Twitter thread by @judijasa
Title: The logarithmic chain complex
Slide Nr: 001

The conventional sum, $+$, and multiplication, $\times$, are abelian products with distributivity. For example, if, $\alpha, \beta, \gamma \in (\mathbb{R}, +, \times)$, then,

$$\alpha \times (\beta + \gamma) = \alpha \times \beta + \alpha \times \gamma. \tag{1}$$

Formally, these two products form a *ring*. As aspiring goldsmiths we shall try to make chains out of these rings. Let, $\overset{0}{\times}$, and, $\overset{1}{\times}$, denote two products that behave like the conventional sum and multiplication, respectively. The set of products[1],

$$\alpha \overset{n+1}{\times} \beta \doteq \log^{-1}\left(\log(\alpha) \overset{n}{\times} \log(\beta)\right), \tag{2}$$

for, $n \in \mathbb{Z}$, forms a sequence of abelian groups and homomorphisms. Note that, $\overset{n+1}{\times}$, is distributive in, $\overset{n}{\times}$. Proof by induction: Assume,

$$\alpha \overset{n}{\times} (\beta \overset{n-1}{\times} \gamma) = (\alpha \overset{n}{\times} \beta) \overset{n-1}{\times} (\alpha \overset{n}{\times} \gamma), \tag{3}$$

then,

$$\begin{aligned}
\alpha \overset{n+1}{\times} (\beta \overset{n}{\times} \gamma) &= \log^{-1}\{\log(\alpha) \overset{n}{\times} \log(\beta \overset{n}{\times} \gamma)\} \\
&= \log^{-1}\{\log(\alpha) \overset{n}{\times} [\log(\beta) \overset{n-1}{\times} \log(\gamma)]\} \\
&\overset{(3)}{=} \log^{-1}\{[\log(\alpha) \overset{n}{\times} \log(\beta)] \overset{n-1}{\times} [\log(\alpha) \overset{n}{\times} \log(\gamma)]\} \\
&= \log^{-1}\{\log(\alpha \overset{n+1}{\times} \beta) \overset{n-1}{\times} \log(\alpha \overset{n+1}{\times} \gamma)\} \\
&= (\alpha \overset{n+1}{\times} \beta) \overset{n}{\times} (\alpha \overset{n+1}{\times} \gamma).
\end{aligned} \tag{4}$$

This covers the proof for, $n \in \mathbb{Z}^+$. Left to the reader is the proof for, $n \in \mathbb{Z}^-$.

In future posts, I will share more trivia about this object and discuss possible applications.

[1] Details such as the base of the log function and its algebraic role are discussed later on.

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Slide Nr: 002

In the latest slide we defined a sequence of abelian products, based on the log function. A complete definition requires a generalisation of the log function to include negative reals as input. Before doing this, let us appreciate the behaviour of these products in, $\mathbb{R}^+$. The definition in Eq. (2) makes it clear these products are commutative. As observed in Coya's law, this is not always obvious, e.g., the equivalent definition of, $\overset{2}{\times}$:

$$\alpha \overset{2}{\times} \beta = \alpha^{\log \beta}. \tag{5}$$

The product, $\overset{2}{\times}$, is interesting, together with, $\overset{-1}{\times}$, are the closest relatives to the conventional products, $\overset{0}{\times}$, and, $\overset{1}{\times}$, for which we hold more intuition. The product, $\overset{2}{\times}$, is distributive in the conventional multiplication ($\overset{1}{\times}$). This makes, $\overset{2}{\times}$, the abelian version of the power product, which is also distributive in the multiplication, i.e., $(\alpha \times \beta)^\gamma = \alpha^\gamma \times \beta^\gamma$.

It is convenient to move into a more abstract characterisation of the sequence of products. Some definitions and conventions:

$$\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\beta} = \overset{n}{\beta} \overset{n}{\times} \overset{n}{\alpha}, \tag{6a}$$

$$\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\mathrm{I}}_0 = \overset{n}{\mathrm{I}}_0 \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{\alpha}, \tag{6b}$$

$$\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\mathrm{I}}_1 = \overset{n}{\mathrm{I}}_1 \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{\mathrm{I}}_1 . \tag{6c}$$

$$(\overset{n}{\mathscr{T}} (\overset{n}{\mathscr{T}} \overset{n}{\alpha})) = \overset{n}{\alpha} \tag{7a}$$

$$\overset{n}{\alpha} \overset{n}{\times} (\overset{n}{\mathscr{T}} \overset{n}{\alpha}) = (\overset{n}{\mathscr{T}} \overset{n}{\alpha}) \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{\mathrm{I}}_0, \tag{7b}$$

$$\overset{n}{\mathscr{T}} \overset{n}{\mathrm{I}}_0 = \overset{n}{\mathrm{I}}_0 \tag{7c}$$

$$\overset{n}{\alpha} \overset{n}{\times} (\overset{n}{\mathscr{T}} \overset{n}{\mathrm{I}}_1) = (\overset{n}{\mathscr{T}} \overset{n}{\mathrm{I}}_1) \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{\mathscr{T}} \overset{n}{\mathrm{I}}_1 . \tag{7d}$$

Abusing of notation on behalf of intuition we keep, $\log(.)$, to denote the generator of homomorphisms. We use, $\log^{+1}(.) = \log(.)$, and, $\log^{-1}(\log(\overset{n}{\alpha})) = \log(\log^{-1}(\overset{n}{\alpha})) = \overset{n}{\alpha}$. The generator of homomorphisms connects nearby groups along the sequence, i.e.,

$$\overset{n-1}{\alpha} = \log(\overset{n}{\alpha}) \in \overset{n}{\mathscr{T}} \overset{n}{\mathrm{I}}_1 \tag{8a}$$

$$\overset{n+1}{\alpha} = \log^{-1}(\overset{n}{\alpha}) \in \overset{n}{\mathrm{I}}_1 . \tag{8b}$$

I imagine the elements, $\overset{n}{\alpha}, \overset{n+1}{\alpha}, ...$, as different scales of a physical degree of freedom. Formally, $\overset{n}{\mathrm{I}}_1 \cup \overset{n}{\mathscr{T}} \overset{n}{\mathrm{I}}_1$, is the *ideal* of the n-group, but it feels as the horizont of too large/small elements, seen from the perspective of the n-scale. In the next slide we identify these abstract elements in the group, $(\mathbb{R}^+, \times)$.