ECON 310 - MIDTERM 2
Version B

Name: ______________________________

Write a short answer and use graphs to support your argument. Use notations used in class and be sure to label everything if using graphs.

Question 1 (25 Points)

Consider a Solow growth model where the population growth rate is 2%, the constant savings rate is 15%, and the capital depreciation rate is 10%. In addition you can assume a Cobb-Douglas production function in capital and labor with a capital share of production of $\alpha = 0.3$. Suppose that the total factor productiviy is normalized to 1.

a) Draw a graph of the law of motion of per capita capital holdings. (**5 points**)

b) Solve for the long-run steady state level of per capita capital k^*. (**5 points**)

c) What is the per capita consumption level in the steady state c^*? (**5 points**)

d) What is the maximum steady state per capita consumption level possible in this economy $c_{\max}$? (**5 points**)

e) What is the savings rate s that would result in this maximum per capita consumption level? (**5 points**)

Question 2 (30 Points)

Consider an one period macroeconomic model which consists of a representative consumer with preferences $u(c_1, c_2) = 2\sqrt{c_1} + 2\sqrt{c_2}$ where c_1 and c_2 are two consumption goods traded in this market with a price of \$4 and \$2, respectively. The consumer has an income of \$600 irrespective of whether she is working or not. In addition, the consumer has to pay lump sum taxes of \$50 to the government.

a) Write down the consumer's budget constraint.**(5 points)**

b) Write down the consumers complete optimization problem (without solving it)? **(5 points)**

c) Write down the consumers optimality condition (without solving for the optimal bundle). **(5 points)**

d) Draw the consumer's budget constraint into a graph then solve for the consumer's optimal consumption bundle (c_1^*, c_2^*) and put those numbers into the graph **(5 points)**

e) What is the total level of utility at this optimal point, i.e., what is the welfare level in utils? **(2.5 points)**

f) The government decides to increase the lump sum tax by 10 dollars. Calculate the new optimal consumption bundle (c_1^{**}, c_2^{**}) and put this bundle into the graph from point (d). **(7.5 points)**

Question 3 (25 Points)

Consider an endogenous growth model and suppose that there are three possible uses of time. Let u denote the fraction of time spent working, s the fraction of time being unemployed and doing nothing, and $1-u-s$ the fraction of time spent accumulating human capital. Assume that the production function is

$$Y = z \times u \times H,$$

where z is the total factor productivity. Human capital is produced according to

$$H' = b(1-u-s)H - d \times H,$$

where $d = 10\%$ is the depreciation rate of human capital in every period and b is a production parameter.

a) Draw a graph of the law of motion for human capital! (**5 points**)

b) Write down the parameter condition under which indefinite growth can occur. **(5 points)**

b) Assume $b = 6.0$ and $u = 0.7$. What is the threshold level of unemployment s above which indefinite growth will stop? **(5 points)**

c) Assume that $z = 2$, $b = 6.0$, $u = 0.7$, and $s = 0.15$. Also assume that the economy begins in period one with $H = 10$ units of human capital. Calculate human capital, consumption, and output for period 4. Be careful with the timing! (**5 points**)

d) Write down the firm problem. Then solve for the firm profit maximizing wage rate. (**5 points**)

Question 4 (20 Points)

Consider the following tax revenue function for a government:

$$REV\left(t\right) = t \times w \times \left(h - l\left(t\right)\right),$$

where $0 \leq t \leq 1$ is the labor tax rate, $w > 0$ is the wage rate, $h > 0$ is the maximum amount of time available to the household, and $l\left(t\right)$ is leisure as an increasing function of the tax rate i.e. if the labor tax rate t increases, leisure increases, so that individuals work less. Assume that

$$l\left(t\right) = \min\left[h, t^{1.5}\right].$$

a) Draw the tax revenue as a function of the labor tax. (**5 points**)

b) Calculate the tax revenue maximizing labor tax rate? (**5 points**)

c) How does the optimal labor tax rate change if w increases to w', where $w' > w$? (**5 points**)

c) Draw the tax revenue as a function of the wage rate. (**5 points**)

