# A Random Search Market with IOUs as Money

Juergen Jung*
Indiana University - Bloomington

13th February 2005

**Abstract**

We demonstrate the existence of an equilibrium in a random search market with divisible goods, pairwise trade and production. In this equilibrium, IOUs are issued by a special agent, the intermediary, and serve as money. The intermediary cannot produce and can only stay in the market because her IOUs are widely held and accepted in trade by all agents.

## 1 Introduction

The theory of money still comes up short in providing a solid answer to the question of why there is fiat money. In many standard models that deal with money, like cash in advance models (CIA), or money in the utility function models (MIU), the economist forces money into the model by assumption. This essentially solves the problem of generating a positive money demand but does not address the issue why money might yield utility or why money has to be used in a transaction.[1] On the other hand, in traditional general equilibrium analysis there is no role for money at all. Duffie (1990) and Ostroy and Starr (1990) both provide an overview about how the general equilibrium literature has tried to cope with the problem. Another interesting dimension has been added by Shubik (1990), who treats the question of money in a game theoretic environment.

Another strand of the literature tries to model the monetarization process endogenously. An early contribution to the money literature by Radford (1945) describes the use of commodity money, mostly cigarettes, in a POW camp during the second world war. A major contribution is a search equilibrium model by Kiyotaki and Wright (1989) using the notion of commodity money. In their seminal paper, which produced its own string of follow-up literature, the authors analyze economies in which individuals specialize in consumption and production. Agents cannot consume their own production good, but they have the possibility to trade their endowments in random, bilateral meetings on a *quid pro quo basis*. This setup typically leads to Jevons (1875) lack of double-coincidence-of-wants problem in a direct barter world. This system suffers particularly from high transactions costs. To ease trading, people begin

*I am grateful for many discussions with my supervisors Klaus Ritzberger and Jamsheed Shorish. Thanks for your patience and time. I would also like to thank Walter Fisher for many helpful comments. Special thanks go to my wife who supported this quest for money as well. All remaining errors are mine.

[1] See (Walsh, 2003, p. 59) for a discussion of the limitations of MIU models.

accepting commodities which they cannot consume, provided that they believe other people will take these goods in later exchange. Kiyotaki and Wright (1989) characterize Nash equilibria in trading strategies and find certain goods emerging endogenously as media of exchange – hence commodity money appears.

In recent years, more models using search theory have been produced (e.g. Kiyotaki and Wright (1991), Kiyotaki and Wright (1993), Aiyagari and Wallace (1991)). Kocherlakota (1998b) delivers an interesting extension stressing the record-keeping role of fiat money. Burdett, Trejos and Wright (2000) directly pick up on Radford's cigarette money idea and provide the accompanying search money model.

A series of related papers derives similar results out of a trading-post environment. In such environments, markets are in separate locations, and typically each location represents a different pair of goods that can be traded at that location. Iwai (1996) and Iwai (1997) identifies them as trading-zones with search frictions. In Howitt (2000), Starr (1999), Starr (1999b), and Hellwig (2001), trading posts are run by intermediaries. In all of these papers, increasing returns to scale implied by the matching process or the intermedieary's transaction technology lead to a concentration of trading posts and thereby foster the use of a common medium of exchange.

One of the main shortcomings of most of the cited papers is a certain negligence to include prices into the models. Combined with the prevalent assumption of indivisible goods, search-models are particularly difficult to integrate into a general equilibrium framework.

The core search money literature also contends itself with highly symmetrical setups. They do not account for trade specialists or institutions and thereby ignore the dominant role they have played in the history of money. Galbraith (1975) historical analysis of money points out the close relationship of money and institutions, mostly banks. Standard search money models also ignore credit money. The idea of markets solely involving barter exchange using a commodity money conflicts with anthropological evidence of the prevalence of individuals and institutions issuing credit in primitive economies. Goldschlager and Baxter (1994) provide a deeper analysis of credit monetary systems.

In this paper we connect some aspects of search money models with general equilibrium theory. There is one monopolist trader, the "intermediary" in an economy otherwise comprised of consumer-producer specialists. The monopolist is not a consumption specialist and may consume any good available, but cannot produce any good. In addition, she can store any type of good for a lower per-unit storage cost than all other consumer-producer agents. Consumer-producer agents can only produce one particular good. Consumer-producer utility functions are such that they prefer consuming one particular commodity, their consumption good. Consuming any other good results in only a "slightly positive" utility.

Every period all agents of this economy are pairwise matched. This includes the intermediary who randomly meets one of the agents per period. Moreover, the intermediary has a technology that allows her to issue identifiable IOU notes. The intermediary states prices of goods and IOUs prior to trade. She issues IOUs whenever a consumer-producer demands one. One might expect that under certain conditions these IOU notes would start circulating as money in the economy.

