# STAT 656: Bayesian Data Analysis
# Fall 2024
# Homework 1

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## Synthetic Data

The *autoregressive model* is frequently used to analyze time series data. The simplest autoregressive model has order 1, and is abbreviated as AR(1). This model assumes that an observation $y_i$ at time point $i$ ($i = 1, \ldots, n$) is generated according to

$$y_i = \rho y_{i-1} + \epsilon_i,$$

where $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$ independently, and $rho$ and $\sigma$ are unknown parameters. For simplicity, we shall assume that $y_0$ is a fixed constant. We will also assume $|\rho| < 1$.

1. (5 points) Write the log-likelihood function $\log L(\rho, \sigma^2 | y_0, y_1, \ldots, y_n)$ for $(\rho, \sigma^2)^\intercal$ for AR(1) model.

   **Solution:**

   Given the formulation above, the log-likelihood function is calculated as follows:

$$\begin{aligned}\log L(\rho, \sigma^2 | y_0, y_1, \ldots, y_n) &= \log \prod_{i=1}^{n} (2\pi\sigma^2)^{-\frac{1}{2}} \cdot \exp\left\{-\frac{(y_i - \rho y_{i-1})^2}{2\sigma^2}\right\} \\ &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \rho y_{i-1})^2 \\ &= -\frac{n}{2}\log(2\pi) - n\log(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \rho y_{i-1})^2.\end{aligned}$$

2. (10 points) Write an R function that computes the log-likelihood function for $(\rho, \log(\sigma))^\intercal$ for this data. Provide a visualziation of this log-likelihood as a contour plot. Hint: The `outer` and `contour` function in `R` can be useful for creating the visualiazation, see also the code of lecture 2 and 3.

   **Solution:**

   The code implementation is as follows:

```r
# Read data
if (file.exists("computation_data_hw_1.csv")) {
    data <- read.csv("computation_data_hw_1.csv")
    y <- data[['x']]
} else {
    stop("Cannot find the data file 'computation_data_hw_1.csv' at ", getwd())
}

# Define the log-likelihood function
```

