APPM 4600
Homework 5

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1 2x2 Non-linear System

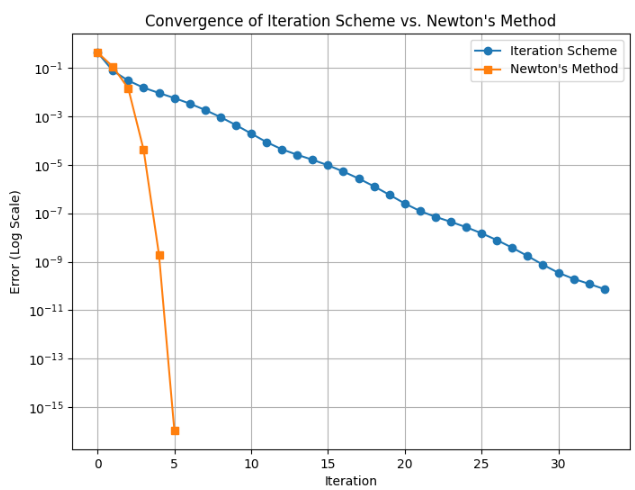
For our Problem, we look to find a solution near $(x, y) = (1, 1)$ of the system:

$$f(x,y) = 3x^2 - y^2 = 0 \tag{1}$$
$$g(x,y) = 3xy^2 - x^3 - 1 = 0 \tag{2}$$

(a) To start, we'll iterate on our system using the following scheme:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} \frac{1}{6} & \frac{1}{18} \\ 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix} ; n = 1, 2, \ldots \tag{3}$$

We'll use the initial conditions $(x_0, y_0) = (1, 1)$. After setting this scheme up in Python, it converges to $(0.5, .866025)$ at a linear convergence rate.



Iteration	Scheme x	Scheme y	Scheme Error	Newton x	Newton y	Newton Error
0	1.000000	1.000000	4.230985e-01	1.000000	1.000000	4.230985e-01
1	0.611111	0.833333	7.386393e-02	0.611111	0.833333	1.091471e-01
2	0.537628	0.825846	2.926832e-02	0.503659	0.852494	1.404593e-02
3	0.509853	0.835075	1.491331e-02	0.499964	0.866046	4.119146e-05
4	0.499764	0.846058	9.018884e-03	0.500000	0.866025	1.967492e-09
5	0.497051	0.854660	5.600220e-03	0.500000	0.866025	1.110223e-16
6	0.497128	0.860259	3.302055e-03	0.500000	0.866025	NaN
7	0.497966	0.863453	1.809810e-03	NaN	NaN	NaN
8	0.498779	0.865071	9.191087e-04	NaN	NaN	NaN
9	0.499352	0.865789	4.334183e-04	NaN	NaN	NaN
10	0.499696	0.866053	1.924401e-04	NaN	NaN	NaN
11	0.499877	0.866117	8.536540e-05	NaN	NaN	NaN
12	0.499962	0.866108	4.306480e-05	NaN	NaN	NaN
13	0.499996	0.866082	2.565267e-05	NaN	NaN	NaN
14	0.500007	0.866058	1.592383e-05	NaN	NaN	NaN
15	0.500007	0.866042	9.442174e-06	NaN	NaN	NaN
16	0.500006	0.866033	5.221663e-06	NaN	NaN	NaN
17	0.500003	0.866028	2.682828e-06	NaN	NaN	NaN
18	0.500002	0.866026	1.281533e-06	NaN	NaN	NaN
19	0.500001	0.866025	5.742303e-07	NaN	NaN	NaN
20	0.500000	0.866025	2.523217e-07	NaN	NaN	NaN
21	0.500000	0.866025	1.224812e-07	NaN	NaN	NaN
22	0.500000	0.866025	7.095923e-08	NaN	NaN	NaN
23	0.500000	0.866025	4.411597e-08	NaN	NaN	NaN
24	0.500000	0.866025	2.650924e-08	NaN	NaN	NaN
25	0.500000	0.866025	1.489240e-08	NaN	NaN	NaN
26	0.500000	0.866025	7.774811e-09	NaN	NaN	NaN
27	0.500000	0.866025	3.771943e-09	NaN	NaN	NaN
28	0.500000	0.866025	1.710572e-09	NaN	NaN	NaN
29	0.500000	0.866025	7.492623e-10	NaN	NaN	NaN
30	0.500000	0.866025	3.516399e-10	NaN	NaN	NaN
31	0.500000	0.866025	1.969076e-10	NaN	NaN	NaN
32	0.500000	0.866025	1.219974e-10	NaN	NaN	NaN
33	0.500000	0.866025	7.418833e-11	NaN	NaN	NaN
34	0.500000	0.866025	NaN	NaN	NaN	NaN

(b) The Above Matrix is equivalent to the inverse of the Jacobian of the vector $[f(x,y), g(x,y)]^T$ evaluated at our initial point., in turn, causes our Fixed point method to effectively being a Lazy Netwon method for a Non-linear system where the Jacobian is evaluated once at the beginning instead of every time the Fixed point iteration is applied to our scheme.

