Trigonometric Fourier Series

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This notebook is intended to give a quick review on the Fourier trigonometric real and complex series as well as provide an easy tool to calculate Fourier Coefficients and generate plots for a given function and periodicity

Fourier Trigonometric Series

The Fourier Series is a method which consist in representing any periodic function or signal $f(t) = f(t + \mathrm{n}T)$ with periodicity T as a superposition (infinite sum) of sine and cosine functions, it also works to turn a generic function into a periodic signal function given the starting and ending points where the function is intended to repeat, that is, two points a, b.

The trick to represent a function as a series of sines and cosines relies on the **Fourier's trigonometric formula** given by:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n * \omega * t) + \sum_{n=1}^{\infty} b_n * \sin(n * \omega * t)$$

where the coefficients a_0, a_n, b_n are called the **Fourier Coefficients** and are calculated by means of the following formulas:

$$a_0 = \frac{2}{T} \int_a^b f(t)\, dt$$
$$a_n = \frac{2}{T} \int_a^b f(t) * \cos(n * \omega * t)\, dt$$
$$b_n = \frac{2}{T} \int_a^b f(t) * \sin(n * \omega * t)\, dt$$

and a_0 represents the midpoint of the oscillatory signal amplitude, a_n the amplitude of the even part of the function, and b_n the amplitude of the odd part of the function.

Fourier complex exponential series

As long as there is a mathematical relation between trigonometric functions and complex numbers, the trigonometric real series can be simplified as a trigonometric **Complex Fourier series** in the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{i n \omega t}$$

where c_n is the **complex fourier coefficient** and can be calculated with the following integral formula:

$$c_n = \frac{1}{T} \int_a^b f(t)\, e^{i n \omega t}\, dt$$

or it can be calculated by using the trigonometric coefficients

$$c_n = \begin{cases} \frac{a_0}{2} & n == 0 \\ \frac{1}{2}(a_n - i\, b_n) & n > 0 \\ \frac{1}{2}(a_n + i\, b_n) & n < 0 \end{cases}$$

Code Construction

Oscillatory parameters

Let’s define the function f(x) and calculate the periodicity interval given initial and final point of one oscillation (as an example $f(x) = x$^2 from 0 to 4):

In[6]:=

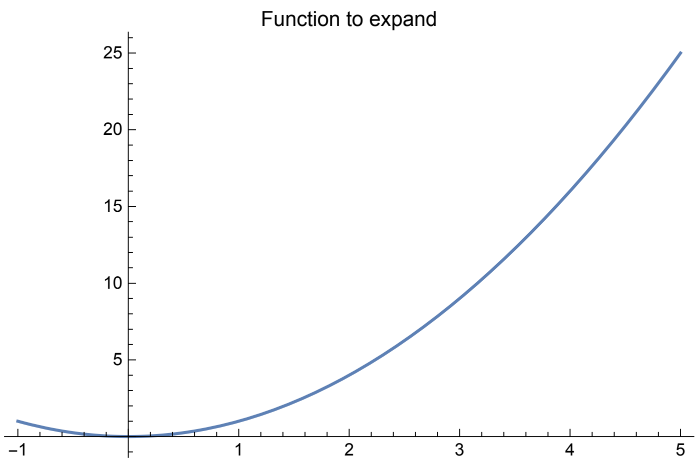
```
f[x_] = x^2;
a = 0;
b = 4;

T = b - a;
ω = 2 π / T;
```

now, show the calculated values and the plot of the original function centered at the given interval

In[11]:=
```
Print["The functions is f(x)=", f[x],
escribe
 " with a period T=", T, " and a frequency ω=", ω]
Plot[{f[x]}, {x, (a - 1), (b + 1)}, PlotLabel → "Function to expand"]
representación gráfica                etiqueta de rep··  función
```

The functions is f(x)=x^2 with a period T=4 and a frequency $\omega = \frac{\pi}{2}$

Out[12]=



Coefficients

The integrals are defined in order to calulate the coefficients explicitly:

