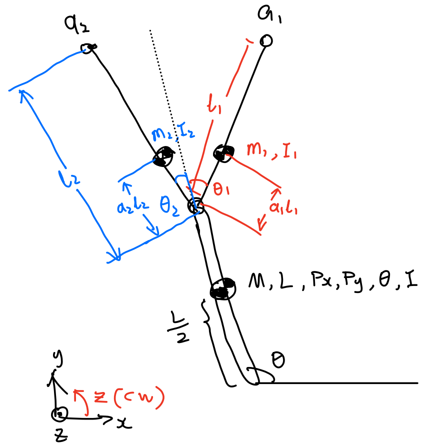

$$T = \frac{1}{2}(\dot{P}_x+\dot{P}_y)^T M (\dot{P}_x+\dot{P}_y) + \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2}\begin{bmatrix}\dot{P}_x & \dot{P}_y & \dot{\theta}\end{bmatrix}\begin{bmatrix} M & & \\ & M & \\ & & I\end{bmatrix}\begin{bmatrix}\dot{P}_x \\ \dot{P}_y \\ \dot{\theta}\end{bmatrix}$$

$$P_{x_1} = P_x - \cos\theta \cdot \frac{L}{2} - \cos(\theta+\theta_1)\, a_1 l_1$$

$$P_{y_1} = P_y + \sin\theta \frac{L}{2} + \sin(\theta+\theta_1)\, a_1 l_1$$

$$\varphi_1 = \theta+\theta_1$$

$$\Rightarrow \begin{bmatrix} V_{x_1} \\ V_{y_1} \\ \dot{\varphi}_1 \end{bmatrix} = \begin{bmatrix}\dot{P}_x \\ \dot{P}_y \\ \dot{\theta}\end{bmatrix} + \underbrace{\begin{bmatrix} \frac{L}{2}\sin\theta\,\dot{\theta} + a_1 l_1 \sin(\theta+\theta_1)(\dot{\theta}+\dot{\theta}) \\ \frac{L}{2}\cos\theta\,\dot{\theta} + a_1 l_1 \cos(\theta+\theta_1)(\dot{\theta}+\dot{\theta}_1) \\ \dot{\theta}_1 \end{bmatrix}}_{V_1}$$

$$P_{x_2} = P_x - \cos\theta \cdot \frac{L}{2} - \cos(\theta+\theta_2)\, a_2 l_2$$

$$P_{y_2} = P_y + \sin\theta \frac{L}{2} + \sin(\theta+\theta_2)\, a_2 l_2$$

$$\varphi_2 = \theta+\theta_2$$

$$\Rightarrow V_{x_2} = \dot{P}_x + \frac{L}{2}\sin\theta\,\dot{\theta} + a_2 l_2 \sin(\theta+\theta_2)(\dot{\theta}+\dot{\theta}_2)$$

$$V_{y_2} = \dot{P}_y + \frac{L}{2}\cos\theta\,\dot{\theta} + a_2 l_2 \cos(\theta+\theta_2)(\dot{\theta}+\dot{\theta}_2)$$

$$\dot{\varphi}_2 = \dot{\theta} + \dot{\theta}_2$$

$$\Rightarrow \begin{bmatrix} V_{x_2} \\ V_{y_2} \\ \dot{\varphi}_2 \end{bmatrix} = \begin{bmatrix}\dot{P}_x \\ \dot{P}_y \\ \dot{\theta}\end{bmatrix} + \underbrace{\begin{bmatrix} \frac{L}{2}\sin\theta\,\dot{\theta} + a_2 l_2 \sin(\theta+\theta_2)(\dot{\theta}+\dot{\theta}_2) \\ \frac{L}{2}\cos\theta\,\dot{\theta} + a_2 l_2 \cos(\theta+\theta_2)(\dot{\theta}+\dot{\theta}_2) \\ \dot{\theta}_2 \end{bmatrix}}_{V_2}$$

