FUNDAMENTAL THEOREM OF CALCULUS

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Part 1

The *integral* from a to x represents the area under the curve $y = f(t)$. For a given $f(t)$, we define the function $F(x)$ as

$$F(x) = \int_a^x f(t)dt, \quad \text{where } a \leq x \leq b$$

The above equation does not define or depend on knowledge of *derivatives*. We define the *derivative* of a $F(x)$ as follows:

$$F\prime(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt}{\Delta x}$$

Definition 1. *The derivative of a function $F(x)$ is equal to the quotient between the area under the curve $f(t)$ from a to $x + \Delta x$ minus the area under the curve $f(t)$ from a to x, and Δx.*

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} f(t)dt$$

Definition 2. *Knowing the above, it can be stated that in the area under the curve $f(t)$ from x to $x + \Delta x$, there exists a point c such that $f(c)\Delta x = \int_x^{x+\Delta x} f(t)dt$. This function $f(c)$ is therefore known as the mean value function over the integral.*

$$f(c) = \frac{1}{\Delta x} \int_x^{x+\Delta x} f(t)dt$$

In other words, there exists a value c in $[x, x + \Delta x]$ where

$$F\prime(x) = \lim_{\Delta x \to 0} f(c)$$

Knowing this, it can then be stated that $f(c) \to f(x) \quad \text{as} \quad \Delta x \to 0.$

Date: 6 February, 2016.

Theorem 3. *Let there be two functions $f_{-}(x)$ and $f_{+}(x)$ such that $f(x)$ is squeezed between the two,*

$$f_{-}(x) \leq f(x) \leq f_{+}(x)$$

if

$$r = \lim_{x \to a} f_{-}(x) = \lim_{x \to a} f_{+}(x)$$

then,

$$\lim_{x \to a} f(x) = r$$

Taking the *squeeze theorem* (defined above) into account, and assuming that

$$x \leq c(\Delta x) \leq x + \Delta x$$

we can then state that if

$$\lim_{\Delta x \to 0} x = x$$

and

$$\lim_{\Delta x \to 0} x + \Delta x = x$$

then,

$$\lim_{\Delta x \to 0} C(\Delta x) = x$$

The function f is continuous at c, so the limit can be taken inside of the function. Thefefore

$$F\prime(x) = f(x)$$

Thus, proving that any continuous function has an anti-derivative.

Q.E.D

Part 2

Let F be an integral (antiderivative) of f, with integral function g defined as

$$g(x) = \int_a^x f(t)dt$$

Using the first part of the *Fundamental Theorem of Calculus*, we can state that g is a continuous function on $[a,b]$ and differentiable on (a,b) and $g\prime(x) = f(x)$ for every x in (a,b)

We now define a new function H as

$$h(x) = g(x) - F(x)$$

h is continuous on $[a,b]$ and differentiable on (a,b) as a difference of two functions with those two properties. If $x \in (a,b), \text{then} h\prime(x) = g\prime(x) - F\prime(x)$. According to the first part of this theorem, $g\prime(x) = f(x)$. By the definition of *integrals being equal to antiderivatives*, $F\prime(x) = f(x)$. Therefore, it can be stated that $h\prime(x) = f(x) - f(x) = 0$ for every $x \in (a,b)$. Because h is continuous at a and b, h is constant on $[a,b]$, making $h(a) = h(b)$.

Therefore

$$\begin{aligned}
h(b) &= h(a) \\
g(b) - F(b) &= g(a) - F(a) \\
g(b) &= g(a) + (F(b) - F(a)) \\
\int_a^b f(t)dt &= \int_a^a f(t)dt + (F(b) - F(a)) \\
\int_a^b f(t)dt &= 0 + F(b) - F(a) \\
\int_a^b f(t)dt &= F(b) - F(a)
\end{aligned}$$

Thus, proving that the antiderivative of a function can be expressed as the area covered by the difference between the integrals of point a and b.

Sources

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