ECON 310 - MIDTERM 1

Write a short answer and use graphs to support your argument. Use notations used in class and be sure to label everything if using graphs.

Question 1 (Consumer's Optimization Problem - 20 Points)

Consider a consumer who is assumed to have h units of time available, which can be allocated between leisure time, l, and time spent working, N^s. The time constraint for the consumer is $l + N^s = h$. The consumer derives utility from consumption C and leisure l, where $C > 0$ and $h \geq l \geq 0$. The preferences over consumption and leisure are defined by $U(C, l)$, which satisfies all three properties specified in the textbook. The consumer's budget constraint is given by

$$C = w(h - l) + \pi - T,$$

where π is dividend income and T is lump-sum taxes. Consider a consumer's utility maximization problem in which the consumer chooses consumption and leisure to maximize utility, taking market wage rate w, dividend incomes π, and tax payments T as given.

a) Draw the consumer's budget constraint in (C,l) space for the case in which taxes are less than dividend income, $(\pi - T) > 0$. **(5 points)**

b) Show that there is an optimal bundle of consumption and leisure that maximizes the consumer's utility given the budget constraint. Draw that bundle into the graph and write down the optimality condition that needs to hold at that point. **(5 points)**

c) Into the graph from question (b) add the following case. Suppose the government cuts taxes, T. What are the effects on the consumer's optimal choice of consumption? **(2.5 points)**

d) What are the effects on the consumer's optimal choice of leisure from case (c)? **(2.5 points)**

e) Go back to case (b) and redraw that picture. Now suppose that the market wage rate, w, increases. What are the effects on the consumer's labor supply? Draw a new copy of the graph from (b) and include this change into the figure. **(5 points)**

Question 2 (30 Points)

Consider a firm having the following production technology:

$$Y = zF(K, N) = zK^{\alpha}N^{1-\alpha},$$

where z is total factor productivity, K is private capital, N is labor and $\alpha \in (0, 1)$ is a parameter governing the income share of capital. Assuming that $K = \overline{K}$ is constant. There are competitive goods and labor markets to which the firm can sell its output at price 1 and from which it can hire labor at real wage rates w.

a) Write down the firm's profit function? **(2.5 points)**

b) Calculate the profit maximization condition for the firm. **(2.5 points)**

c) Draw the firms maximization problem by drawing a revenues and costs as two separate 'lines' into one graph. Mark the level of labor in this graph where profit is maximized. **(2.5 points)**

d) Show that the firm has a constant returns to scale (CRS) production technology. Either do this mathematically or use a numerical example. **(5 points)**

e) Show that the marginal product of labor MP_N is a decreasing function in N. Show this mathematically or use a numerical example. **(2.5 points)**

f) Suppose that the government subsidizes employment. That is, the government pays the firm x \$ for each unit of labor that the firm hires. Write down the new maximization problem of the firm. In the graph from part (c) draw the change caused by this policy and analyze how labor and profits are affected. **(5 points)**

g) Assume $z = 1, K = 20$, $\alpha = 0.3$, and $w = \$2$. Solve for the profit maximizing labor input. **(5 points)**

h) Assume $z = 1, K = 20$, $\alpha = 0.3$, and $w = \$2$. Solve for maximum profit in \$ when the government subsidizes labor with 1\$ per hired unit of labor. **(5 points)**

Question 3 (15 Points)

Consider an one period macroeconomic model which consists of a representative consumer with preferences $u(c_1, c_2) = 2\sqrt{c_1} + 2(c_2)^{\frac{1}{2}}$ where c_1 and c_2 are two consumption goods traded in this market with a price of p_1 and p_2. The consumer has an income of I irrespective of whether she is working or not. In addition the consumer has to pay lump sum taxes T to the government.

a) Write down the consumer's budget constraint.**(5 points)**

b) Write down the consumers complete optimization problem (without solving it)? **(5 points)**

c) Write down the consumers optimality condition (without solving for the optimal bundle). **(5 points)**

Question 4 (35 Points)

Consider an one period macroeconomic model which consists of a representative consumer with preferences $u(c_1, c_2) = 2\sqrt{c_1} + 2\sqrt{c_2}$ where c_1 and c_2 are two consumption goods traded in this market with a price of \$2 and \$1, respectively. The consumer has an income of \$100 irrespective of whether she is working or not. In addition, the consumer has to pay lump sum taxes of \$20 to the government.

a) Draw the consumer's budget constraint into a graph. **(5 points)**

b) Solve for the consumer's optimal consumption bundle (c_1^*, c_2^*) and put those numbers into the graph from part (a) **(5 points)**

c) What is the total level of utility at this optimal point, i.e., what is the welfare level in utils? **(2.5 points)**

d) The government decides to increase the lump sum tax by 5 dollars. Calculate the new optimal consumption bundle (c_1^{**}, c_2^{**}) and put this bundle into the graph from point (b). **(5 points)**

e) Draw a new graph with the consumer's budget constraint and the optimal bundle from part (a) and (b). Next assume that the government taxes consumption good c_2 with a proportional tax of 20 percent so that for every dollar spent on consumption good c_2 the government receives a share of 20 percent.

