

The Log chain: A sequence of abelian groups based on the logarithm

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Slide Nr. 1/9

The conventional sum, $+$, and multiplication, \times , are abelian products with distributivity. For example, if, $\alpha, \beta, \gamma \in (\mathbb{R}, +, \times)$, then,

$$\alpha \times (\beta + \gamma) = \alpha \times \beta + \alpha \times \gamma. \quad (1)$$

Formally, these two products form a *ring*. As aspiring goldsmiths we shall try to make chains out of these rings. Let, $\overset{0}{\times}$, and, $\overset{1}{\times}$, denote two products that behave like the conventional sum and multiplication, respectively. The products¹, $n \in \mathbb{Z}$:

$$\alpha \overset{n+1}{\times} \beta \doteq \log^{-1} \left(\log(\alpha) \overset{n}{\times} \log(\beta) \right), \quad (2)$$

form a sequence of abelian groups. Note that, $\overset{n+1}{\times}$, is distributive in, $\overset{n}{\times}$. Proof by induction: Assume,

$$\alpha \overset{n}{\times} (\beta \overset{n-1}{\times} \gamma) = (\alpha \overset{n}{\times} \beta) \overset{n-1}{\times} (\alpha \overset{n}{\times} \gamma), \quad (3)$$

then,

$$\begin{aligned} \alpha \overset{n+1}{\times} (\beta \overset{n}{\times} \gamma) &= \log^{-1} \{ \log(\alpha) \overset{n}{\times} \log(\beta \overset{n}{\times} \gamma) \} \\ &= \log^{-1} \{ \log(\alpha) \overset{n}{\times} [\log(\beta) \overset{n-1}{\times} \log(\gamma)] \} \\ &\stackrel{(?)}{=} \log^{-1} \{ [\log(\alpha) \overset{n}{\times} \log(\beta)] \overset{n-1}{\times} [\log(\alpha) \overset{n}{\times} \log(\gamma)] \} \\ &= \log^{-1} \{ \log(\alpha \overset{n+1}{\times} \beta) \overset{n-1}{\times} \log(\alpha \overset{n+1}{\times} \gamma) \} \\ &= (\alpha \overset{n+1}{\times} \beta) \overset{n}{\times} (\alpha \overset{n+1}{\times} \gamma). \end{aligned} \quad (4)$$

This covers the proof for, $n \in \mathbb{Z}^+$. Left to the reader is the proof for, $n \in \mathbb{Z}^-$.

In future posts, I will share more trivia about this structure and discuss possible applications.

¹Details such as the base of the log function and its algebraic role are discussed later on.

In the latest slide we defined a sequence of abelian products, based on the log function. A complete definition requires a generalisation of the log function to operate on negative reals. Before doing this, let us appreciate the behaviour of these products in, \mathbb{R}^+ . The definition in Eq. (??) makes it clear these products are commutative. As observed in Coya's law (@johncarlosbaez), this is not always obvious, e.g., the equivalent definition of, $\overset{2}{\times}$, as,

$$\alpha \overset{2}{\times} \beta = \alpha^{\log \beta}. \quad (5)$$

The product, $\overset{2}{\times}$, is interesting, together with, $\overset{-1}{\times}$, are the closest relatives to the conventional products, $\overset{0}{\times}$, and, $\overset{1}{\times}$, for which we hold more intuition. The product, $\overset{2}{\times}$, is distributive in the conventional multiplication ($\overset{1}{\times}$). This makes, $\overset{2}{\times}$, the abelian version of the power product, which is also distributive in the multiplication, i.e., $(\alpha \times \beta)^\gamma = \alpha^\gamma \times \beta^\gamma$.