In that sense an IOUs serve two purposes. First, they are a *store of value* from one period to the next. Compared to storing the production good (or any other good) directly, an IOU is costless to store and does not depreciate. Second, IOUs serve as a *medium of exchange*. A consumer-producer can first use her production good to obtain an IOU, and then use the IOU in one of the following periods to trade for her consumption good.

IOUs can only actively function as money if we are able to postulate that they are accepted by all consumer-producers and that the intermediary does not default on her IOUs. However, consumer-producers prefer to consume any good to consuming nothing at all. This is enough to prevent any consumer-producer from credibly committing to honour an IOU issued by herself in the past. Consequently, in this model there exist only IOUs issued by the intermediary. The comparably lower storage costs of the intermediary attracts enough trade to guarantee the survival of the intermediary.

We show that under given assumptions an equilibrium exists in which IOUs are widely held and used for trade among all agents. This equilibrium is not unique.

In the next section we introduce the formal model. In section 3 and 4 we explain the trading mechanism. Section 5 describes the equilibrium. We conclude in section 7. The Appendix contains the more technical proofs of our results.

# 2 The Model

## 2.1 Consumer-Producers

We consider an economy in which units of durable and perfectly divisible, heterogenous goods are produced. Time is discrete. There are $n \geq 3$ different types of goods and $n$ different types of infinitely lived consumer-producers in the economy ($n$ being an odd and finite number).[2] We refer to consumer-producers as the agents throughout the rest of the paper. Every agent of type $i$ is endowed with one unit of good $i+1$ at the beginning of the first trading period. Agent type $n$ gets commodity 1. There is only one agent of every type.[3]

Consumer-producer $i$ is a consumption specialist and derives utility $u(c_i) = \sqrt{c_i}$ from consuming good (of type) $i$. In addition, consumer-producer $i$ can also derive utility $u(c_{j\neq i}) = \varepsilon\sqrt{c_j}$ from consuming any other good $j \neq i$ that she is currently storing.[4] The constant $\varepsilon$ is positive, close to zero and the same for all agents.[5] We call goods $j \neq i$ the non-consumption goods for agent $i$. The per-period utility function $u^i(c^i) : \mathbb{R}^n_+ \to \mathbb{R}_+$ is identical for all agent types and given by:

$$u^i(c^i) = \sqrt{c^i_i} + \sum\nolimits_{j\neq i=1}^{n} \varepsilon\sqrt{c^i_j}$$

where $c^i_j$ is the $j$th entry in the consumption vector $c^i \in \mathbb{R}^n_+$ of agent $i$.

In period $t$ agent $i$ holds a vector $w^i_t \in \mathbb{R}^n_+$ of goods in storage. She can store any good for as long as she likes, incurring storage costs in the form of a depreciation rate $\delta$. Depreciation is defined in terms of units of commodities which are subtracted from an agent's endowment.

[2] We use infinitely lived agents to allow for a tractable solution.

[3] This is not an essential assumption but makes the exposition somewhat easier.

[4] We use this restriction to prohibite credit arrangements among consumer-producers. We also see that there are no gifts in this economy.

[5] We assume that the net utility of consuming plus producing is large enough that agents will not want to drop out of the economy and that they always offer to trade for their consumption good once they meet another consumer-producer holding that good.

The maximum amount of depreciation is the entire endowment holding. Depreciation is an increasing function of the endowment $\delta(w)$: $\mathbb{R}_+^n \to \mathbb{R}_+^n$ and defined as

$$\delta(w) = \left\{ x \in \mathbb{R}_+^n \mid x_j = \min\left[\frac{(w_j)^2}{2\theta}, w_j\right] \text{ for all } j \in \{1, 2, ..., n\} \right\},$$

where $\theta$ is a positive constant and the same for all agent types, $x = (x_1, ..., x_n) \in \mathbb{R}_+^n$ and $w = (w_1, ..., w_n) \in \mathbb{R}_+^n$. The single entries $x_j$define the amount of good $j$ lost for that period.

We see that the marginal rate of depreciation for a particular good grows with higher levels of storage of that good, and that storage costs are the same for any type of commodity. However, if the agent has a certain quantity of good $j$ in storage, say $w_j$, then the per unit storage costs for $j$ turn out to be higher than for any other good $k \neq j$ that is stored in lower quantities. In addition, we introduce a storage capacity restriction on all agents, denoted by $\bar{w} \in \mathbb{R}_+$. Then $\sum_{j=1}^n w_j \leq \bar{w}$ for every agent.

