A Lagrangian is dependent on a given function f:

$$L[f] = L\left(x, f(x), f'(x)\right) \tag{1}$$

The Action is defined as a path integral for the Lagrangian:

$$A[f] = \int_a^b L[f]dx \tag{2}$$

Suppose $f(x)$ extremizes A. Then let's define ζ as small arbitrary perturbation of f:

$$\zeta(x) = f(x) + \epsilon\eta(x) \tag{3}$$

- The $\eta(x)$ is the difference between f and its perturbation ζ. Since ζ must start on a and end on b just as f, this requires that $\eta(a) = \eta(b) = 0$.
- The ϵ here is a small constant. By definition, $\zeta \to f$ as $\epsilon \to 0$.

If we analyze the perturbed Action

$$A[\zeta] = \int_a^b L\left(\zeta\right) dx \tag{4}$$

then we see as $\epsilon \to 0$, $A[\zeta] \to A[f]$.

Since $A[f]$ is an extremum, then it must be the case that:

$$\lim_{\epsilon \to 0} \frac{dA[\zeta]}{d\epsilon} = 0 \tag{5}$$

Let's see how we can evaluate the total derivative via chain rule:

$$\frac{dA[\zeta]}{d\epsilon} = \frac{d}{d\epsilon}\int_a^b L[\zeta]dx \tag{6}$$

$$= \int_a^b \frac{dL}{d\epsilon}dx \tag{7}$$

$$= \int_a^b \left[\frac{dx}{d\epsilon}\frac{\partial L}{\partial x} + \frac{d\zeta}{d\epsilon}\frac{\partial L}{\partial \zeta} + \frac{d\zeta'}{d\epsilon}\frac{\partial L}{\partial \zeta'}\right] dx \tag{8}$$

We can evaluate each total derivative:

$\dfrac{dx}{d\epsilon} = 0$	$\dfrac{d\zeta}{d\epsilon} = \eta(x)$	$\dfrac{d\zeta'}{d\epsilon} = \eta'(x)$
since x doesn't depend on ϵ	since $\zeta(x) = f(x) + \epsilon\eta(x)$	since $\zeta'(x) = f'(x) + \epsilon\eta'(x)$

Altogether then,

$$\frac{dA[\zeta]}{d\epsilon} = \int_a^b \left[\eta(x)\frac{\partial L}{\partial \zeta} + \eta'(x)\frac{\partial L}{\partial \zeta'}\right] dx \tag{9}$$

$$\lim_{\epsilon \to 0} \frac{dA[\zeta]}{d\epsilon} = \int_a^b \left[\eta(x)\frac{\partial L}{\partial f} + \eta'(x)\frac{\partial L}{\partial f'}\right] dx = 0 \tag{10}$$

where the last line goes to zero following from (5).

Now, we have information about η and its axes, but not η'.