It is convenient to move to a more abstract characterisation. Each product, $\overset{n}{\times}$, forms a group denoted, \mathfrak{g}_n , with 'finite' elements, $\overset{n}{\alpha}, \overset{n}{\beta}, \dots$. Arbitrary elements in the sequence of groups are denoted without the upper index, i.e., α . Some definitions and conventions:

$$\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\beta} = \overset{n}{\beta} \overset{n}{\times} \overset{n}{\alpha}, \quad (6a)$$

$$\overset{n}{\alpha} \overset{n}{\times} \overset{n}{I}_0 = \overset{n}{I}_0 \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{\alpha}, \quad (6b)$$

$$\overset{n}{\alpha} \overset{n}{\times} \overset{n}{I}_1 = \overset{n}{I}_1 \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{I}_1. \quad (6c)$$

$$(\overset{n}{\mathcal{J}}(\overset{n}{\mathcal{J}}\overset{n}{\alpha})) = \overset{n}{\alpha} \quad (7a)$$

$$\overset{n}{\alpha} \overset{n}{\times} (\overset{n}{\mathcal{J}}\overset{n}{\alpha}) = (\overset{n}{\mathcal{J}}\overset{n}{\alpha}) \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{I}_0, \quad (7b)$$

$$\overset{n}{\mathcal{J}}\overset{n}{I}_0 = \overset{n}{I}_0 \quad (7c)$$

$$\overset{n}{\alpha} \overset{n}{\times} (\overset{n}{\mathcal{J}}\overset{n}{I}_1) = (\overset{n}{\mathcal{J}}\overset{n}{I}_1) \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{\mathcal{J}}\overset{n}{I}_1. \quad (7d)$$

Abusing of notation on behalf of intuition we keep, $\log(\cdot)$, to denote the generator of homomorphisms. We use, $\log^{+1}(\cdot) = \log(\cdot)$, and, $\log^{-1}(\log(\overset{n}{\alpha})) = \log(\log^{-1}(\overset{n}{\alpha})) = \overset{n}{\alpha}$. The logarithm (and its inverse) connects nearby groups along the sequence, i.e.,

$$\overset{n-1}{\alpha} = \log(\overset{n}{\alpha}) \in \overset{n}{\mathcal{J}}\overset{n}{I}_1 \quad (8a)$$

$$\overset{n+1}{\alpha} = \log^{-1}(\overset{n}{\alpha}) \in \overset{n}{I}_1. \quad (8b)$$

I imagine the elements, $\overset{n}{\alpha}, \overset{n+1}{\alpha}, \dots$, as different scales of a physical degree of freedom. Formally, $\overset{n}{\mathcal{J}}\overset{n}{I}_1$, and, $\overset{n}{I}_1$, are elements of an *ideal*, but they feel as the horizon of too large/small elements from the perspective of the n -scale. Next we identify these elements in, (\mathbb{R}^+, \times) .

As a complement to Eqs. (??) are the relations,

$$\log(\overset{n}{\mathcal{F}} \overset{n}{I}_1) = \overset{n-1}{\mathcal{F}} \overset{n-1}{I}_1, \quad \log(\overset{n}{I}_1) = \overset{n-1}{I}_1, \quad \text{and}, \quad \log(\overset{n}{I}_0) = \overset{n-1}{I}_0. \quad (9)$$

Example. Taking, (\mathbb{R}^+, \times) , as a representation of the n -group (\mathfrak{g}_n) one obtains that,

$$0^+ \equiv \overset{n}{\mathcal{F}} \overset{n}{I}_1, \quad +\infty \equiv \overset{n}{I}_1, \quad 1 \equiv \overset{n}{I}_0, \quad \text{and}, \quad 1/\overset{n}{\alpha} \equiv \overset{n}{\mathcal{F}} \overset{n}{\alpha}. \quad (10)$$

