# 1 Question 1

Solve the following model for equilibrium. Preferences are given as

$$u(c_1, c_2) = \ln(c_1) + \beta \ln(c_2),$$

where $c_1$ is consumption today and $c_2$ is consumption tomorrow, and $\beta$ is the time preference factor. The young household supplies one unit of labor inelastically and earns the wage rate $w$ for it. In addition, the household pays a lump sum tax of $t_1$ in period one and $t_2$ in period 2. There are $N_1$ young households in the economy and $N_2$ old households.

There is a competitive firm that produces output using the following production function:

$$F(K, L) = A\left(K^{\alpha} + L^{1-\alpha}\right),$$

where $A$ is the total factor productivity, $K$ is aggregate stock of capital, and $L$ is aggregate labor. Assume that capital depreciates fully between the two periods so that $\delta = 100\%$.

- Set up the budget constraint of a household.
- Solve the household problem

# 2 Solution:

$$\begin{aligned} c_1 + s &= (1-t)\,w, \\ c_2 &= (1+r)\,s. \end{aligned}$$

The Lagrangian is:

$$\begin{aligned} L(c_1, c_2, s, \lambda, \mu) &= \ln(c_1) + \beta \ln(c_2) \\ &\quad +\lambda\left((1-t)\,w - c_1 - s\right) \\ &\quad +\mu\left((1+r)\,s - c_2\right). \end{aligned}$$

The FOCSs are:

$$\begin{aligned} \partial c_1 &: \frac{1}{c_1} = \lambda, \\ \partial c_2 &: \frac{\beta}{c_2} = \mu, \\ \partial s &: \lambda = \mu(1+r), \\ \partial \lambda &: c_1 + s = (1-t)\,w, \\ \partial \mu &: c_2 = (1+r)\,s. \end{aligned}$$

This is a system of 5 equations in 5 unknowns: $c_1, c_2, s, \lambda, \mu$. Combine the first two equations and put them into the third:

$$\begin{aligned}
\frac{1}{c_1} &= \frac{\beta}{c_2}(1+r), \\
c_1 + s &= (1-t)\,w, \\
c_2 &= (1+r)\,s.
\end{aligned}$$

We have now 3 equations in 3 unknowns. The first equation is called the Euler equation and it relates today's to tomorrow's consumption - it's an intertemporal optimality condition. Next substitute savings out using the second and third equation. From (2) we have

$$s = (1-t)\,w - c_1.$$

Plug this into (3) and get:

$$\begin{aligned}
c_2 &= c_1\beta(1+r) \\
c_2 &= (1+r)\left((1-t)\,w - c_1\right).
\end{aligned}$$

This is a system of 2 equations in 2 unknowns. Now set the two equations equal to each other and solve for $c_1$ :

$$c_1\beta(1+r) = (1+r)\left((1-t)\,w - c_1\right).$$

Bring all $c_1$ on one side and cancel $(1+r)$:

$$c_1\beta + c_1 = (1-t)\,w.$$

The collect $c_1$ and solve for it:

$$\begin{aligned}
c_1 &= \frac{(1-t)\,w}{\beta+1}, \\
&= \frac{1}{1+\beta}(1-t)\,w.
\end{aligned}$$

So that

$$\begin{aligned}
c_2 &= c_1 \times \beta(1+r), \\
&= \frac{(1-t)\,w}{1+\beta} \times \beta(1+r), \\
&= \frac{\beta}{1+\beta}(1-t)(1+r)\,w,
\end{aligned}$$

and savings is

$$\begin{aligned}
s &= (1-t)\,w - c_1 \\
&= (1-t)\,w - \frac{1}{1+\beta}(1-t)\,w, \\
&= \left(1 - \frac{1}{1+\beta}\right)(1-t)\,w,, \\
&= \frac{\beta}{1+\beta}(1-t)\,w.
\end{aligned}$$

Since there is no other income than savings income in the second period, savings is independent of the interest rate $r$.

From the firm problem

$$\max_{K,L} = A\left(K^{\alpha} + L^{1-\alpha}\right) - qK - wL,$$

where $q$ is the cost of capital and $w$ is the cost of labor. We get the following FOCs:

$$\begin{aligned} \partial K &: \alpha A K^{\alpha-1} = q, \\ \partial L &: (1-\alpha) A L^{-\alpha} = w. \end{aligned}$$

In equilibrium we then have:

$$\begin{aligned} c_1 &= \frac{1}{1+\beta}(1-t)w, \\ c_2 &= \frac{\beta}{1+\beta}(1-t)(1+r)w, \\ s &= \frac{\beta}{1+\beta}(1-t)w, \\ N_1 s &= K, \\ \alpha A K^{\alpha-1} &= q, \\ (1-\alpha) A L^{-\alpha} &= w, \\ q &= r+\delta, \\ L &= N_1, \\ Y &= A\left(K^{\alpha} + L^{1-\alpha}\right), \\ G &= tw. \end{aligned}$$

which we solve for $c_1, c_2, s, K, q, w, r, L, Y, G$ because $N_1, N_2, \delta, t, \beta, \alpha, A$ are all given exogenous variables. We next solve this system of 10 equations in 10 unknowns. Remember that this has to be consistent with the aggregate resource constraint:

$$N_1 c_1 + N_2 c_2 + N_1 s + G = Y.$$

So let's solve this. Take the following subsystem in $s, K, q, w, r, L$:

