Twitter thread by @judijasa
Title: The logarithmic chain complex
Slide Nr: 001

The conventional sum, $+$, and multiplication, $\times$, are abelian products with distributivity. For example, if, $\alpha, \beta, \gamma \in (\mathbb{R}, +, \times)$, then,

$$\alpha \times (\beta + \gamma) = \alpha \times \beta + \alpha \times \gamma. \tag{1}$$

Formally, these two products form a *ring*. As aspiring goldsmiths we shall try to make chains out of these rings. Let, $\overset{0}{\times}$, and, $\overset{1}{\times}$, denote two products that behave like the conventional sum and multiplication, respectively. The set of products[1],

$$\alpha \overset{n+1}{\times} \beta \doteq \log^{-1}\left(\log(\alpha) \overset{n}{\times} \log(\beta)\right), \tag{2}$$

for, $n \in \mathbb{Z}$, forms a sequence of abelian groups and homomorphisms. Note that, $\overset{n+1}{\times}$, is distributive in, $\overset{n}{\times}$. Proof by induction: Assume,

$$\alpha \overset{n}{\times} (\beta \overset{n-1}{\times} \gamma) = (\alpha \overset{n}{\times} \beta) \overset{n-1}{\times} (\alpha \overset{n}{\times} \gamma), \tag{3}$$

then,

$$\begin{aligned}
\alpha \overset{n+1}{\times} (\beta \overset{n}{\times} \gamma) &= \log^{-1}\{\log(\alpha) \overset{n}{\times} \log(\beta \overset{n}{\times} \gamma)\} \\
&= \log^{-1}\{\log(\alpha) \overset{n}{\times} [\log(\beta) \overset{n-1}{\times} \log(\gamma)]\} \\
&\overset{(3)}{=} \log^{-1}\{[\log(\alpha) \overset{n}{\times} \log(\beta)] \overset{n-1}{\times} [\log(\alpha) \overset{n}{\times} \log(\gamma)]\} \\
&= \log^{-1}\{\log(\alpha \overset{n+1}{\times} \beta) \overset{n-1}{\times} \log(\alpha \overset{n+1}{\times} \gamma)\} \\
&= (\alpha \overset{n+1}{\times} \beta) \overset{n}{\times} (\alpha \overset{n+1}{\times} \gamma).
\end{aligned} \tag{4}$$

This covers the proof for, $n \in \mathbb{Z}^+$. Left to the reader is the proof for, $n \in \mathbb{Z}^-$.

In future posts, I will share more trivia about this object and discuss possible applications.

[1]Details such as the base of the log function and its algebraic role are discussed later on.