Denoting, $\log[y](x) = \log_y(x)$, and letting, $\overset{n}{\alpha}, \overset{n}{\beta} \in \mathbb{R}^+$, note that,

$$\log[\overset{n}{\beta}](\overset{n}{\alpha}) \notin \overset{n}{\mathcal{F}} \overset{n}{I}_1, \quad (11)$$

which contradicts, Eq. (??). An instance of this observation is, $\log_{2.56}(11.4) \notin 0^+$. Is there a base that can fulfill Eqs. (??)? Understood as a limit, note that,

$$\log[\overset{n}{I}_1](\overset{n}{\alpha}) \in \overset{n}{\mathcal{F}} \overset{n}{I}_1. \quad (12)$$

An instance of this observation is, $(+\infty)^x = 23.71 \Rightarrow x \in 0^+$. This suggests the local identification,² $\log(.) \equiv \log[\overset{n}{I}_1](.)$. Comparing the Eqs. (??) with the limits,

$$\log[+\infty](0^+) = -1^<, \quad \log+\infty = 1^<, \quad \text{and}, \quad \log[+\infty](1) = 0, \quad (13)$$

implies,

$$-1^< \equiv \overset{n-1}{\mathcal{F}} \overset{n-1}{I}_1, \quad 1^< \equiv \overset{n-1}{I}_1, \quad \text{and}, \quad 0 \equiv \overset{n-1}{I}_0. \quad (14)$$

Whenever the group, \mathfrak{g}_n , is associated to, (\mathbb{R}^+, \times) , the open interval, $(-1, +1)$, provides the elements for the abelian group, \mathfrak{g}_{n-1} , associated to the product, $\overset{n-1}{\times}$, with group identity, $\overset{n-1}{I}_0 \equiv 0$, and inverse, $\overset{n-1}{\mathcal{F}}(.) \equiv (-1) \times (.)$. Recalling that, $\overset{n}{\times}$, is distributive in, $\overset{n-1}{\times}$, completes the similarities of, $\overset{n-1}{\times}$, with the conventional sum, $+$.

²An alternative is, $\log(.) \equiv \log[\overset{n}{\mathcal{F}} \overset{n}{I}_1](.)$.

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The definition of the log action can be extended beyond the transformation of elements defined in Eqs. (??). From Eq. (??), it follows that,

$$\log(\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\beta}) = \log(\overset{n}{\alpha}) \overset{n-1}{\times} \log(\overset{n}{\beta}). \quad (15)$$

Define the log action on products as,

$$\log(\overset{n}{\times}) = \overset{n-1}{\times}. \quad (16)$$

The Eq. (??) now follows from distributivity of the log action in elements and products, i.e.,

$$\log(\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\beta}) = \log(\overset{n}{\alpha}) \log(\overset{n}{\times}) \log(\overset{n}{\beta}). \quad (17)$$

Similarly, one can define the log action on the inverse as,

$$\log(\overset{n}{\mathcal{I}}) = \overset{n-1}{\mathcal{I}}. \quad (18)$$

The familiar relation, $\overset{n}{\alpha} \in (\mathbb{R}^+, \times)$:

$$\log(1/\overset{n}{\alpha}) = -\log(\overset{n}{\alpha}), \quad (19)$$

motivates the general form, $n \in \mathbb{Z}$:

$$\log(\overset{n}{\mathcal{I}} \overset{n}{\alpha}) = \overset{n-1}{\mathcal{I}} \log(\overset{n}{\alpha}). \quad (20)$$

The latter equation also follows from distributivity of the log action on elements and inverse operations, i.e.,

$$\log(\overset{n}{\mathcal{I}} \overset{n}{\alpha}) = \log(\overset{n}{\mathcal{I}}) \log(\overset{n}{\alpha}). \quad (21)$$

Recall from slide Nr. 003, that if, $\overset{n}{\alpha} \in (\mathbb{R}^+, \times)$, then, $\overset{n-1}{\alpha} = \log(\overset{n}{\alpha})$, is an element of a sum-like group with inverse, $\overset{n-1}{\mathcal{I}} (\cdot) \equiv (-1) \times (\cdot)$, then,

$$\log(\overset{n-1}{\mathcal{I}} \overset{n-1}{\alpha}) \equiv \log(-\overset{n-1}{\alpha}). \quad (22)$$

Together with Eq. (??), one obtains,

$$\log(-\overset{n-1}{\alpha}) = \overset{n-2}{\mathcal{I}} \log(\overset{n-1}{\alpha}). \quad (23)$$

This describes the log action on the negative numbers as a particular case. In the next slide we identify the inverse, $\overset{n-2}{\mathcal{I}}$.

