Math 301: Function Theory

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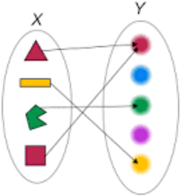
What Are Functions?

Definition

- In mathematics, a function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.



- General format of functions:

$$f(x) = x^2$$

History of Functions

Concept emerged in the 17th century in connection with the development of calculus. For example the slope $\frac{dy}{dx}$ of a graph at a point was regarded as a function of the x-coordinate of the point.

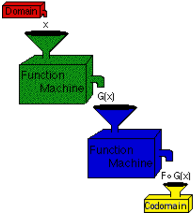
- Went from being defined as *analytic expression* to *single-valued mapping from one set to another*.
- Term *function* was introduced by **Gottfried Leibniz** in 1673 to describe a quantity related to a curve, such as a curve's slope at a specific point.
- Later re-defined by Leibniz as *any expression made up of a variable and some constants*.
- The familiar notation $f(x)$ was later introduced by *Leonhard Euler*.
- Functions from those times are called today *differentiable functions*.

Function Composition

Composition is the operation of taking the output from one function and using that as the input to a second function. It, together with the algebraic operations on functions, is what allows us to take the basic building blocks of "pure" function types and build large varieties of functions which is the focus of calculus.

Definition

- The composition $f \circ g$ of the two functions f and g is the function which feeds an input to g and feeds the output of g to f



Why Functions?

- They are an integral part of mathematics and computational world
  - Theory is foundation for functional programming paradigm seen in languages such as Haskell and Python
- Allow for use of rules more than once in an expression
- Allow for different parameter values
- Useful for recognizing similarities in a set of observations

Components – Domain and Range

Functions can be seen as relations that uniquely associate members of one set with members of another set. *A function from* $A \rightarrow B$ is an object f such that every $\alpha \in A$ is uniquely associated with an object $f(\alpha) \in B$. A function is therefore a many-to-one relation.

- The set A of values at which a function is defined is called its domain
- The set $f(A) \subset B$ of values that the function can produce is called the *range*. It is denoted as follows:

$$D(f) = \{x \in \mathbb{R}; f(x)\}$$

- In other words, range of a function is the set of all values obtained by substituting arguments from its domain

Components – Real Functions

A function is defined as "Real" when $f : A \subset \mathbb{R} \in \mathbb{R}$

- Domain elements in real functions are mapped to unique range elements
  - Can be expressed graphically by performing the *vertical line test*
- They are the most important type of *mapping*.
  - Can be thought of as any mapping from some subset of the set of real numbers to the set of real numbers

Basic Notions of Real Functions

- Can be thought of as a perscription which assigns values to arguments
  - For example: $y = f(x)$ means that to the value x of the argument, the function f assigns the value y
  - Second notation: $f : x \to y$ implies that the function f sends the value x to y
- Usual way of specifying assignments is that the function value y can be obtained by substituting x to a specific formula:

$$f(x) = 2x + 3$$

Substitution and Visualization

$$f(a) = f|_a = f|_{x=a}$$

- Used when function needs adjustment, but not yet ready for substitution. Example:

$$\left.\frac{x^2-1}{x-1}\right|_{x=3} = \left.\frac{(x-1)(x+1)}{x-1}\right|_{x=3} = 4.$$

- Graphs are common way of visualizing functions in mathematics.
- Used to mark in the two-dimensional plane (x, y) all couples $x, f(x)$

Types of Functions

Functions are primary objects of study in calculus, consisting of various types:

