Finite differences and quadrature

Question 1 **10 marks**

Consider the function

$$f(x) = \tan x. \tag{1}$$

(**a**) Find the symbolic expression for $f'(x)$ as a function of x.

$$f'(x) = 1 + \tan^2(x)$$

(**b**) Write a MATLAB script `A5Q1.m` to approximate $f'(x)$ at $x = 1.0$ using each of the following finite-difference schemes:

$$(\delta_+ f)(x; h) = \frac{f(x+h) - f(x)}{h}; \qquad (\delta f)(x; h) = \frac{f(x+h) - f(x-h)}{2h}$$

(centred and forward finite-differences) for $h = 10^0, 10^{-1}, 10^{-2}, \ldots, 10^{-14}, 10^{-15}$. For each value of h, compute the associated absolute approximation errors

$$E_+(x; h) = |f'(x) - (\delta_+ f)(x; h)|; \qquad E(x; h) = |f'(x) - (\delta f)(x; h)|$$

Display the results neatly in a table using **`fprintf`** with h decreasing, i.e., complete the following table:

h	$(\delta_+ f)(1.0; h)$	$E_+(1.0; h)$	$(\delta f)(1.0; h)$	$E(1.0; h)$
10^0				
10^{-1}				
10^{-2}				
$\vdots$				
10^{-15}				

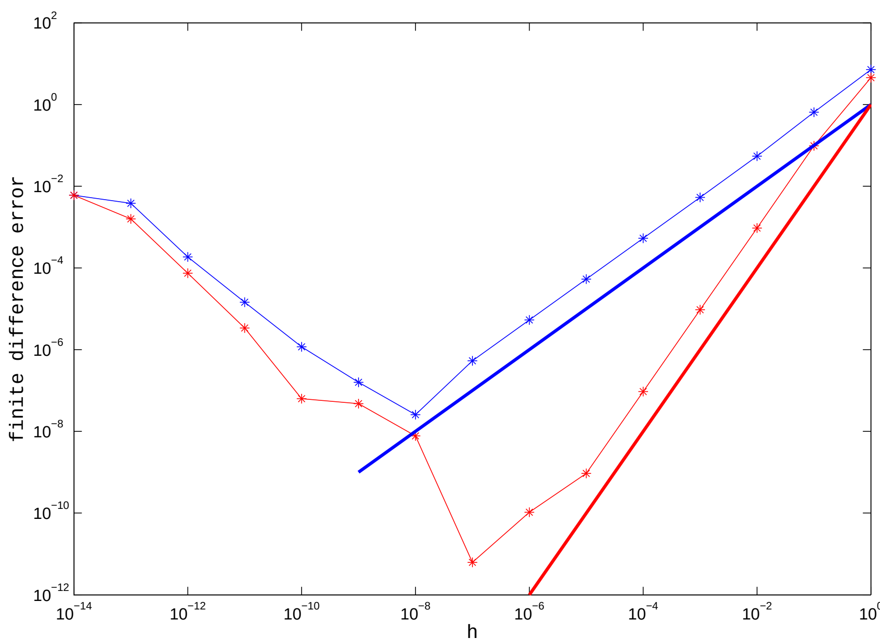
```
f=@(x) tan(x)
fp=@(x) 1+(tan(x))^2;
h=1;x=1;
fprintf('   h           fd       fd error       cd      cd error\n');
for i=1:15
    fd=(f(x+h)-f(x))/h;
    cd=(f(x+h)-f(x-h))/(2*h);
    efd=abs(fd-fp(1));
    ecd=abs(cd-fp(1));
    fprintf('%3.1e %12.9e %3.1e %12.9e %3.1e\n',h,fd,efd,cd,ecd);
    h=h/10;
end;
```

gives the following output:

h	fd	fd error	cd	cd error
1.0e+00	-3.742447588e+00	7.2e+00	-1.092519932e+00	4.5e+00
1.0e-01	4.073519326e+00	6.5e-01	3.523007198e+00	9.7e-02
1.0e-02	3.479829956e+00	5.4e-02	3.426464160e+00	9.5e-04
1.0e-03	3.430863217e+00	5.3e-03	3.425528271e+00	9.5e-06
1.0e-04	3.426052408e+00	5.3e-04	3.425518915e+00	9.5e-08
1.0e-05	3.425572171e+00	5.3e-05	3.425518822e+00	9.4e-10
1.0e-06	3.425524155e+00	5.3e-06	3.425518821e+00	1.0e-10
1.0e-07	3.425519355e+00	5.3e-07	3.425518821e+00	6.2e-12
1.0e-08	3.425518846e+00	2.6e-08	3.425518813e+00	7.8e-09
1.0e-09	3.425518980e+00	1.6e-07	3.425518869e+00	4.8e-08
1.0e-10	3.425517647e+00	1.2e-06	3.425518758e+00	6.3e-08
1.0e-11	3.425504325e+00	1.4e-05	3.425515427e+00	3.4e-06
1.0e-12	3.425704165e+00	1.9e-04	3.425593142e+00	7.4e-05
1.0e-13	3.421707362e+00	3.8e-03	3.423927808e+00	1.6e-03
1.0e-14	3.419486916e+00	6.0e-03	3.419486916e+00	6.0e-03

(**c**) Plot the error of the two schemes in one figure with a logarithmic scale. Also plot straight lines indicating a decrease as $\mathrm{O}(h)$ and as $\mathrm{O}(h^2)$.

The following graph shows the forward difference error in blue and the centered difference error in red. The solid blue and red lines indicate h and h^2, respectively. Note, that the error increases if h gets too small. This is due to round-off error.



Question 2 **10 marks**

For $f(x) = \sin(x)$, consider the integral

$$\int_{x=0}^{2} f(x)\,\mathrm{d}x$$

(**a**) Approximate this integral using the composite midpoint formula on two intervals, with $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$. Compare the result to the exact value of the integral.

$$\int_{x=0}^{2} \sin(x)\,\mathrm{d}x \approx \sin(0.5) + \sin(1.5) \approx 1.4769$$

so the error is about 6×10^{-2}.

(**b**) Approximate this integral using the composite Simpson formula on two intervals, with $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$. Compare the result to the exact value of the integral.

The exact value is $1 - \cos(2) \approx 1.4161468\ldots$

$$\int_{x=0}^{2} \sin(x)\,\mathrm{d}x \approx \frac{1}{6}\left[\sin(0) + 4\sin(0.5) + \sin(1)\right] + \frac{1}{6}\left[\sin(1) + 4\sin(1.5) + \sin(2)\right] \approx 1.41665$$

so the error is about 5×10^{-4}.