APPM 4600
Homework 1

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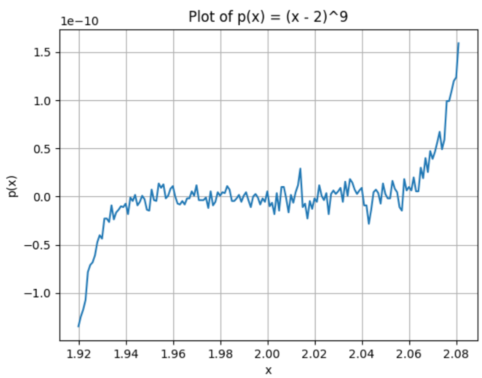
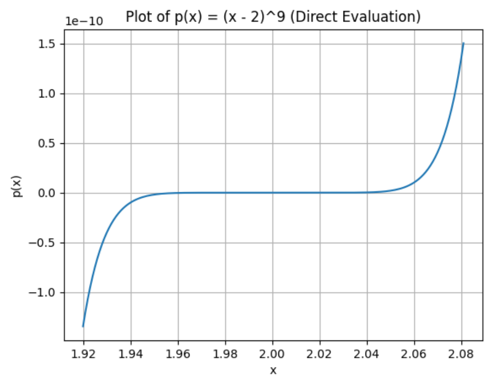
1. Consider the polynomial $p(x)$:
$(x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$

Plotting $p(x)$ as $(x-2)^9$ vs via $p(x)$ in its expanded form:





Ans: The difference in the two plots is substantial; plotting $(x-2)^9$ we gain our desired plot compared to plotting the expanded form in which we see the line is far more jagged and jumpy, which does not line up with what we expect. while plotting the function via its coefficients we gain an extreme amount of error due to the number of times we subtract similar values relative to their magnitude.

2. Evaluate The following Equations while avoiding cancellation.
   (a) $\sqrt{x+1} - 1$; $x \approx 0$

   To avoid cancellation we must "get rid" of the subtraction taking place. the easiest way to do this is to bring back a helpful trick from calc 1 in multiplying by a convenient form of 1. This will take the form of: $\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$

$$(\sqrt{x+1} - 1)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) = \frac{x+1-1}{\sqrt{x+1}+1} = \frac{x}{\sqrt{x+1}+1}$$

we could now plug in our small value of x and evaluate to gain a new value which would have a much higher degree of accuracy.

(b) $sin(x) - sin(y)$; $x \approx y$

first assume we can calculate $x - y = h$ to full precision which because we are adding and not subtracting is a reasonable assumption. as well we will take advantage of a Trig Identity for $sin(x) - sin(y) = 2cos(\frac{x+y}{2})sin(\frac{x-y}{2})$ we can now rewrite the form of the expression:

$$sin(x) - sin(y) = 2cos(\tfrac{x+y}{2})sin(\tfrac{x-y}{2})$$
$$= 2cos(\tfrac{x+(h-x)}{2})sin(\tfrac{x-(h-x)}{2}) = 2cos(\tfrac{h}{2})sin(\tfrac{h-2x}{2})$$

This Form of the answer reduces the cancellation effect but is not perfect. Another way to do this is via a Taylor series:
$sin(x) - sin(y)$

$$sin(x) = x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!}$$
$$sin(y) = y - \tfrac{y^3}{3!} + \tfrac{y^5}{5!} - \tfrac{y^7}{7!}$$
$$sin(x) - sin(y) = (x - y) - (\tfrac{x^3}{3!} + \tfrac{y^3}{3!}) + (\tfrac{x^5}{5!} + \tfrac{y^5}{5!}) - (\tfrac{x^7}{7!} + \tfrac{y^7}{7!})$$

Which as an approach will also suffer from some loss of precession but in comparison to the original problem is much less.