In[13]:=
```
a0[n_] = Refine[2/T ∫_a^b f[t] ⅆt, Element[n, Integers]];
              refina                  pertenece a  números enteros

an[n_] = Refine[2/T ∫_a^b f[t] * Cos[n * ω * t] ⅆt, Element[n, Integers]];
              refina                 coseno              pertenece a  números enteros

bn[n_] = Refine[2/T ∫_a^b f[t] * Sin[n * ω * t] ⅆt, Element[n, Integers]];
              refina                 seno                pertenece a  números enteros
```

Now the coefficient values are printed:

In[16]:=
```
Print[ "The coefficients obtained are:"]
Print["a0=", a0[n]]
Print["an=", an[n]]
Print["bn=", bn[n]]
```

The coefficients obtained are:

$a_0 = \frac{32}{3}$

$a_n = \frac{16}{n^2 \pi^2}$

$b_n = -\frac{16}{n \pi}$

Trigonometric Series

Now that the needed components where found, the series is completely defined with the general formula.

In[20]:=
```
Print["Then, the Fourier Series expansion of f(t)=", f[t], " is given by: "]
Print["f(t)=", a0[n]/2, "+Sum[n=1..∞](", an[n]*Cos[n*ω*t], ")+(", bn[n]*Sin[n*ω*t], "))"]
```

Then, the Fourier Series expansion of $f(t) = t^2$ is given by:

$$f(t) = \frac{16}{3} + \sum_{n=1}^{\infty} \left(\frac{16 \operatorname{Cos}\left[\frac{n \pi t}{2}\right]}{n^2 \pi^2} \right) + \left(-\frac{16 \operatorname{Sin}\left[\frac{n \pi t}{2}\right]}{n \pi} \right))$$

In order to generate the plot of the result, the general formula is used to define a slightly modified version that introduces a maximum value to truncate the calculation up to the maximum (for this example there are calculated only the first 5 terms of the summation) since the original form has a summation that goes to infinite.

In[54]:=
```
F[n_,t_]=a0[n]/2+Sum[k=1..n]((an[k]*Cos[k*ω*t])+(bn[k]*Sin[k*ω*t]));
NN=5;
Expanse=Simplify[ExpToTrig[F[NN,t]]];

Print["As an example, the first ", NN, " terms of the expansion are: "]
Print[f_NN[t],"≈",Expanse]
```

As an example, the first 5 terms of the expansion are:

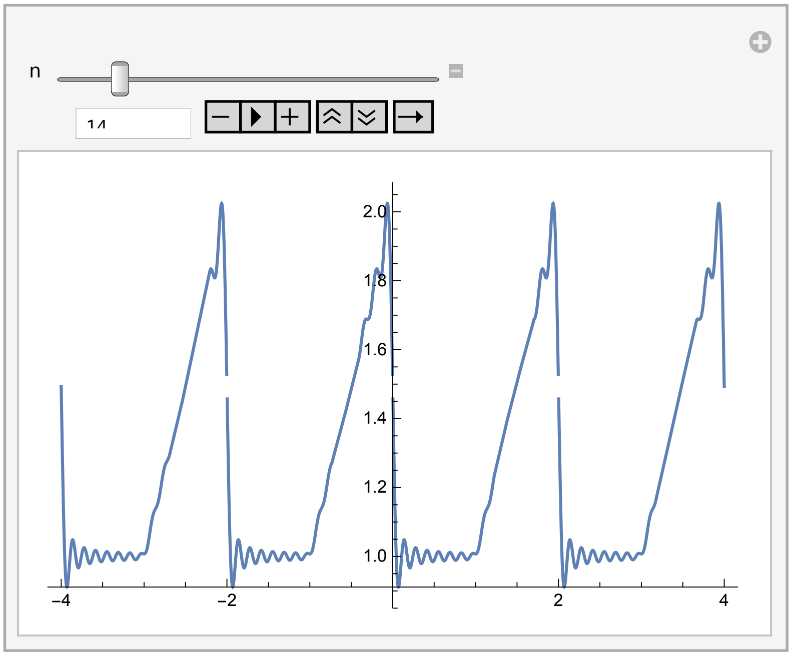
$$f_5[t] \approx \frac{1}{225 \pi^2} \left(1200 \pi^2 + 3600 \operatorname{Cos}\left[\frac{\pi t}{2}\right] + 900 \operatorname{Cos}[\pi t] + 400 \operatorname{Cos}\left[\frac{3 \pi t}{2}\right] + 225 \operatorname{Cos}[2 \pi t] + 144 \operatorname{Cos}\left[\frac{5 \pi t}{2}\right] - 3600 \pi \operatorname{Sin}\left[\frac{\pi t}{2}\right] - 1800 \pi \operatorname{Sin}[\pi t] - 1200 \pi \operatorname{Sin}\left[\frac{3 \pi t}{2}\right] - 900 \pi \operatorname{Sin}[2 \pi t] - 720 \pi \operatorname{Sin}\left[\frac{5 \pi t}{2}\right] \right)$$