(c) Judging off of our results from both methods, the exact solution is $(\frac{1}{5}, \frac{\sqrt{3}}{2})$ which, when plugged into our system, comes out as the correct answer.

2 Application of Theorem 10.6

Consider The following Non-Linear system:

$$\begin{bmatrix} x_{n+1} = \frac{1}{\sqrt{2}}\sqrt{1+(x_n+y_n)^2} - \frac{2}{3} \\ y_{n+1} = \frac{1}{\sqrt{2}}\sqrt{1+(x_n-y_n)^2} - \frac{2}{3} \end{bmatrix} \tag{4}$$

we aim to find a region D such that our system converges to a unique fixed point solution.

To start, let's find the Jacobian of our system to relate it to a constant $K \leq 1$ where $|\frac{\partial g_i(x)}{\partial x_j}| \leq \frac{K}{n}$ for our system $n = 2$. Therefore, we are looking for an upper bound of which the values of x, y keep $|\frac{\partial g_i(x)}{\partial x_j}| \leq \frac{1}{n}$

$$J_g(x) = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{(x+y)}{\sqrt{1+(x+y)^2}} & \frac{(x+y)}{\sqrt{1+(x+y)^2}} \\ \frac{(x-y)}{\sqrt{1+(x-y)^2}} & \frac{-(x-y)}{\sqrt{1+(x-y)^2}} \end{bmatrix} \tag{5}$$

From our Jacobian, we see that the indices of $J_g(x) \leq \frac{1}{\sqrt{2}}$ stay within the bounds formed from theorem 10.6. Which further implies our region D:

$$D = \{(x, y) \in \mathbb{R}^2 \mid |x+y| \leq 1, |x-y| \leq 1\} \tag{6}$$

From this conclusion, we can state that with $(x_0, y_0) \in D$, applying fixed point iteration is guaranteed to find a unique solution.

3 Fixed Point to find the line $f(x, y) = 0$

(a) First, we need to derive an iteration scheme that we can apply through fixed point iteration, which will appear similar to the following:

$$\begin{Bmatrix} x_{n+1} = x_n - df_x \\ y_{n+1} = y_n - df_y \end{Bmatrix}; d = \frac{f}{f_x^2 + f_y^2} \tag{7}$$

First, Let's start by setting up what iterating on the gradient would look like via parametrization:

$$\begin{aligned} x &= x_0 - tf_x \\ y &= y_0 - tf_y \end{aligned} \tag{8}$$

$$f(x, y) = 0 \tag{9}$$

$$f(x_0 - tf_x, y_0 - tf_y) \approx f(x_0, y_0) + t(f_x^2 + f_y^2) = 0 \tag{10}$$

We can pull out the t in line ten because this scheme is analogous to a Newton step for our parameter t. No solving for t we get:

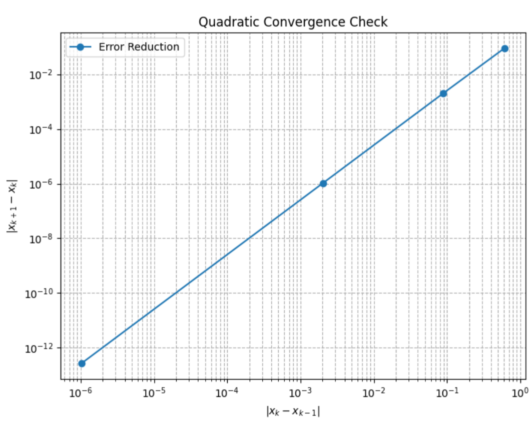
$$t = \frac{f(x_0, y_0)}{f_x^2 + f_y^2} \tag{11}$$

which returns us to our starting point where $d = t$; therefore, in our original scheme, d parameterize our curve.

(b) We now aim to generalize this scheme from 2-D to 3-D, Specifically on the surface $f(x, y, z) = x^2 + 4y^2 + 4z^2 = 16$. This can be done intuitively with the scheme from the starting point $(x_0, y_0, z_0) = (1, 1, 1)$:

$$\left\{ \begin{aligned} x_{n+1} &= x_n - df_x \\ y_{n+1} &= y_n - df_y \\ z_{n+1} &= z_n - dz_y \end{aligned} \right\}; d = \frac{f(x_n, y_n, z_n)}{f_x^2 + f_y^2 + f_z^2} \tag{12}$$

After applying this scheme, we get the following graph and table; by analyzing the error, we see $e_{k+1} \approx e_k^2$:



Iteration	x	y	z	f(x,y,z)	Error
0	1.000000	1.000000	1.000000	-7.000000e+00	0.000000e+00
1	1.106061	1.424242	1.424242	1.451102e+00	6.092718e-01
2	1.093926	1.361742	1.361742	3.139797e-02	8.921842e-02
3	1.093642	1.360329	1.360329	1.604374e-05	2.017745e-03
4	1.093642	1.360328	1.360328	4.199308e-12	1.032082e-06
5	1.093642	1.360328	1.360328	-3.552714e-15	2.702246e-13