The Log chain: A sequence of abelian groups based on the logarithm

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The conventional sum, $+$, and multiplication, $\times$, are abelian products with distributivity. For example, if, $\alpha, \beta, \gamma \in (\mathbb{R}, +, \times)$, then,

$$\alpha \times (\beta + \gamma) = \alpha \times \beta + \alpha \times \gamma. \tag{1}$$

Formally, these two products form a *ring*. As aspiring goldsmiths we shall try to make chains out of these rings. Let, $\overset{0}{\times}$, and, $\overset{1}{\times}$, denote two products that behave like the conventional sum and multiplication, respectively. The products[1], $n \in \mathbb{Z}$:

$$\alpha \overset{n+1}{\times} \beta \doteq \log^{-1}\left(\log(\alpha) \overset{n}{\times} \log(\beta)\right), \tag{2}$$

form a sequence of abelian groups. Note that, $\overset{n+1}{\times}$, is distributive in, $\overset{n}{\times}$. Proof by induction: Assume,

$$\alpha \overset{n}{\times} (\beta \overset{n-1}{\times} \gamma) = (\alpha \overset{n}{\times} \beta) \overset{n-1}{\times} (\alpha \overset{n}{\times} \gamma), \tag{3}$$

then,

$$\begin{aligned}
\alpha \overset{n+1}{\times} (\beta \overset{n}{\times} \gamma) &= \log^{-1}\{\log(\alpha) \overset{n}{\times} \log(\beta \overset{n}{\times} \gamma)\} \\
&= \log^{-1}\{\log(\alpha) \overset{n}{\times} [\log(\beta) \overset{n-1}{\times} \log(\gamma)]\} \\
&\overset{(3)}{=} \log^{-1}\{[\log(\alpha) \overset{n}{\times} \log(\beta)] \overset{n-1}{\times} [\log(\alpha) \overset{n}{\times} \log(\gamma)]\} \\
&= \log^{-1}\{\log(\alpha \overset{n+1}{\times} \beta) \overset{n-1}{\times} \log(\alpha \overset{n+1}{\times} \gamma)\} \\
&= (\alpha \overset{n+1}{\times} \beta) \overset{n}{\times} (\alpha \overset{n+1}{\times} \gamma).
\end{aligned} \tag{4}$$

This covers the proof for, $n \in \mathbb{Z}^+$. Left to the reader is the proof for, $n \in \mathbb{Z}^-$.

In future posts, I will share more trivia about this structure and discuss possible applications.

[1] Details such as the base of the log function and its algebraic role are discussed later on.

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In the latest slide we defined a sequence of abelian products, based on the log function. A complete definition requires a generalisation of the log function to operate on negative reals. Before doing this, let us appreciate the behaviour of these products in, $\mathbb{R}^+$. The definition in Eq. (2) makes it clear these products are commutative. As observed in Coya's law (@johncarlosbaez), this is not always obvious, e.g., the equivalent definition of, $\overset{2}{\times}$, as,

$$\alpha \overset{2}{\times} \beta = \alpha^{\log \beta}. \tag{5}$$

The product, $\overset{2}{\times}$, is interesting, together with, $\overset{-1}{\times}$, are the closest relatives to the conventional products, $\overset{0}{\times}$, and, $\overset{1}{\times}$, for which we hold more intuition. The product, $\overset{2}{\times}$, is distributive in the conventional multiplication ($\overset{1}{\times}$). This makes, $\overset{2}{\times}$, the abelian version of the power product, which is also distributive in the multiplication, i.e., $(\alpha \times \beta)^\gamma = \alpha^\gamma \times \beta^\gamma$.