Plot of the signal

Now, an interactive plot is created in order to manipulate the truncation value to get more precission as this value is increased. The plot also considers a parameter M which means the number of oscillations that are going to be displayed as well as the maximum value B to modify the slider

In[31]:=
```
B=100;
M=4;
Manipulate[Plot[{F[n,t]},{t,-(T*M)/2,(T*M)/2}],{n,1,B,1}]
```

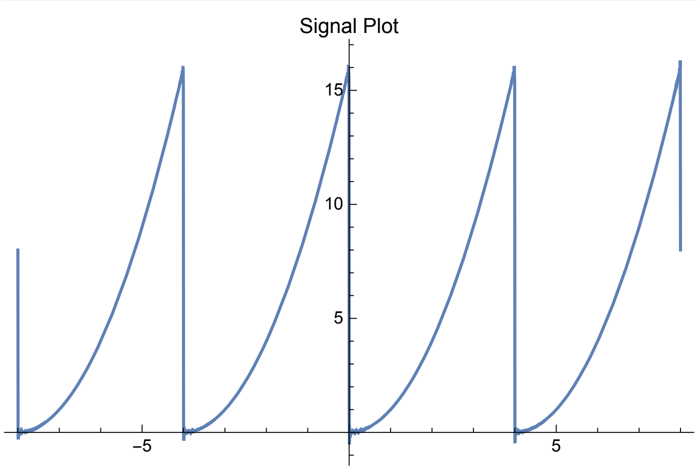
Out[33]=



as well as a comparative plot with a very high truncation order (B=1000) to visualize the smoothness of the signal for higher precision

In[ ]:=
```
Plot[{F[1000,t]},{t,-(T*M)/2,(T*M)/2},PlotLabel→"Signal Plot"]
```

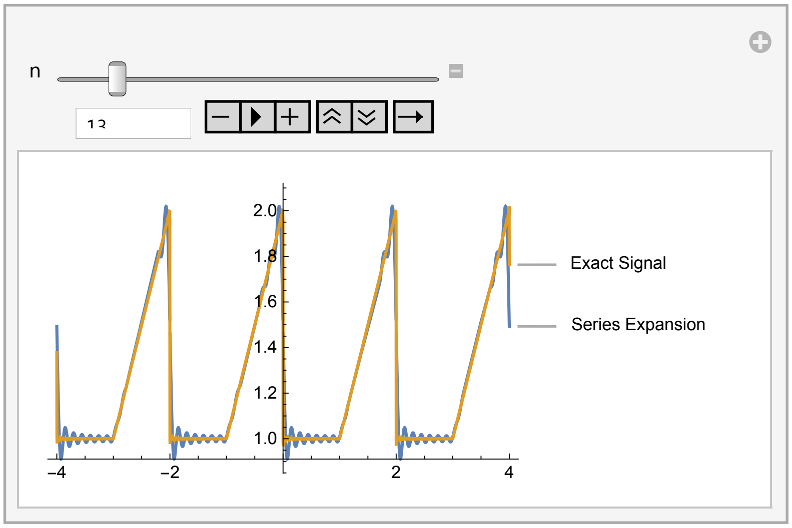
Out[ ]=



Both plots also can be merged into one in order to see clearly the comparison between the (almost) exact plot Vs the aproximations

In[ ]:=
```
Manipulate[Plot[{F[n,t],F[1000,t]},{t,-(T*M)/2,(T*M)/2},PlotLabels→{"Series Expansion","Exact S
```

Out[ ]=



Complex Coefficient

Complex superposition of trigonometric coefficients is used to obtain the complex coefficient c_n:

