Assignment 2

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1 Neoclassical Growth Model

1.1 Competitive equilibrium

Given a sequence of prices $\{p_t, r_t, w_t\}$ a competitive equilibrium is defined by the firm allocations $\{k_t^d, l_t^d, y_t\}$ and the household allocations $\{c_t, l_t^s, x_{t+1}, i_t, k_t^s\}$ such that the firm solves the following problem:

$$\pi = Max_{y,k^d,n^d} \sum_{t=0}^{\infty} p_t(y_t - w_t n_t - r_t k_t)$$

subject to

$$y_t = zF(k_t, n_t) \ \wedge y_{t,} k_t, n_t \geq 0 \ \forall t$$

And the household solves the following problem

$$Max_{c,i,x',k^s,n^s} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta} \right)$$

subject to

$$\sum p_t \left(c_t + i_t\right) = \sum p_t \left(wl_t + r_t k_t\right) + \pi$$

$$x_{t+1} = x_t(1-\delta) + i_t$$

Notice that in the problem, we are assuming that the capital depreciation is complete. Hence, $\delta = 1$ and previous expression turns into

$$x_{t+1} = i_t$$

$$n_t \in [0, 1]$$

$$k_t \in [0, x_t]$$

$$c_t, x_{t+1} \geq 0$$

Here it is important to precise that x_t is associated to the capital stock that the household has available to rent. The variable k_t is related with the capital services that the household decide to rent to the firm.

Finally, the market clearing conditions are given by:

$$c_t + i_t = y_t \,\forall t$$

$$l_t^s = l_t^d$$

$$k_t^s = k_t^d$$

1.2 Steady State

To find the steay state I am going to use the competitive setting:

First, it is important to notice that under the assumptions given in the problem, we have the following equivalences in the competitive equilibrium:

1. As the firm's production function is homogeneous of degree 1, $\pi = 0$
2. The household does not retrieve any utility form holding unproductive capital. Hence $k_t = x_t$

Under this particular setting, the househoild problem changes as follows:

And the household solves the following problem

$$Max_{c,i,x',k^s,n^s} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta} \right)$$

subject to

$$\sum p_t \left(c_t + k_{t+1}\right) = \sum p_t \left(w_t l_t + r_t k_t\right)$$

$$l_t, c_t, k_{t+1} \geq 0 \;\forall t$$

$$c_t + k_{t+1} = f(k_t) \,\forall t$$

$$n_t^s = n_t^d$$

$$k_t^s = k_t^d$$

I can solve this problem using a lagrangian and introducing the constraints:

$$[c_t] : \beta^t c_t^{-\sigma} - \lambda p_t = 0 \tag{1}$$

$$[c_{t+1}] : \beta^{t+1} c_{t+1}^{-\sigma} - \lambda p_{t+1} = 0 \tag{2}$$

$$[k_{t+1}] : -\lambda p_t + \lambda p_{t+1} r_{t+1} = 0 \tag{3}$$

$$[l_t] : -\beta^t \chi l_t^{\eta} + \lambda p_t w_t = 0 \tag{4}$$

From this FOC I can get

$$\frac{\beta c_{t+1}^{-\sigma}}{c_t^{-\sigma}} = \frac{p_{t+1}}{p_t} = \frac{1}{r_{t+1}} \tag{5}$$

$$\lambda = \frac{\beta^t c_t^{-\sigma}}{p_t}$$

Replacing λ into (1)

$$\chi l_t^{\eta} = c_t^{-\sigma} w_t \tag{6}$$

Using the FOC of the firm's problem with respect to w

$$w_t = z(1-\alpha) l_t^{-\alpha} k_t^{\alpha} \tag{7}$$

Replacing (7) into (6)

$$\chi l_t^{\eta} = c_t^{-\sigma} z(1-\alpha) l_t^{-\alpha} k^{\alpha} \tag{8}$$

$$\chi l^{\eta} = (z k^{\alpha} l^{1-\alpha} - k'(1-\delta)^{-\sigma} z(1-\alpha) l^{-\alpha} k^{\alpha}$$

Now, using the steady state equilibrium condition from (5) we have:

$$k = \left(\frac{1}{z\beta\alpha l^{1-\alpha}}\right)^{\frac{1}{\alpha-1}} = l\,(z\beta\alpha)^{\frac{1}{1-\alpha}} \tag{9}$$

Replacing (9) into (8)

$$\chi l^{\eta} = c^{-\sigma} z(1-\alpha)\,(z\beta\alpha)^{\frac{\alpha}{1-\alpha}}$$