It is convenient to move to a more abstract characterisation. Each product, $\overset{n}{\times}$, forms a group denoted, $\mathfrak{g}_n$, with 'finite' elements, $\overset{n}{\alpha}, \overset{n}{\beta}, ...$ Arbitrary elements in the sequence of groups are denoted without the upper index, i.e., α. Some definitions and conventions:

$$\begin{aligned} \overset{n}{\alpha} \overset{n}{\times} \overset{n}{\beta} &= \overset{n}{\beta} \overset{n}{\times} \overset{n}{\alpha}, & \text{(6a)} \\ \overset{n}{\alpha} \overset{n}{\times} \overset{n}{\mathrm{I}}_0 &= \overset{n}{\mathrm{I}}_0 \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{\alpha}, & \text{(6b)} \\ \overset{n}{\alpha} \overset{n}{\times} \overset{n}{\mathrm{I}}_1 &= \overset{n}{\mathrm{I}}_1 \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{\mathrm{I}}_1 \,. & \text{(6c)} \end{aligned}$$

$$\begin{aligned} (\overset{n}{\mathscr{T}} (\overset{n}{\mathscr{T}} \overset{n}{\alpha})) &= \overset{n}{\alpha} & \text{(7a)} \\ \overset{n}{\alpha} \overset{n}{\times} (\overset{n}{\mathscr{T}} \overset{n}{\alpha}) &= (\overset{n}{\mathscr{T}} \overset{n}{\alpha}) \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{\mathrm{I}}_0, & \text{(7b)} \\ \overset{n}{\mathscr{T}} \overset{n}{\mathrm{I}}_0 &= \overset{n}{\mathrm{I}}_0 & \text{(7c)} \\ \overset{n}{\alpha} \overset{n}{\times} (\overset{n}{\mathscr{T}} \overset{n}{\mathrm{I}}_1) &= (\overset{n}{\mathscr{T}} \overset{n}{\mathrm{I}}_1) \overset{n}{\times} \overset{n}{\alpha} = \overset{n}{\mathscr{T}} \overset{n}{\mathrm{I}}_1 \,. & \text{(7d)} \end{aligned}$$

Abusing of notation on behalf of intuition we keep, log(.), to denote the generator of homomorphisms. We use, $\log^{+1}(.) = \log(.)$, and, $\log^{-1}(\log(\overset{n}{\alpha})) = \log(\log^{-1}(\overset{n}{\alpha})) = \overset{n}{\alpha}$. The logarithm (and its inverse) connects nearby groups along the sequence, i.e.,

$$\begin{aligned} \overset{n-1}{\alpha} &= \log(\overset{n}{\alpha}) \in \overset{n}{\mathscr{T}} \overset{n}{\mathrm{I}}_1 & \text{(8a)} \\ \overset{n+1}{\alpha} &= \log^{-1}(\overset{n}{\alpha}) \in \overset{n}{\mathrm{I}}_1 \,. & \text{(8b)} \end{aligned}$$

I imagine the elements, $\overset{n}{\alpha}, \overset{n+1}{\alpha}, ...$, as different scales of a physical degree of freedom. Formally, $\overset{n}{\mathscr{T}} \overset{n}{\mathrm{I}}_1$, and, $\overset{n}{\mathrm{I}}_1$, are elements of an *ideal*, but they feel as the horizon of too large/small elements from the perspective of the n-scale. Next we identify these elements in, $(\mathbb{R}^+, \times)$.

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As a complement to Eqs. (8) are the relations,

$$\log(\overset{n}{\mathscr{T}}\ \overset{n}{\mathrm{I}}_1) = \overset{n-1}{\mathscr{T}}\ \overset{n-1}{\mathrm{I}}_1, \quad \log(\overset{n}{\mathrm{I}}_1) = \overset{n-1}{\mathrm{I}}_1, \quad \text{and,} \quad \log(\overset{n}{\mathrm{I}}_0) = \overset{n-1}{\mathrm{I}}_0 . \tag{9}$$

Example. Taking, $(\mathbb{R}^+, \times)$, as a representation of the n-group $(\mathfrak{g}_n)$ one obtains that,

$$0^+ \equiv \overset{n}{\mathscr{T}}\ \overset{n}{\mathrm{I}}_1, \quad +\infty \equiv \overset{n}{\mathrm{I}}_1, \quad 1 \equiv \overset{n}{\mathrm{I}}_0, \quad \text{and,} \quad 1/\ \overset{n}{\alpha} \equiv \overset{n}{\mathscr{T}}\ \overset{n}{\alpha} . \tag{10}$$

