Exercise 1

a) $T(n) = 4T(n/2) + n \Rightarrow T(n) = \Theta(n^2)$ - the first case of the Master theorem.
b) $T(n) = 4T(n/2) + n^2 \Rightarrow T(n) = \Theta(n^2 \lg n)$ - the second case of the Master theorem.
c) $T(n) = 4T(n/2) + n^3 \Rightarrow T(n) = \Theta(n^3)$ - the third case of the Master theorem.

From best to worst

- $\Theta(1)$
- $\Theta(log\ n)$
- $\Theta(\sqrt{n})$
- $\Theta(n^c)$, if $0 < c < 1$
- $\Theta(n)$
- $\Theta(n\ log\ n)$
- $\Theta(n^c)$
- $\Theta(c^n)$, if $c < 1$

Logarithm rules:

$$lg * n = log_2 n\ (binary\ \ logarithm)$$
$$ln * n = log_e n\ (naturallogarithm)$$
$$lg^k n = (lg * n)^k\ (exponentiation)$$
$$lg * lg * n = lg(lg * n)\ (composition)$$
$$n * lg^2\ n = n * lg^2(n)$$

CLRS 4.5-4

Here, $a = 4, b = 2$, and thus $n^{\lg_b a} = n^{\lg_2 4} = n^2$. Next, $f(n) = n^2 \lg n$.

Although $f(n) = n^2 \lg n$ grows faster than n^2, but not polynomially faster. Equivalently, there does not exist a constant ϵ such that $f(n) = \Omega(n^{2+\epsilon})$. Thus, case 3 cannot be applied here. We need to use either the repeated substitution method or the recursion tree method to solve the recurrence.

$T(n) = 4T(\frac{n}{2}) + n^2 \lg n$ (step 1)

$= 4(4T(\frac{n}{2^2}) + (\frac{n}{2})^2 \lg \frac{n}{2}) + n^2 \lg n = 4^2 T(\frac{n}{2^2}) + n^2(\lg n - 1) + n^2 \lg n$

$= 4^2 T(\frac{n}{2^2}) + 2n^2 \lg n - n^2$ (step 2)

$= 4^2(4T(\frac{n}{2^3}) + (\frac{n}{2^2})^2 \lg \frac{n}{2^2}) + 2n^2 \lg n - n^2 = 4^3 T(\frac{n}{2^3}) + n^2(\lg n - 2) + 2n^2 \lg n - n^2$

$= 4^3 T(\frac{n}{2^3}) + 3n^2 \lg n - (2+1)n^2$ (step 3)

$= 4^3(4T(\frac{n}{2^4}) + (\frac{n}{2^3})^2 \lg \frac{n}{2^3}) + 3n^2 \lg n - (2+1)n^2 = 4^4 T(\frac{n}{2^4}) + n^2(\lg n - 3) + 3n^2 \lg n - (2+1)n^2$

$= 4^4 T(\frac{n}{2^4}) + 4n^2 \lg n - (3+2+1)n^2$ (step 4)

$\ldots = 4^i T(\frac{n}{2^i}) + i * n^2 \lg n - n^2(1 + 2 + 3 + \ldots + (i-1))$ (step i)

$= 4^i T(\frac{n}{2^i}) + i * n^2 \lg n - n^2 \sum_{k=1}^{i-1} k = 4^i T(\frac{n}{2^i}) + i * n^2 \lg n - n^2 \frac{i*(i-1)}{2}$

The last step uses Eq.A.1 CLRS p 1146, where you need to replace n by i-1.

We need to identify a i such that $n/2^i = 1$, and we have $i = \lg n$. Next, we take $i = \lg n$ back to $T(n)$, we have $4^i T(\frac{n}{2^i}) + i * n^2 \lg n - n^2 \frac{i*(i-1)}{2} = 4^{\lg n} T(\frac{n}{2^{\lg n}}) + \lg n * n^2 \lg n - n^2 \frac{\lg n*(\lg n - 1)}{2}$
$= n^2 T(1) + 0.5 * n^2 \lg^2 n + 0.5 * n^2 \lg n = \Theta(n^2 \lg^2 n)$.

Or, you can check CLRS 4.6-2.