Iteration and recursion

Due: Friday, January 24th, 8:10. Hand in your assignment in at the lecture.
Write your answers out carefully and clearly. Upload the following Matlab
files to Blackboard:

- `iteration.m` and `recursion.m` for Question 1;

- `steffensen.m` for Question 2;

- a text file called `README` explaining any other files submitted. If you
  submit no other files, a `README` file is not needed.

Be sure all MATLAB files submitted include a comment line with your name
and student number, e.g.,

```
% Peter Parker
% 123456789
```

Also, be a **good programmer** and include comments with a brief descrip-
tion of the functionality, input and output arguments and usage of each
function or script. Also, add some comments that explain what steps are
taken.

Question 1   **10 marks**
Consider the nonlinear equation
$$f(x) = \sin(\pi x) - x^2 = 0$$

(**a**) Can you "solve the equation by hand", i.e. express the solution x^* explicitly in terms of
elementary functions and the constant π?

(**b**) Show that the equation $f(x) = 0$ has the same solution(s) as $g(x) = x$ if

$$g(x) = \frac{1}{2\pi} \sin(\pi x) - \frac{1}{2\pi} x^2 + x$$

In fact, we can show that for any initial point in $(0, 2]$ the sequence $x_k = g(x_{k-1})$ converges to a
unique solution x^*.

(**c**) Write a function that computes this sequence. Your function should:

    – Take for input the an initial point, a maximal number of iterations and a tolerance for
$|x_k - x_{k-1}|$ (the estimated error) and for $|f(x_k)|$ (the residual).

    – Print to the commend window the list of iterates, stopping when either the maximal
number of iterations is reached or the estimated error **and** the residual are below their
respective tolerance.

    – Output the approximate solution, its estimated error and its residual.

(**d**) If you used a `for` or `while` loop in (**c**), program a function with the same functionality using
recursion (i.e. no explicit loops). If you used recursion in (**c**) then program a function with
the same functionality using loops (i.e. no recursion).
Name your functions `iteration.m` for the version with a loop and `recursion.m` for the version
without.

(e) What happens if you take $x_0 = 0$? What happens if you take $x_0 < 0$?

Question 2   **10 marks**

(a) Write a function that implements the following pseudo-code:
Input: f, f', x, ϵ, N.
Output: x^*.

1. Repeat N times:
   (a) Set $y_1 = x$.
   (b) Take one Newton step, starting from y_1. Call the result y_2.
   (c) Take one Newton step, starting from y_2. Call the result y_3.
   (d) Set
   $$x = y_1 - \frac{(y_2 - y_1)^2}{y_3 - 2y_2 + y_1}$$
   (e) Display $|f(x)|$.
   (f) If $|f(x)| < \epsilon$ print "converged!", break.
2. Output $x^* = x$.

This algorithm is called Steffensen's iteration.

(b) Test your routine on the problem

$$\exp(-x^2 + x) - \frac{1}{2}x = 1.0836 \quad \text{(with initial guess } x = 1\text{)}$$

Show that Newton iteration does not converge quadratically, but your new iterative algorithm does.