Denoting, $\log[y](x) = \log_y(x)$, and letting, $\overset{n}{\alpha}, \overset{n}{\beta} \in \mathbb{R}^+$, note that,

$$\log[\overset{n}{\beta}](\overset{n}{\alpha}) \notin \overset{n}{\mathscr{T}}\ \overset{n}{\mathrm{I}}_1, \tag{11}$$

which contradicts, Eq. (8a). An instance of this observation is, $\log_{2.56}(11.4) \notin 0^+$. Is there a base that can fulfill Eqs. (8)? Understood as a limit, note that,

$$\log[\overset{n}{\mathrm{I}}_1](\overset{n}{\alpha}) \in \overset{n}{\mathscr{T}}\ \overset{n}{\mathrm{I}}_1 . \tag{12}$$

An instance of this observation is, $(+\infty)^x = 23.71 \Rightarrow x \in 0^+$. This suggests the local identification,[2] $\log(.) \equiv \log[\overset{n}{\mathrm{I}}_1](.)$. Comparing the Eqs. (9) with the limits,

$$\log[+\infty](0^+) = -1^<, \quad \log[+\infty](+\infty) = 1^<, \quad \text{and,} \quad \log[+\infty](1) = 0, \tag{13}$$

implies,

$$-1^< \equiv \overset{n-1}{\mathscr{T}}\ \overset{n-1}{\mathrm{I}}_1, \quad 1^< \equiv \overset{n-1}{\mathrm{I}}_1, \quad \text{and,} \quad 0 \equiv \overset{n-1}{\mathrm{I}}_0 . \tag{14}$$

Whenever the group, $\mathfrak{g}_n$, is associated to, $(\mathbb{R}^+, \times)$, the open interval, $(-1, +1)$, provides the elements for the abelian group, $\mathfrak{g}_{n-1}$, associated to the product, $\overset{n-1}{\times}$, with group identity, $\overset{n-1}{\mathrm{I}}_0 \equiv 0$, and inverse, $\overset{n-1}{\mathscr{T}}(.) \equiv (-1)\times(.)$. Recalling that, $\overset{n}{\times}$, is distributive in, $\overset{n-1}{\times}$, completes the similarities of, $\overset{n-1}{\times}$, with the conventional sum, $+$.

[2]An alternative is, $\log(.) \equiv \log[\overset{n}{\mathscr{T}}\ \overset{n}{\mathrm{I}}_1](.)$.

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The definition of the log action can be extended beyond the transformation of elements defined in Eqs. (8). From Eq. (2), it follows that,

$$\log(\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\beta}) \;=\; \log(\overset{n}{\alpha}) \;\overset{n-1}{\times}\; \log(\overset{n}{\beta}). \tag{15}$$

Define the log action on products as,

$$\log(\overset{n}{\times}) = \overset{n-1}{\times} . \tag{16}$$

The Eq. (15) now follows from distributivity of the log action in elements and products, i.e.,

$$\log(\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\beta}) = \log(\overset{n}{\alpha}) \; \log(\overset{n}{\times}) \; \log(\overset{n}{\beta}). \tag{17}$$

Similarly, one can define the log action on the inverse as,

$$\log(\overset{n}{\mathscr{T}}) = \overset{n-1}{\mathscr{T}} . \tag{18}$$

The familiar relation, $\overset{n}{\alpha} \in (\mathbb{R}^+, \times)$:

$$\log(1/\overset{n}{\alpha}) = -\log(\overset{n}{\alpha}), \tag{19}$$

motivates the general form, $n \in \mathbb{Z}$:

$$\log(\overset{n}{\mathscr{T}} \overset{n}{\alpha}) = \overset{n-1}{\mathscr{T}} \log(\overset{n}{\alpha}). \tag{20}$$

The latter equation also follows from distributivity of the log action on elements and inverse operations, i.e.,

$$\log(\overset{n}{\mathscr{T}} \overset{n}{\alpha}) = \log(\overset{n}{\mathscr{T}}) \log(\overset{n}{\alpha}). \tag{21}$$

