Seminar on Modular Forms and Sphere Packings

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Winter Semester 2025/26

Time and Place: Tuesdays 2–4pm in SR 4.

Practicalities

1. **Prerequisites:** Students are expected to have a good understanding of Complex Analysis 1 and Modular Forms 1. Modular forms that we need will be recalled in Talk 3.

2. **Requirements for participants:**

   - Talk: You are expected to give a 90-minute talk on your chosen topic. Definitions and results must be stated clearly, and when possible illustrated with concrete examples. It is expected that you have studied and understood the proofs in detail.
   - Please create a handout for the seminar participants to accompany your talk. This document should include the key definitions and results from your talk. You are also welcome to explain details of proofs and examples there, which you do not have time to cover in the talk. A template for the handout is available here.
   - Start preparing your talk well in advance (about 4 weeks before your talk). Some general guidance on how to give a good seminar talk can be found here. Two weeks before your talk, meet Dr. Xu for a preliminary discussion in his office 3/332; this helps in identifying gaps in understanding. Please talk to Dr. Xu to schedule a time in advance.

     Please bring a draft of the handout to the preliminary discussion. Please send the final version of your handout by email to Dr. Xu no later than Monday morning before your presentation, so that the materials can be made available to the other seminar participants in advance.
   - Attendance of the talks is mandatory.
   - Language: Your talk and your handout must be in English.

Talks

1. Preliminaries

0) Overview (October 14th) Speaker: Junyan Xu
1) Sphere packings in Euclidean spaces (October 21st)
Reference: [Romik] A.1-3.
Define and explain the following notions: sphere packings in a Euclidean space, the density of a packing, lattice packings, and periodic packings. Show that one can approximate the density of an arbitrary packing by periodic packings. Describe optimal packings in dimensions 1, 2 and 3 and compute their densities.
If time allows, you are free to cover the resolution of Kepler's conjecture by Hales, the kissing number problem and coverings (dual to packings). Even though this seminar will be focused on the upper bounds on the optimal densities in specific dimensions, you are welcome to discuss recent progress on the lower bound [Klartag].

2) Construction and properties of the E_8 and Leech lattices (October 28th)
References: [Romik] A.7, [dLV] §2, [SPLAG] §4.8 and §4.11.
Construct the E_8 and Leech lattices and show they are self-dual, even lattices. Present basic properties of the lattices, such as the automorphism groups, possible distances between lattice points, and minimal distances. Compute the densities of the associated lattice packings. Provide more example of lattices if time allows, for example the optimal lattice packings in dimensions ≤ 8 and their uniqueness following [CohnKumar] and [SPLAG] §4.

3) Theta series of lattices and modular forms (November 4th)
References: [CohnICM] §3, [Romik] §5.1-7, 5.12-14 and A.7, [CohnAMS] "Modular Forms", [Serre] §VII.6.
Define the (weakly holomorphic quasi-)modular forms used to construct magic functions, including the Eisenstein series, the modular discriminant, and the Jacobi thetanull functions, and describe their transformation properties ("functional equations") under generators of the modular group. Define the theta series of a lattice and describe the theta series of the E_8 and Leech lattices.

4) Fourier transform and Poisson summation (November 11th)
Reference: [Romik] Theorem 2.6-7, A.6.
Define Fourier transform on Euclidean spaces and state basic properties such as linearity and behavior under translation, scaling, and differentiation; compute the Fourier transform of the Gaussian. Define Schwartz functions, and show that Fourier transform is an automorphism on the space of Schwartz functions, and an involution

when restricted to even Schwartz functions. Prove the Poisson summation formula for lattices in Euclidean spaces, derive the functional equation for the Jacobi theta function, and show that theta series give rise to modular forms under appropriate conditions.

5) The linear programming bound of Cohn and Elkies and the search for magic functions (November 18th)
References: [Romik] A.8, [CohnAMS] "Linear Programming Bounds" and "The Hunt for the Magic Functions", [Okounkov] §3.
Prove the linear programming bound first for lattice packings using the Poisson summation formula, then prove it for periodic packings, and finally arbitrary packings. Show that the auxiliary function can be assumed to be radial. If a lattice packing achieves the bound, show that the auxiliary function must vanish at the lattice points, and its Fourier transform at the dual lattice points. Explain how these observations led people to seek radial Fourier eigenfunctions with double roots at specific points in dimensions 8 and 24.
(The linear programming bound is also conjectured to be sharp in dimension 2, but the construction of that magic auxiliary function is still open and appears hard due to the arithmetic nature, though numerical evidence suggests it exists.)