The sequence of products defined in Eq. (??) requires a consistent definition for the log action on negative numbers. Such definition is given by Eq. (??) if, (\mathbb{R}^+, \times) , represents the n -group³. To fully identify Eq. (??) in the present representation reduces to identify the product inverse, $\overset{n-2}{\mathcal{F}}$. Comparing the Eqs. (??) and (??) with the limits,

$$\log[+\infty](-1^<) = -\infty, \quad \log[+\infty](+1^<) = 0^-, \quad \text{and,} \quad \log[+\infty](0) = -1, \quad (24)$$

implies,

$$-\infty \equiv \overset{n-2}{\mathcal{F}} \overset{n-2}{I_1}, \quad 0^- \equiv \overset{n-2}{I_1}, \quad \text{and,} \quad -1 \equiv \overset{n-2}{I_0}. \quad (25)$$

The set, \mathbb{R}^- , provides the elements for the abelian group, \mathfrak{g}_{n-2} , with associated product, $\overset{n-2}{\times}$, group identity, -1 , and inverse, $\overset{n-2}{\mathcal{F}} \equiv 1/(\cdot)$. Comparing with Eqs. (??) one obtains that, $\overset{n-2}{\mathcal{F}} \equiv \overset{n}{\mathcal{F}}$. The latter identification is not a general property of the sequence of products, it results from the limited description of the representation chosen. In general, the native representation of the inverse is given by, $n \in \mathbb{Z}$:⁴

$$\overset{n}{\mathcal{F}}(\cdot) \equiv (\overset{n}{\mathcal{F}} \overset{n+1}{I_0}) \overset{n+1}{\times} (\cdot). \quad (26)$$

Despite, \mathbb{R}^- , is not closed under the conventional multiplication, \times , it is closed under the product, $\overset{n-2}{\times}$, as defined in Eq. (??). Proof: Using Eq. (??) one proceeds by induction,

$$\begin{aligned} \overset{n-2}{\alpha} \overset{n-2}{\times} \overset{n-2}{\beta} &= \log \left(\log^{-1}(\overset{n-2}{\alpha}) \overset{n-1}{\times} \log^{-1}(\overset{n-2}{\beta}) \right) \\ &= \log(\overset{n-1}{\alpha} \overset{n-1}{\times} \overset{n-1}{\beta}) = \log(\overset{n-1}{\gamma}) = \overset{n-2}{\gamma}. \end{aligned} \quad (27)$$

As explained in more detail in slide Nr. 8, the elements in, \mathfrak{g}_n , and, \mathfrak{g}_{n+4m} , for, $m, n \in \mathbb{Z}$, are related by an automorphism. Under this representation, it is therefore sufficient to identify four consecutive groups to describe the sequence of groups. After setting, $\mathfrak{g}_n \equiv (\mathbb{R}^+, \times)$, we already identified the consecutive groups, \mathfrak{g}_{n-1} , and, \mathfrak{g}_{n-2} . In slide Nr. 8 we complete the analysis by identifying, \mathfrak{g}_{n-3} , but before we shall refine some concepts.

³The choice of representation for the n -group constrains the representation of nearby groups.

⁴To test involution of inverse in the native rep, note that, $\alpha \overset{m}{\times} (\overset{n}{\mathcal{F}} \beta) = (\overset{n}{\mathcal{F}} \alpha) \overset{m}{\times} \beta = \overset{n}{\mathcal{F}} (\alpha \overset{n}{\times} \beta)$, if, $m = n + 1$.