Recall from slide Nr. 003, that if, $\overset{n}{\alpha} \in (\mathbb{R}^+, \times)$, then, $\overset{n-1}{\alpha} = \log(\overset{n}{\alpha})$, is an element of a sum-like group with inverse, $\overset{n-1}{\mathscr{T}}(.) \equiv (-1) \times (.)$, then,

$$\log(\overset{n-1}{\mathscr{T}} \overset{n-1}{\alpha}) \equiv \log(-\overset{n-1}{\alpha}). \tag{22}$$

Together with Eq. (20), one obtains,

$$\log(-\overset{n-1}{\alpha}) = \overset{n-2}{\mathscr{T}} \; \log(\overset{n-1}{\alpha}). \tag{23}$$

This describes the log action on the negative numbers as a particular case. In the next slide we identify the inverse, $\overset{n-2}{\mathscr{T}}$.

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The sequence of products defined in Eq. (2) requires a consistent definition for the log action on negative numbers. Such definition is given by Eq. (23) if, $(\mathbb{R}^+, \times)$, represents the n-group[3]. To fully identify Eq. (23) in the present representation reduces to identify the product inverse, $\overset{n-2}{\mathscr{T}}$. Comparing the Eqs. (9) and (14) with the limits,

$$\log[+\infty](-1^<) = -\infty, \quad \log[+\infty](+1^<) = 0^-, \quad \text{and,} \quad \log[+\infty](0) = -1, \tag{24}$$

implies,

$$-\infty \equiv \overset{n-2}{\mathscr{T}} \overset{n-2}{\mathrm{I}_1}, \quad 0^- \equiv \overset{n-2}{\mathrm{I}_1}, \quad \text{and,} \quad -1 \equiv \overset{n-2}{\mathrm{I}_0}. \tag{25}$$

The set, $\mathbb{R}^-$, provides the elements for the abelian group, $\mathfrak{g}_{n-2}$, with associated product, $\overset{n-2}{\times}$, group identity, -1, and inverse, $\overset{n-2}{\mathscr{T}} \equiv 1/(.)$. Comparing with Eqs. (10) one obtains that, $\overset{n-2}{\mathscr{T}} \equiv \overset{n}{\mathscr{T}}$. The latter identification is not a general property of the sequence of products, it results from the limited description of the representation chosen. In general, the native representation of the inverse is given by, $n \in \mathbb{Z}$:[4]

$$\overset{n}{\mathscr{T}}(.) \equiv (\overset{n}{\mathscr{T}} \overset{n+1}{\mathrm{I}_0}) \overset{n+1}{\times} (.). \tag{26}$$

Despite, $\mathbb{R}^-$, is not closed under the conventional multiplication, $\times$, it is closed under the product, $\overset{n-2}{\times}$, as defined in Eq. (2). Proof: Using Eq. (8) one proceeds by induction,

$$\begin{aligned} \overset{n-2}{\alpha} \overset{n-2}{\times} \overset{n-2}{\beta} &= \log\left(\log^{-1}(\overset{n-2}{\alpha}) \overset{n-1}{\times} \log^{-1}(\overset{n-2}{\beta})\right) \\ &= \log(\overset{n-1}{\alpha} \overset{n-1}{\times} \overset{n-1}{\beta}) = \log(\overset{n-1}{\gamma}) = \overset{n-2}{\gamma}. \end{aligned} \tag{27}$$

As explained in more detail in slide Nr. 8, the elements in, $\mathfrak{g}_n$, and, $\mathfrak{g}_{n+4m}$, for, $m, n \in \mathbb{Z}$, are related by an automorphism. Under this representation, it is therefore sufficient to identify four consecutive groups to describe the sequence of groups. After setting, $\mathfrak{g}_n \equiv (\mathbb{R}^+, \times)$, we already identified the consecutive groups, $\mathfrak{g}_{n-1}$, and, $\mathfrak{g}_{n-2}$. In slide Nr. 8 we complete the analysis by identifying, $\mathfrak{g}_{n-3}$, but before we shall refine some concepts.

[3] The choice of representation for the n-group constrains the representation of nearby groups.

[4] To test involution of inverse in the native rep, note that, $\alpha \overset{m}{\times} (\overset{n}{\mathscr{T}} \beta) = (\overset{n}{\mathscr{T}} \alpha) \overset{m}{\times} \beta = \overset{n}{\mathscr{T}} (\alpha \overset{n}{\times} \beta)$, if, $m = n + 1$.

