Assignment 1

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1 Neoclassical Growth Model

1.1 Competitive equilibrium

Given a sequence of prices $\{p_t, r_t, w_t\}$ a competitive equilibrium is defined by the firm allocations $\{k_t^d, n_t^d, y_t\}$ and the household allocations $\{c_t, n_t^s, x_{t+1}, i_t, k_t^s\}$ such that the firm solves the following problem:

$$\pi = Max_{y,k^d,n^d} \sum_{t=0}^{\infty} p_t(y_t - w_t n_t - r_t k_t)$$

subject to

$$y_t = zF(k_t, n_t) \ \wedge y_{t,} k_t, n_t \geq 0 \ \forall t$$

And the household solves the following problem

$$Max_{c,i,x',k^s,n^s} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$\sum p_t \left(c_t + i_t\right) = \sum p_t \left(w n_t + r_t k_t\right) + \pi$$

$$x_{t+1} = x_t(1 - \delta) + i_t$$

Notice that in the problem, we are assuming that the capital depreciation is complete. Hence, $\delta = 1$ and previous expression turns into

$$x_{t+1} = i_t$$

$$n_t \in [0, 1]$$

$$k_t \in [0, x_t]$$

$$c_t, x_{t+1} \geq 0$$

Here it is important to precise that x_t is associated to the capital stock that the household has available to rent. The variable k_t is related with the capital services that the household decide to rent to the firm.

Finally, the market clearing conditions are given by:

$$c_t + i_t = y_t \, \forall t$$

$$n_t^s = n_t^d$$

$$k_t^s = k_t^d$$

1.2 Social Planner Problem

The social planner is maximazing the lifetime utility of the representative agent:

$$Max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

Subjecto to the resource constraint:

$$c_t + k_{t+1} = F(k_t, n_t)$$

$$c_t, k_t \geq 0 \ n_t \in [0, 1]$$

$$k_0 \leq \bar{k}_0$$

I can redefine $f(k_t) = F(k_t, 1) + k_t(1 - \delta)$

1.3 Equilibrium allocations and planner's allocations coincide

To show this, I am going to derive the Euler's equation for both competitve equilibrium and planner's problem. I am going to show that the Euler's equations are equivalent which leads to the same equilibrium allocations.

First, it is important to notice that under the assumptions given in the problem, we have the following equivalences in the competitive equilibrium:

1. As the firm's production function is homogeneous of degree 1, $\pi = 0$
2. The household does not retrieve any utility form holding unproductive capital. Hence $k_t = x_t$

3. Same applies for labor supply $n_t = 1$

Under this particular setting, the househoild problem changes as follows:

And the household solves the following problem

$$Max_{c,i,x',k^s,n^s} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$\sum p_t (c_t + k_{t+1}) = \sum p_t (w_t + r_t k_t)$$

$$c_t, k_{t+1} \geq 0 \; \forall t$$

$$c_t + k_{t+1} = f(k_t) \, \forall t$$

$$n_t^s = n_t^d$$

$$k_t^s = k_t^d$$

I can solve this problem using a lagrangian and introducing the constraints:

$$[c_t] : \beta^t U'(c_t) - \lambda p_t = 0$$

$$[c_{t+1}] : \beta^{t+1} U'(c_{t+1}) - \lambda p_{t+1} = 0$$

$$[k_{t+1}] : -\lambda p_t + \lambda p_{t+1} r_{t+1} = 0$$

From this FOC I can get

$$\frac{\beta U'(c_{t+1})}{U'(c_t)} = \frac{p_{t+1}}{p_t} = \frac{1}{r_{t+1}}$$

As the rate at which the household can rent the capital increases, the household can save more and obtain higher returns over those savings, which reduce the rate of substitution between c_{t+1} and c_t (i.e the household now prefers better to consume less today, save and consume more tomorrow).