In addition, an agent can also hold an endowment vector $z^i \in \mathbb{R}_+^n$ of IOUs with zero storage cost. IOUs do not depreciate. The initial endowment of IOUs is zero. The total endowment vector of agent $i$ is $\left(w^i, z^i\right) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$.

Consumer-producer $i$ is also a production specialist and can only produce good $i+1$ at the end of every period and after depreciation is deducted. In order to produce the agent has to invest some of good $i+1$ into the procution process. The production function $F^i$: $\mathbb{R}_+ \to \mathbb{R}_+$ is concave and fulfills Inada conditions. The law of transition for commodities is therefore

$$\varphi^i(y) = \left\{x \in \mathbb{R}_+^n : x_{i+1} = F^i\left(y_{i+1} - \delta\left(y_i\right)\right); x_j = y_{i+1} - \delta\left(y_i\right) \forall j \neq (i+1)\right\}$$

where $y$ is the commodity investment vector.

The consumer-producer is restricted to producing and trading once per trading period. This restriction basically excludes the uninteresting autarky case, where every producer would always consume her own production good and reproduce it immediately.

## 2.2 Intermediaries

We now introduce a second type of agent, a trader or middleman. We refer to this type of agent as the intermediary, who is indexed by 0. The intermediary cannot produce any commodity, but derives utility, $u^0(c_j) = \sqrt{c_j}$ from consuming any good $j = 1, ..., n$. The utility function $u^0(c) : \mathbb{R}_+^n \to \mathbb{R}_+$ over all goods defined as:

$$u^0(c) = \sum_{j=1}^n \sqrt{c_j}$$

Furthermore, the intermediary can store any type of good. Her endowment vector is denoted as $\left(w^0, z^0\right) \in \mathbb{R}_+^n \times \mathbb{R}_-^n$. Depreciation, $\delta^0(w^0)$:$\mathbb{R}_+^n \to \mathbb{R}_+^n$ is again a function of the endowment $w^0$.

$$\delta^0(w^0) = \left\{ x \in \mathbb{R}_+^n \mid x_j = \min\left[\frac{\left(w_j^0\right)^2}{2\theta^0}, w_j^0\right] \text{ for all } j \in \{1, 2, ..., n\} \right\}$$

As a specialist in trade she incurs lower storage costs than consumer-producer types, so that the constant $\theta^0 > \theta$. In addition, the intermediary has a higher storage capacity than consumer-producers, $\bar{w}^0 > \bar{w}$. The transition function of the intermediary is

$$\varphi^0(y) = \left\{x \in \mathbb{R}^n_+ : x_i = y_i - \delta^0(y_i)\, \forall i = \{1, ..., n\}\right\}$$

Moreover, the intermediary has a technology to produce identifiable IOU notes that cannot be forged by anybody in the economy. The maximum amount of IOUs that the intermediary is able to issue is $\bar{z}^0$. This restriction ensures boudedness of the dynamic optimization problem. Her total endowment vector is denoted by $\left(w^0, z^0\right) \in \mathbb{R}^n_+ \times \mathbb{R}^n_-$. The negative entries in her endowment vector stem from the feact that she is issuing these IOU notes.

The intermediary trades in all goods. In every period $t$, the intermediary, being the monopoly trader in the economy, states her optimal cross-price matrix $P_t \in \mathcal{P} = [\varepsilon, \infty)^{2n \times 2n}$ for all commodities and IOUs in the economy prior to trading:[6]

$$P_{(2n \times 2n)} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}, \text{ where}$$

$$P_1 = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}, \; P_2 = \begin{pmatrix} p_{1,IOU_1} & \cdots & p_{1,IOU_n} \\ p_{2,IOU_1} & \cdots & p_{2,IOU_n} \\ \vdots & \ddots & \vdots \\ p_{n,IOU_1} & \cdots & p_{n,IOU_n} \end{pmatrix}$$

$$P_3 = \begin{pmatrix} p_{IOU_1,1} & p_{IOU_1,2} & \cdots & p_{IOU_1,n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{IOU_n,1} & p_{IOU_n,2} & \cdots & p_{IOU_n,n} \end{pmatrix}, \; P_4 = \begin{pmatrix} p_{IOU_1,IOU_1} & \cdots & p_{IOU_1,IOU_n} \\ \vdots & \ddots & \vdots \\ p_{IOU_n,IOU_1} & \cdots & p_{IOU_n,IOU_n} \end{pmatrix}$$

and all prices are positive.

$P_1$ is the submatrix that contains all good for good exchange rates, $P_2$ contains the goods for IOU exchange rates, $P_3$ contains the IOU for goods rates and $P_4$ the IOU for IOU rates.