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Define the n-group's 'absolute value' as,

$$|\,.\,|_n = |\,.\,|_{n-1} \overset{n}{\times} \overset{n}{\mathscr{T}}\,(.), \tag{28}$$

where, $|\,.\,|_0$, is the conventional absolute value. The instants,

$$|-\overset{0}{\alpha}|_0 = |\overset{0}{\alpha}|_0, \qquad |1/\overset{1}{\alpha}|_1 = |\overset{1}{\alpha}|_1, \tag{29}$$

and,

$$|1/\overset{0}{\alpha}|_0 = 1/|\overset{0}{\alpha}|_0, \qquad |-\overset{1}{\alpha}|_1 = -|\overset{1}{\alpha}|_1, \tag{30}$$

motivate the general assumptions,[5]

$$|\overset{n\pm1}{\mathscr{T}}\overset{n}{\alpha}|_n = \overset{n\pm1}{\mathscr{T}}|\overset{n}{\alpha}|_n, \quad \text{and,} \qquad |\overset{n}{\mathscr{T}}\overset{n}{\alpha}|_n = |\overset{n}{\alpha}|_n. \tag{31}$$

An important property is that, $|\mathfrak{g}_n|_n$, is isomorphic to the open interval, $(\overset{n}{\mathrm{I}}_0, \overset{n}{\mathrm{I}}_1)$, while, $|\mathfrak{g}_n|_{n+1}$, is isomorphic to, $\mathbb{Z}_2$. The proof of the latter is the following: Any element in, $\mathfrak{g}_n$, can be parametrised as, $\overset{n}{\alpha} = (\overset{n}{\mathscr{T}})^\nu\,|\overset{n}{\alpha}|_n$, where, $(\overset{n}{\mathscr{T}})^1 = \overset{n}{\mathscr{T}}$, and, $(\overset{n}{\mathscr{T}})^2(.) = \overset{n}{\mathrm{I}}_0 \overset{n}{\times} (.)$. Then,

$$\begin{aligned} |\overset{n}{\alpha}|_{n+1} &= |\overset{n}{\alpha}|_n \overset{n+1}{\times} \overset{n+1}{\mathscr{T}}\overset{n}{\alpha} \\ &= |\overset{n}{\alpha}|_n \overset{n+1}{\times} \overset{n+1}{\mathscr{T}}\,(\overset{n}{\mathscr{T}})^\nu\,|\overset{n}{\alpha}|_n \\ &= (\overset{n}{\mathscr{T}})^\nu(|\overset{n}{\alpha}|_n \overset{n+1}{\times} \overset{n+1}{\mathscr{T}}\,|\overset{n}{\alpha}|_n) = (\overset{n}{\mathscr{T}})^\nu\,\overset{n}{\mathrm{I}}_0\,. \end{aligned} \tag{32}$$

In general, $k \in \mathbb{Z}^+$:

$$|\overset{n}{\alpha}|_{n-k} = |\overset{n}{\alpha}|_n \overset{n}{\times} \overset{n}{\alpha} \overset{n-1}{\times} \overset{n}{\alpha} \overset{n-2}{\times} \cdots \overset{n-k+1}{\times} \overset{n}{\alpha}, \tag{33}$$

and,

$$|\overset{n}{\alpha}|_{n+k} = |\overset{n}{\alpha}|_n \overset{n+1}{\times} \overset{n+1}{\mathscr{T}} \overset{n+1}{\alpha} \overset{n+2}{\times} \overset{n+2}{\mathscr{T}} \overset{n}{\alpha} \overset{n+3}{\times} \cdots \overset{n+k}{\times} \overset{n+k}{\mathscr{T}} \overset{n}{\alpha}\,. \tag{34}$$

[5]Despite, $(\overset{n}{\mathscr{T}}\,\overset{n}{\alpha}) \overset{n}{\times} \overset{n}{\beta} \neq \overset{n}{\alpha} \overset{n}{\times} (\overset{n}{\mathscr{T}}\,\overset{n}{\beta})$, it holds that, $(\overset{n}{\mathscr{T}}\,|\overset{n}{\alpha}|_{n\pm1}) \overset{n}{\times} \overset{n}{\alpha} = |\overset{n}{\alpha}|_{n\pm1} \overset{n}{\times} (\overset{n}{\mathscr{T}}\,\overset{n}{\alpha})$. Some of these assumptions can be derived from a minimal set, so there is work to do for a more concise presentation.