Using the market clearing condition, our Euler's equation is reduced to:

$$\frac{\beta U'(f(k_{t+1}) - k_{t+2})}{U'(f(k_t) - k_{t+1})} = \frac{1}{r_{t+1}}$$

Using the FOC of the firm:

$$\frac{\beta U'(f(k_{t+1})-k_{t+2})}{U'(f(k_t)-k_{t+1})}=\frac{1}{f'(k_{t+1})}$$

Now, for the social planner problem we have the following Euler's equation (replacing the constraint into the objective function and maximizing with respect k_{t+1}):

$$-\beta^t U'(f(k_t)-k_{t+1})+\beta^{t+1}U'(f(k_{t+1})-k_{t+2})f'(k_{t+1})=0$$

Which lead to an identical Euler's equation as in the competitive problem. This means that the optimal allocation of capital $\{k_{t+1}\}_{t=0}^{\infty}$ satisfies the competitive equyilibrium allocations if and only if it is Pareto efficient (i.e it satisfies the social planner's problem optimality conditions)

1.4 Dynamic Social Planner's problem

The social planner is solving a dynamic problem using as a state variable k. The corresponding Bellman equation is:

$$\nu(k)=max_{0\leq k'\leq f(k)}\left\{U(f(k)-k')\right\}+\beta\nu(k')$$

1.5 Solving the problem

Assuming that $u(c)=ln(c)$ and that $F(k,n)=zk_t^{\alpha}n_t^{1-\alpha}$

Under this assumptions, I can use the undetermined coefficients method (Assume that $\nu(k)=A+Bln(k)$). In this case, the maximization problem is reduced to:

$$\nu(k)=max_{0\leq k'\leq f(k)}\left\{ln(zk^{\alpha}-k')\right\}+\beta\left(A+Bln(k')\right)$$

FOC:

$$\frac{1}{zk^{\alpha}-k'}=\frac{\beta B}{k'}$$

$$k'=\beta B(zk^{\alpha}-k')$$

$$k'=\frac{\beta Bzk^{\alpha}}{(1+\beta B)}$$

Hence:

$$A+Bln(k)=\left\{ln\left(zk^{\alpha}-\frac{\beta Bzk^{\alpha}}{(1+\beta B)}\right)\right\}+\beta\left(A+Bln\left(\frac{\beta Bzk^{\alpha}}{(1+\beta B)}\right)\right)$$

$$A + Bln(k) = \left\{ ln\left(\frac{zk^{\alpha}}{1+\beta B}\right)\right\} + \beta\left(A + Bln\left(\frac{\beta Bzk^{\alpha}}{(1+\beta B)}\right)\right)$$

$$A{+}Bln(k) = \{ln(z) + \alpha ln(k) - ln(1+\beta B)\}{+}\beta A{+}\beta Bln\left(\frac{\beta Bz}{(1+\beta B)}\right){+}\beta B\alpha ln(k)$$

$$A{+}Bln(k) = \{ln(z) - ln(1+\beta B)\}{+}\beta A{+}\beta Bln\left(\frac{\beta Bz}{(1+\beta B)}\right){+}(\beta B\alpha + \alpha)\, ln(k)$$

Hence

$$B = \alpha(\beta B + 1)$$

$$B = \frac{\alpha}{1-\alpha\beta}$$

$$k' = \frac{\beta zk^{\alpha}}{(1+\beta\frac{\alpha}{1-\alpha\beta})} \times \frac{\alpha}{1-\alpha\beta}$$

$$k' = \frac{\beta zk^{\alpha}}{(\frac{1}{1-\alpha\beta})} \times \frac{\alpha}{1-\alpha\beta} = \alpha\beta zk^{\alpha}$$

1.6 Steady state variables

$$k^* = \frac{\beta zk^{*\alpha}}{(\frac{1}{1-\alpha\beta})} \times \frac{\alpha}{1-\alpha\beta} = \alpha\beta zk^{*\alpha}$$

$$k^* = (\alpha\beta z)^{\frac{1}{1-\alpha}}$$

$$y = z^{\frac{1}{1-\alpha}}\,(\alpha\beta)^{\frac{\alpha}{1-\alpha}}$$

$$c = z^{\frac{1}{1-\alpha}}\left((\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}\right)$$

