TRANSPOSE OF LORENTZ TRANSFORMATION

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Good notations often play central roles for making theoretical advances in the physical sciences, where calculations can easily become lengthy. However, certain notations create more confusion than clarity for beginners.

A quick search reveals widespread confusion regarding index notation and the Einstein summation convention among students. In particular, the property

$$(\Lambda^t)^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu} \tag{1}$$

of the transpose of a matrix representing a Lorentz transformation is often misunderstood (see, for example, Stack Exchange). Of course, (1) is not the usual definition for the transpose of a matrix. To my surprise, I couldn't find any simple proof of this equality — just interminable discussions in forums. To make things worse, in many places, the discussions ended up being even more confusing. Unsatisfied with the answers out there, I decided to write and share the proof below, which I believe is accessible to physics students.

Before we prove (1), remember that the transpose of any square matrix M, usually denoted by M^t, is defined by

$$(M^t)_{\alpha\beta} = M_{\beta\alpha}. \quad (*)$$

Proposition. *Let Λ be the matrix of a Lorentz transformation. Then,*

$$(\Lambda^t)^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}$$

and

$$(\Lambda^t)_{\nu}{}^{\mu} = \Lambda^{\mu}{}_{\nu}.$$

Proof. Let η denote the Minkowski metric. We'll prove the first one, since the second is analogous. For that, we simple compute

$$\begin{aligned}(\Lambda^t)^{\mu}{}_{\nu} &= \eta^{\mu\alpha}(\Lambda^t)_{\alpha\nu} \\ &= \eta^{\mu\alpha}\Lambda_{\nu\alpha} \quad (*) \\ &= \Lambda_{\nu}{}^{\mu}.\end{aligned}$$

□

*This is a first draft. Future versions could evolve into lecture notes; who knows.