1 Solutions to question 1

The law of motion is

$$k' = \frac{szf(k)}{1+n} + \frac{1-d}{1+n}k.$$

At the steady state $\bar{k} = k' = k$ so that the law of motion becomes

$$\bar{k} = \frac{szf(\bar{k})}{1+n} + \frac{1-d}{1+n}\bar{k}.$$

Some cancellations result in

$$\bar{k}(d+n) = szf(\bar{k}).$$

Replacing the production function $f(k) = k^{\alpha}$ we get

$$\bar{k}(d+n) = sz\bar{k}^{\alpha},$$

which can be solved for

$$\bar{k} = \left(\frac{sz}{(d+n)}\right)^{\frac{1}{1-\alpha}}. \tag{1}$$

Steady state output is

$$\bar{y} = zf(\bar{k}) = z\bar{k}^{\alpha}$$

and steady state per capita consumption is a fixed fraction of income

$$\bar{c} = (1-s)\bar{y}$$

as is savings/investment per capita

$$\bar{\imath} = s\bar{y}.$$

The numerical solution is therefore: $\bar{k} = \left(\frac{0.15*1}{(0.1+0.02)}\right)^{\frac{1}{1-0.3}} = 1.3754$
$\bar{y} = 1.3754^{0.3} = 1.1003$
$\bar{c} = (1-0.15)*1.1003 = 0.93526$
$\bar{\imath} = 0.15*1.1003 = 0.16505$
To solve for $c_{\max}$ we need to maximize

$$\begin{aligned} c &= (1-s)y \\ &= y - sy \\ &= zf(k) - szf(k) \\ &= zf(k) - k(d+n) \end{aligned}$$

so

$$\max_k zf(k) - k(d+n)$$

results in FOC

$$zf'(k) = d+n$$

which using Cobb-Douglas is

$$z\alpha k^{\alpha-1} = d+n$$

which we can solve for

$$k_{gr}=\left(\frac{d+n}{z\times\alpha}\right)^{\frac{1}{\alpha-1}}.$$

The numerical solution is: $\left(\frac{0.1+0.02}{1*0.3}\right)^{\frac{1}{0.3-1}}=3.7024$ so that

$y_{gr}=3.7024^{0.3}=1.4810$

$c_{gr}=zf\left(k\right)-k\left(d+n\right)=1.4810-3.7024*\left(0.1+0.02\right)=1.0367$

Since

$$c=\left(1-s\right)y$$

we get

$$s=1-\frac{c}{y}$$

which is: s=$1-\frac{1.0367}{1.4810}=0.3$

Check using equation (1)

$$\bar{k}=\left(\frac{sz}{(d+n)}\right)^{\frac{1}{1-\alpha}}=\left(\frac{0.3*1}{(0.1+0.02)}\right)^{\frac{1}{1-0.3}}=3.7024=k_{gr}$$

$\left(\frac{0.3*1}{(0.1+0.02)}\right)^{\frac{1}{1-0.3}}=3.7024$ which is k_{gr}.

2 Solutions to question 2

$$\begin{aligned}&\max_{c_1,c_2}\left\{2\times c_1^{\frac{1}{2}}+2\times c_2^{\frac{1}{2}}\right\}\\&s.t.\\p_1c_1+p_2c_2&=I-T\end{aligned}$$

2.1 Method 1: Substitute into preferences.

$$c_2=\frac{I-T}{p_2}-\frac{p_1}{p_2}c_1$$

$$\max_{c_1}\left\{2\times c_1^{\frac{1}{2}}+2\times\left[\frac{I-T}{p_2}-\frac{p_1}{p_2}c_1\right]^{\frac{1}{2}}\right\}$$

The FOC is

$$\frac{1}{\sqrt{c_1}}+\frac{1}{\sqrt{\frac{I-T}{p_2}-\frac{p_1}{p_2}c_1}}\left(-\frac{p_1}{p_2}\right)=0.$$

Solve further

$$\frac{1}{\sqrt{c_1}}=\left(\frac{p_1}{p_2}\right)\frac{1}{\sqrt{\frac{I-T}{p_2}-\frac{p_1}{p_2}c_1}}$$

Square both sides

$$\begin{aligned}
\frac{1}{c_1} &= \frac{\left(\frac{p_1}{p_2}\right)^2}{\frac{I-T}{p_2} - \frac{p_1}{p_2}c_1} \\
\frac{I-T}{p_2} - \frac{p_1}{p_2}c_1 &= \left(\frac{p_1}{p_2}\right)^2 c_1 \\
\frac{I-T}{p_2} &= \left(\frac{p_1}{p_2}\right)^2 c_1 + \frac{p_1}{p_2}c_1 \\
c_1 &= \frac{\frac{I-T}{p_2}}{\left(\frac{p_1}{p_2}\right)^2 + \frac{p_1}{p_2}} \\
c_1 &= \frac{\frac{I-T}{p_2}}{\left(\frac{p_1}{p_2}+1\right)\frac{p_1}{p_2}} \\
c_1^* &= \frac{I-T}{p_1\left(1+\frac{p_1}{p_2}\right)}
\end{aligned}$$

