APPM 4600
Homework 2

February 7, 2025

Jude Gogolewski

---

1 $2x - 1 = sin(x)$

(a) Find a closed interval $[a, b]$ on which this equation has a root and prove its existence:

To start, let's rewrite this equation to see how a root could exist:

$$2x - 1 = sin(x) \tag{1}$$
$$2x - 1 - sin(x) = 0 \tag{2}$$

Looking at the pieces of equation (2), we will use some analysis to find our interval. $2x - 1$ gives a negative value at $x = 0$ as well $sin(x) = 0$ at $x = 0$ therefore we have found our negative bound. $2x - 1$ becomes positive at $x = 0.5$ and stays positive after that. $sin(x) \geq 0; x \in [0, \pi]$. therefore our bounds will be $[0, \pi]$
We will use the Intermediate value theorem to prove a root r exists. We know our

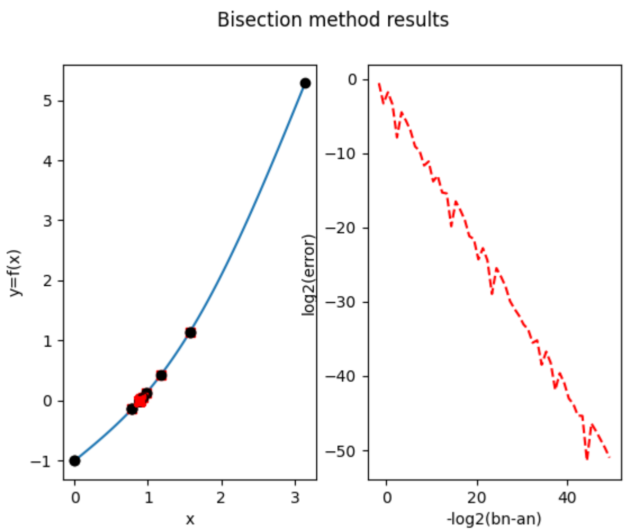
function (2) is continuous as it is the sum of continuous functions. We also know that at the bound $x = 0$, function (2) is negative, and at $x = \pi$, function (2) is positive. Therefore, By the Intermediate value theorem, there exists a point r at with $2r - 1 = sin(r)$ or $2r - 1 - sin(r) = 0$

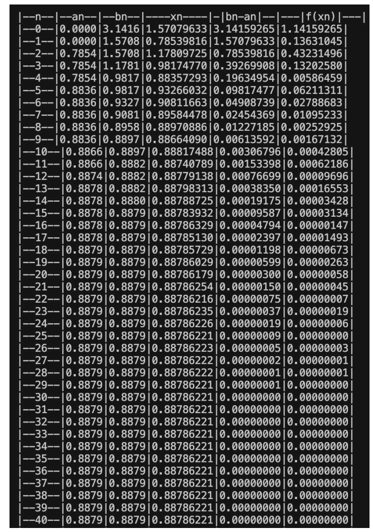
(b) Prove r is the only root for the function
We will examine our chosen bounds and the first derivative to prove this.

$$f(x) = 2x - 1 - sin(x) \tag{3}$$
$$f'(x) = 2 - cos(x) \tag{4}$$

$f'(x)$ is positive for all real numbers as $-1 \leq cos(x) \leq 1$ therefore $1 \leq f'(x) \leq 3$. because we know the first derivative is always positive, we know the function is always growing, and in turn, only one root can exist.

(c) Use the bisection code to approximate r to 8 decimals.
After applying the code, the final root given was: 0.88786221. attached below are some images to show the code in action:



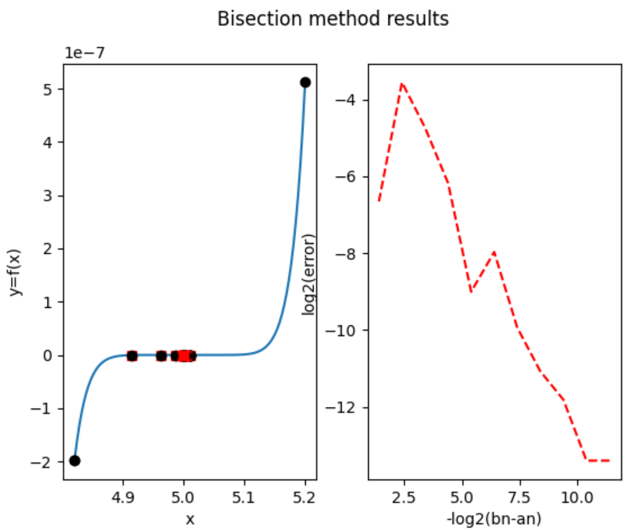


n	an	bn	xn	|bn-an|	f(xn)
0	0.0000	3.1416	1.57079633	3.14159265	1.14159265
1	0.0000	1.5708	0.78539816	1.57079633	0.13631045
2	0.7854	1.5708	1.17809725	0.78539816	0.43231496
3	0.7854	1.1781	0.98174770	0.39269908	0.13202580
4	0.7854	0.9817	0.88357293	0.19634954	0.00586459
5	0.8836	0.9817	0.93266032	0.09817477	0.06211311
6	0.8836	0.9327	0.90811663	0.04908739	0.02788683
7	0.8836	0.9081	0.89584478	0.02454369	0.01095233
8	0.8836	0.8958	0.88970886	0.01227185	0.00252925
9	0.8836	0.8897	0.88664090	0.00613592	0.00167132
10	0.8866	0.8897	0.88817488	0.00306796	0.00042805
11	0.8866	0.8882	0.88740789	0.00153398	0.00062186
12	0.8874	0.8882	0.88779138	0.00076699	0.00009696
13	0.8878	0.8882	0.88798313	0.00038350	0.00016553
14	0.8878	0.8880	0.88788725	0.00019175	0.00003428
15	0.8878	0.8879	0.88783932	0.00009587	0.00003134
16	0.8878	0.8879	0.88786329	0.00004794	0.00000147
17	0.8878	0.8879	0.88785130	0.00002397	0.00001493
18	0.8879	0.8879	0.88785729	0.00001198	0.00000673
19	0.8879	0.8879	0.88786029	0.00000599	0.00000263
20	0.8879	0.8879	0.88786179	0.00000300	0.00000058
21	0.8879	0.8879	0.88786254	0.00000150	0.00000045
22	0.8879	0.8879	0.88786216	0.00000075	0.00000007
23	0.8879	0.8879	0.88786235	0.00000037	0.00000019
24	0.8879	0.8879	0.88786226	0.00000019	0.00000006
25	0.8879	0.8879	0.88786221	0.00000009	0.00000000
26	0.8879	0.8879	0.88786223	0.00000005	0.00000003
27	0.8879	0.8879	0.88786222	0.00000002	0.00000001
28	0.8879	0.8879	0.88786222	0.00000001	0.00000001
29	0.8879	0.8879	0.88786221	0.00000001	0.00000000
30	0.8879	0.8879	0.88786221	0.00000000	0.00000000
31	0.8879	0.8879	0.88786221	0.00000000	0.00000000
32	0.8879	0.8879	0.88786221	0.00000000	0.00000000
33	0.8879	0.8879	0.88786221	0.00000000	0.00000000
34	0.8879	0.8879	0.88786221	0.00000000	0.00000000
35	0.8879	0.8879	0.88786221	0.00000000	0.00000000
36	0.8879	0.8879	0.88786221	0.00000000	0.00000000
37	0.8879	0.8879	0.88786221	0.00000000	0.00000000
38	0.8879	0.8879	0.88786221	0.00000000	0.00000000
39	0.8879	0.8879	0.88786221	0.00000000	0.00000000
40	0.8879	0.8879	0.88786221	0.00000000	0.00000000

2 $f(x) = (x-5)^9$

expanded form:
$x^9 - 45x^8 + 900x^7 - 10500x^6 + 78750x^5 - 393750x^4 + 1312500x^3 - 2812500x^2 + 3515625x - 1953125$