In[34]:=
```
cn[n]=Piecewise[{{a0[n]/2,n==0},{Refine[(an[n]-(i*bn[n]))/2,Element[n,Integers]],n>0},{Refine[(
Print["The complex Fourier coefficient is given by cₙ=", cn[n]]
```

The complex Fourier coefficient is given by $c_n = \begin{cases} \frac{16}{3} & n == 0 \\ \frac{1}{2}\left(\frac{16}{n^2 \pi^2} + \frac{16\, i}{n\, \pi}\right) & n > 0 \\ \frac{1}{2}\left(\frac{16}{n^2 \pi^2} - \frac{16\, i}{n\, \pi}\right) & n < 0 \\ 0 & \text{True} \end{cases}$

Or it can be determined with the direct form also

In[36]:=
$$\texttt{Simplify}\left[\texttt{ExpToTrig}\left[\texttt{Refine}\left[\frac{1}{T}\int_a^b f[t]*E^{-I*n*\omega*t}\,dt\right]\right],\texttt{Assumptions}\to n\in\texttt{Integers}\right]$$

Out[36]=
$$\frac{8 + 8\, i\, n\, \pi}{n^2\, \pi^2}$$

Complex Fourier Series

Finally, the complex series can be written as:

In[37]:=
```
CompF[n_,t_] = Sum[(cn[k]*E^(-I*k*ω*t)), {k,-n,n}];
Simplify[ExpToTrig[CompF[2,t]]]
```

Out[38]= $cn[0] + (cn[-1] + cn[1]) \operatorname{Cos}\left[\frac{\pi t}{2}\right] + (cn[-2] + cn[2]) \operatorname{Cos}[\pi t] + i\, cn[-1] \operatorname{Sin}\left[\frac{\pi t}{2}\right] - i\, cn[1] \operatorname{Sin}\left[\frac{\pi t}{2}\right] + i\, cn[-2] \operatorname{Sin}[\pi t] - i\, cn[2] \operatorname{Sin}[\pi t]$

Script

Merging the relevant parts of the code developed before it is obtained the following global code:

```
(*Define the function and the starting and ending points*)
f[x_]=x^2;
a=0;
b=4;


(*The output will be the plot, the period and the frequency desired*)
T=b-a;
ω=2π/T;
Print["The functions is f(x)=", f[x], " with a period T=", T, " and a frequency ω=",ω]
Plot[{f[x]},{x,(a-1),(b+1)},PlotLabel→"Function to expand"]


(*Calculation of the fourier Coefficients*)
a0[n_]=Refine[2/T ∫_a^b f[t]dt,Element[n,Integers]];
an[n_]=Refine[2/T ∫_a^b f[t]*Cos[n*ω*t]dt,Element[n,Integers]];
bn[n_]=Refine[2/T ∫_a^b f[t]*Sin[n*ω*t]dt,Element[n,Integers]];


(*Output with the calculated values of the fourier Coefficients*)
Print[ "The coefficients obtained are:"]
Print["a_0=", a0[n]]
Print["a_n=", an[n]]
Print["b_n=", bn[n]]


(*Fourier Series result*)
Print["Then, the Fourier Series expansion of f(t)=", f[t], " is given by: "]
Print["f(t)=", a0[n]/2, "+Sum_{n=1}^∞(", an[n]*Cos[n*ω*t],")+(", bn[n]*Sin[n*ω*t],"))"]


(*Truncated function defined to generate the plot*)
```

```
Print["Whith the following corresponding plot:"]
F[n_,t_]=a0[n]/2+Sum[((an[k]*Cos[k*ω*t])+(bn[k]*Sin[k*ω*t])),{k,1,n}];
(*B is the maximum truncation value, M is the number of oscillations displayed*)
B=100;
M=4;
Manipulate[Plot[{F[n,t],F[1000,t]},{t,-(T*M)/2,(T*M)/2},PlotLabels→{"Series Expansion","Exact S


(*Complex coefficients calculated*)
cn[n]=Piecewise[{{a0[n]/2,n==0},{Refine[(an[n]-(i*bn[n]))/2,Element[n,Integers]],n>0},{Refine[ (
Print["The complex Fourier coefficient is given by cn=", cn[n]]


(*Expansion up to NN terms*)
NN=5;
Expanse=Simplify[ExpToTrig[F[NN,t]]];
Print["As an example, the first ", NN, " terms of the expansion are: "]
Print[fNN[t],"≈",Expanse]
```