(c) $\frac{1-cos(x)}{sin(x)}$; $x \approx 0$

To solve I will use the trig Identities
$1 - cos(x) = 2sin^2(\frac{x}{2})$ and $sin(x) = 2sin(\frac{x}{2})cos(\frac{x}{2})$

$$\frac{1-cos(x)}{sin(x)} = \frac{2sin^2(\frac{x}{2})}{2sin(\frac{x}{2})cos(\frac{x}{2})} = tan(\tfrac{x}{2})$$

Ans:

(a) Multiply by complex conjugate

(b) $sin(x) - sin(y) = 2cos(\frac{x+y}{2})sin(\frac{x-y}{2})$
$sin(x) - sin(y)$ via Taylor series

(c) $cos(x) = 2sin^2(\frac{x}{2})$
$sin(x) = 2sin(\frac{x}{2})cos(\frac{x}{2})$

3. Consider the polynomial: $F(x) = (1 + x + x^3)cos(x)$
Find $P_2(x)$ the second order taylor series polynomial of $F(x)$ around $x_0 = 0$

$$F'(x) = cos(x)(1 + 3x^2) - (1 + x + x^3)sin(x)$$
$$F''(x) = 6xcos(x) - Cos(x)(1 + x + x^3) - 2(1 + 3x^2)sin(x)$$

$$P_2(x) = 1 + x - \tfrac{1}{2}(x)^2$$

   (a) Evaluating $P_2(0.5)$ & $F(x)$ then comparing the upper bound on the error using the error formula to the actual error we find:

$$F(0.5) = 1.42607, P_2(0.5) = 1.25,$$
$$|F(0.5) - P_2(0.5)| = 0.176072$$

   (b) Find an upper bound on the error for $|F(x) - P - 2(x)|$ as a function of x

$$E(x) = |F(x) - P_2(x)| = (1 + x + x^3)cos(x) - (1 + x - \tfrac{1}{2}x^2)$$

   (c) approximate the integral of $F(x)$ from $[0, 1]$ using $P_2(x)$

$$\textstyle\int_0^1 1 + x - \frac{1}{2}x^2 = \frac{7}{6}$$

   (d) Estimate the Error in the Integral

$$|\textstyle\int_0^1 (1 + x + x^3)cos(x) - \int_0^1 1 + x - \frac{1}{2}x^2| = |5 - 2cos(1) - 3sin(1) - \frac{7}{6}| = 0.22831576717336444$$

Ans:

(a) $|F(0.5) - P_2(0.5)| = 0.176072$

(b) $(1 + x + x^3)cos(x) - (1 + x - \frac{1}{2}x^2)$

(c) $\frac{7}{6}$

(d) 0.22831576717336444

4. Consider the quadratic equation $ax^2 + bx + c = 0$ with $a = 1, b = -56, c = 1$

   (a) assuming we can compute roots up to 3 decimals:

$$r_{1,2} = \frac{56 \pm \sqrt{(-56)^2 - 4(1)(1)}}{2(1)} = 55.98, 0.018$$

If we now reorganize our formula to remove the subtraction taking place, we get:

$$r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} = 55.982137159266443$$
$$r_2 = \frac{2c}{(-b+\sqrt{b^2-4ac})} = .017862840733554864$$

This is done by multiplying by the conjugate.

(b) now to find the roots via $(x-r_1)(x-r_2)=0$:

$$(x-r_1)(x-r_2) = x^2-(r_1+r_2)x+r_1r_2 = 0$$
$$-(r_1+r_2) = \frac{b}{a} = -56$$
$$r_1r_2 = \frac{c}{a} = 1$$

now to reorganize to find the "bad" root r_2:

$$r_2 = \frac{c}{ar_1} = \frac{1}{55.98} = .01786$$

Ans:

(a) $r_1 = 55.982137159266443$
$r_2 = .017862840733554864$

(b) $r_2 = \frac{c}{ar_1} = .01786$

5. **Cancellation of Terms** Consider $y = x_1 - x_2$

(a) Upper bounds on the absolute and relative error of y

$$y = x_1 - x_2$$
$$\tilde{y} = \tilde{x_1} - \tilde{x_2} = (x_1+\Delta x_1) - (x_2+\Delta x_2) = y - (\Delta x_1 + \Delta x_2)$$