$$w = F_L = z(1-\alpha)\,(\alpha\beta z)^{\frac{\alpha}{1-\alpha}}$$

$$r = F_K = z(\alpha)\,(\alpha\beta z)^{\frac{-(1-\alpha)}{1-\alpha}}$$

1.7 Graphs

1.7.1 Point a:

Figure 1: Capital

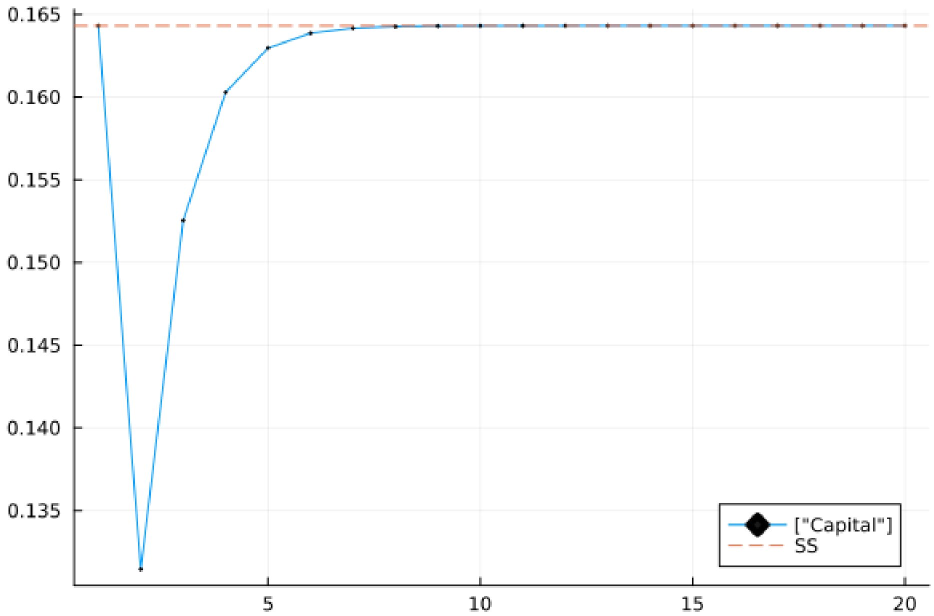