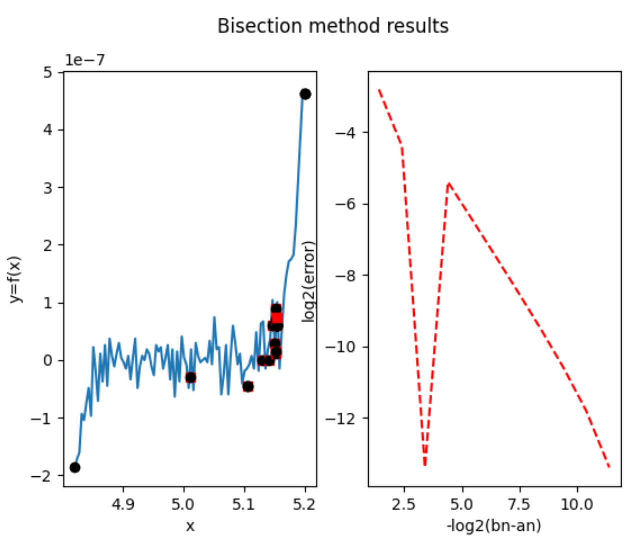
(a) we will apply the bisection code from our previous section to our function in the non-expanded form:
we get a root of: 5.0000732




```
Bisection method with nmax=100 and tol=1.0e-04


|--n--|--an--|--bn--|----xn----|-|bn-an|--|---|f(xn)|---|
|--0--|4.8200|5.2000|5.01000000|0.38000000|0.00000000|
|--1--|4.8200|5.0100|4.91500000|0.19000000|0.00000000|
|--2--|4.9150|5.0100|4.96250000|0.09500000|0.00000000|
|--3--|4.9625|5.0100|4.98625000|0.04750000|0.00000000|
|--4--|4.9863|5.0100|4.99812500|0.02375000|0.00000000|
|--5--|4.9981|5.0100|5.00406250|0.01187500|0.00000000|
|--6--|4.9981|5.0041|5.00109375|0.00593750|0.00000000|
|--7--|4.9981|5.0011|4.99960938|0.00296875|0.00000000|
|--8--|4.9996|5.0011|5.00035156|0.00148437|0.00000000|
|--9--|4.9996|5.0004|4.99998047|0.00074219|0.00000000|
|--10--|5.0000|5.0004|5.00016602|0.00037109|0.00000000|
|--11--|5.0000|5.0002|5.00007324|0.00018555|0.00000000|
```

(b) Repeating this for the expanded form of our function:
we get a root of: 5.1524072




```
Bisection method with nmax=100 and tol=1.0e-04


|--n--|--an--|--bn--|----xn----|-|bn-an|--|---|f(xn)|---|
|--0--|4.8200|5.2000|5.01000000|0.38000000|0.00000003|
|--1--|5.0100|5.2000|5.10500000|0.19000000|0.00000004|
|--2--|5.1050|5.2000|5.15250000|0.09500000|0.00000006|
|--3--|5.1050|5.1525|5.12875000|0.04750000|0.00000000|
|--4--|5.1288|5.1525|5.14062500|0.02375000|0.00000000|
|--5--|5.1406|5.1525|5.14656250|0.01187500|0.00000006|
|--6--|5.1466|5.1525|5.14953125|0.00593750|0.00000003|
|--7--|5.1495|5.1525|5.15101562|0.00296875|0.00000001|
|--8--|5.1510|5.1525|5.15175781|0.00148438|0.00000001|
|--9--|5.1518|5.1525|5.15212891|0.00074219|0.00000009|
|--10--|5.1521|5.1525|5.15231445|0.00037109|0.00000007|
|--11--|5.1523|5.1525|5.15240723|0.00018555|0.00000007|
```

(c) The Difference in the two plots and estimations is due to the cancellation effect of subtracting values of similar magnitude. This can be seen in both plots as both approximations have areas where we subtract two values of similar size; in the beginning of the non-expanded form, we can see a spike in the differences between b_n and a_n. Similar phenomena can be seen in the expanded form as well. The entire function is subject to this cancellation effect in the expanded form. As seen from the plots of our functions, as far as the computer is concerned, the non-expanded and expanded versions of the same function are entirely different.