1) Write down the new budget constraint. **(2.5 points)**
2) Draw this new budget constraint into the picture. **(2.5 points)**
3) Re-optimize this problem and solve for the new optimal level of consumption for c_1 and c_2? **(10 points)**
4) How much does the government collect from this new tax? **(2.5 points)**

1 Solutions to question 2

The profit function is given by

$$\pi = zK^{\alpha}\left(N^{d}\right)^{(1-\alpha)} - wN^{d}.$$

The profit maximization condition for the firm is

$$MP_N = w.$$

So that the max problem of the firm is

$$\max_{N_d}\left\{zK^{\alpha}\left(N^{d}\right)^{(1-\alpha)} - wN^{d}\right\}$$

The FOC is

$$(1-\alpha)zK^{\alpha}N_d^{-\alpha)} = w$$

which solves to

$$N_d = \left(\frac{w}{(1-\alpha)zK^{\alpha}}\right)^{\frac{-1}{\alpha}}.$$

So that given $z = 1, K = 20$, $\alpha = 0.3$, and $w = \$2$

$\left(\frac{2}{(1-0.3)*20^{0.3}}\right)^{\frac{-1}{0.3}} = 0.604\,31$

So that profits are: $20^{0.3} * 0.60431^{0.7} - 2 * 0.60431 = 0.517\,98$

With the subsidy the firm maximizes

$$\max_{N_d}\left\{zK^{\alpha}\left(N^{d}\right)^{(1-\alpha)} - wN^{d} + xN^{d}\right\}$$

so that the FOC is

$$MP_L = w - x.$$

And the optimal labor is

$$N_d = \left(\frac{w-x}{(1-\alpha)zK^{\alpha}}\right)^{\frac{-1}{\alpha}}.$$

$\left(\frac{2-1}{(1-0.3)*20^{0.3}}\right)^{\frac{-1}{0.3}} = 6.091$

So that profits are: $1 * 20^{0.3} * 6.091^{0.7} - 2 * 6.091 + 1 * 6.091 = 2.610\,4$

2 Solutions to question 3

$$\begin{aligned} & \max_{c_1,c_2}\left\{2 \times c_1^{\frac{1}{2}} + 2 \times c_2^{\frac{1}{2}}\right\} \\ & s.t. \\ p_1c_1 + p_2c_2 &= I - T \end{aligned}$$

The optimality condition is

$$MRS = \frac{MU_{c_1}}{MU_{c2}} = \frac{p_1}{p_2},$$

which is

$$\frac{c_1^{-\frac{1}{2}}}{c_2^{-\frac{1}{2}}} = \frac{p_1}{p_2}.$$

In other words, the "bang-per-buck" has to be equal across the two goods at the optimal solution:

$$\frac{MU_{c_1}}{p_1} = \frac{MU_{c_2}}{p_2}.$$

Or you substitute and take the FOC

$$\max_{c_1} \left\{ 2 \times c_1^{\frac{1}{2}} + 2 \times \left[\frac{I-T}{p_2} - \frac{p_1}{p_2} c_1 \right]^{\frac{1}{2}} \right\},$$

so that FOC:

$$\frac{\partial \left(2 \times c_1^{\frac{1}{2}} + 2 \times \left[\frac{I-T}{p_2} - \frac{p_1}{p_2} c_1 \right]^{\frac{1}{2}} \right)}{\partial c_1} = 0.$$

3 Solutions to question 4

$$\begin{aligned} & \max_{c_1, c_2} \left\{ 2 \times c_1^{\frac{1}{2}} + 2 \times c_2^{\frac{1}{2}} \right\} \\ & s.t. \\ p_1 c_1 + p_2 c_2 \quad = \quad & I - T \end{aligned}$$

3.1 Method 1: Substitute into preferences.

$$c_2 = \frac{I-T}{p_2} - \frac{p_1}{p_2} c_1$$

$$\max_{c_1} \left\{ 2 \times c_1^{\frac{1}{2}} + 2 \times \left[\frac{I-T}{p_2} - \frac{p_1}{p_2} c_1 \right]^{\frac{1}{2}} \right\}$$

The FOC is

$$\frac{1}{\sqrt{c_1}} + \frac{1}{\sqrt{\frac{I-T}{p_2} - \frac{p_1}{p_2} c_1}} \left(-\frac{p_1}{p_2} \right) = 0.$$