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Define the proper metric in, $\mathfrak{g}_n$, as,

$$g_n(\alpha,\beta) = |\,\alpha \overset{n}{\times} \overset{n}{\mathscr{T}}\,\beta\,|_n. \tag{35}$$

The metric, g_{n+1}, distinguishes attributes between elements, $\overset{n}{\alpha}, \overset{n}{\beta} \in \mathfrak{g}_n$, that go 'under the radar' in the metric, g_n. These attributes are generated by the group inverse, $\overset{n}{\mathscr{T}}$.

Recall the identifications, $0^+ \equiv \overset{n}{\mathscr{T}}\,\overset{n}{\mathrm{I}}_1$, and, $0 \equiv \overset{n-1}{\mathrm{I}_0}$. The instant, $0^+ \in 0$, motivates the general assumption, $n \in \mathbb{Z}$:

$$\overset{n}{\mathscr{T}}\,\overset{n}{\mathrm{I}}_1 \in \overset{n-1}{\mathrm{I}_0}\ . \tag{36}$$

Moreover, since, $\overset{n-1}{\mathscr{T}}\,\overset{n-1}{\mathrm{I}_0} = \overset{n-1}{\mathrm{I}_0}$, then also, $\overset{n-1}{\mathscr{T}}\,\overset{n}{\mathscr{T}}\,\overset{n}{\mathrm{I}}_1 \in \overset{n-1}{\mathrm{I}_0}$. Recalling that, $\overset{n-1}{\mathscr{T}} \equiv (-1)\times(.)$, the latter case corresponds to the instant, $0^- \in 0$, .

Two elements, $\overset{n}{\alpha}$, and, $\overset{n}{\beta}$, are said to be *infinitesimally close* if,

$$g_n(\overset{n}{\alpha},\overset{n}{\beta}) = \overset{n+1}{\mathscr{T}}\,\overset{n+1}{\mathrm{I}_1}, \tag{37}$$

and said to have the *same phase* if,

$$g_{n+1}(\overset{n}{\alpha},\overset{n}{\beta}) = \overset{n+2}{\mathscr{T}}\,\overset{n+2}{\mathrm{I}_1}\ . \tag{38}$$

If two elements of the same group are infinitesimally close and have the same phase they are said to be equal elements in the group, denoted, $\overset{n}{\alpha} \overset{n}{\doteq} \overset{n}{\beta}$. Two elements, α, β, are said to be equal if, $g_n(\alpha,\beta) = \overset{n+1}{\mathscr{T}}\,\overset{n+1}{\mathrm{I}_1}$, for all, $n \in \mathbb{Z}$.

Away of the identity, being infinitesimally close is sufficient criteria for equality in the group, i.e., if, $g_n(\overset{n}{\alpha},\overset{n}{\beta}) = \overset{n+1}{\mathscr{T}}\,\overset{n+1}{\mathrm{I}_1}$, and, $\overset{n}{\alpha} \notin \overset{n}{\mathrm{I}}_0$, then, $\overset{n}{\alpha} \overset{n}{\doteq} \overset{n}{\beta}$. Proof: $g_n(\overset{n}{\alpha},\overset{n}{\beta}) = \overset{n+1}{\mathscr{T}}\,\overset{n+1}{\mathrm{I}_1}$, implies, $\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\mathscr{T}}\,\overset{n}{\beta} \in \overset{n}{\mathrm{I}}_0$, therefore, $\overset{n}{\alpha} = \overset{n}{\beta} \overset{n}{\times} \overset{n}{\gamma}$, $\overset{n}{\gamma} \in \overset{n}{\mathrm{I}}_0$. Note that, $\overset{n}{\alpha} \notin \overset{n}{\mathrm{I}}_0$, if and only if, $\overset{n}{\beta} \notin \overset{n}{\mathrm{I}}_0$, otherwise, $\overset{n}{\alpha} \overset{n}{\times} \overset{n}{\mathscr{T}}\,\overset{n}{\beta} \notin \overset{n}{\mathrm{I}}_0$, which is a contradiction. Since, $\overset{n}{\beta} \notin \overset{n}{\mathrm{I}}_0$, then, $\overset{n}{\beta} \overset{n}{\times} \overset{n}{\gamma} \overset{n}{\doteq} \overset{n}{\beta}$, therefore, $\overset{n}{\alpha} \overset{n}{\doteq} \overset{n}{\beta}$. From now on, the equality, $=$, in Eqs. (6) and (7) is replaced by, $\overset{n}{\doteq}$.