3 Upper Bound on Iterations of Bisection

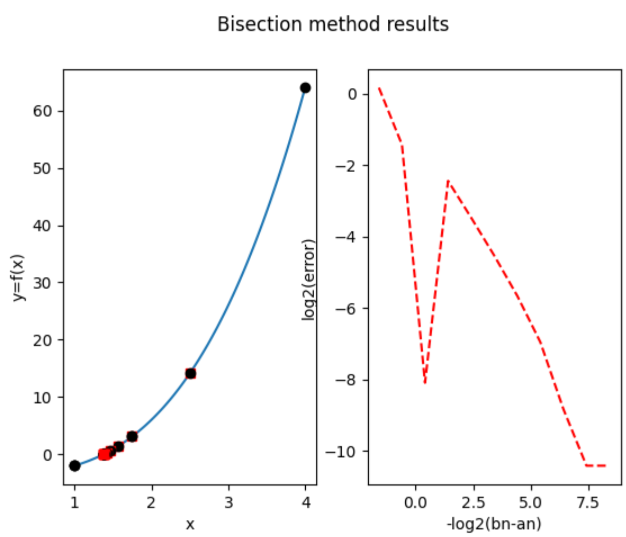
(a) Let our function of interest be: $x^3 + x - 4 = 0$, $x \in [1, 4]$

We seek to find an upper bound on the number of iterations to get within the tolerance level of 10^{-3}:

Theorem states: $n \geq log_2(\frac{b-a}{\epsilon}) - 1$
for our interval we see that: $n \geq log_2(\frac{4-1}{10^{-3}}) - 1 = log_2(3000) - 1 \approx 10.55$
Therefore we gain an upper bound of $n = 11$ iterations

(b) To Test let Input our function, bounds, and error tolerance into our python code and get out the number of iterations:

As seen from our code, we got 11 iterations, which is our upper bound.




```
 Bisection method with nmax=100 and tol=1.0e-03


|--n--|--an--|--bn--|----xn----|-|bn-an|--|---|f(xn)|---|
|--0--|1.0000|4.0000|2.50000000|3.00000000|14.12500000|
|--1--|1.0000|2.5000|1.75000000|1.50000000|3.10937500|
|--2--|1.0000|1.7500|1.37500000|0.75000000|0.02539062|
|--3--|1.3750|1.7500|1.56250000|0.37500000|1.37719727|
|--4--|1.3750|1.5625|1.46875000|0.18750000|0.63717651|
|--5--|1.3750|1.4688|1.42187500|0.09375000|0.29652023|
|--6--|1.3750|1.4219|1.39843750|0.04687500|0.13326025|
|--7--|1.3750|1.3984|1.38671875|0.02343750|0.05336350|
|--8--|1.3750|1.3867|1.38085938|0.01171875|0.01384421|
|--9--|1.3750|1.3809|1.37792969|0.00585938|0.00580869|
|--10--|1.3779|1.3809|1.37939453|0.00292969|0.00400888|
|--11--|1.3779|1.3794|1.37866211|0.00146484|0.00090212|
```

4 Order of convergence and error constant

Given a sequence $[p_n]_{n=0}^{\infty}$ where $p_n \neq p$, $\forall n$ we can say:

$$lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^{\alpha}} = \lambda \tag{5}$$

With an order of convergence α and rate λ

(a) given the sequence: $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}, x_* = 2$

we can find the order and error constant via the limit above, but first, we have to check if the fixed point in question is attainable:

$$f(x) = -16 + 6x + \frac{12}{x} \tag{6}$$

$$f'(x) = 6 - \frac{12}{x^2} \tag{7}$$

$$f'(x_*) = f(2) = 3 > 1 \tag{8}$$

Because the derivative at the fixed point x_* is greater than one, the series will not converge no matter how close we start, assuming we do not start at the root. Therefore, this series diverges at the point x_*, which has no α and λ.