Solve further

$$\frac{1}{\sqrt{c_1}} = \left(\frac{p_1}{p_2} \right) \frac{1}{\sqrt{\frac{I-T}{p_2} - \frac{p_1}{p_2} c_1}}$$

Square both sides

$$\begin{aligned}
\frac{1}{c_1} &= \frac{\left(\frac{p_1}{p_2}\right)^2}{\frac{I-T}{p_2} - \frac{p_1}{p_2}c_1} \\
\frac{I-T}{p_2} - \frac{p_1}{p_2}c_1 &= \left(\frac{p_1}{p_2}\right)^2 c_1 \\
\frac{I-T}{p_2} &= \left(\frac{p_1}{p_2}\right)^2 c_1 + \frac{p_1}{p_2}c_1 \\
c_1 &= \frac{\frac{I-T}{p_2}}{\left(\frac{p_1}{p_2}\right)^2 + \frac{p_1}{p_2}} \\
&= \frac{\frac{I-T}{p_2}}{\left(\frac{p_1}{p_2}+1\right)\frac{p_1}{p_2}} \\
&= \frac{I-T}{p_1\left(1+\frac{p_1}{p_2}\right)}
\end{aligned}$$

3.2 Method 2: Second way of solving this is

$$\begin{aligned}
MRS &= -\frac{MU_{c_1}}{MU_{c_2}} = -\frac{p_1}{p_2} \\
&\rightarrow \frac{MU_{c_1}}{MU_{c_2}} = \frac{p_1}{p_2}
\end{aligned}$$

which is

$$\begin{aligned}
& \max_{c_1,c_2}\left\{2 \times c_1^{\frac{1}{2}} + 2 \times c_2^{\frac{1}{2}}\right\} \\
& s.t. \\
p_1c_1 + p_2c_2 &= I - T
\end{aligned}$$

$$\begin{aligned}
\frac{c_2^{\frac{1}{2}}}{c_1^{\frac{1}{2}}} &= \frac{p_1}{p_2} \\
c_2 &= c_1\left(\frac{p_1}{p_2}\right)^2
\end{aligned}$$

Using the budget constraint

$$p_1c_1 + p_2c_1\left(\frac{p_1}{p_2}\right)^2 = I - T$$

we can solve for

$$\begin{aligned}
c_1 &= \frac{I-T}{p_1 + p_2\left(\frac{p_1}{p_2}\right)^2} \\
&= \frac{I-T}{p_1\left(1+\frac{p_1}{p2}\right)}
\end{aligned}$$

and

$$c_2 = \frac{I-T}{p_1\left(1+\frac{p_1}{p_2}\right)}\left(\frac{p_1}{p_2}\right)^2$$

3.3 Numerical solution

Using the numbers from above we get:

$c_1 = \frac{100-20}{2\left(1+\frac{2}{1}\right)} = 13.333$

and $c_2 = \frac{100-20}{2\left(1+\frac{2}{1}\right)}\left(-\frac{2}{1}\right)^2 = 53.333$

Check: $13.33333 * 2 + 53.33333 * 1 = 80.000$

Welfare is equal to:

$2 * \sqrt{13.33333} + 2 * \sqrt{53.333333} = 21.909$

Increase lump-sum tax by \$5 from \$20 to \$25 :

$c_1 = \frac{100-25}{2\left(1+\frac{2}{1}\right)} = 12.5$

and $c_2 = \frac{100-25}{2\left(1+\frac{2}{1}\right)}\left(-\frac{2}{1}\right)^2 = 50.0$

Check: $12.5 * 2 + 50 * 1 = 75.0$

With a 20% tax on good 2, the new maximization problem is:

$$\begin{aligned} & \max_{c_1,c_2}\left\{2\times c_1^{\frac{1}{2}} + 2\times c_2^{\frac{1}{2}}\right\} \\ & s.t. \\ p_1c_1 + 1.2\times p_2c_2 &= I-T \end{aligned}$$

The solution is

$$\begin{aligned} c_1 &= \frac{I-T}{p_1\left(1+\frac{p_1}{1.2\times p2}\right)}, \\ c_2 &= \frac{I-T}{p_1\left(1+\frac{p_1}{1.2\times p_2}\right)}\left(\frac{p_1}{1.2\times p_2}\right)^2 \end{aligned}$$

$c_1 = \frac{100-20}{2\left(1+\frac{2}{1.2}\right)} = 15.0$

and $c_2 = \frac{100-20}{2\left(1+\frac{2}{1.2}\right)}\left(\frac{2}{1.2}\right)^2 = 41.667$