Any element in a group can be reinterpreted as a set of infinitesimally close elements in the group metric. Denote, $\overset{n}{\beta} \in \overset{n}{\alpha}$, if the element, $\overset{n}{\beta}$, is infinitesimally close to the element, $\overset{n}{\alpha}$. For example, $0^{\pm} \in 0$, reads as: the elements, 0^-, and, 0^+, are infinitesimally close to zero, in their proper group metric.

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In analogy to the limits, $\pm\infty = 1/0^{\pm}$, define,

$$\infty = 1/0. \tag{39}$$

We are no longer in the real line, the element, ∞, is the point at infinity in the *real projective line*.

Comparing the Eqs. (9) and (25) with the limits,

$$\log[+\infty](-\infty) = +1^{>}, \quad \log[+\infty](0^{-}) = -1^{>}, \quad \text{and,} \quad \log[+\infty](-1) = \infty, \tag{40}$$

implies,

$$+1^{>} \equiv \overset{n-3}{\mathscr{T}} \overset{n-3}{\mathrm{I}_1}, \quad -1^{>} \equiv \overset{n-3}{\mathrm{I}_1}, \quad \text{and,} \quad \infty \equiv \overset{n-3}{\mathrm{I}_0}. \tag{41}$$

The set, $\mathbb{R} - (-1, +1)$, provides the elements for the abelian group, $\mathfrak{g}_{n-3}$, with group identity, ∞, and inverse, $\overset{n-3}{\mathscr{T}}(.) \equiv (-1) \times (.)$. As a result of the representation note that, $\overset{n-3}{\mathscr{T}} \equiv \overset{n-1}{\mathscr{T}}$. Finally, comparing the Eqs. (9) and (43) with the limits,

$$\log[+\infty](+1^{>}) = 0^{+}, \quad \log[+\infty](-1^{>}) = +\infty, \quad \text{and,} \quad \log[+\infty](\infty) = +1, \tag{42}$$

implies,

$$0^{+} \equiv \overset{n-4}{\mathscr{T}} \overset{n-4}{\mathrm{I}_1}, \quad +\infty \equiv \overset{n-4}{\mathrm{I}_1}, \quad \text{and,} \quad +1 \equiv \overset{n-4}{\mathrm{I}_0}. \tag{43}$$

Comparing the latter with Eq. (10) it follows that, $\mathfrak{g}_{n-4}$, and, $\mathfrak{g}_n$, are related by an automorphism. In particular, $\overset{n}{\mathrm{I}}_{0(1)} = \overset{n-4}{\mathrm{I}}_{0(1)}$, and, $\overset{n}{\mathscr{T}} \overset{n}{\mathrm{I}}_{0(1)} = \overset{n-4}{\mathscr{T}} \overset{n-4}{\mathrm{I}}_{0(1)}$.

A *chain complex* is a sequence of abelian groups (or modules) connected by homomorphisms, such that the composition of any two consecutive homomorphisms, restricted to finite elements of the domain, is a projection into the identity.[6] There is a *chain complex* structure underlying the sequence of products. This is shown in the next slide.

[6] The identity of a group forms a trivial subgroup.

