Twitter thread by @judijasa
Title: The logarithmic chain complex
Slide Nr: 001

The conventional sum, $+$, and multiplication, $\times$, are abelian products with distributivity. For example, if, $\alpha, \beta, \gamma \in (\mathbb{R}, +, \times)$, then,

$$\alpha \times (\beta + \gamma) = \alpha \times \beta + \alpha \times \gamma. \tag{1}$$

Formally, these two products form a *ring*. As aspiring goldsmiths we shall try to make chains out of these rings. Let, $\overset{0}{\times}$, and, $\overset{1}{\times}$, denote two products that behave like the conventional sum and multiplication, respectively. The set of products[1],

$$\alpha \overset{n+1}{\times} \beta \doteq \log^{-1}\left(\log(\alpha) \overset{n}{\times} \log(\beta)\right), \tag{2}$$

for, $n \in \mathbb{Z}$, forms a sequence of abelian groups and homomorphisms. Note that, $\overset{n+1}{\times}$, is distributive in, $\overset{n}{\times}$. Proof by induction: Assume,

$$\alpha \overset{n}{\times} (\beta \overset{n-1}{\times} \gamma) = (\alpha \overset{n}{\times} \beta) \overset{n-1}{\times} (\alpha \overset{n}{\times} \gamma), \tag{3}$$

then,

$$\begin{aligned}
\alpha \overset{n+1}{\times} (\beta \overset{n}{\times} \gamma) &= \log^{-1}\{\log(\alpha) \overset{n}{\times} \log(\beta \overset{n}{\times} \gamma)\} \\
&= \log^{-1}\{\log(\alpha) \overset{n}{\times} [\log(\beta) \overset{n-1}{\times} \log(\gamma)]\} \\
&\overset{(3)}{=} \log^{-1}\{[\log(\alpha) \overset{n}{\times} \log(\beta)] \overset{n-1}{\times} [\log(\alpha) \overset{n}{\times} \log(\gamma)]\} \\
&= \log^{-1}\{\log(\alpha \overset{n+1}{\times} \beta) \overset{n-1}{\times} \log(\alpha \overset{n+1}{\times} \gamma)\} \\
&= (\alpha \overset{n+1}{\times} \beta) \overset{n}{\times} (\alpha \overset{n+1}{\times} \gamma).
\end{aligned} \tag{4}$$

This covers the proof for, $n \in \mathbb{Z}^+$. Left to the reader is the proof for, $n \in \mathbb{Z}^-$.

In future posts, I will share more trivia about this object and discuss possible applications.

[1]Details such as the base of the log function and its algebraic role are discussed later on.

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Slide Nr: 002

In the latest slide we defined a sequence of abelian products, based on the log function. A complete definition requires a generalisation of the log function to include negative reals as input. Before doing this, let us appreciate the behaviour of these products in, $\mathbb{R}^+$. The definition in Eq. (2) makes it clear these products are commutative. As observed in Coya's law, this is not always obvious, e.g., the equivalent definition of, $\overset{2}{\times}$:

$$\alpha \overset{2}{\times} \beta = \alpha^{\log \beta}. \tag{5}$$

The product, $\overset{2}{\times}$, is interesting, together with, $\overset{-1}{\times}$, are the closest relatives to the conventional products, $\overset{0}{\times}$, and, $\overset{1}{\times}$, for which we hold more intuition. The product, $\overset{2}{\times}$, is distributive in the conventional multiplication ($\overset{1}{\times}$). This makes, $\overset{2}{\times}$, the abelian version of the power product, which is also distributive in the multiplication, i.e., $(\alpha \times \beta)^\gamma = \alpha^\gamma \times \beta^\gamma$.

It is convenient to move to a more abstract characterisation of the sequence of products. Some definitions and conventions:

$$\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\beta} = \overset{n}{\beta} \overset{n}{\times} \overset{n}{\alpha}, \tag{6a}$$

$$\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\mathrm{I}}_0 = \overset{n}{\mathrm{I}}_0 \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{\alpha}, \tag{6b}$$

$$\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\mathrm{I}}_1 = \overset{n}{\mathrm{I}}_1 \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{\mathrm{I}}_1 . \tag{6c}$$

$$(\overset{n}{\mathscr{T}}\ (\overset{n}{\mathscr{T}} \overset{n}{\alpha})) = \overset{n}{\alpha} \tag{7a}$$

$$\overset{n}{\alpha} \overset{n}{\times}\ (\overset{n}{\mathscr{T}}\ \overset{n}{\alpha}) = (\overset{n}{\mathscr{T}}\ \overset{n}{\alpha})\ \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{\mathrm{I}}_0, \tag{7b}$$

$$\overset{n}{\mathscr{T}}\ \overset{n}{\mathrm{I}}_0 = \overset{n}{\mathrm{I}}_0 \tag{7c}$$

$$\overset{n}{\alpha} \overset{n}{\times}\ (\overset{n}{\mathscr{T}}\ \overset{n}{\mathrm{I}}_1) = (\overset{n}{\mathscr{T}}\ \overset{n}{\mathrm{I}}_1)\ \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{\mathscr{T}}\ \overset{n}{\mathrm{I}}_1 . \tag{7d}$$