Figure 2: Production

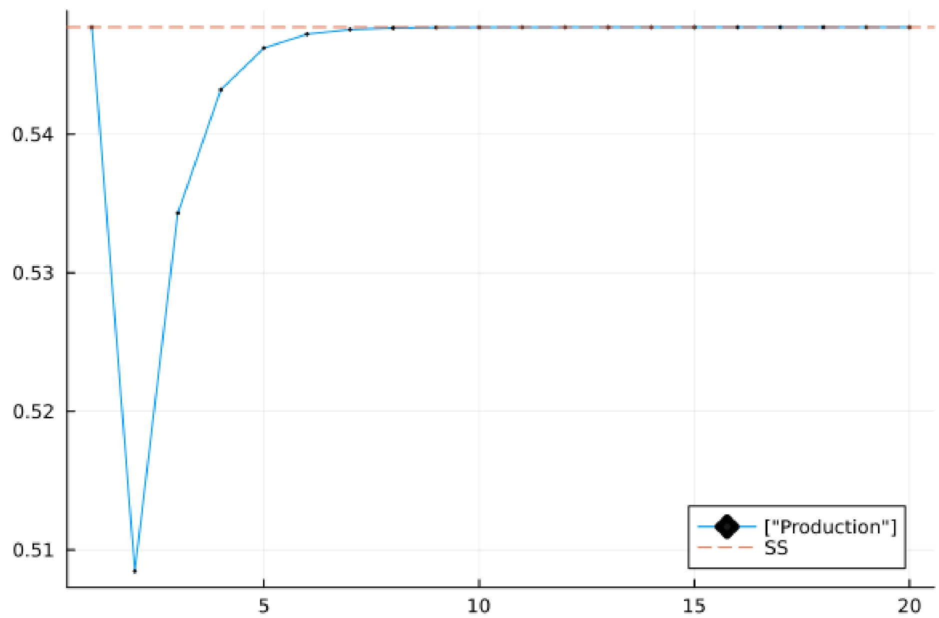


Figure 3: Consumption

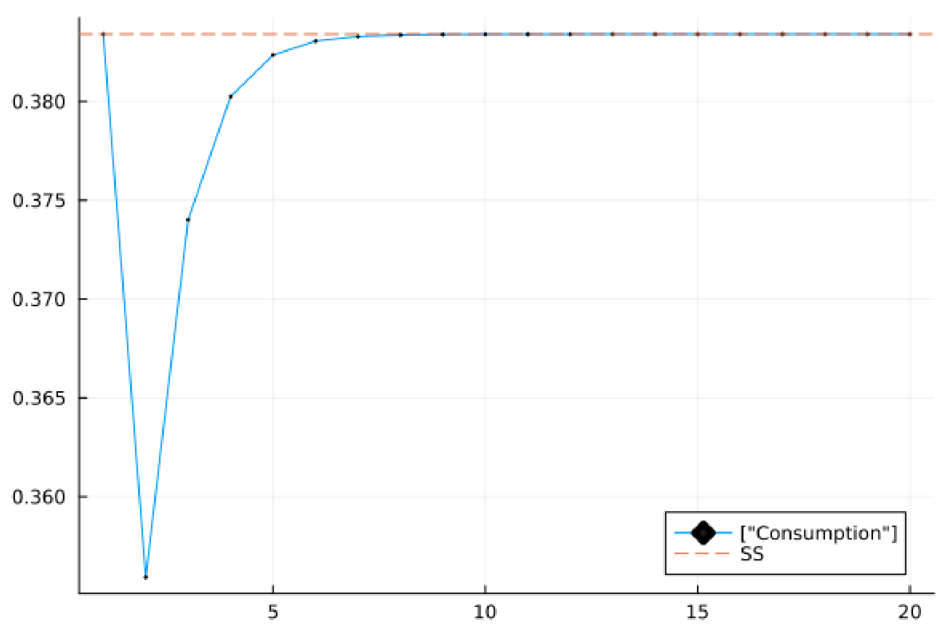


Figure 4: Rate of capital

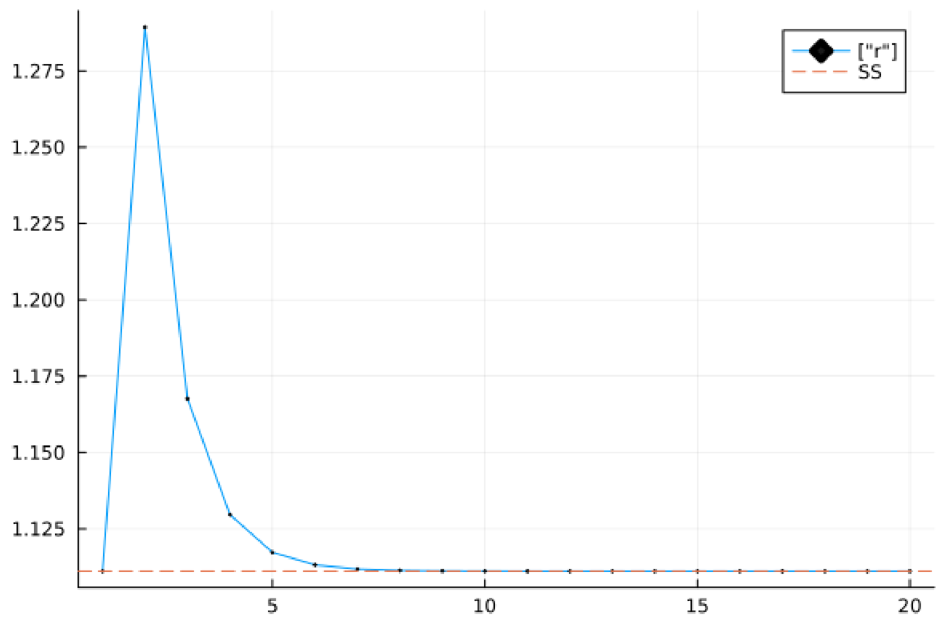
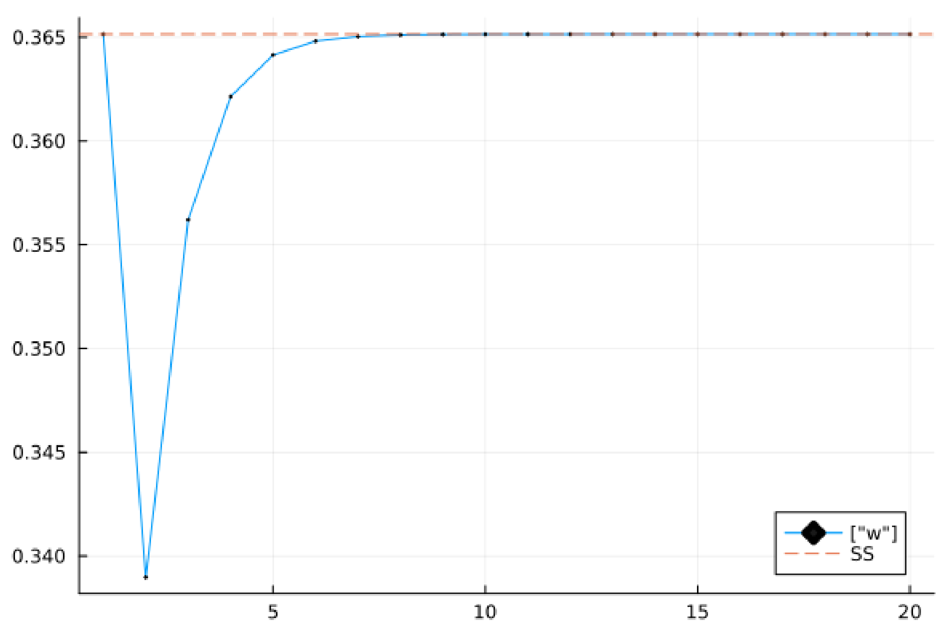