$$T = \underbrace{\frac{1}{2}\begin{bmatrix}\dot{P}_x & \dot{P}_y & \dot{\theta}\end{bmatrix}\begin{bmatrix} M+m_1+m_2 & & \\ & M+m_1+m_2 & \\ & & I+I_1+I_2\end{bmatrix}\begin{bmatrix}\dot{P}_x \\ \dot{P}_y \\ \dot{\theta}\end{bmatrix}}_{T_0} + \underbrace{\frac{1}{2} V_2^T \begin{bmatrix} m_2 & & \\ & m_2 & \\ & & I_2\end{bmatrix} V_2}_{T_2} + \underbrace{\frac{1}{2} V_1^T \begin{bmatrix} m_1 & & \\ & m_1 & \\ & & I_1\end{bmatrix} V_1}_{T_1} \quad (1)$$

$$\frac{1}{2} V_2^T \begin{bmatrix} m_2 & & \\ & m_2 & \\ & & I_2\end{bmatrix}$$

$$L = T - V$$

$$q = \begin{bmatrix}\theta_1 \\ \theta_2 \\ P_x \\ P_y \\ \theta\end{bmatrix} \qquad \tau = \begin{bmatrix}\tau_1 \\ \tau_2 \\ 0 \\ 0 \\ 0\end{bmatrix} \qquad \tau = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + A^T \lambda$$

$$V = Mg\,P_y + m_2 g\left(P_y + \frac{2}{L}\sin\theta + a_2 l_2 \sin(\theta+\theta_2)\right) + m_1 g\left(P_y + \frac{2}{L}\sin\theta + a_1 l_1 \sin(\theta+\theta_1)\right)$$

$$\frac{\partial V}{\partial q} = \begin{bmatrix} m_1 g\, a_1 l_1 \cos(\theta+\theta_1) \\ m_2 g\, a_2 l_2 \cos(\theta+\theta_2) \\ 0 \\ Mg + m_2 g + m_1 g \\ m_1 g\left[\frac{2}{L}\cos\theta + a_1 l_1 \cos(\theta+\theta_1)\right] + m_2 g\left[\frac{2}{L}\cos\theta + a_2 l_2 \cos(\theta+\theta_2)\right] \end{bmatrix}$$

$$\frac{\partial V}{\partial \dot{q}} = 0$$

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$$\underbrace{\begin{bmatrix} \frac{L}{2}\sin\theta\,\dot\theta + a_1 l_1 \sin(\theta+\theta_1)(\dot\theta_1+\dot\theta) \\ \frac{L}{2}\cos\theta\,\dot\theta + a_1 l_1 \cos(\theta+\theta_1)(\dot\theta+\dot\theta_1) \\ \dot\theta_1 \end{bmatrix}}_{V_1}$$

$$T_1 = \frac{1}{2} m_1 \left[\left[\frac{L}{2}\sin\theta\,\dot\theta + a_1 l_1 \sin(\theta+\theta_1)(\dot\theta+\dot\theta_1)\right]^2 + \left[\frac{L}{2}\cos\theta\,\dot\theta + a_1 l_1 \cos(\theta+\theta_1)(\dot\theta+\dot\theta_1)\right]^2 \right] + \frac{1}{2} I_1 \dot\theta_1^2$$

$\cos(\theta_1)(\dot\theta_1+\dot\theta)\,\dot\theta$

$$= \frac{1}{2} m_1 \left(\frac{L^2}{4}\dot\theta^2 + a_1^2 l_1^2 (\dot\theta+\dot\theta_1)^2 + L a_1 l_1 \underline{\left[\sin\theta\,\dot\theta \sin(\theta+\theta_1)(\dot\theta+\dot\theta_1) + \cos\theta\,\dot\theta\cos(\theta+\theta_1)(\dot\theta+\dot\theta_1)\right]} \right) + \frac{1}{2} I_1 \dot\theta_1^2$$

$$\frac{\partial T_1}{\partial q} = \begin{bmatrix} \frac{1}{2} m_1 L a_1 l_1 \left[\sin\theta\,\dot\theta\cos(\theta+\theta_1)(\dot\theta+\dot\theta_1) - \cos\theta\,\dot\theta \sin(\theta+\theta_1)(\dot\theta+\dot\theta_1)\right] \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} m_1 L a_1 l_1 \cdot \sin\theta_1 (\dot\theta+\dot\theta_1)\dot\theta \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\frac{\partial T_1}{\partial \dot{q}} = \begin{bmatrix} I_1 \dot\theta_1 + m_1 a_1^2 l_1^2 (\dot\theta+\dot\theta_1) + m_1 L a_1 l_1 \left(\sin\theta\,\dot\theta\sin(\theta+\theta_1) + \cos\theta\,\dot\theta\cos(\theta_1+\theta)\right) \\ 0 \\ 0 \\ 0 \\ m_1 a_1^2 l_1^2 (\dot\theta+\dot\theta_1) + m_1 L a_1 l_1 \left(\cos(\theta_1)\dot\theta_1 + 2\cos(\theta_1)\dot\theta\right) \end{bmatrix}$$