Examples:

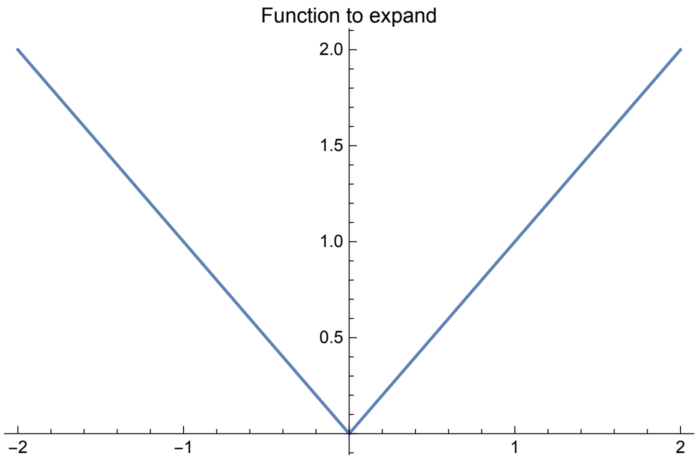
Even triangle signal

An even possitive triangle signal can be defined easily by using the absolute value of a linear function $f(x) = |x|$ from -1 to 1

In[86]:=
```
f[x_]=Abs[x];
a=-1;
b=1;


T=b-a;
ω=2π/T;
Print["The functions is f(x)=", f[x], " with a period T=", T, " and a frequency ω=",ω]
Plot[{f[x]},{x,(a-1),(b+1)},PlotLabel→"Function to expand"]

a0[n_]=Refine[2/T ∫_a^b f[t]ⅆt,Element[n,Integers]];
an[n_]=Refine[2/T ∫_a^b f[t]*Cos[n*ω*t]ⅆt,Element[n,Integers]];
bn[n_]=Refine[2/T ∫_a^b f[t]*Sin[n*ω*t]ⅆt,Element[n,Integers]];


Print[ "The coefficients obtained are:"]
Print["a_0=", a0[n]]
Print["a_n=", an[n]]
Print["b_n=", bn[n]]


Print["Then, the Fourier Series expansion of f(t)=", f[t], " is given by: "]
Print["f(t)=", a0[n]/2,"+ ∑_{n=1}^∞(", an[n]*Cos[n*ω*t],")+(", bn[n]*Sin[n*ω*t],"))"]


Print["Whith the following corresponding plot:"]
F[n_,t_]= a0[n]/2 + ∑_{k=1}^n((an[k]*Cos[k*ω*t])+(bn[k]*Sin[k*ω*t]));
B=100;
M=4;
Manipulate[Plot[{F[n,t],F[1000,t]},{t,-(T*M)/2,(T*M)/2},PlotLabels→{"Series Expansion","Exact S


cn[n]=Piecewise[{{a0[n]/2,n==0},{Refine[(an[n]-(i*bn[n]))/2,Element[n,Integers]],n>0},{Refine[
Print["The complex Fourier coefficient is given by c_n=", cn[n]]


NN=5;
Expanse=Simplify[ExpToTrig[F[NN,t]]];

Print["As an example, the first ", NN, " terms of the expansion are: "]
Print[f_NN[t],"≈",Expanse]
```

The functions is f(x)=Abs[x] with a period T=2 and a frequency ω=π

Out[92]=



The coefficients obtained are:

$a_0 = 1$

$$a_n = \frac{2\left(-1 + (-1)^n\right)}{n^2 \pi^2}$$

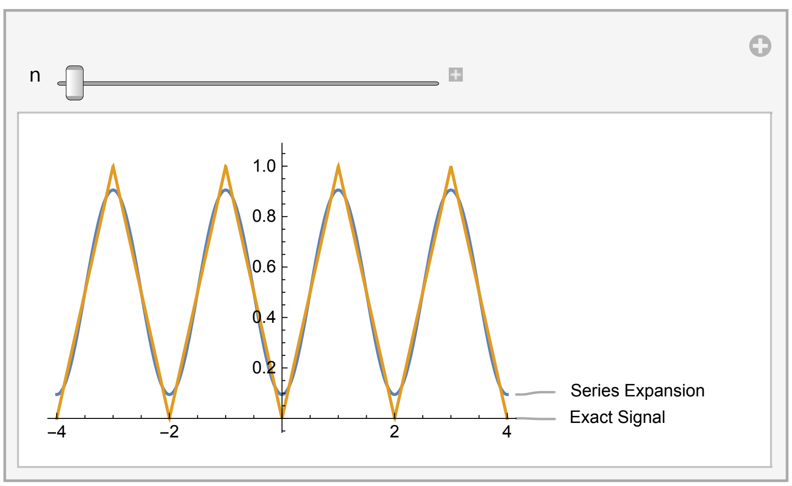
$b_n = 0$

Then, the Fourier Series expansion of f(t)=Abs[t] is given by:

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{2\left(-1 + (-1)^n\right) \mathrm{Cos}[n \pi t]}{n^2 \pi^2} \right) + (0))$$

Whith the following corresponding plot:

Out[106]=



The complex Fourier coefficient is given by $c_n = \begin{cases} \frac{1}{2} & n == 0 \\ \frac{-1 + (-1)^n}{n^2 \pi^2} & n > 0 \,||\, n < 0 \\ 0 & \text{True} \end{cases}$

As an example, the first 5 terms of the expansion are:

$$f_5[t] \approx \frac{1}{2} - \frac{4\,\mathrm{Cos}[\pi t]}{\pi^2} - \frac{4\,\mathrm{Cos}[3 \pi t]}{9 \pi^2} - \frac{4\,\mathrm{Cos}[5 \pi t]}{25 \pi^2}$$

Piecewise function

This code of course works even with weird piecewise functions like, for example

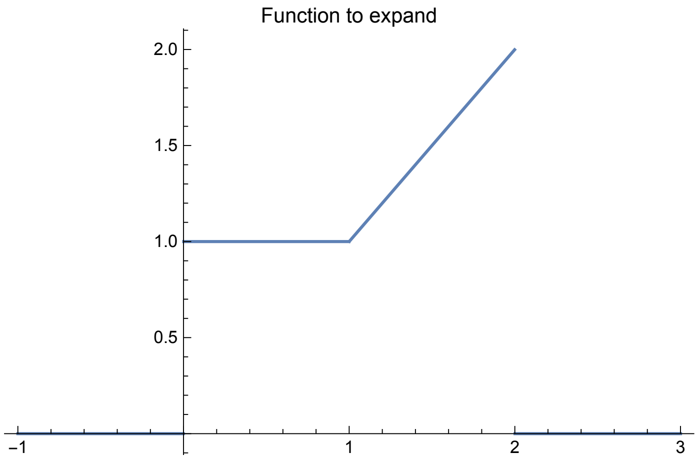
$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ x & 1 \leq x \leq 2 \end{cases}$ from 0 to 2

In[222]:=

```
f[x_]=Piecewise[{{1,0≤x≤1},{x,1≤x≤2}}];
a=0;
b=2;


T=b-a;
ω=2π/T;
Print["The functions is f(x)=", f[x], " with a period T=", T, " and a frequency ω=",ω]
Plot[{f[x]},{x,(a-1),(b+1)},PlotLabel→"Function to expand"]

a0[n_]=Refine[2/T ∫_a^b f[t]ⅆt,Element[n,Integers]];
an[n_]=Refine[2/T ∫_a^b f[t]*Cos[n*ω*t]ⅆt,Element[n,Integers]];
bn[n_]=Refine[2/T ∫_a^b f[t]*Sin[n*ω*t]ⅆt,Element[n,Integers]];


Print[ "The coefficients obtained are:"]
Print["a0=", a0[n]]
Print["an=", an[n]]
Print["bn=", bn[n]]


Print["Then, the Fourier Series expansion of f(t)=", f[t], " is given by: "]
Print["f(t)=", a0[n]/2,"+∑_{n=1}^∞(", an[n]*Cos[n*ω*t],")+(", bn[n]*Sin[n*ω*t],"))"]


Print["Whith the following corresponding plot:"]
F[n_,t_]=a0[n]/2+∑_{k=1}^n((an[k]*Cos[k*ω*t])+(bn[k]*Sin[k*ω*t]));
B=100;
M=4;
Manipulate[Plot[{F[n,t],F[1000,t]},{t,-(T*M)/2,(T*M)/2},PlotLabels→{"Series Expansion","Exact S

cn[n]=Piecewise[{{a0[n]/2,n==0},{Refine[(an[n]-(i*bn[n]))/2,Element[n,Integers]],n>0},{Refine[
Print["The complex Fourier coefficient is given by cn=", cn[n]]


NN=5;
Expanse=Simplify[ExpToTrig[F[NN,t]]];

Print["As an example, the first ", NN, " terms of the expansion are: "]
Print[fNN[t],"≈",Expanse]
```