Moreover, considering that $c = zk^{\alpha} l^{1-\alpha} - k$

$$\frac{c}{l} = \frac{zk^{\alpha} l^{1-\alpha}}{l^{\alpha} l^{1-\alpha}} - \frac{k}{l}$$

$$\frac{c}{l} = z\,(z\beta\alpha)^{\frac{\alpha}{1-\alpha}} - (z\beta\alpha)^{\frac{1}{1-\alpha}} \tag{10}$$

$$\chi l^{\eta+\sigma} = \left(z^{\frac{1}{1-\alpha}}\left((\beta\alpha)^{\frac{\alpha}{1-\alpha}} - (1-\delta) * (\beta\alpha)^{\frac{1}{1-\alpha}}\right)\right)^{-\sigma} z(1-\alpha)\left(z\beta\alpha\right)^{\frac{\alpha}{1-\alpha}}$$

$$l = \left(\frac{\left(z^{\frac{1}{1-\alpha}}\left((\beta\alpha)^{\frac{\alpha}{1-\alpha}} - (\beta\alpha)^{\frac{1}{1-\alpha}}\right)\right)^{-\sigma} z(1-\alpha)\left(z\beta\alpha\right)^{\frac{\alpha}{1-\alpha}}}{\chi}\right)^{\frac{1}{\eta+\sigma}}$$

From this expression I can get the capital SS expression:

$$k = \left(\frac{\left(z^{\frac{1}{1-\alpha}}\left((\beta\alpha)^{\frac{\alpha}{1-\alpha}} - (\beta\alpha)^{\frac{1}{1-\alpha}}\right)\right)^{-\sigma} z(1-\alpha)\left(z\beta\alpha\right)^{\frac{\alpha}{1-\alpha}}}{\chi}\right)^{\frac{1}{\eta+\sigma}} \left(z\beta\alpha\right)^{\frac{1}{1-\alpha}}$$

From k and l I can get y, w and r

1.3 Social Planner Problem

The social planner is maximazing the lifetime utility of the representative agent:

$$Max_{l_t,c_t,k_{t+1}} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}\right)$$

Subjecto to the resource constraint:

$$c_t + k_{t+1} = F(k_t, n_t)$$

$$c_t, k_t \geq 0 \ l_t \in [0,1]$$

$$k_0 \leq \bar{k}_0$$

1.4 Dynamic Social Planner's problem

The social planner is solving a dynamic problem using as a state variable k. The corresponding Bellman equation is:

$$\nu(k) = max_{0\leq k' \leq f(k)} \left\{U(f(k) - k')\right\} + \beta\nu(k')$$

1.5 Solving the problem

Assuming that $u(c,l) = \left(\frac{c^{1-\sigma}}{1-\sigma} - \chi\frac{l^{1+\eta}}{1+\eta}\right)$ and that $F(k,n) = zk^{\alpha}n^{1-\alpha}$

$$\nu(k) = max_{l,0\leq k' \leq f(k,l)} \left\{\left(\frac{(zk^{\alpha}l^{1-\alpha} - k')^{1-\sigma}}{1-\sigma} - \chi\frac{l^{1+\eta}}{1+\eta}\right)\right\} + \beta\nu(k')$$

FOC

$$[l_t] : (zk^{\alpha}l^{1-\alpha} - k')^{-\sigma} z(1-\alpha)l^{-\alpha}k^{\alpha} - \chi l^{\eta} = 0$$

l

1.6 Steady state variables

$$k^* = \frac{\beta z k^{*\alpha}}{(\frac{1}{1-\alpha\beta})} \times \frac{\alpha}{1-\alpha\beta} = \alpha\beta z k^{*\alpha}$$

$$k^* = (\alpha\beta z)^{\frac{1}{1-\alpha}}$$

$$y = z^{\frac{1}{1-\alpha}} (\alpha\beta)^{\frac{\alpha}{1-\alpha}}$$

$$c = z^{\frac{1}{1-\alpha}} \left((\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}} \right)$$

$$w = F_L = z(1-\alpha) (\alpha\beta z)^{\frac{\alpha}{1-\alpha}}$$

$$r = F_K = z(\alpha) (\alpha\beta z)^{\frac{-(1-\alpha)}{1-\alpha}}$$

1.7 Numerical Solutions , time and iterations

Table 1: Solution Performance (100 Grid Points)

Method	Time	Iterations
VFI	4.6 scs	834
Howard	4.4 scs	777
McQueen	0.06	12

2 Value Functions

Figure 1: K' Policy Function

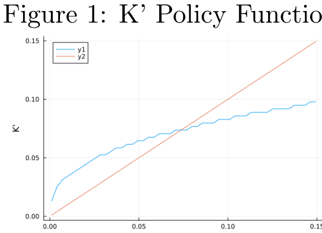

Figure 2: Labor Policy Function

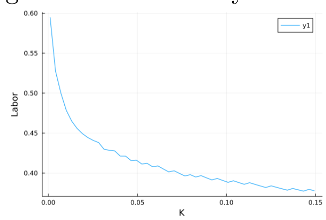

Figure 3: Consumption Policy Function