$$\frac{\partial T_2}{\partial q} = \begin{bmatrix} 0 \\ \frac{1}{2} m_2 L a_2 l_2 \left[\sin\theta\, \dot\theta \cos(\theta+\theta_2)(\dot\theta+\dot\theta_2) - \cos\theta\, \dot\theta \sin(\theta+\theta_2)(\dot\theta+\dot\theta_2) \right] \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} m_2 L a_2 l_2 \cdot \sin\theta\, (\dot\theta+\dot\theta_2)\dot\theta \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (4)$$

$$\frac{\partial T_2}{\partial \dot q} = \begin{bmatrix} 0 \\ I_2 \dot\theta_2 + m_2 a_2^2 l_2^2 (\dot\theta+\dot\theta_2) + m_2 L a_2 l_2 \big(\underbrace{\sin\theta\, \dot\theta \sin(\theta+\theta_2) + \cos\theta\, \dot\theta \cos(\theta_2+\theta)}_{\cos\theta_2 \dot\theta} \big) \\ 0 \\ 0 \\ m_2 a_2^2 l_2^2 (\dot\theta+\dot\theta_2) + m_2 L a_2 l_2 \big(\underbrace{\cos(\theta_2)\dot\theta_2 + 2\cos(\theta_2)\dot\theta}_{\cos\theta_2 (2\dot\theta+\dot\theta_2)} \big) \end{bmatrix}$$

$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot q}\right) = \frac{d}{dt} \begin{bmatrix} I_1 \dot\theta_1 + m_1 a_1^2 l_1^2 (\dot\theta+\dot\theta_1) + m_1 L a_1 l_1 \big(\underbrace{\sin\theta\, \dot\theta \sin(\theta+\theta_1) + \cos\theta\, \dot\theta \cos(\theta_1+\theta)}_{\cos\theta_1 \dot\theta} \big) \\ 0 \\ 0 \\ 0 \\ m_1 a_1^2 l_1^2 (\dot\theta+\dot\theta_1) + m_1 L a_1 l_1 \big(\cos(\theta_1)\dot\theta_1 + 2\cos(\theta_1)\dot\theta \big) \end{bmatrix}$$

$$= \begin{bmatrix} I_1 \ddot\theta_1 + m_1 a_1^2 l_1^2 (\ddot\theta+\ddot\theta_1) + m_1 a_1 l_1 L (\cos\theta_1)\ddot\theta - m_1 a_1 l_1 L \dot\theta \dot\theta_1 \sin\theta_1 \\ 0 \\ 0 \\ 0 \\ m_1 a_1^2 l_1^2 (\ddot\theta+\ddot\theta_1) + m_1 a_1 l_1 L \left[\cos(\theta_1)(2\ddot\theta+\ddot\theta_1) - (2\dot\theta+\dot\theta_1)\sin\theta_1 \dot\theta_1 \right] \end{bmatrix} \quad (5)$$

$$\frac{d}{dt}\left(\frac{\partial T_2}{\partial \dot q}\right) = \frac{d}{dt} \begin{bmatrix} 0 \\ I_2 \dot\theta_2 + m_2 a_2^2 l_2^2 (\dot\theta+\dot\theta_2) + m_2 L a_2 l_2 \big(\underbrace{\sin\theta\, \dot\theta \sin(\theta+\theta_2) + \cos\theta\, \dot\theta \cos(\theta_2+\theta)}_{\cos\theta_2 \dot\theta} \big) \\ 0 \\ 0 \\ m_2 a_2^2 l_2^2 (\dot\theta+\dot\theta_2) + m_2 L a_2 l_2 \big(\underbrace{\cos(\theta_2)\dot\theta_2 + 2\cos(\theta_2)\dot\theta}_{\cos\theta_2 (2\dot\theta+\dot\theta_2)} \big) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ I_2 \ddot\theta_2 + m_2 a_2^2 l_2^2 (\ddot\theta+\ddot\theta_2) + m_2 L a_2 l_2 \cos\theta_2 \ddot\theta - m_2 a_2 l_2 L \sin\theta_2 \dot\theta \\ 0 \\ 0 \\ m_2 a_2^2 l_2^2 (\ddot\theta+\ddot\theta_2) + m_2 L a_2 l_2 \left(\cos(\theta_2)(\ddot\theta_2 + 2\ddot\theta) - \sin\theta_2 \dot\theta_2 (2\dot\theta+\dot\theta_2) \right) \end{bmatrix} \quad (6)$$