- linear
- polynomial
- trigonometric
- logarithmic

Linear Functions

Allow us to approximate more complicated functions in differential calculus. There are three standad forms of linear functions:

$$f(x) = mx + b \qquad \text{(Slope-intercept form)}$$

$$y - y_o = m(x - x_0) \qquad \text{(The "point-slope" form)}$$

$$Ax + By = C \qquad \text{(The "general" form)}$$

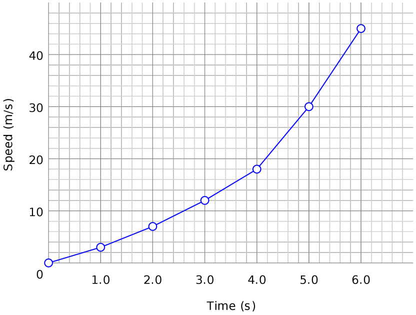
Linear Functions

Frequently used to solve for the following calculations:

- **Intercepts**
  - Finding the slope, x-intercept, and y-intercept of the line given its equation
- **Intersections**
  - Finding the intersection of two lines from their equations
- **Equations**
  - Finding the equation of the line through a series of points given one of the following three pairs of data:
    - the slope of the line and the y-intercept
    - the slope of the line and a point (x_0, y_0) on the line
    - the coordinates of two points on the line

Graphs

If $f(x)$ is linear, the graph of $y = f(x)$ is a straight line. The parameter m in the formula below represents the slope of a line. In general form, the slope is $-A/B$ if $B \neq 0$, and indefinite if $B = 0$.



Polynomial Functions

Can be represented by the formula:

$$f(x) = a_n x^n + a_{n-1} x n - 1 + \cdots a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers. These are called the coefficients. It is assumed that $a_n \neq 0$. The number n is called the degree of the polynomial function.

Polynomial Functions

Standard Forms

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

- The a_i are real numbers and are called **coefficients**.
- The term a_n is assumed to be non-zero and is called the **leading term.**
- The degree of the polynomial is the largest exponent of x
- A Polynomial with one term is called a **monomial**
- A degree-zero polynomial is a constant
- A degree 1 polynomial is a linear function
- A degree 2 polynomial is a quadratic function
- a degree 3 polynomial is a cubic

Power Functions

Power functions are of the form:

$$f(x) = kx^p$$

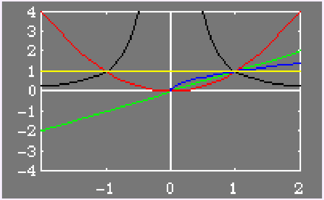
where p is any real number and k is non-zero. Rules are:

- $x^{(p+q)} = (x^p)(x^q)$
- $x^{pq} = (x^p)^q$
- $x^{-p} = \frac{1}{x^p}$
- $x^{\frac{1}{p}}$ is the p^{th} root of x
- $x^0 = 1$ for any $x \neq 0$. 0^0 is undefined
- $(xy)^p = (x^p)(y^p)$

Power Functions

Graphs of Power Functions

$p > 1$	Concave up. Grows as x grows large.
$p = 1$	The straight line $y = x$.
$0 < p < 1$	Concave down Grows as x grows large.
$p = 0$	The straight line $y = 1$.
$p < 0$	Concave up Approaches 0 as x grows large



$y = x^2$ (RED), $y = x$ (GREEN)

$y = x^0$ (YELLOW)

$y = x^{1/2}$ (BLUE), $y = x^{-2}$ (BLACK)

Rational Functions

Functions that can be represented as the quotient of polynomials

- Typical form of $\frac{p(x)}{a(x)}$ where p and q are polynomials.
- $p(x)$ is called the numerator and $q(x)$ is the denominator

$$g(x) = \frac{x^2 - 4}{x^2 - 5x + 6}$$

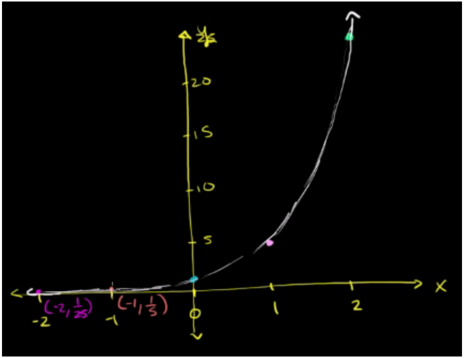
In the example above, $x^2 - 4$ is the numerator and $x^2 - 5x + 6$ is the denominator. A polynomial is a rational function whose denominator is 1.