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Define the n -group's 'absolute value' as,

$$|\cdot|_n = |\cdot|_{n-1} \times^{\frac{n}{n-1}} \mathcal{F}(\cdot), \quad (28)$$

where, $|\cdot|_0$, is the conventional absolute value. The instants,

$$|-\frac{0}{\alpha}|_0 = |\frac{0}{\alpha}|_0, \quad |1/\frac{1}{\alpha}|_1 = |\frac{1}{\alpha}|_1, \quad (29)$$

and,

$$|1/\frac{0}{\alpha}|_0 = 1/|\frac{0}{\alpha}|_0, \quad |-\frac{1}{\alpha}|_1 = -|\frac{1}{\alpha}|_1, \quad (30)$$

motivate the general assumptions,⁵

$$|\frac{n\pm 1}{\mathcal{F}}\tilde{\alpha}|_n = \frac{n\pm 1}{\mathcal{F}}|\tilde{\alpha}|_n, \quad \text{and}, \quad |\frac{n}{\mathcal{F}}\tilde{\alpha}|_n = |\tilde{\alpha}|_n. \quad (31)$$

An important property is that, $|\mathfrak{g}_n|_n$, is isomorphic to the open interval, $(\tilde{\mathbf{I}}_0, \tilde{\mathbf{I}}_1)$, while, $|\mathfrak{g}_n|_{n+1}$, is isomorphic to, \mathbb{Z}_2 . The proof of the latter is the following: Any element in, \mathfrak{g}_n , can be parametrised as, $\tilde{\alpha} = (\mathcal{F})^\nu |\tilde{\alpha}|_n$, where, $(\mathcal{F})^1 = \mathcal{F}$, and, $(\mathcal{F})^2(\cdot) = \tilde{\mathbf{I}}_0 \times^{\frac{n}{n-1}} (\cdot)$. Then,

$$\begin{aligned} |\frac{n}{\tilde{\alpha}}|_{n+1} &= |\frac{n}{\tilde{\alpha}}|_n \times^{\frac{n+1}{n}} \mathcal{F}^{\frac{n+1}{n}} \tilde{\alpha} \\ &= |\frac{n}{\tilde{\alpha}}|_n \times^{\frac{n+1}{n}} \mathcal{F}^{\frac{n+1}{n}} (\mathcal{F})^\nu |\tilde{\alpha}|_n \\ &= (\mathcal{F})^\nu (|\frac{n}{\tilde{\alpha}}|_n \times^{\frac{n+1}{n}} \mathcal{F}^{\frac{n+1}{n}} |\tilde{\alpha}|_n) = (\mathcal{F})^\nu \tilde{\mathbf{I}}_0. \end{aligned} \quad (32)$$

In general, $k \in \mathbb{Z}^+$:

$$|\frac{n}{\tilde{\alpha}}|_{n-k} = |\frac{n}{\tilde{\alpha}}|_n \times^{\frac{n}{n-1}} \tilde{\alpha} \times^{\frac{n-1}{n-2}} \tilde{\alpha} \times^{\frac{n-2}{n-3}} \dots \times^{\frac{n-k+1}{n-k}} \tilde{\alpha}, \quad (33)$$

and,

$$|\frac{n}{\tilde{\alpha}}|_{n+k} = |\frac{n}{\tilde{\alpha}}|_n \times^{\frac{n+1}{n}} \mathcal{F}^{\frac{n+1}{n}} \tilde{\alpha} \times^{\frac{n+2}{n+3}} \mathcal{F}^{\frac{n+2}{n+3}} \tilde{\alpha} \times^{\frac{n+3}{n+4}} \dots \times^{\frac{n+k}{n+k+1}} \mathcal{F}^{\frac{n+k}{n+k+1}} \tilde{\alpha}. \quad (34)$$