The functions is $f(x)=\begin{cases} 1 & 0 \le x \le 1 \\ x & 1 \le x \le 2 \\ 0 & \text{True} \end{cases}$ with a period T=2 and a frequency $\omega=\pi$

Out[228]=



The coefficients obtained are:

$a_0 = \frac{5}{2}$

$a_n = \frac{1 + (-1)^{1+n}}{n^2 \pi^2}$

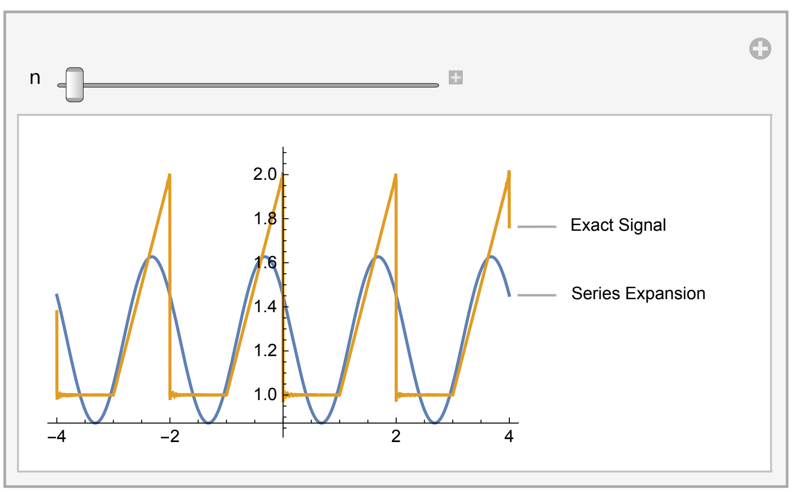
$b_n = -\frac{1}{n \pi}$

Then, the Fourier Series expansion of $f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ t & 1 \le t \le 2 \\ 0 & \text{True} \end{cases}$ is given by:

$$f(t) = \frac{5}{4} + \sum_{n=1}^{\infty} \left(\frac{\left(1 + (-1)^{1+n}\right) \mathrm{Cos}[n \pi t]}{n^2 \pi^2} \right) + \left(-\frac{\mathrm{Sin}[n \pi t]}{n \pi} \right))$$

Whith the following corresponding plot:

Out[242]=



The complex Fourier coefficient is given by $c_n = \begin{cases} \frac{5}{4} & n == 0 \\ \frac{1}{2}\left(\frac{1+(-1)^{1+n}}{n^2 \pi^2} + \frac{i}{n \pi}\right) & n > 0 \\ \frac{1}{2}\left(\frac{1+(-1)^{1+n}}{n^2 \pi^2} - \frac{i}{n \pi}\right) & n < 0 \\ 0 & \text{True} \end{cases}$

As an example, the first 5 terms of the expansion are:

$$f_5[t] \approx \frac{1}{900\pi^2}\big(1800\,\mathrm{Cos}[\pi t] + 200\,\mathrm{Cos}[3\pi t] + 72\,\mathrm{Cos}[5\pi t] - 15\pi\,(-75\pi + 60\,\mathrm{Sin}[\pi t] + 30\,\mathrm{Sin}[2\pi t] + 20\,\mathrm{Sin}[3\pi t] + 15\,\mathrm{Sin}[4\pi t] + 12\,\mathrm{Sin}[5\pi t])\big)$$

References:

Hsu, H. P. (1984). *Applied Fourier Analysis*. Harcourt Brace Jovanovich.