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An example of a chain complex is the de Rham cohomology, a sequence of *differential forms* connected by the *exterior derivative*. Let, $\mathrm{d}x^1, ..., \mathrm{d}x^n$, be the generators of 1-forms. Consider the differential form, $\sigma = u\ \mathrm{d}x^1 \wedge \mathrm{d}x^2$. Applying the exterior derivative, one obtains,

$$\mathrm{d}\sigma = \mathrm{d}u \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 = \sum_{i=3}^{n} \frac{\partial u}{\partial x^i}\ \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2. \tag{44}$$

The de Rham cohomology is a sequence of additive groups[7], therefore the identity element is always zero. Consistently, after two consecutive maps one obtains zero, i.e., $\mathrm{d}^2 = 0$. In particular,

$$\mathrm{d}^2\sigma = \sum_{i=3}^{n} \left(\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x^k \partial x^i}\ \mathrm{d}x^k \right) \wedge \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 = 0. \tag{45}$$

The sequence of products defined in Eq. (2) contains two chain complexes, associated to odd and even indexed products, respectively. In both cases (odd/even) the connecting homomorphisms (analogue of the external derivative) are generated by, $\log^2[\overset{n}{\mathrm{I}}_1](.)$, for a fixed, $n \in \mathbb{Z}$. Recall any ideal is a valid base for the logarithm, as long as it remains fixed. Unlike the de Rham cohomology, the group identity along these chain complexes is not preserved, it alternates between, $\overset{m}{\mathrm{I}}_0$, and, $\overset{m-2}{\mathrm{I}_0}$, for any parity fixed, $m \in \mathbb{Z}$. This property doesn't compromise the projection into the identity after two consecutive maps, i.e.,

$$\log^4[\overset{n}{\mathrm{I}}_1](\overset{m}{\alpha}) \in \overset{m}{\mathrm{I}}_0\ . \tag{46}$$

Proof. From Eq. (8a) and taking subsequent logarithms,

$$\log[\overset{n}{\mathrm{I}}_1](\overset{m}{\alpha}) \in \overset{m}{\mathscr{T}}\ \overset{m}{\mathrm{I}}_1, \tag{47a}$$

$$\log^2[\overset{n}{\mathrm{I}}_1](\overset{m}{\alpha}) \in \log[\overset{n}{\mathrm{I}}_1](\overset{m}{\mathscr{T}}\ \overset{m}{\mathrm{I}}_1) = \overset{n-1}{\mathscr{T}} \log[\overset{n}{\mathrm{I}}_1](\overset{m}{\mathrm{I}}_1) = \overset{m-1}{\mathscr{T}}\ \overset{m-1}{\mathrm{I}_1}, \tag{47b}$$

$$\log^3[\overset{n}{\mathrm{I}}_1](\overset{m}{\alpha}) \in \log[\overset{n}{\mathrm{I}}_1](\overset{m-1}{\mathscr{T}}\ \overset{m}{\mathrm{I}}_0) = \overset{n-2}{\mathscr{T}} \log[\overset{n}{\mathrm{I}}_1](\overset{m-1}{\mathrm{I}_1}) = \overset{m-2}{\mathscr{T}}\ \overset{m-2}{\mathrm{I}_1}, \tag{47c}$$

$$\log^4[\overset{n}{\mathrm{I}}_1](\overset{m}{\alpha}) \in \log[\overset{n}{\mathrm{I}}_1](\overset{m-2}{\mathscr{T}}\ \overset{m-2}{\mathrm{I}_1}) = \overset{m-3}{\mathscr{T}} \log[\overset{n}{\mathrm{I}}_1](\overset{m-2}{\mathrm{I}_1}) = \overset{m-3}{\mathscr{T}}\ \overset{m-3}{\mathrm{I}_1}\ . \tag{47d}$$

From Eq. (36) it follows that,

$$\log^4[\overset{n}{\mathrm{I}}_1](\overset{m}{\alpha}) \in \overset{m-4}{\mathrm{I}_0}\ . \tag{48}$$

Since, $\log^4[\overset{n}{\mathrm{I}}_1](.)$, is an automorphism then, $\overset{m-4}{\mathrm{I}_0} = \overset{m}{\mathrm{I}}_0$, proving Eq. (46).

[7] Vector spaces are additive groups.