⁵Despite, $(\frac{n}{\mathcal{F}}\tilde{\alpha}) \times^{\frac{n}{n-1}} \tilde{\beta} \neq \tilde{\alpha} \times^{\frac{n}{n-1}} (\mathcal{F}^{\frac{n}{n-1}}\tilde{\beta})$, it holds that, $(\mathcal{F}|\frac{n}{\tilde{\alpha}}|_{n\pm 1}) \times^{\frac{n}{n-1}} \tilde{\alpha} = |\frac{n}{\tilde{\alpha}}|_{n\pm 1} \times^{\frac{n}{n-1}} (\mathcal{F}^{\frac{n}{n-1}}\tilde{\alpha})$. Some of these assumptions can be derived from a minimal set, so there is work to do for a more concise presentation.

Define the proper metric in, \mathfrak{g}_n , as,

$$g_n(\alpha, \beta) = |\alpha \times \overset{n}{\mathcal{F}} \beta|_n. \quad (35)$$

The metric, g_{n+1} , distinguishes attributes between elements, $\overset{n}{\alpha}, \overset{n}{\beta} \in \mathfrak{g}_n$, that go ‘under the radar’ in the metric, g_n . These attributes are generated by the group inverse, $\overset{n}{\mathcal{F}}$.

Recall the identifications, $0^+ \equiv \overset{n}{\mathcal{F}} \overset{n}{I}_1$, and, $0 \equiv \overset{n-1}{I}_0$. The instant, $0^+ \in 0$, motivates the general assumption, $n \in \mathbb{Z}$:

$$\overset{n}{\mathcal{F}} \overset{n}{I}_1 \in \overset{n-1}{I}_0. \quad (36)$$

Moreover, since, $\overset{n-1}{\mathcal{F}} \overset{n-1}{I}_0 = \overset{n-1}{I}_0$, then also, $\overset{n-1}{\mathcal{F}} \overset{n}{\mathcal{F}} \overset{n}{I}_1 \in \overset{n-1}{I}_0$. Recalling that, $\overset{n-1}{\mathcal{F}} \equiv (-1) \times (\cdot)$, the latter case corresponds to the instant, $0^- \in 0$.

Two elements, $\overset{n}{\alpha}$, and, $\overset{n}{\beta}$, are said to be *infinitesimally close* if,

$$g_n(\overset{n}{\alpha}, \overset{n}{\beta}) = \overset{n+1}{\mathcal{F}} \overset{n+1}{I}_1, \quad (37)$$

and said to have the *same phase* if,

$$g_{n+1}(\overset{n}{\alpha}, \overset{n}{\beta}) = \overset{n+2}{\mathcal{F}} \overset{n+2}{I}_1. \quad (38)$$

If two elements of the same group are infinitesimally close and have the same phase they are said to be equal elements in the group, denoted, $\overset{n}{\alpha} \overset{n}{=} \overset{n}{\beta}$. Two elements, α, β , are said to be equal if, $g_n(\alpha, \beta) = \overset{n+1}{\mathcal{F}} \overset{n+1}{I}_1$, for all, $n \in \mathbb{Z}$.

Away of the identity, being infinitesimally close is sufficient criteria for equality in the group, i.e., if, $g_n(\overset{n}{\alpha}, \overset{n}{\beta}) = \overset{n+1}{\mathcal{F}} \overset{n+1}{I}_1$, and, $\overset{n}{\alpha} \notin \overset{n}{I}_0$, then, $\overset{n}{\alpha} \overset{n}{=} \overset{n}{\beta}$. Proof: $g_n(\overset{n}{\alpha}, \overset{n}{\beta}) = \overset{n+1}{\mathcal{F}} \overset{n+1}{I}_1$, implies, $\overset{n}{\alpha} \times \overset{n}{\mathcal{F}} \overset{n}{\beta} \in \overset{n}{I}_0$, therefore, $\overset{n}{\alpha} = \overset{n}{\beta} \times \overset{n}{\gamma}$, $\overset{n}{\gamma} \in \overset{n}{I}_0$. Note that, $\overset{n}{\alpha} \notin \overset{n}{I}_0$, if and only if, $\overset{n}{\beta} \notin \overset{n}{I}_0$, otherwise, $\overset{n}{\alpha} \times \overset{n}{\mathcal{F}} \overset{n}{\beta} \notin \overset{n}{I}_0$, which is a contradiction. Since, $\overset{n}{\beta} \notin \overset{n}{I}_0$, then, $\overset{n}{\beta} \times \overset{n}{\gamma} \overset{n}{=} \overset{n}{\beta}$, therefore, $\overset{n}{\alpha} \overset{n}{=} \overset{n}{\beta}$. From now on, the equality, $=$, in Eqs. (??) and (??) is replaced by, $\overset{n}{=}$.