The price $p_{ij}$ is the quantity of good $i$ the intermediary offers to pay for one unit of $j$. Stated differently $p_{12}$ transfers commodity 2 into $p_{12}$ units of commodity 1. Changing a good into itself, $p_{ii}$, is a neutral exchange because exchanges are cost-free. Prices on the main diagonal are therefore equal to one by definition.

Trade in goods and IOUs is not restricted in any other way. Also, trading IOUs for other IOUs is possible. The specific IOU technology is reasonable to assume, once we claim that the intermediary is a specialist in trade who needs a device that allows her to stay in business, although she cannot produce herself.

She lives on price spreads created by the bid and ask differences for the traded bundles. The intermediary maximizes lifetime consumption with respect to prices, given the expected consumer-producer demand.

Trade that cannot be executed in commodities can be executed on the symbolic basis of IOU-issuing, which also serves as a record of trade for the intermediary. The latter is represented by negative entries in the intermediary's $z^0$-component of the endowment vector. The total number of issued IOUs present in the economy must equal the intermediary's $z^0$-component in

[6] We will drop the time subscript to not clutter the notation.

all periods $t$:

$$\sum\nolimits_{i=1}^{n} z_t^i = -z_t^0.$$

IOUs do not get lost, nor are they reduced during trade. The intermediary is the sole creator and destructor of IOUs. IOUs are destroyed once agents trade back goods for IOUs with the intermediary.

# 3 Trade between Agents

Trade occurs between pairs of individuals. During a unit interval of time (a trading period), all agents are matched simultaneously and pairwise. In this way, trade between an agent pair is limited by their respective endowment holdings. Agents are matched in the following way.[7]

Let $D$ be the set of permutations of the set $M = \{0, 1, 2, ..., n\}$ with the property that for each $d \in D$, $d(i) = j$ if and only if $d(j) = i$ for all $i \in M$. A random pairing, $\pi$, is a random variable taking on values in $D$. If $\pi$ is a random pairing, let $\pi^i$ be the random variable whose value is $d(i)$ if $\pi$ takes on the value $d \in D$. The random pairing $\pi$ will sometimes be denoted by $(\pi^0, \pi^1, ..., \pi^n)$. Note that since $\pi$ takes on values in $D$, $\pi^i = j$ if and only if $\pi^j = i$. Agent $i$ is matched with the intermediary whenever $d(i) = j = 0$.

This matching technology suggests that traders are paired each period by the realization of the sequence $\{\pi_t\}_{t=0}^{\infty}$ of random pairings. Hence, agent $i$'s period $t$ trading partner is the realization of $\pi_t^i$.

## 3.1 Opportunity Sets of Consumer-Producers

We assume that every agent enters a possible trade with her endowment vector $(w, z) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$. No agent knows in advance her opportunity set in a given trade. However, once an agent is matched, she learns the endowment and the type of the agent she is matched with. Based on this information her opportunity set $\Gamma^i$ is formed. The set determines the range of the "net-trade" (demand) for commodities and IOUs of the agent, after meeting another consumer-producer type (or after meeting the intermediary).

The net-trade vector is bounded by the amount that the agents carry into trade. They can hold a maximum amount of $\bar{w}$. We defined the endowment capacity of the intermediary $\bar{w}^0$ as being greater than the agents maximum possible endowment, that is $\bar{w}^0 > \bar{w}$. The maximum amount that an agent can aquire in trade is exactly this quantity $\bar{w}^0$. A similar argument holds for the maximum amount of IOUs which is denoted as $\bar{z}^0$. The maximum amount of investment $y$ that an agent can carry over into the next period is the sum of the agent's maximum endowment and the intermediary's maximum endowment, $\bar{w} + \bar{w}^0$.

**Definition 1** *We define the net-trade vector for agent $i$ meeting agent $j$ as* $\left(\Delta w_t^{ij}, \Delta z_t^{ij}\right) = \left(\hat{w}_t^i - w_t^i, \hat{z}_t^i - z_t^i\right) \in \mathbb{R}^n \times \mathbb{R}^n$. *The vector* $\left(\hat{w}_t^i, \hat{z}_t^i\right) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ *is the post-trade vector of agent $i$.*

[7] See Harris (1979) for a similar mechanism.

We stress the fact that the net trade vector can have negative entries. The opportunity set will depend on the characteristics of agent $i$'s period $t$ trading partner $j$, and the endowment bundles $\left(w_t^i, z_t^i\right)$ and $\left(w_t^j, z_t^j\right)$ that she and her trading partner have ready for trade.