$$\begin{aligned} s &= \frac{\beta}{1+\beta}(1-t)w, \\ N_1 s &= K, \\ \alpha A K^{\alpha-1} &= q, \\ (1-\alpha) A L^{-\alpha} &= w, \\ q &= r+\delta, \\ L &= N_1, \end{aligned}$$

in Since $\delta = 1$ we have $q = r + \delta$, and substituting $L = N_1$:

$$\begin{aligned} N_1 \times s &= K, \\ \alpha A K^{\alpha-1} - \delta &= r, \\ (1-\alpha) A N_1^{-\alpha} &= w, \\ s &= \frac{\beta}{1+\beta}(1-t) w, \end{aligned}$$

which we solve for $K, s, w, r$. Subsitute savings into the first equation

$$\begin{aligned} N_1 \times \frac{\beta}{1+\beta}(1-t) w &= K, \\ \alpha A K^{\alpha-1} &= r, \\ (1-\alpha) A N_1^{-\alpha} &= w, \end{aligned}$$

jjjwhich we solve for $K, w, r$. We can immediately solve this for the wage rate

$$\bar{w} = (1-\alpha) A N_1^{-\alpha}$$

so that the system reduces to

$$\begin{aligned} N_1 \frac{\beta}{1+\beta}(1-t) \bar{w} &= K, \\ \alpha A K^{\alpha-1} - \delta &= r, \end{aligned}$$

which we solve for $r$ and $K$. So that $r$ is

$$\begin{aligned} r &= \alpha A K^{\alpha-1} - \delta, \\ r &= \alpha A \left( N_1 \frac{\beta}{1+\beta}(1-t) \bar{w} \right)^{a-1} - \delta. \end{aligned}$$

We can now express all other endogenous variables in terms of $r$ and solve for everything.

## 2.1 Numerical example:

$\beta = 0.99, \alpha = 0.3, A = 1, N_1 = 1, N_2 = 1, t = 0.1$

$\bar{w} = (1-\alpha) A N_1^{-\alpha} = (1-0.3) = 0.7$
$r = \alpha A \left( N_1 \frac{\beta}{1+\beta}(1-t) \bar{w} \right)^{a-1} - \delta = 0.3 * 1 * \left( 1 * \frac{0.99}{1+0.99} * (1-0.1) * 0.7 \right)^{(0.3-1)} - 1 = -0.324\,17$
$s = \frac{\beta}{1+\beta}(1-t) w = \frac{0.99}{1+0.99} * (1-0.1) * 0.7 = 0.31342$
$c_1 = \frac{1}{1+\beta}(1-t) w = \frac{1}{1+0.99} * (1-0.1) * 0.7 = 0.31658$
$c_2 = \frac{\beta}{1+\beta}(1+r)(1-t) w = \frac{0.99}{1+0.99} * (1-0.324\,17) * (1-0.1) * 0.7 = 0.211\,82$
$N_1 \times s = K = 0.31342$

$G = tw = 0.1 * 0.7 = 0.07$
$Y = A\left(K^{\alpha} + L^{1-\alpha}\right) = 0.31342^{0.3} + 1^{0.7} = 1.706\,1$
Profits $= Y - qK - wL = 1.7061 - 0.7 * 1 - 0.67583 * 0.31342 = 0.794\,28$
$N_1 c_1 + N_2 c_2 + N_1 s + G +$Profits$= Y$
$0.31658 + 0.21182 + 0.31342 + 0.07 + 0.79428 = 1.706\,1$

# 3 Example in book

$$\begin{aligned} & \max u\left(c, c', l, l'\right) \\ & s.t. \\ c + \frac{c'}{1+r} &= w\left(1-l\right) + \pi - T + \frac{w'\left(1-l'\right) + \pi' - T'}{1+r} \end{aligned}$$

The Lagrangian is:

$$\begin{aligned} L\left(c, c', l, l', \lambda\right) &= u\left(c, c', l, l'\right) \\ &\quad + \lambda\left(w\left(1-l\right) + \pi - T + \frac{w'\left(1-l'\right) + \pi' - T'}{1+r} - c - \frac{c'}{1+r}\right), \end{aligned}$$

so that the FOCs are:

$$\begin{aligned} \partial c &: u_c = \lambda, \\ \partial c' &: u_{c'} = \frac{\lambda}{1+r}, \\ \partial l &: u_l = \lambda w, \\ \partial l' &: u_{l'} = \frac{\lambda w'}{1+r}, \\ \partial \lambda &: c + \frac{c'}{1+r} = w\left(1-l\right) + \pi - T + \frac{w'\left(1-l'\right) + \pi' - T'}{1+r}. \end{aligned}$$

Which results in:

$$\begin{aligned} u_l &= u_c w, \\ u_{c'} &= \frac{u_c}{1+r}, \\ u_{l'} &= u_{c'} w', \end{aligned}$$

which can be expressed as

$$MRS_{l,c} = \frac{u_l}{u_c} = \frac{\lambda w}{\lambda} = w,$$

$$
\begin{aligned}
MRS_{c,c'} &= \frac{u_c}{u_{c'}} = \frac{\lambda}{\frac{\lambda}{1+r}} = 1 + r, \\
MRS_{l',c'} &= \frac{u_{l'}}{u_{c'}} = \frac{\frac{\lambda w'}{1+r}}{\frac{\lambda}{1+r}} = w', \\
MRS_{l,l'} &= \frac{u_l}{u_{l'}} = \frac{w\left(1+r\right)}{w'}.
\end{aligned}
$$