Any element in a group can be reinterpreted as a set of infinitesimally close elements in the group metric. Denote, $\overset{n}{\beta} \in \overset{n}{\alpha}$, if the element, $\overset{n}{\beta}$, is infinitesimally close to the element, $\overset{n}{\alpha}$. For example, $0^\pm \in 0$, reads as: the elements, 0^- , and, 0^+ , are infinitesimally close to zero, in their proper group metric.

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In analogy to the limits, $\pm\infty = 1/0^\pm$, define,

$$\infty = 1/0. \quad (39)$$

We are no longer in the real line, the element, ∞ , is the point at infinity in the *real projective line*.

Comparing the Eqs. (??) and (??) with the limits,

$$\log[+\infty](-\infty) = +1^>, \quad \log[+\infty](0^-) = -1^>, \quad \text{and}, \quad \log[+\infty](-1) = \infty, \quad (40)$$

implies,

$$+1^> \equiv \overset{n-3}{\mathcal{F}} \overset{n-3}{I_1}, \quad -1^> \equiv \overset{n-3}{I_1}, \quad \text{and}, \quad \infty \equiv \overset{n-3}{I_0}. \quad (41)$$

The set, $\mathbb{R} - (-1, +1)$, provides the elements for the abelian group, \mathfrak{g}_{n-3} , with group identity, ∞ , and inverse, $\overset{n-3}{\mathcal{F}}(.) \equiv (-1) \times (.)$. As a result of the representation note that, $\overset{n-3}{\mathcal{F}} \equiv \overset{n-1}{\mathcal{F}}$. Finally, comparing the Eqs. (??) and (??) with the limits,

$$\log[+\infty](+1^>) = 0^+, \quad \log[+\infty](-1^>) = +\infty, \quad \text{and}, \quad \log[+\infty](\infty) = +1, \quad (42)$$

implies,

$$0^+ \equiv \overset{n-4}{\mathcal{F}} \overset{n-4}{I_1}, \quad +\infty \equiv \overset{n-4}{I_1}, \quad \text{and}, \quad +1 \equiv \overset{n-4}{I_0}. \quad (43)$$

Comparing the latter with Eq. (??) it follows that, \mathfrak{g}_{n-4} , and, \mathfrak{g}_n , are related by an automorphism. In particular, $\overset{n}{I_{0(1)}} = \overset{n-4}{I_{0(1)}}$, and, $\overset{n}{\mathcal{F}} \overset{n}{I_{0(1)}} = \overset{n-4}{\mathcal{F}} \overset{n-4}{I_{0(1)}}$.

A *chain complex* is a sequence of abelian groups (or modules) connected by homomorphisms, such that the composition of any two consecutive homomorphisms, restricted to finite elements of the domain, is a projection into the identity.⁶ There is a *chain complex* structure underlying the sequence of products. This is shown in the next slide.

⁶The identity of a group forms a trivial subgroup.