Domain and Range of Rational Functions

- Domain of a rational function $\frac{p(X)}{q(x)}$ consists of all ponints where $q(x)$ is non-zero
- Domain depends on the way in which $p(x)$ and $q(x)$ are chosen. By expanding the function above, $x = 2$ will now appear in its domain, whereas it did not before:

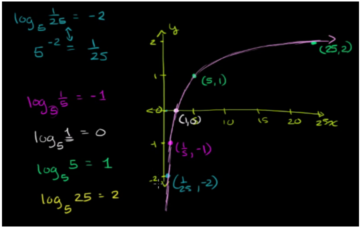
$$g(x) = \frac{(x-2)(x+2)}{(x-3)(x-2)}$$

- Exponential Functions
  - Functions of the form $f(x) = b^x$ for any positive real number b.
  - Characterized by their rate of growth being proportional to their value



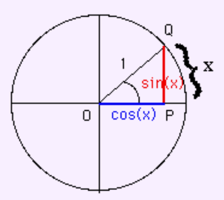
Other Functions

- Logarithmic Functions
  - Defined as any function of the form $\log_b x$ being the inverse function of the exponential function b^x
  - Three bases for logarithms:
    - **Natural logarithms** written as $\ln x$, expressed as the logarithm to the base e. Thus e^x and $\ln(x)$ are inverse functions
    - **Common logarithms** written as $\log x$, expressed as the logarithm to the base 10. Thus $10^{\log x} = x$
    - **Binary logarithms** used in communications and computer science with a base of 2. Expressed as $\log(x)$ or $\log_2 x$ where x is an integer measuring the number of bits it takes to write x.

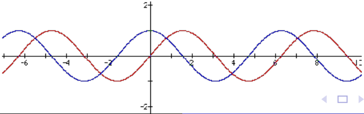


Other Functions

- Trigonometric Functions
  - Examples of these include $\sin(x)$ and $\cos(x)$ defined in the image below:



  - Other trigonometric functions defined below:
    - $\tan(x) = \frac{\sin(x)}{\cos(x)}$
    - $\cot(x) = \frac{\cos(x)}{\sin(x)}$
    - $\sec(x) = \frac{1}{\cos(x)}$
    - $\csc(x) = \frac{1}{\sin(x)}$

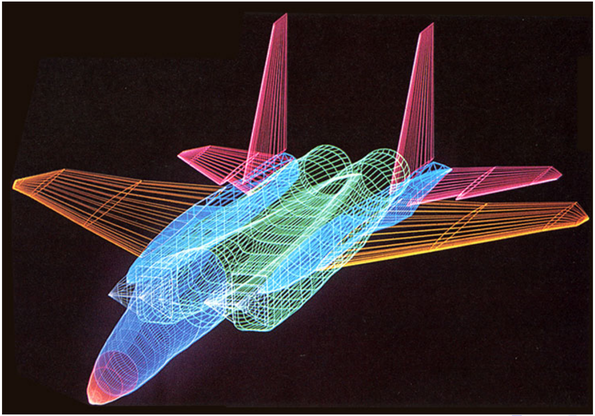


Real-World Applications

- Mathematical function theory is used in the design of various programming languages. The intent of using such concept as a basis in a different approach to the design of programming languages gives way to several advantages and disadvantages not seen with procedural and object-oriented programming. Examples of purely functional programming languages include:
    - Haskell
    - Agda
    - Mercury
- It is further used in every-day life when computing things such as **miles per gallon**, or **weekly salaries**, or **compound interest**. These are all functions of specific rates such as houerly pay for salaries, or initial investment and interest rate over time for compound interest.

Conclusion

Function theory is an important part of mathematics and of the world around us. It is used every day in applications like the stock market, to calculating monthly budgets, to rendering computer graphics.



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