8. $S_{max}(a,b) = max\{a,b\}$.

- $S(1,1) = max\{1,1\} = 1$ ✓
- $S(0,a) = max\{0,a\} = a$
- $\therefore S(a,b) \overset{?}{\leq} S(c,d)$ $a \leq c$,

Supongamos $a \leq b$. y $c \leq d$

$max\{a,b\} = b \leq d = max\{c,d\}$.

Análogo $a \geq b$ y $c \geq d$.

Supongamos $a \leq b$ y $c \geq d$.

$max\{a,b\} = b \leq d \leq c = max\{c,d\}$

Supongamos $a \geq b$ y $c \leq d$

$max\{a,b\} = a \leq c \leq d = max\{c,d\}$

$\because S(a,b) = max\{a,b\} = max\{b,a\} = S(b,a)$.

$\therefore S(S(a,b),c) = max\{max\{a,b\}, c\}$

Supongamos $a \leq b \leq c$

$\Delta = max\{b,c\} = c = max\{a,c\}$

$= max\{a, max\{b,c\}\}$

Análogo a $b \leq a \leq c$.

Supongamos $c \leq a \leq b$ $\Delta = max\{b,c\} = b$

$= max\{a,b\}$

$= max\{a, max\{b,c\}\}$

Análogo a $c \leq b \leq a$.

Supongamos $a \leq c \leq b$, $\Delta = max\{b,c\} = b$

$= max\{a,b\}$

$= max\{a, max\{b,c\}\}$

Análogo a $b \leq c \leq a$.

$S(a,b) = a+b-ab$

- $S(1,1) = 1+1-1 = 1$ ✓
- $\because S(0,a) = 0+a-0\cdot a = a$
- $\therefore S(a,b) \overset{?}{\leq} S(c,d)$, $a \leq c$ y $b \leq d$.

$a \leq c$ y $b \leq d$

$1-d \leq 1-b$

$1-c \leq 1-a$

$(1-d)(1-c) \leq (1-a)(1-b)$

$1-(d+c)+dc \leq 1-(a+b)+ab$

$a+b-ab \leq c+d-cd$ ✓

$\because S(a,b) = a+b-ab = b+a-ba = S(b,a)$

$\therefore S(S(a,b),c) = [a+b-ab]+c-c[a+b-ab]$

$= a+(b+c)-ab-ca-cb+abc$

$= a+(b+c)-a(b+c-bc)-cb$

$= a+(b+c-cb)-a(b+c-bc)$

$= S(a,S(b,c))$

$S(a,b) = min\{1, a+b\}$

- $S(1,1) = min\{1,1+1\}$
- $\because S(0,a) \overset{?}{=} min\{1,a\} = a$
- $\therefore S(a,b) \overset{?}{\leq} S(c,d)$, $a \leq c$, $b \leq d$.

Supongamos $a+b \leq 1$ } $a+b \leq min\{c+d, 1\}$.
$S(a,b) = a+b \leq c+d$ }

Supongamos $a+b > 1 \Rightarrow c+d > 1$
$S(a,b) = 1 \leq 1 = min\{c+d, 1\}$

$\because S(S(a,b),c) = min\{1, min\{1, a+b\}+c\}$

Supongamos $a+b > 1$, $b+c > 1$

$\Delta = min\{1, 1+c\} = 1 = min\{1, 1+a\}$

$= min\{1, min\{1, b+c\}+a\}$

Supongamos $a+b > 1$, $b+c \leq 1$

$\Delta = min\{1, 1+c\} = 1 = min\{1, a+b+c\}$

$= min\{1, a+min\{1, b+c\}\}$

Supongamos $a+b \leq 1$, $b+c > 1$

$\Delta = min\{1, a+b+c\} = 1 = min\{1, 1+a\}$

$= min\{1, a+min\{1, b+c\}\}$

Supongamos $a+b \leq 1$ y $b+c \leq 1$

$\Delta = min\{1, a+b+c\} = min\{1, a+min\{1, b+c\}\}$.

$S(a,b) = \begin{cases} a, & b=0 \\ b, & a=0 \\ 1, & a,b>0 \end{cases}$

- $S(1,1) = 1$
- $\because S(0,a) = a$
- $\therefore S(a,b) \overset{?}{\leq} S(c,d)$. $a \leq c$, $b \leq d$

Supongamos $c=0 \Rightarrow a=0$

$S(a,b) = b \leq d = S(c,d)$

Análogo a $d=0$.

Supongamos $c,d>0$

$S(a,b) \leq 1 = S(c,d)$

$\because S(S(a,b),c) = \begin{cases} S(a,b), & c=0 \\ c, & S(a,b)=0 \\ 1, & S(a,b), c>0 \end{cases}$

$S(a,b), c=0$: a, $c=0, b=0$; b, $c=0, a=0$; 1, $c=0, a,b>0$

c, $S(a,b)=0$: c, $a=b=0$

1, $S(a,b), c>0$: 1, $c>0 \wedge (a>0 \vee b>0)$

$= \begin{cases} a, & c=0, b=0 \\ b, & c=0, a=0 \\ c, & a=b=0 \\ 1, & otherwise \end{cases}$ $\parallel \rightarrow$ ✓

$S(a,S(b,c)) = \begin{cases} a, & S(b,c)=0 \\ S(b,c), & a=0 \\ 1, & S(b,c), a>0 \end{cases}$

$S(b,c)=0$: a, $b=0 \wedge c=0$

$a=0$: b, $a=0 \wedge c=0$; c, $a=0 \wedge b=0$; 1, $a=0 \wedge b,c>0$

1, $a>0 \wedge (b>0 \vee c>0)$

$= \begin{cases} a, & b=0 \wedge c=0 \\ b, & a=0 \wedge c=0 \\ c, & a=0 \wedge b=0 \\ 1, & otherwise \end{cases}$