Figure 5: Wages



1.7.2 Point B

Figure 6: Capital

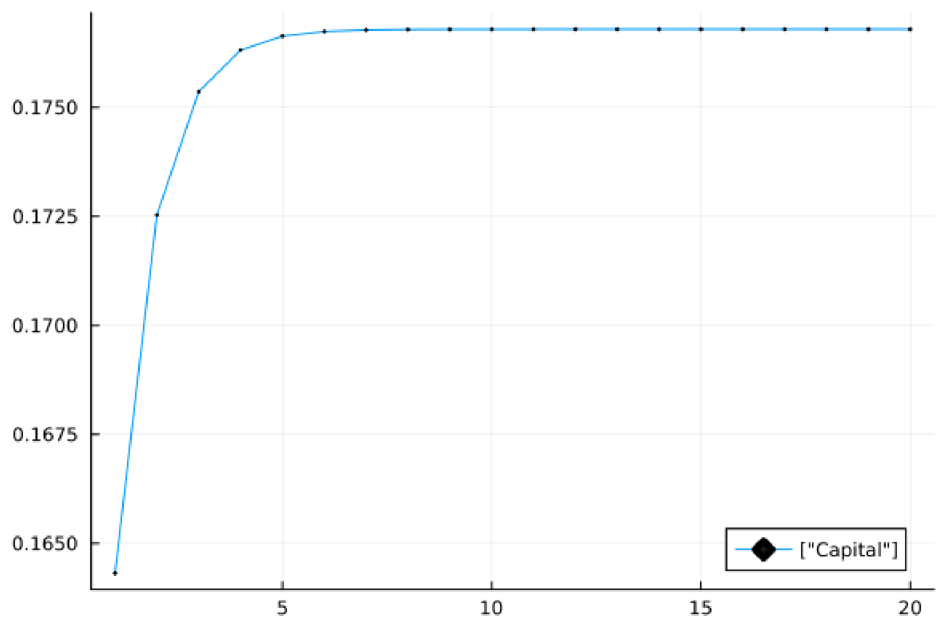


Figure 7: Production

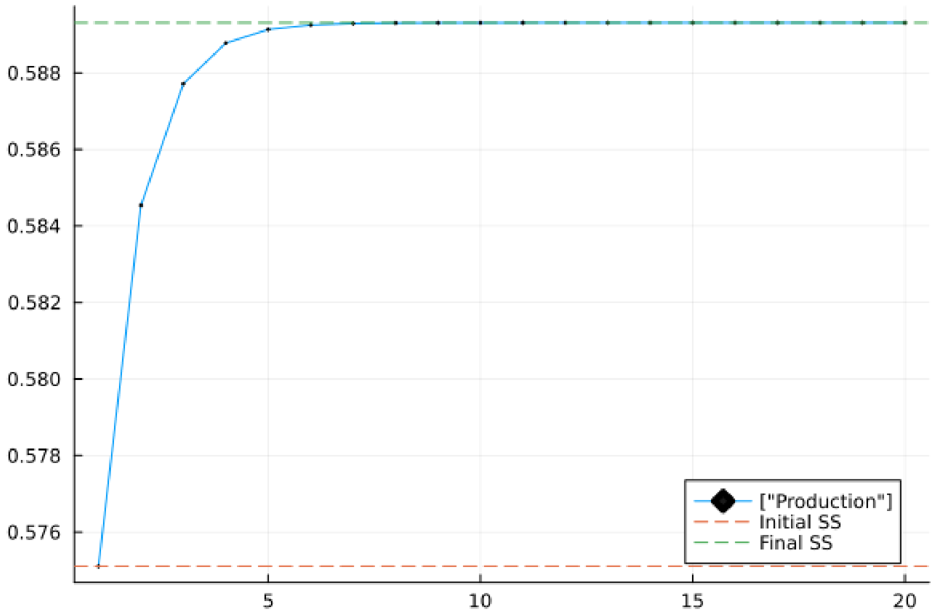