2.2 Method 2: Second way of solving this is

$$\begin{aligned}
MRS &= -\frac{MU_{c_1}}{MU_{c_2}} = -\frac{p_1}{p_2} \\
&\rightarrow \frac{MU_{c_1}}{MU_{c_2}} = \frac{p_1}{p_2}
\end{aligned}$$

which is

$$\begin{aligned}
&\max_{c_1,c_2}\left\{2\times c_1^{\frac{1}{2}} + 2\times c_2^{\frac{1}{2}}\right\} \\
&s.t. \\
p_1c_1 + p_2c_2 &= I - T
\end{aligned}$$

$$\begin{aligned}
\frac{c_2^{\frac{1}{2}}}{c_1^{\frac{1}{2}}} &= \frac{p_1}{p_2} \\
c_2 &= c_1\left(\frac{p_1}{p_2}\right)^2
\end{aligned}$$

Using the budget constraint

$$p_1c_1 + p_2c_1\left(\frac{p_1}{p_2}\right)^2 = I - T$$

we can solve for

$$\begin{aligned} c_1 &= \frac{I-T}{p_1 + p_2\left(\frac{p_1}{p_2}\right)^2} \\ &= \frac{I-T}{p_1\left(1+\frac{p_1}{p2}\right)} \end{aligned}$$

and

$$c_2 = \frac{I-T}{p_1\left(1+\frac{p_1}{p_2}\right)}\left(\frac{p_1}{p_2}\right)^2$$

2.3 Numerical solution

\$4 and \$2, respectively. The consumer has an income of \$600 irrespective of whether she is working or not. In addition, the consumer has to pay lump sum taxes of \$50 to the government.

Using the numbers from above we get:
$c_1 = \frac{600-50}{4\left(1+\frac{4}{2}\right)} = 45.833$
and $c_2 = \frac{600-50}{4\left(1+\frac{4}{2}\right)}\left(-\frac{4}{2}\right)^2 = 183.33$
Check: $45.833 * 4 + 183.33 * 2 = 549.99$
Welfare is equal to:
$2 * \sqrt{45.833} + 2 * \sqrt{183.33} = 40.620$
Increase lump-sum tax by \$5 from \$50 to \$55 :
$c_1 = \frac{600-55}{4\left(1+\frac{4}{2}\right)} = 45.417$
and $c_2 = \frac{600-55}{4\left(1+\frac{4}{2}\right)}\left(-\frac{4}{2}\right)^2 = 181.67$
Check: $45.417 * 4 + 181.67 * 2 = 545.01$

3 Solutions to question 3

$$\begin{aligned} H' &= b(1-u-s)H - d \times H \\ H' &= (b(1-u-s)-d)\,H \end{aligned}$$

Growth occurs when

$$(b(1-u-s)-d) > 1.$$

So that growth stops whenever

$$\begin{aligned} (b(1-u-s)-d) &< 1 \\ 6(1-0.7-s)-0.1 &< 1 \\ 6*0.3-6s-0.1 &< 1 \\ 6*0.3-0.1-1 &< 6s \\ \frac{6*0.3-0.1-1}{6} &< s \\ 0.117 &< s \end{aligned}$$

$\frac{6*0.3-0.1-1}{6} = 0.11667$

$$\begin{aligned} H_4' &= (b(1-u-s)-d)^3 H \\ &= (6*(1-0.7-0.15)-0.1)^3 10 \end{aligned}$$

$H_4' = (6*(1-0.7-0.15)-0.1)^3 10 = 5.12$
$(6*(1-0.7-0.15)-0.1)^3 = 0.512$

4 Solutions to question 4

$$\begin{aligned} REV(t) &= t \times w \times (h - l(t)), \\ &= t \times w \times \left(h - t^{1.5}\right) \\ &= \left(t \times w \times h - w \times t^{2.5}\right) \end{aligned}$$

The FOC is

$$w \times h - 2.5wt^{1.5} = 0,$$

so that

$$\begin{aligned} t^* &= \left(\frac{wh}{2.5w}\right)^{\frac{1}{1.5}} \\ t^* &= \left(\frac{h}{2.5}\right)^{\frac{1}{1.5}} \end{aligned}$$