Interpolation error and least squares

Question 1 **15 marks**

Consider the function

$$f(x) = \ln \frac{x+2}{x+1}$$

(**a**) Compute the second order interpolating polynomial in the Lagrange form which is equal to $f(x)$ at the points $x_0 = 0$, $x_1 = 1$ and $x_2 = 3$. Produce a plot of the function and the interpolant and a separate plot of the interpolation error.

The coefficients are approximately

$$\Pi_4 f(x) = 0.693147 - 0.353189\,x + 0.0655071\,x^2$$

(**b**) Find an upper bound for the interpolation error $E(x) = |P(x) - f(x)|$. Show that, on the domain $x \in [0,3]$, we have $E(x) < 1$.

The error is given in lecture 13 as

$$|P(x) - f(x)| = \left| \frac{1}{3!} f^{(3)}(\xi)\,(x-x_0)(x-x_1)(x-x_2) \right| = \frac{1}{6} \left| \left(\frac{2}{(\xi+2)^3} - \frac{2}{(\xi+1)^3} \right) x(x-1)(x-3) \right|$$

so we have to find the maximum of

$$\left| \left(\frac{2}{(\xi+2)^3} - \frac{2}{(\xi+1)^3} \right) \right| \quad \text{for } \ \xi \in (0,3)$$

The maximum is located either at $\xi = 0$ or $\xi = 3$ or there where the derivative is zero, however

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \left(\frac{2}{(\xi+2)^3} - \frac{2}{(\xi+1)^3} \right) = \frac{-6}{(\xi+2)^4} + \frac{6}{(\xi+1)^4} > 0$$

so the maximum is located at $\xi = 0$ and takes the value 14/8. So we find that

$$|P(x) - f(x)| \le \frac{7}{24} |x(x-1)(x-3)|$$

Because the error is zero at $x = 0$ and $x = 3$ its maximum has to be located at a point where the derivative is zero. There are two such points, namely

$$\frac{\mathrm{d}}{\mathrm{d}x} x(x-1)(x-3) = 3x^2 - 8x + 3 = 0 \quad \Rightarrow \quad x = \frac{8 \pm \sqrt{28}}{6}$$

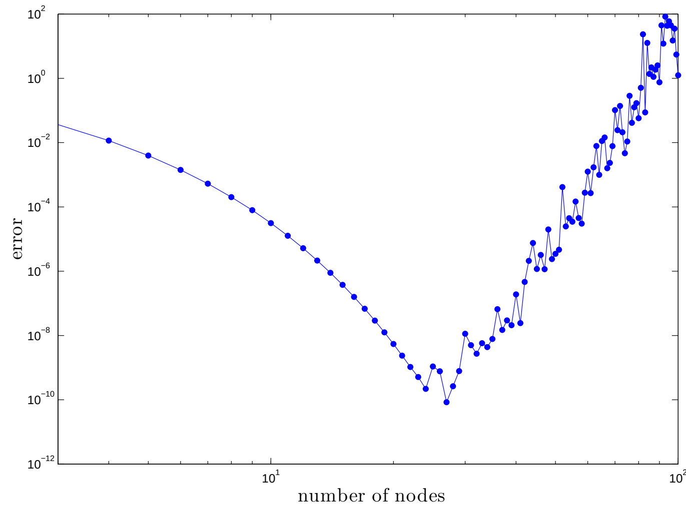
The value of the error at these points is approximately 0.184 and 0.616, smaller then 1 as required.

(**c**) Write a script to find the interpolating polynomial on an equidistant grid with k nodes on $[0,3]$, where k ranges from 3 to 100. For each interpolant, estimate the error by comparing the polynomial to f on a fine grid. Plot the error as a function of k on a logarithmic scale. What behaviour do you see? Explain the behaviour in terms of the two sources of error identified in the lectures.

Here is a script:

```
f=@(x) log((x+2)./(x+1));
err=[];
xfine=linspace(0,3,10000); % fine grid to compute error
ffine=f(xfine);
for k=2:100
    xs=linspace(0,3,k);
    ys=f(xs);
    p=polyfit(xs,ys,k-1);  % or call your own interpolation function!
    yfine=polyval(p,xfine);
    err=[err;[k norm(ffine-yfine)]];
end;
loglog(err(:,1),err(:,2),'-.');
```

produces a plot like this (after adding labels):



Up to about 23 nodes, the error decreases as we expect from the upper bound for interpolation error, but then the linear problem we need to solve to find the interpolant becomes ill-conditioned (Matlab probably warns you!) and the error goes up because of round-off error in the coefficients.

Question 2 **5 marks**

Show that the least-squares objective function

$$\phi(\vec{y}) = \frac{1}{N}\|\vec{b} - A\vec{y}\|_2^2$$

is minimal if and only if

$$A^T A\vec{y} = A^T\vec{b}$$

where $A \in \mathbb{R}^{N,M}$, $\vec{y} \in \mathbb{R}^M$, $\vec{b} \in \mathbb{R}^N$ and $N > M$.

The function has an extremum when

$$\frac{\partial \phi(\vec{y})}{\partial y_k} = 0 \text{ for } k = 1, \ldots, M$$

and this gives

$$\frac{\partial}{\partial y_k}\sum_{i=1}^{N}(b_i - \sum_{j=1}^{M} A_{ij}y_j)^2 = \sum_{i=1}^{N} 2(b_i - \sum_{j=1}^{M} A_{ij}y_j)\sum_{j=1}^{M}\frac{\partial A_{ij}y_j}{\partial y_k} = \sum_{i=1}^{N} 2(b_i - \sum_{j=1}^{M} A_{ij}y_j)A_{ik} =$$
$$2\sum_{i=1}^{N} A_{ki}^T(b_i - \sum_{j=1}^{M} A_{ij}y_j) = 0$$

and the last expression is just the k^{th} element of $A^T(\vec{b} - A\vec{y})$, from which the normal equation follows. Thus, there is only one extremum, and because ϕ is quadratic and non-negative, this is a minimum.