STAT 656: Bayesian Data Analysis
Fall 2024
Homework 1

Juanwu Lu*

Synthetic Data

The *autoregressive model* is frequently used to analyze time series data. The simplest autoregressive model has order 1, and is abbreviated as AR(1). This model assumes that an observation y_i at time point i ($i = 1, \ldots, n$) is generated according to

$$y_i = \rho y_{i-1} + \epsilon_i,$$

where $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$ independently, and rho and σ are unknown parameters. For simplicity, we shall assume that y_0 is a fixed constant. We will also assume $|\rho| < 1$.

1. (5 points) Write the log-likelihood function $\log L(\rho, \sigma^2 | y_0, y_1, \ldots, y_n)$ for $(\rho, \sigma^2)^\intercal$ for AR(1) model.

   Solution:

   Given the formulation above, the log-likelihood function is calculated as follows:

   $$\begin{aligned}
   \log L(\rho, \sigma^2 | y_0, y_1, \ldots, y_n) &= \log \prod_{i=1}^{n} (2\pi\sigma^2)^{-\frac{1}{2}} \cdot \exp\left\{ -\frac{(y_i - \rho y_{i-1})^2}{2\sigma^2} \right\} \\
   &= -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \rho y_{i-1})^2 \\
   &= -\frac{n}{2} \log(2\pi) - n \log(\sigma) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \rho y_{i-1})^2.
   \end{aligned}$$

2. (10 points) Write an R function that computes the log-likelihood function for $(\rho, \log(\sigma))^\intercal$ for this data. Provide a visualziation of this log-likelihood as a contour plot. Hint: The `outer` and `contour` function in `R` can be useful for creating the visualiazation, see also the code of lecture 2 and 3.

3. For the purposes of this problem, suppose we specify $\rho \sim \text{Uniform}(-1, 1)$, $\log(\sigma) \sim \mathcal{N}(0, 10^2)$ independently *a priori* (note that this may not be an appropriate prior for the parameters of an AR(1) model in general). Write an R function that computes the log of the posterior density (up to a constant) for $(\rho, \log(\sigma))^\intercal$ under this prior. Provide a visualization of this function as above. How

*College of Engineering, Purdue University, West Lafayette, IN, USA