An example of a chain complex is the de Rham cohomology, a sequence of *differential forms* connected by the *exterior derivative*. Let, dx^1, \dots, dx^n , be the generators of 1-forms. Consider the differential form, $\sigma = u \, dx^1 \wedge dx^2$. Applying the exterior derivative, one obtains,

$$d\sigma = du \wedge dx^1 \wedge dx^2 = \sum_{i=3}^n \frac{\partial u}{\partial x^i} dx^i \wedge dx^1 \wedge dx^2. \quad (44)$$

The de Rham cohomology is a sequence of additive groups⁷, therefore the identity element is always zero. Consistently, after two consecutive maps one obtains zero, i.e., $d^2 = 0$. In particular,

$$d^2\sigma = \sum_{i=3}^n \left(\sum_{k=1}^n \frac{\partial^2 u}{\partial x^k \partial x^i} dx^k \right) \wedge dx^i \wedge dx^1 \wedge dx^2 = 0. \quad (45)$$

The sequence of products defined in Eq. (??) contains two chain complexes, associated to odd and even indexed products, respectively. In both cases (odd/even) the connecting homomorphisms (analogue of the external derivative) are generated by, $\log^2[\mathbb{I}_1]^n(\cdot)$, for a fixed, $n \in \mathbb{Z}$. Recall any ideal is a valid base for the logarithm, as long as it remains fixed. Unlike the de Rham cohomology, the group identity along these chain complexes is not preserved, it alternates between, $\overset{m}{\mathbb{I}}_0$, and, $\overset{m-2}{\mathbb{I}}_0$, for any parity fixed, $m \in \mathbb{Z}$. This property doesn't compromise the projection into the identity after two consecutive maps, i.e.,

$$\log^4[\mathbb{I}_1]^n(\overset{m}{\alpha}) \in \overset{m}{\mathbb{I}}_0. \quad (46)$$

Proof. From Eq. (??) and taking subsequent logarithms,

$$\log[\mathbb{I}_1]^n(\overset{m}{\alpha}) \in \overset{m}{\mathcal{F}} \overset{m}{\mathbb{I}}_1, \quad (47a)$$

$$\log^2[\mathbb{I}_1]^n(\overset{m}{\alpha}) \in \log[\mathbb{I}_1]^n(\overset{m}{\mathcal{F}} \overset{m}{\mathbb{I}}_1) = \overset{n-1}{\mathcal{F}} \log[\mathbb{I}_1]^n(\overset{m}{\mathbb{I}}_1) = \overset{m-1}{\mathcal{F}} \overset{m-1}{\mathbb{I}}_1, \quad (47b)$$

$$\log^3[\mathbb{I}_1]^n(\overset{m}{\alpha}) \in \log[\mathbb{I}_1]^n(\overset{m-1}{\mathcal{F}} \overset{m}{\mathbb{I}}_0) = \overset{n-2}{\mathcal{F}} \log[\mathbb{I}_1]^n(\overset{m-1}{\mathbb{I}}_1) = \overset{m-2}{\mathcal{F}} \overset{m-2}{\mathbb{I}}_1, \quad (47c)$$

$$\log^4[\mathbb{I}_1]^n(\overset{m}{\alpha}) \in \log[\mathbb{I}_1]^n(\overset{m-2}{\mathcal{F}} \overset{m-2}{\mathbb{I}}_1) = \overset{m-3}{\mathcal{F}} \log[\mathbb{I}_1]^n(\overset{m-2}{\mathbb{I}}_1) = \overset{m-3}{\mathcal{F}} \overset{m-3}{\mathbb{I}}_1. \quad (47d)$$

From Eq. (??) it follows that,

$$\log^4[\mathbb{I}_1]^n(\overset{m}{\alpha}) \in \overset{m-4}{\mathbb{I}}_0. \quad (48)$$

Since, $\log^4[\mathbb{I}_1]^n(\cdot)$, is an automorphism then, $\overset{m-4}{\mathbb{I}}_0 = \overset{m}{\mathbb{I}}_0$, proving Eq. (??).

⁷Vector spaces are additive groups.