Figure 8: Consumption

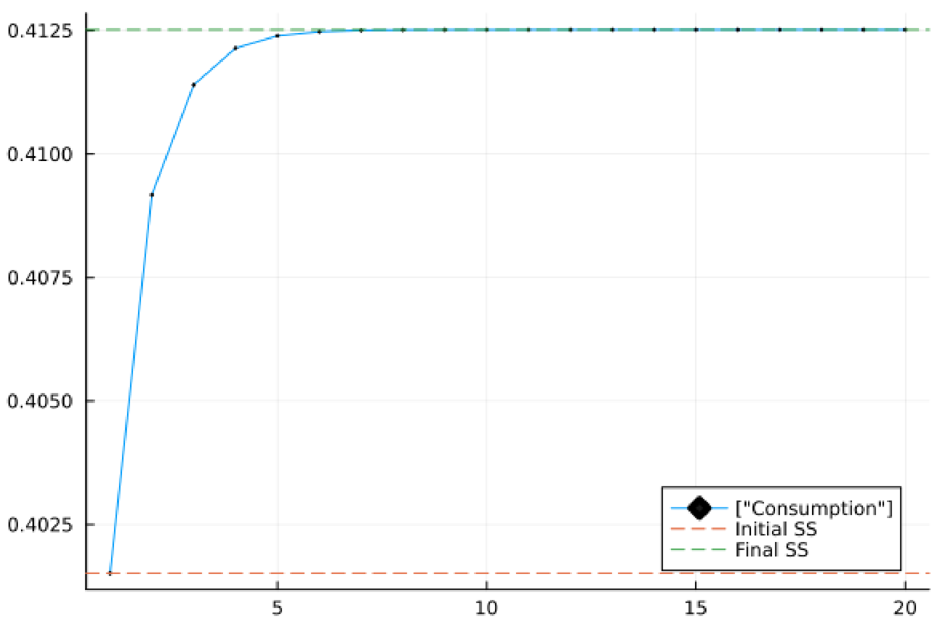


Figure 9: Rate of capital

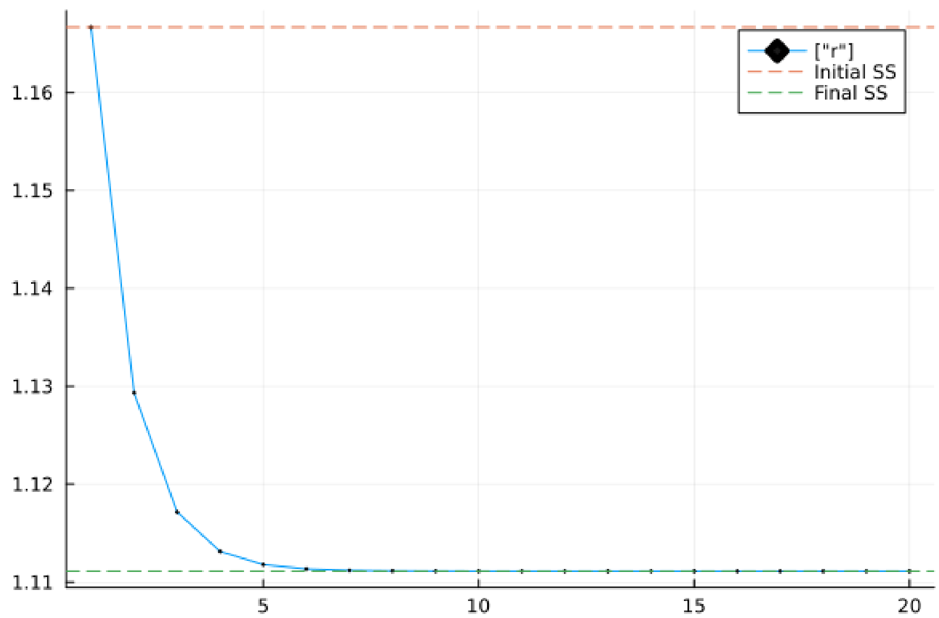


Figure 10: Wages