Practice Problems 1 - Bayesian Decision theory

Two-class classification problem $\begin{cases} C_1 \\ C_2 \end{cases}$

Universe $\rightarrow x \in [0, 2]$

Class C_1 pdf $p(x|C_1) = \frac{2-x}{2}$ | Class C_2 pdf $p(x|C_2) = \frac{1}{2}$

And we have $P(C_1) = 3/4$; $P(C_2) = 1/4$

a) Values of x that should be classified in C_1 or C_2

$$\boxed{\text{Bayes rule: } p(C|x) = \frac{P(C)\,P(x|C)}{P(x)}}$$

We can calculate $P(x)$ as:

$$P(x) = P(x|C_1)\,P(C_1) + P(x|C_2)\,P(C_2) = \left[\frac{2-x}{2}\right]\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)$$

$$= \frac{6-3x}{8} + \frac{1}{8} = \frac{7-3x}{8}$$

So now, we can use it to calculate the posterior probability

$$P(C_1|x) = \frac{P(C_1)\,P(x|C_1)}{P(x)} = \frac{\left(\frac{3}{4}\right)\left[\frac{2-x}{2}\right]}{\frac{7-3x}{8}} = \frac{\frac{6-3x}{\cancel{8}}}{\frac{7-3x}{\cancel{8}}} = \frac{6-3x}{7-3x}$$

$$P(C_2|x) = \frac{P(C_2)\,P(x|C_2)}{P(x)} = \frac{\left(\frac{1}{\cancel{4}}\right)\left(\frac{1}{\cancel{2}}\right)}{\frac{7-3x}{\cancel{8}}} = \frac{1}{7-3x}$$

Having the posterior probability in the 2 classes, we can get the intervals with this inequation

- We classify in C_1 if $P(C_1|x) > 0{,}5$

$$\frac{6-3x}{7-3x} > 0{,}5 \rightarrow 12-6x > 7-3x \rightarrow 12-7 > 6x-3x \rightarrow 3x < 5 \rightarrow \boxed{x < \frac{5}{3}}$$

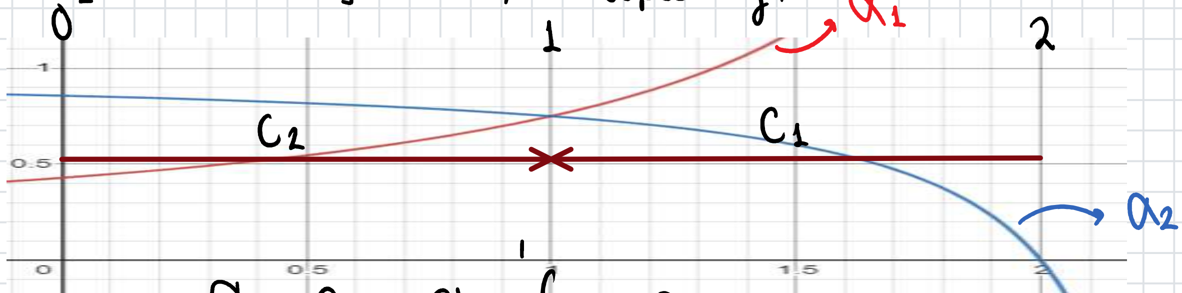
- We classify in C_2 if $P(C_2|x) > 0{,}5$, this is the complement of the previous case and if the math has been done rigth this will be $\boxed{x \geq \frac{5}{3}}$

So, finally we get:
$$\begin{cases} \text{choose } C_1 \text{ if } 0 \leq x < 5/3 \\ \text{choose } C_2 \text{ if } 5/3 \leq x \leq 2 \end{cases}$$

b) Now we have a different situation, because of now we have a cost λ_{ij} associated with the classification of an example x from class C_j into class C_i

Suppose that $\boxed{\lambda_{11} = \lambda_{22} = 0}$

What values λ_{12} and λ_{21} will make the classification intervals of x for C_1 and C_2 were [0, 1] and [1, 2] respectively?



$K = 2$ → $\alpha_1 = C_1$ → Classify in C_1
→ $\alpha_2 = C_2$ → Classify in C_2

	C_1	C_2
α_1	0	λ_{12}
α_2	λ_{21}	0

$$R(\alpha_1 | x) = \sum_{k=1}^{2} \lambda_{1k} P(C_k | x)$$

$$R(\alpha_2 | x) = \lambda_{21} P(C_1 | x)$$
$$= \cancel{\lambda_{11} P(C_1 | x)} + \lambda_{12} P(C_2 | x)$$

$$R(\alpha_1 | x) = \lambda_{12} P(C_2 | x)$$

$$\boxed{P(C_2 | x) = \frac{1}{7-3x} \qquad P(C_1 | x) = \frac{6-3x}{7-3x}}$$

$$R(\alpha_1 | x) = \lambda_{12} \left[\frac{1}{7-3x}\right] \qquad R(\alpha_2 | x) = \lambda_{21} \left[\frac{6-3x}{7-3x}\right]$$

We choose α_1 if:

$$\lambda_{12} \left[\frac{1}{7-3x}\right] < \lambda_{21} \left[\frac{6-3x}{7-3x}\right]$$

and we choose α_2 if:

$$\lambda_{12} \left[\frac{1}{7-3x}\right] > \lambda_{21} \left[\frac{6-3x}{7-3x}\right]$$

Lets evaluate in $x = 1$, that it have to be the border, so the risk will be the same

$$\lambda_{12} \left(\frac{1}{4}\right) = \lambda_{21} \left(\frac{3}{4}\right) \qquad \frac{\lambda_{12}}{\lambda_{21}} = \left(\frac{3 \cdot \cancel{4}}{\cancel{4}}\right) = 3$$

We choose C_1 in [0, 1], if the ratio of λ_{12} and λ_{21} is

$$\boxed{\lambda_{12} / \lambda_{21} = 3}$$