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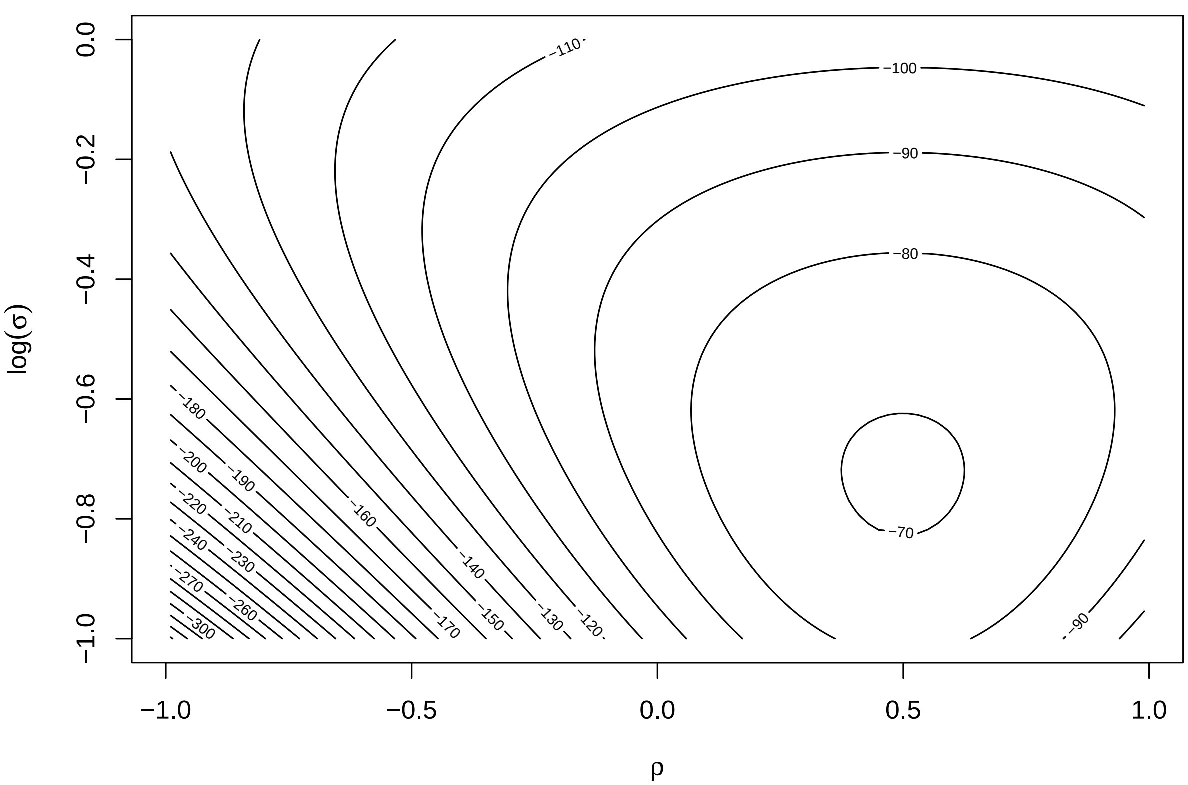
```r
ar_loglik <- function(rho, log_sig) {
    n <- length(y)
    sig <- exp(log_sig)
    rho <- rep(as.numeric(rho), times = n - 1)
    log_lik <- -0.5 * (
        n * log(2 * pi)
        + 2 * n * log_sig
        + (y[1]^2 + sum((y[2:n] - rho * y[1:(n-1)])^2)) / sig^2
    )
    return(log_lik)
}

# Visualization
rho <- seq(-0.99, 0.99, length=100)
log_sig <- seq(-1.0, 0.0, length=100)
loglik <- outer(rho, log_sig, Vectorize(ar_loglik))
contour(
    x=rho,
    y=log_sig,
    z=loglik,
    xlab=expression(rho),
    ylab=expression(log(sigma)),
    nlevels=20,
)
```




3. (10 points) For the purposes of this problem, suppose we specify $\rho \sim \text{Uniform}(-1, 1)$, $\log(\sigma) \sim \mathcal{N}(0, 10^2)$ independently *a priori* (note that this may not be an appropriate prior for the parameters of an AR(1) model in general). Write an R function that computes the log of the posterior density (up to a constant) for $(\rho, \log(\sigma))^\intercal$ under this prior. Provide a visualization of this function as above. How does this

function compare to the log-likelihood function? Would you say that this prior specification is overly informative? Why or why not?

4. (10 points) Draw 1000 values of $(\rho, \log(\sigma))^\intercal$ from a discrete grid approximation to the posterior. Be sure to describe your choice of discrete grid. Hint: The previous step can be helpful in this regard, together with the R function sample. Again, look at the code associated with the lectures.
5. (5 points) Use these draws to calculate the following summaries for each of $\rho$ and $\log(\sigma)$: $0.025, 0.25, 0.5, 0.75, 0.975$ quantiles, mean, standard deviation, skewness, and kurtosis. You can use the library `moments` if you want.
6. (10 points) Write an `R` function that takes parameters $(\rho, \log(\sigma))^intercal$ and simulates a new dataset yrep according to the AR process. Recall that we can simulate from the posterior predictive distribution of new datasets $y^{\text{rep}}$ given todays dataset as follows: first simulate a parameter set from the posterior distribution, and use this to simulate a new dataset. Use your two earlier `R` functions to generate 1000 such posterior predictive samples. Summarize your draws.
7. (10 points) Compare the observed data to these posterior predictive summaries. Also create a plot where the observed data is suporposed on these posterior predictive trajectories. What can you say about the model fit? Does the model appear appropriate?