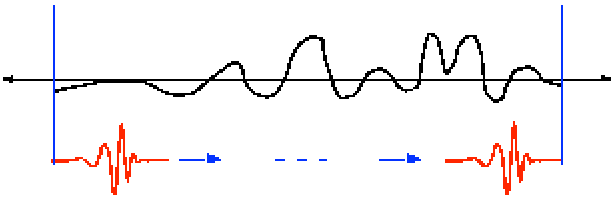
What's Continuous About the Continuous Wavelet Transform?

Any signal processing performed on a computer using real-world data must be performed on a discrete signal — that is, on a signal that has been measured at discrete time. So what exactly is "continuous" about it?

What's "continuous" about the CWT, and what distinguishes it from the discrete wavelet transform (to be discussed in the following section), is the set of scales and positions at which it operates.

Unlike the discrete wavelet transform, the CWT can operate at every scale, from that of the original signal up to some maximum scale that you determine by trading off your need for detailed analysis with available computational horsepower.

The CWT is also continuous in terms of shifting: during computation, the analyzing wavelet is shifted smoothly over the full domain of the analyzed function.



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Was this topic helpful? Yes No

From Fourier Analysis to Wavelet Analysis Critically-Sampled Discrete Wavelet Transform