An agent $i$ faces two possible scenarios. In the first scenario the agent meets the intermediary, that is $d(i) = j = 0$. We know that for a net-trade allocation to be feasible, it cannot exceed the total quantity of goods carried by both agents. Hence $-w_t^i \leq \Delta w_t^{i0} \leq w_t^0$, with $w_t^0$ being the intermediary's pre-trade commodity endowment vector.[8] This establishes the size of an implicit Edgeworth box that the agents are forming.

The post-trade endowment in IOUs is somewhat more complicated. Recall that the intermediary is the sole creater of IOU notes. Any additional IOU note issued by the intermediary increases the debt recordings of the intermediary, as represented in her $z^0$-endowment vector. So a positive change in the agent's post-trade IOU vector, $\left(\Delta z_t^{i0}\right)_+$, is a negative entry in the IOU vector of the intermediary. The size of the negative entry depends on prices set by the intermediary prior to trade. In addition, the agent can trade an IOU for a combination of goods (that the intermediary is holding) at a specific price posted by the intermediary. It is therefore difficult to establish the size of the Edgeworth box in the IOU case. We will restrict the size to a maximum amount of IOUs, $-\bar{z}^0$, that the intermediary can issue.

However, we can still put more structure upon the opportunity set. In the following we introduce a union of budget restrictions that incorporates the exchange rate prices $P$. Here we are interested in what net-trade combinations are feasible for a given cross-price matrix. Similar to perfect competition general equilibrium models with given prices for every good, we claim that an agent cannot walk out of a trade with more than she carried into it (expressed in terms of some unifying wealth measure). A consumer-producer cannot borrow.

In order to compare arbitrary allocations, we express the entire bundle that the agent is buying in terms of a numeraire good, say commodity $j$, and call it the *net-purchases-bundle* in $j$ (or $NPB_j$). Then we compare the $NPB_j$ to the entire bundle that the agent is selling, *net-sales-bundle* (or $NSB_j$) which is expressed in the same numeraire good $j$.

Since different net-trade outcomes are possible depending on which numeraire good agents are using for their trading decisions, we have to take all possible numeraire cases into account. Accordingly, we can use every good as numeraire and write $NSB \in \mathbb{R}_+^{2n}$, with $NSB_j$ as its elements, and $NPB \in \mathbb{R}_+^{2n}$, with $NPB_j$ as its elements, $j \in \{1, ..., n\}$.

**Definition 2** *Feasible net-trade vector*
*Given a certain net-trade vector, we denote a feasible net-trade vector as a vector for which we can find at least one numeraire good such that the NPB and NSB are feasible, i.e. $NPB_j \leq NSB_j$ for at least one $j$.*

We develop this idea more formally in two steps. In the first step, we simply show that all net-sales expressed in various numeraire bundles must exceed or be equal to the net-purchase

[8] We use the standard vector inequality notation: if $x, y \in R^n$, we define:
$x = y$ if and only if $x_i = y_i$ for $i = 1, ..., n$;
$x \geq y$ if and only if $x_i \geq y_i$ for $i = 1, ..., n$;
$x > y$ if and only if $x \geq y$ and $x \neq y$;
$x \gg y$ if and only if $x_i > y_i$ for $i = 1, ..., n$.

in the same good:

$$\overbrace{\begin{pmatrix} \Delta w_t^{ij} \\ \Delta z_t^{ij} \end{pmatrix}_+}^{\text{goods purchased}} \leq \overbrace{\left| P \begin{pmatrix} \Delta w_t^{ij} \\ \Delta z_t^{ij} \end{pmatrix}_- \right|}^{NSB}$$

Example 3 shows explicitly how the $NSB$ is calculated

**Example 3** *Consider a case with* 3 *goods. If we want to convert the variety of goods sold (this is recorded in the negative entries of* $\left(\Delta w_t^{ij}, \Delta z_t^{ij}\right)$) *into one particular (numeraire) good, we use the following procedure:*

$$NSB = \left| \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,IOU_3} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,IOU_3} \\ \vdots & \vdots & \ddots & \vdots \\ p_{IOU_3,1} & p_{IOU_3,2} & \cdots & p_{IOU_3,IOU_3} \end{pmatrix} \begin{pmatrix} \Delta w_1 \\ \Delta w_2 \\ \vdots \\ \Delta z_3 \end{pmatrix}_- \right| = \begin{pmatrix} \begin{bmatrix} \textit{all income from sales} \\ \textit{transferred into good } 1 \end{bmatrix} \\ \textit{all in } 2 \\ \vdots \\ \textit{all in } IOU_3 \end{pmatrix}$$