2. Viazovska's constructions and proofs

6) Viazovka's Ansatz (November 25th)
References: [CohnICM] §4-5, [Romik] §6.3, [CohnAMS] "Viazovska's Proof".
Explain Viazovska's idea of constructing the eigenfunctions as the product of a squared sine factor (which imposes double roots at lattice points) and a Laplace transform (nicely behaved under Fourier transform). Describe the shift of contour (to a pitchfork shape) and the change of variable that allow us to derive functional equations that need to be satisfied by the integrands, which strongly suggest modular forms should be used to construct them; explain how the equations depend on the dimension.

7) Construction of the +1-eigenfunctions (December 2nd)
References: [Romik] §6.4, [Viazovska] §4, [CKMRV] §2.
Construct the 8-dimensional +1-eigenfunction using Eisenstein series: estimate its growth and perform analytic continuation to show it is a Schwartz function; verify the functional equations to show it is indeed a +1-eigenfunction. Sketch the construction in 24 dimensions and compare, focusing on methods to find the modular form satisfying the functional equations. (Table 1 of [dLV] shows the functions side-by-side.)

8) Construction of the -1-eigenfunctions (December 9th)
References: [Romik] §6.5, [Viazovska] §4, [CKMRV] §3.
Construct the 8-dimensional -1-eigenfunction using the Jacobi thetanull series and prove its properties as in the previous talk. Sketch the construction in 24 dimensions and compare. Derive inequalities between quasimodular forms that need to be satisfied for the magic functions to be valid auxiliary functions for the linear programming bound. Explain why there is an extra "harder" inequality in the 24-dimensional case. (Conjecture 8.1 of [CohnElkies] is easily verified for the constructed magic functions, and uniqueness of E_8 and Leech lattices among periodic packings follows (uniqueness among lattice packings was known since [CohnKumar] (2009)).)

9) Modular form inequalities (December 16th)
References: [Lee], [Romik] §6.6-7, [Viazovska] §5, [CKMRV] §4 and Appendix A.
(The three references correspond to three generations of proofs of the key modular form inequalities that shows the magic functions constructed verify the conditions to serve as an auxiliary function for the linear programming bound. We shall focus on the algebraic proofs found by Seewoo Lee in 2024.)
Define Serre derivatives of quasimodular forms, present the theory of positivity, and outline the proofs of modular form inequalities, focusing on different treatment of the "easy", "hard" and "harder" inequalities.

3. Advanced topics

10) Fourier interpolation and uniqueness of magic functions (January 13th)
References: [CohnBull] §3, [RV].
Viazovska's 2016 papers showed the existence of magic functions which have double roots at specific points together with their Fourier transforms. It has been subsequently conjectured that one can in fact arbitrarily specify values of the (radial Schwartz) function, its Fourier transform, its derivative, and the derivative of its Fourier transform at those points. The interpolation basis is again constructed using modular forms. A consequence is the uniqueness of magic functions ([CKMRV22], Corollary 1.8). We will first investigate the prototype first-order 1-dimensional Fourier interpolation result in this talk.

11) Fourier interpolation and universal optimality (January 20th)
References: [ViazovskaEMS] §5, [CKMRV22] §1.
Sketch the construction of interpolation basis for the 8- and 24-dimensional interpolation results. Define the notions of completely monotonic functions and universal optimality. Describe the linear programming bound for energy and explain how interpolation implies universal optimality.

12) Dual linear programming bound and non-sharpness of linear programming bound (January 27th)
References: [Li], [CT], [CDV].
Outline the method of Cohn–Triantafillou using modular forms to prove that the linear programming bound for sphere packing is not sharp in dimensions 3–6, 12, and 16. (It is conjectured that the bound is only sharp in dimensions 1, 2, 8, and 24.)

References

[Viazovska] Maryna Viazovska, The sphere packing problem in dimension 8, Annals of Mathematics 185 (2017), 991–1015, `https://doi.org/10.4007/annals.2017.185.3.7`.

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[Lee] Seewoo Lee, *Algebraic proof of modular form inequalities for optimal sphere packings*, `https://arxiv.org/abs/2406.14659`.

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[CKMRV22] Henry Cohn, Abhinav Kumar, Stephen Miller, Danylo Radchenko, Maryna Viazovska, *Universal optimality of the E_8 and Leech lattices and interpolation formulas*, Annals of Mathematics 196 (2022), 983–1082, https://doi.org/10.4007/annals.2022.196.3.3.

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