Abusing of notation on behalf of intuition we keep, log(.), to denote the generator of homomorphisms. We use, $\log^{+1}(.) = \log(.)$, and, $\log^{-1}(\log(\overset{n}{\alpha})) = \log(\log^{-1}(\overset{n}{\alpha})) = \overset{n}{\alpha}$. The generator of homomorphisms connects nearby groups along the sequence, i.e.,

$$\overset{n-1}{\alpha} = \log(\overset{n}{\alpha}) \in \overset{n}{\mathscr{T}}\ \overset{n}{\mathrm{I}}_1 \tag{8a}$$

$$\overset{n+1}{\alpha} = \log^{-1}(\overset{n}{\alpha})\ \in \overset{n}{\mathrm{I}}_1 . \tag{8b}$$

I imagine the elements, $\overset{n}{\alpha}, \overset{n+1}{\alpha}, \ldots$, as different scales of a physical degree of freedom. Formally, $\overset{n}{\mathscr{T}}\ \overset{n}{\mathrm{I}}_1$, and, $\overset{n}{\mathrm{I}}_1$, are elements of the *ideal* of the n-group, but they feel as the horizont of too large/small elements, from the perspective of the n-scale. The ideal forms a subgroup with identity, $\overset{n}{\mathscr{T}}\ \overset{n}{\mathrm{I}}_1 \overset{n}{\times} \overset{n}{\mathrm{I}}_1$, not to be confused with the group identity, $\overset{n}{\mathrm{I}}_0$, which can be expressed similarly as, $\overset{n}{\mathrm{I}}_0 = \overset{n}{\mathscr{T}}\ \overset{n}{\alpha} \overset{n}{\times} \overset{n}{\alpha}$. Next we identify these elements in, $(\mathbb{R}^+, \times)$.

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Slide Nr: 003

As a complement to Eqs. (8) are the relations,

$$\log(\overset{n}{\mathscr{T}}\,\overset{n}{\mathrm{I}}_1) = \overset{n-1}{\mathscr{T}}\,\overset{n-1}{\mathrm{I}}_1, \quad \log(\overset{n}{\mathrm{I}}_1) = \overset{n-1}{\mathrm{I}}_1, \quad \text{and,} \quad \log(\overset{n}{\mathrm{I}}_0) = \overset{n-1}{\mathrm{I}}_0 . \tag{9}$$

Example. Taking, $(\mathbb{R}^+, \times)$, as the n-group, one obtains that,

$$1 \equiv \overset{n}{\mathrm{I}}_0, \quad +\infty \equiv \overset{n}{\mathrm{I}}_1, \quad 0^+ \equiv \overset{n}{\mathscr{T}}\,\overset{n}{\mathrm{I}}_1, \quad \text{and,} \quad 1/\,\overset{n}{\alpha} \equiv \overset{n}{\mathscr{T}}\,\overset{n}{\alpha} . \tag{10}$$

Denoting, $\log[y](x) = \log_y(x)$, and letting, $\overset{n}{\alpha}, \overset{n}{\beta} \in \mathbb{R}^+ - (0^+ \cup +\infty)$, note that,

$$\log[\overset{n}{\beta}](\overset{n}{\alpha}) \notin \overset{n}{\mathscr{T}}\,\overset{n}{\mathrm{I}}_1, \tag{11}$$

which contradicts, Eq. (8a). An instance of this observation is, $\log_{2.56}(11.4) \notin 0^+$. Is there a base that can fulfill Eqs. (8)? Understood as a limit, note that,

$$\log[\overset{n}{\mathrm{I}}_1](\overset{n}{\alpha}) \in \overset{n}{\mathscr{T}}\,\overset{n}{\mathrm{I}}_1 . \tag{12}$$

An instance of this observation is, $(+\infty)^x = 23.71 \Rightarrow x \in 0^+$. This suggests the local identification,[2] $\log(.) \equiv \log[\overset{n}{\mathrm{I}}_1](.)$. Comparing the Eqs. (9) with the limits,

$$\log[+\infty](0^+) = -1^<, \quad \log[+\infty](+\infty) = +1^<, \quad \text{and,} \quad \log[+\infty](1) = 0, \tag{13}$$

implies,

$$-1^< \equiv \overset{n-1}{\mathscr{T}}\,\overset{n-1}{\mathrm{I}}_1, \quad +1^< \equiv \overset{n-1}{\mathrm{I}}_1, \quad \text{and,} \quad 0 \equiv \overset{n-1}{\mathrm{I}}_0 . \tag{14}$$

Whenever the group, $\overset{n}{\times}$, is associated to the conventional multiplication, $\times$, in, $\mathbb{R}^+$, the open interval, $(-1, +1)$, provides the elements for the abelian group, $\overset{n-1}{\times}$, with group identity, 0, and inverse, $\overset{n-1}{\mathscr{T}}(.) \equiv (-1) \times (.)$. Recalling that, $\overset{n}{\times}$, is distributive in, $\overset{n-1}{\times}$, completes the similarities of, $\overset{n-1}{\times}$, with the conventional sum, $+$.