Definition 2 is very coarse and does not yet tell us anything about combinations of purchases and sales that the agents might agree upon. In the next step, we calculate how much of a specific good the agent must sell in order to buy the quantities specified in the positive net-trade vector. We again express these net-purchases in terms of the diverse numeraire bundles that we have already seen. What we get out of this are the various entries for the $NPB$. We are now able to compare the two vectors:

$$\overbrace{\left[P^{*T}\right]_k \begin{pmatrix} \Delta w_t^{ij} \\ \Delta z_t^{ij} \end{pmatrix}_+}^{NPB_k} \leq \overbrace{\left| P_k \begin{pmatrix} \Delta w_t^{ij} \\ \Delta z_t^{ij} \end{pmatrix}_- \right|}^{NSB_k} \text{ for at least one } k \in \{1, ..., n\}.$$

The vector $P_k$ is the $k$th row of the cross-price matrix $P$. If we invert all entries in matrix $P$ we get matrix $P^*$ of the same dimension. The vector $P_k^*$ is the $k$th row of this matrix. Vector $\left[P^{*T}\right]_k$ is the $k$th row of the transposed matrix $P^*$. Multiplying the inverse column prices with the positive parts of the net-trade vector results in the quantity of commodity $k$ that is needed in order to finance these purchases. Multiplying the $k$th row in the cross-price matrix $P$ with the negative parts of the net-trade vector transforms all sales into units of the particular good $k$ (a scalar), now available for payment.

This quantity reflects the total income of all sales in units of $k$. The whole expression compares the amount of commodity $k$ needed (to be sold) in order to finance all purchases expressed in commodity $k$. Note that if we find one $k$ for which the inequality holds, the trade becomes feasible. It does not matter if this trade would be prohibited if another commodity were to be used as numeraire. Clearly, we expect agents to be able to find the "best" numeraire good for a specific trade. We illustrate this mechanism in the following example.

**Example 4** *Consider again the case with* 3 *goods. We first transform all purchases into nu-*

*meraire goods and get the $NPB$.*

$$\left[\overbrace{\begin{pmatrix} \frac{1}{p_{1,1}} & \frac{1}{p_{1,2}} & \cdots & \frac{1}{p_{1,IOU_3}} \\ \frac{1}{p_{2,1}} & \frac{1}{p_{2,2}} & \cdots & \frac{1}{p_{2,IOU_3}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_{IOU_3,1}} & \frac{1}{p_{OU_3,2}} & \cdots & \frac{1}{p_{IOU_3,IOU_3}} \end{pmatrix}}^{P^*}\right]^T \begin{pmatrix} \Delta w_1 \\ \Delta w_2 \\ \vdots \\ \Delta z_3 \end{pmatrix}_+ = \begin{pmatrix} \textit{units of } 1 \textit{ needed to finance purchases} \\ \textit{units of } 2 \textit{ needed} \\ \vdots \\ \textit{units of } IOU_3 \textit{ needed} \end{pmatrix}_+$$

*Then we transform all sales into numeraire goods and get the $NSB$.*

$$\left| \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,IOU_3} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,IOU_3} \\ \vdots & \vdots & \ddots & \vdots \\ p_{IOU_3,1} & p_{IOU_3,2} & \cdots & p_{IOU_3,IOU_3} \end{pmatrix} \begin{pmatrix} \Delta w_1 \\ \Delta w_2 \\ \vdots \\ \Delta z_3 \end{pmatrix}_- \right| = \begin{pmatrix} \textit{all income transferred into good } 1 \\ \textit{all in } 2 \\ \vdots \\ \textit{all in } IOU_3 \end{pmatrix}$$

*Then for at least one entry of the above vectors the inequality $NPB_j \leq NSB_j$ has to hold in order for there to be feasible trade.*

We summarize this result in the following definition:

**Definition 5** *We define the feasible trade-opportunity set, $\Gamma^i_{0t}$, of consumer-producer $i$ meeting the intermediary as:*

$$\Gamma^i_{0t} = \Gamma^i_0(\pi^i_t = 0, w^i_t, z^i_t, P_t) = \left\{ \begin{array}{c} (\Delta w^{i0}_t, \Delta z^{i0}_t) \in \mathbb{R}^n \times \mathbb{R}^n \mid -w^i_t \leq \Delta w^{i0}_t \leq w^0_t; \\ -z^i_t \leq \Delta z^{i0}_t \leq \bar{z}^0 \\ \left[P^{*T}\right]_k \begin{pmatrix} \Delta w^{i0}_t \\ \Delta z^{i0}_t \end{pmatrix}_+ \leq \left| P_k \begin{pmatrix} \Delta w^{i0}_t \\ \Delta z^{i0}_t \end{pmatrix}_- \right| \textit{ for at least one } k \in \{1, ..., n\} \end{array} \right\}$$