Constraint: 2 arms on bar

$$a_1: \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \left(\begin{bmatrix} P_x \\ P_y \end{bmatrix} + \begin{bmatrix} -\cos\theta \frac{L}{2} - \cos(\theta+\theta_1)\, l_1 \\ \sin\theta \frac{L}{2} + l_1 \sin(\theta+\theta_1) \end{bmatrix} \right) \cdot u_1$$

$u_1 = 0 \Rightarrow$ not active

$u_1 = 1 \Rightarrow$ active

$$a_2: \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \left(\begin{bmatrix} P_x \\ P_y \end{bmatrix} + \begin{bmatrix} -\cos\theta \frac{L}{2} - \cos(\theta+\theta_2)\, l_2 \\ \sin\theta \frac{L}{2} + l_1 \sin(\theta+\theta_2) \end{bmatrix} \right) \cdot u_2$$

$u_2 = 0 \Rightarrow$ not active

$u_2 = 1 \Rightarrow$ active

$$\frac{\partial a_1}{\partial t} = 0 \Rightarrow \begin{bmatrix} \dot{P}_x + \frac{L}{2}\sin\theta\dot\theta + l_1 \sin(\theta+\theta_1)(\dot\theta+\dot\theta_1) \\ \dot{P}_y - \frac{L}{2}\cos\theta\dot\theta - l_1\cos(\theta+\theta_1)(\dot\theta+\dot\theta_1) \end{bmatrix} = 0 \Rightarrow \overbrace{\begin{bmatrix} l_1\sin(\theta+\theta_1) & 0 & 1 & 0 & \frac{L}{2}\sin\theta + l_1\sin(\theta+\theta_1) \\ -l_1\cos(\theta+\theta_1) & 0 & 0 & 1 & -\frac{L}{2}\cos\theta - l_1\cos(\theta+\theta_1) \end{bmatrix}}^{(A_1)} \dot{q} = 0$$

$$\frac{\partial a_2}{\partial t} = 0 \Rightarrow \begin{bmatrix} \dot{P}_x + \frac{L}{2}\sin\theta\dot\theta + l_2 \sin(\theta+\theta_2)(\dot\theta+\dot\theta_2) \\ \dot{P}_y - \frac{L}{2}\cos\theta\dot\theta - l_2\cos(\theta+\theta_2)(\dot\theta+\dot\theta_2) \end{bmatrix} = 0 \Rightarrow \overbrace{\begin{bmatrix} 0 & l_2\sin(\theta+\theta_2) & 1 & 0 & \frac{L}{2}\sin\theta + l_2\sin(\theta+\theta_2) \\ 0 & -l_2\cos(\theta+\theta_2) & 0 & 1 & -\frac{L}{2}\cos\theta - l_2\cos(\theta+\theta_2) \end{bmatrix}}^{(A_2)} \dot{q} = 0$$

$$\Rightarrow A^T = \begin{bmatrix} l_1\sin(\theta+\theta_1) & -l_1\cos(\theta+\theta_1) & 0 & 0 \\ 0 & 0 & l_2\sin(\theta+\theta_2) & -l_2\cos(\theta+\theta_2) \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \frac{L}{2}\sin\theta + l_1\sin(\theta+\theta_1) & -\frac{L}{2}\cos\theta - l_1\cos(\theta+\theta_1) & \frac{L}{2}\sin\theta + l_2\sin(\theta+\theta_2) & -\frac{L}{2}\cos\theta - l_2\cos(\theta+\theta_2) \end{bmatrix}$$

$$\lambda = \begin{bmatrix} u_1 \lambda_{x_1} \\ u_1 \lambda_{y_1} \\ u_2 \lambda_{x_2} \\ u_2 \lambda_{y_2} \end{bmatrix}, \quad u_i = \begin{cases} 1, & \text{active} \\ 0, & \text{inactive} \end{cases}$$