[2] An alternative is, $\log(.) \equiv \log[\overset{n}{\mathscr{T}}\,\overset{n}{\mathrm{I}}_1](.)$, which modifies by, $\overset{n-1}{\mathscr{T}}$, the output.

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Slide Nr: 004

The definition of the log action can be extended beyond the transformation of elements defined in Eqs. (8). From Eq. (2), it follows that,

$$\log(\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\beta}) \;=\; \log(\overset{n}{\alpha}) \;\overset{n-1}{\times}\; \log(\overset{n}{\beta}). \tag{15}$$

Define the log action on products as,

$$\log(\overset{n}{\times}) = \overset{n-1}{\times}\;. \tag{16}$$

The Eq. (15) now follows from distributivity of the log action in elements and products, i.e.,

$$\log(\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\beta}) = \log(\overset{n}{\alpha})\;\log(\overset{n}{\times})\;\log(\overset{n}{\beta}). \tag{17}$$

Similarly, one can define the log action on the product inverse as,

$$\log(\overset{n}{\mathscr{T}}) = \overset{n-1}{\mathscr{T}}\;. \tag{18}$$

The familiar relation, $\overset{n}{\alpha} \in (\mathbb{R}^+, \times)$:

$$\log(1/\,\overset{n}{\alpha}) = -\log(\overset{n}{\alpha}), \tag{19}$$

motivates the general form $(n \in \mathbb{Z})$,

$$\log(\overset{n}{\mathscr{T}}\,\overset{n}{\alpha}) = \overset{n-1}{\mathscr{T}}\log(\overset{n}{\alpha}). \tag{20}$$

The latter equation also follows from distributivity of the log action on elements and inverse operations, i.e.,

$$\log(\overset{n}{\mathscr{T}}\,\overset{n}{\alpha}) = \log(\overset{n}{\mathscr{T}})\log(\overset{n}{\alpha}). \tag{21}$$

Recall from slide Nr. 003, that if, $\overset{n}{\alpha} \in (\mathbb{R}^+, \times)$, then, $\overset{n-1}{\alpha} = \log(\overset{n}{\alpha})$, is an element of a sum-like group with inverse, $\overset{n-1}{\mathscr{T}}(.) \equiv (-1) \times (.)$, then,

$$\log(-\,\overset{n-1}{\alpha}) \equiv \log(\overset{n-1}{\mathscr{T}}\,\overset{n-1}{\alpha}). \tag{22}$$

From Eq. (20), one obtains,

$$\log(\overset{n-1}{\mathscr{T}}\,\overset{n-1}{\alpha}) = \overset{n-2}{\mathscr{T}}\;\log(\overset{n-1}{\alpha}). \tag{23}$$

This describes the log action on the negative numbers as a particular case. In the next slide we identify the inverse, $\overset{n-2}{\mathscr{T}}$.

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Slide Nr: 005

The sequence of products defined in Eq. (2) requires a consistent definition for the log action on negative numbers. Such definition is given by Eq. (24) if, $(\mathbb{R}^+, \times)$, represents the n-group[3]. To fully identify Eq. (24) in the present representation reduces to identify the product inverse, $\overset{n-2}{\mathscr{T}}$. Comparing the Eqs. (9) and (14) with the limits,

$$\log[+\infty](-1^<) = -\infty, \quad \log[+\infty](+1^<) = 0^-, \quad \text{and,} \quad \log[+\infty](0) = -1, \tag{24}$$

implies,

$$-\infty \equiv \overset{n-2}{\mathscr{T}} \overset{n-2}{\mathrm{I}_1}, \quad 0^- \equiv \overset{n-2}{\mathrm{I}_1}, \quad \text{and,} \quad -1 \equiv \overset{n-2}{\mathrm{I}_0}. \tag{25}$$

The set, $\mathbb{R}^-$, provides the elements for the abelian group, $\overset{n-2}{\times}$, with group identity, -1, and inverse, $\overset{n-2}{\mathscr{T}} \equiv 1/(.)$. Despite, $\mathbb{R}^-$, is not closed under the conventional multiplication, $\times$, it is closed under the product, $\overset{n-2}{\times}$, as defined in Eq. (2). Using Eq. (8) one proceeds by induction,

$$\begin{aligned} \overset{n-2}{\alpha} \overset{n-2}{\times} \overset{n-2}{\beta} &= \log\left(\log^{-1}(\overset{n-2}{\alpha}) \overset{n-1}{\times} \log^{-1}(\overset{n-2}{\beta})\right) \\ &= \log(\overset{n-1}{\alpha} \overset{n-1}{\times} \overset{n-1}{\beta}) = \log(\overset{n-1}{\gamma}) = \overset{n-2}{\gamma}. \end{aligned} \tag{26}$$

[3]The choice of representation for the n-group constrains the representation of nearby groups.