Absolute error:

$$|\tilde{y} - y| = |y + (\Delta x_1 + \Delta x_2) - y| = |\Delta x_1 + \Delta x_2|$$

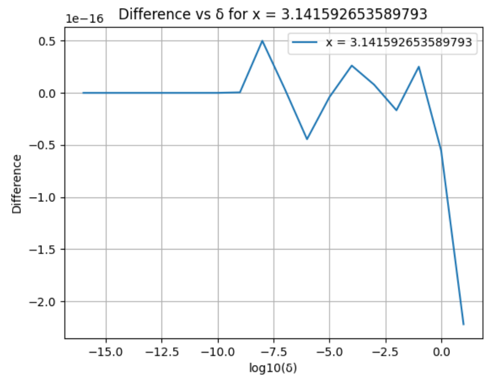
Relative error:

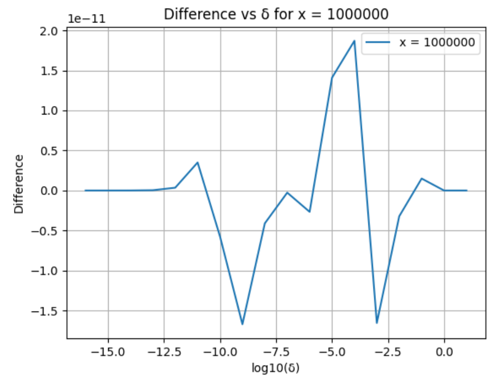
$$\frac{\tilde{y}-y}{y} = \frac{|\Delta x_1 + \Delta x_2|}{x_1 - x_2}$$

(b) $cos(x+\delta) - cos(x)$

(i) Manipulate to remove the subtraction

$$cos(x+\delta) - cos(x) = -2sin(\frac{x+\delta+x}{2})sin(\frac{x+\delta-x}{2}) = -2sin(x+\frac{\delta}{2})sin(\frac{\delta}{2})$$



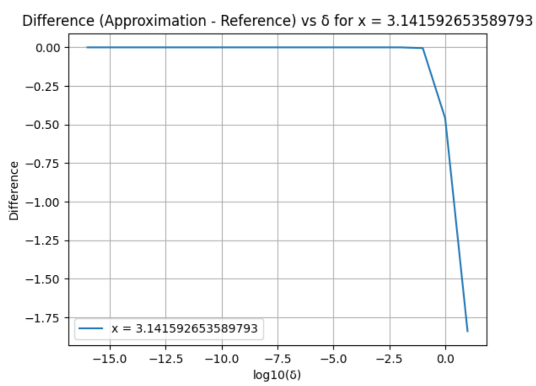


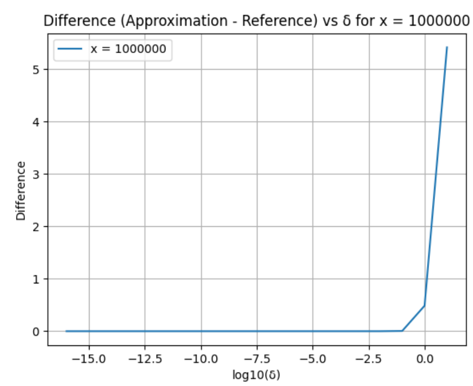
(ii)

(c) Taylor expanding $cos(x+\delta) - cos(x)$

$$cos(x+\delta) - cos(x) = -\delta sin(x) - \frac{\delta^2}{2!}cos(x) + \frac{\delta^3}{3!}sin(x)... \approx -\delta sin(x)$$

this is because of the scale of δ relative to x. The higher order terms of δ will drop off, which will be shown to be more accurate than $cos(x+\delta) - cos(x)$. I choose this algorithm because of the scale of our values of δ. The largest value of δ we will work with is 1 which compared to our values of x is still fairly small. As seen here Both are much more accurate for lower values of δ and only become





inaccurate as δ approaches 0