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$$5 = \begin{bmatrix} I_1\ddot\theta_1 + m_1 a_1^2 l_1^2(\ddot\theta+\ddot\theta_1) + m_1 a_1 l_1 L(\cos\theta_1)\ddot\theta - m_1 a_1 l_1 L\dot\theta\dot\theta_1\sin\theta_1 \\ 0 \\ 0 \\ 0 \\ m_1 a_1^2 l_1^2(\ddot\theta+\ddot\theta_1) + m_1 a_1 l_1 L\left[\cos(\theta_1)(2\ddot\theta+\ddot\theta_1) - (2\dot\theta+\dot\theta_1)\sin\theta_1\dot\theta_1\right] \end{bmatrix}$$

$$= \begin{bmatrix} I_1 + m_1 a_1^2 l_1^2 & 0 & 0 & 0 & (m_1 a_1^2 l_1^2 + m_1 a_1 l_1 L\cos\theta_1) \\ 0_{4\times1} & 0_{4\times1} & 0_{4\times1} & 0_{4\times1} & 0_{4\times1} \end{bmatrix} \ddot{q} +$$

$$\begin{bmatrix} m_1 a_1 l_1 L\sin\theta_1\dot\theta & 0 & 0 & 0 & m_1 a_1 l_1 L\sin\theta_1\dot\theta_1 \\ 0_{4\times1} & 0_{4\times1} & 0_{4\times1} & 0_{4\times1} & 0_{4\times1} \end{bmatrix} \dot{q}$$

$$
⑥ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I_2 + m_2 a_2^2 l_2^2 & 0 & 0 & \left(m_2 a_2^2 l_2^2 + m_2 a_2 l_2 L \cos\theta_2\right) \\ 0_{3\times1} & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} \end{bmatrix} \ddot{q}
$$

$$
+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & m_2 a_2 l_2 L \sin\theta_2 \dot{\theta} & 0 & 0 & m_2 a_2 l_2 L \sin\theta_2 \dot{\theta}_2 \\ 0_{3\times1} & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} \end{bmatrix} \dot{q}
$$

$$
\frac{\partial T_0}{\partial q} = 0 \qquad \frac{d}{dt}\frac{\partial T_0}{\partial \dot{q}} = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & M+m_1+m_2 & & \\ & & & M+m_1+m_2 & \\ & & & & I+I_1+I_2 \end{bmatrix} \ddot{q} \qquad ⑧
$$

$$
③ = \begin{bmatrix} -\frac{1}{2} m_1 L a_1 l_1 \cdot \sin\theta_1 (\dot{\theta}+\dot{\theta}_1)\dot{\theta} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
= \left[\begin{array}{c:c:c:c:c} -\frac{1}{2} m_1 L a_1 l_1 \sin\theta_1 \dot{\theta}^2 & 0 & 0 & 0 & -\frac{1}{2} m_1 L a_1 l_1 \sin\theta_1 (\dot{\theta}+\dot{\theta}_1) \\ 0_{4\times1} & 0_{4\times1} & 0_{4\times1} & 0_{4\times1} & 0_{4\times1} \end{array}\right] \dot{q}
$$

$$
④ = \begin{bmatrix} 0 \\ -\frac{1}{2} m_2 L a_2 l_2 \cdot \sin\theta (\dot{\theta}+\dot{\theta}_2)\dot{\theta} \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
= \left[\begin{array}{c:c:c:c:c} 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} m_2 L a_2 l_2 \sin\theta_2 \dot{\theta}^2 & 0 & 0 & -\frac{1}{2} m_2 L a_2 l_2 \sin\theta_2 (\dot{\theta}+\dot{\theta}_2) \\ 0_{3\times1} & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} \end{array}\right] \dot{q}
$$

EOM:

$$\underline{\tau} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + A^T\lambda$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 5 + 6 + 8 - (3 + 4 + 2) + \underbrace{A^T\lambda}_{\text{from } 7}$$

$$q = \begin{bmatrix} \theta_1 \\ \theta_2 \\ p_x \\ p_y \\ \theta \end{bmatrix}$$