(b) Given the sequence: $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}, x_* = 3^{\frac{1}{3}}$

Repeat the steps from part A to see if this is an attainable fixed point.

$$f(x) = \frac{2}{3}x + \frac{1}{x^2} \tag{9}$$

$$f'(x) = \frac{2}{3} - \frac{2}{x^3} \tag{10}$$

$$f'(x_*) = f'(3^{\frac{1}{3}}) = \frac{2}{3} - \frac{2}{3^{\frac{3}{3}}} = 0 \tag{11}$$

$$f''(x) = \frac{6}{x^4} \tag{12}$$

$$f''(x_*) = f''(3^{\frac{1}{3}}) = \frac{6}{3^{\frac{4}{3}}} \neq 0 \tag{13}$$

Now, because our first derivative is zero but our second is not, we can say that the order of convergence for this series around x_* is two or $\alpha = 2$.

(c) Given the sequence: $x_{n+1} = \frac{12}{1+x_n}, x_* = 3$

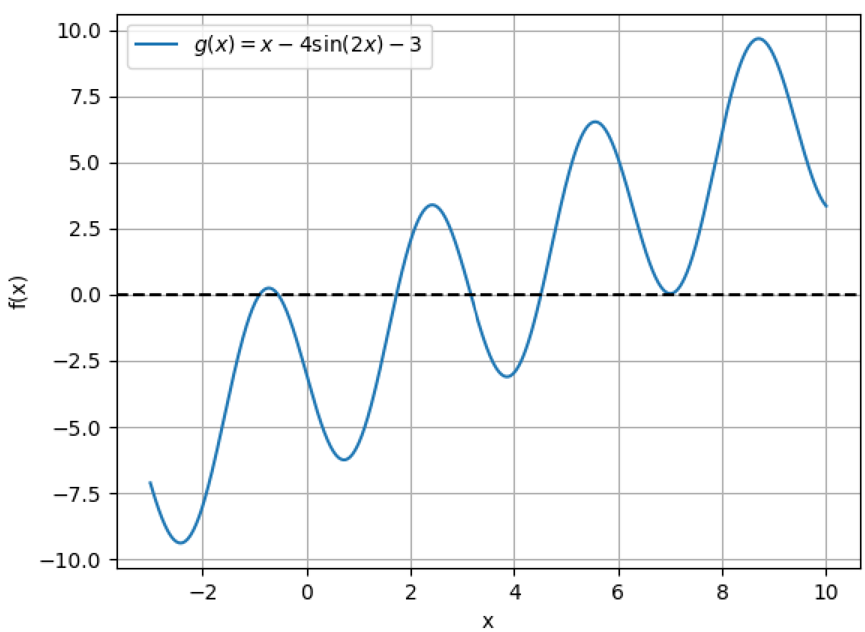
$$f(x) = \frac{12}{1+x}, f(x_*) = 3 \tag{14}$$

$$f'(x) = -\frac{12}{(1+x)^2}, f'(x_*) = -\frac{3}{4} \tag{15}$$

Because the magnitude of our first derivative is less than 1, we can say that this series converges linearly at a rate of $\frac{3}{4}$.

5 $x - 4sin(2x) - 3 = 0$

(a) After using Python to plot our function, we can see there are five total roots for this function (it appears like six, but the value of our most positive "root" is not $g(r_{max}) \neq 0$):



(b) Before We use code to determine our roots Through fixed point iteration, we must first check which of these roots can be found through fixed point iteration. To do this we will check $|g'(x)|$ and its relation to 1:

$$g(x) = -sin(2x) + \frac{5}{4}x - \frac{3}{4} \tag{16}$$

$$g'(x) = -2cos(x) + \frac{5}{4} \tag{17}$$

from the graph above we can see the approximate values of our roots: $r \approx -.9, -.5, 1.7, 3, 4.5$

$$g'(-0.9) \approx 1.7 > 1 \tag{18}$$
$$g'(-0.5) \approx |-.3| < 1 \tag{19}$$
$$g'(1.7) \approx 3 > 1 \tag{20}$$
$$g'(3) \approx 1.1 > 1 \tag{21}$$
$$g'(4.5) \approx 3 > 1 \tag{22}$$

Therefore, because the derivative at all of these points except for $r \approx -0.5$ is greater than one, we can say these points are expansive and not contractive, and in turn, fixed point iteration will be ineffective in finding the roots. For the single root we can find we find the value of $r = -0.54444$