The mapping $\pi^i_t = j$ indicates the type of agent the consumer-producer is going to meet - in this case she meets with agent 0, the intermediary. The vector $\left(w^i_t, z^i_t\right)$ is her endowment vector, $P_t$ is the cross-price matrix set by the intermediary, with $P^*_k$ being the $k$th column of the cross-price matrix $P$ using its inverse elements. The vector $P_k$ is the $k$th row of the cross-price matrix, $\left(\Delta w^{i0}_t, \Delta z^{i0}_t\right)$ is the respective net-trade demand vector of agent $i$, and $\left(w^0_t, z^0_t\right)$ is the endowment of the intermediary.

The opportunity set $\Gamma_0$ is the union of all possible budget combinations. Here we mean that we can express the entire endowment vector in units of a particular good, the numeraire. We can then repeat this procedure using each good as the numeraire and finally take the union of the resulting budget sets. We end up with the maximum (or feasible) budget set that accounts for all possible trade combinations of the agent.

In the second step, we describe what happens if an agent meets another consumer-producer. We know that for bilateral trade to be feasible, the post-trade allocation of the agent cannot exceed the total quantity of goods and IOUs held by both agents. From the agent's point of view we want to make sure that her net-trade vector lies within the limits of what is feasible in terms of the total amount of tradeable items. We also want to make sure that any resulting net-trade allocation does not exceed the agent's initial endowment evaluated at the endogenous prices that are negotiated by the two traders. In short, we end up with a restriction defining the size of the implicit Edgeworth box that the agents are setting up, and with a series of budget constraints on net-trades. We define this agent-meets-agent part of the opportunity set as $\Gamma^i_{1t}$.

**Definition 6** *We define the feasible trade opportunity set, $\Gamma^i_{jt}$, of consumer-producer i meeting another consumer-producer as:*

$$\begin{aligned}
\Gamma^i_{jt}(\pi^i_t &= j \neq 0, w^i_t, z^i_t, P^{ij}_t) \\
&= \left\{ \begin{array}{c} (\Delta w^{ij}_t, \Delta z^{ij}_t) \in \mathbb{R}^n \times \mathbb{R}^n \mid -w^i_t \leq \Delta w^{ij}_t \leq w^j_t; \\ -z^i_t \leq \Delta z^{ij}_t \leq z^j_t; \\ \exists P^{ij}_t : \left[P^{ij*T}\right]_k \begin{pmatrix} \Delta w^{ij}_t \\ \Delta z^{ij}_t \end{pmatrix}_+ \leq \left| P^{ij}_k \begin{pmatrix} \Delta w^{ij}_t \\ \Delta z^{ij}_t \end{pmatrix}_- \right| \text{ for at least one } k \in \{1,...,n\} \end{array} \right\}
\end{aligned}$$

We are not interested in how this works in detail, as long as the bargaining process does not result in any waste of the gains to trade. We postulate that agents $i$ and $j$ are able to negotiate an endogenous price matrix $P^{ij}_t$ such that the given allocations are feasible. Here we will follow the exposition in Harris (1979) and Feldman (1973).[9]

We denote the agent consumption space $X = \left[0, \bar{w} + \bar{w}^0\right]^n$. The consumption vector $c^i$ for consumer producer $i$ and the consumption vector $c^0$ for the intermediary will be elemets of $X$. Since we need this set to be compact we introduce the respective maximum endowments for the consumer-producer as $\bar{w}$ and for the intermediary as $\bar{w}^0$. We then define the follwing states and actions and their respective spaces:

**Definition 7** *The endowmet state vector $s$ of a consumer-producer is the tuple $\{(w,z,P)\} \in S = [0, \bar{w}]^n \times \left[0, \bar{z}^0\right]^n \times (0, \infty]^{2n \times 2n}$. The intermediary endowment state is $s^0 = \left\{\left(w^0, z^0\right)\right\} \in S^0 = \left[0, \bar{w}^0\right]^n \times \left[-\bar{z}^0, 0\right]^n$.*

**Definition 8** *The action vector $a$ of a consumer-producer is the tuple $\{(\Delta w, \Delta z, y)\} \in A = \left[-\bar{w}, \bar{w}^0\right]^n \times \left[-\bar{z}^0, \bar{z}^0\right]^n \times \left[0, \bar{w} + \bar{w}^0\right]^n$. The intermediary action is $a^0 = \{(\hat{y}, P)\} \in A^0 = \left[0, \bar{w}^0 + \bar{w}\right]^n \times [0, \bar{p}]$ where $\bar{p}$ is the maximum price that the intermediary is able to set.*

[9] Harris (1979) has shown that agents trade in a core set along the contract curve. The whole economy moves gradually towards the competitive equilibrium along a path of temporary pairwise competitive equilibria which are the results of the random pairings $\{\boldsymbol{\pi}_t\}$ for all agents. The endogenous prices are a function of the pairing device and the initial economy-wide endowment. However, this result depends crucially on the existence of a "money" good that is held and desired by all agents! Furthermore, holding this good exceeds the immediate consumption value and fosters future trade that would otherwise not be possible.

Feldman (1973) shows that myopic trading will lead to a pairwise optimal allocation. When two agents meet in a period they engage in an optimizing bilateral trade move that makes no trader worse off. His results depend crucially on a continuous and strictly concave utility function that allows for levelling out marginal utility rates of additional consumption in some goods.

Both, endowment state spaces $S$ and $S^0$ as well as action spaces $A$ and $A^0$ are closed and bounded and therefore compact. The net-trade vector is bounded by the amount that the agents carry into trade. The maximum endowment that agents can hold is $\bar{w}$. Since we defined the maximum endowment capacity of the intermediary as greater than that of the consumer-producer, $\bar{w}^0 > \bar{w}$, the maximum amount that an agent can acquire in any trade is bounded by $\bar{w}^0$. The maximum amount of investment $y$ that any agent can carry over into the next period is the sum of the agent's maximum endowment and the intermediary's maximum endowment, i.e. $\bar{w} + \bar{w}^0$.

We finally combine both feasible actions sets $\Gamma_0^i$ and $\Gamma_j^i$, add the investment constraint on $y$ and define the action correspondence for agent $i$ as follows:

**Definition 9** *The set $\Gamma_t^i : M \times S \times \mathcal{P} \to A$ is the feasible action set of agent $i$ in period $t$, i.e., agent $i$ expects that $(\Delta w_t^i, \Delta z_t^i, y_t^i) \in \Gamma_t^i$ and is a correspondence. Furthermore,*

$$\begin{aligned}
\Gamma_t^i &= \Gamma^i(\pi_t^i = j, w_t^i, z_t^i, P_t) \\
&= \left\{ \begin{array}{c}
(\Delta w_t^{ij}, \Delta z_t^{ij}, y_t^i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+^n \mid -w_t^i \leq \Delta w_t^{ij} \leq w_t^j; \\
y_t^i \leq w_t^i + \Delta w_t^i; \\
\left\{ \begin{array}{c}
-z_t^i \leq \Delta z_t^{ij} \leq \bar{z}^0; \\
\left[ \begin{array}{c} \left[P^{*T}\right]_k \begin{pmatrix} \Delta w_t^{ij} \\ \Delta z_t^{ij} \end{pmatrix}_+ \leq \left| P_k \begin{pmatrix} \Delta w_t^{ij} \\ \Delta z_t^{ij} \end{pmatrix}_- \right| \\ \textit{for at least one } k \in \{1, ..., n\}; \end{array} \right] \quad \textit{if } \pi^i = j = 0
\end{array} \right. \\
\left\{ \begin{array}{c}
-z_t^i \leq \Delta z_t^{ij} \leq z_t^j; \\
\left[ \begin{array}{c} \exists P_t^{ij} : \left[p^{ij*T}\right]_k \begin{pmatrix} \Delta w_t^{ij} \\ \Delta z_t^{ij} \end{pmatrix}_+ \leq \left| p_k^{ij} \begin{pmatrix} \Delta w_t^{ij} \\ \Delta z_t^{ij} \end{pmatrix}_- \right| \\ \textit{for at least one } k \in \{1, ..., n\}. \end{array} \right] \quad \forall\ \pi_t^i = j \neq 0
\end{array} \right.
\end{array} \right\}
\end{aligned}$$

## 3.2 The Maximization Problem of the Consumer-Producers

At the beginning of each period $t$ a consumer-producer enters into a trading round with her total endowment vector $\left(w_t^i, z_t^i\right)$. She either trades with another consumer-producer or with the intermediary and leaves the trading platform with her post-trade endowment vector $\left(\hat{w}_t^i, \hat{z}_t^i\right)$. She then engages in a consumption and savings decision.

We formulate this problem as a net-trade plus consumption-savings decision. The net-trade vectors are chosen out of the opportunity set $\Gamma^i$. The agent then decides the quantity of goods she wants to carry over into the next period, $y_t^i$. After that, this period's storage costs are deducted based on holdings $y_t^i$ invested into the next period. The resulting amount in $i+1$ is used for production. Hence, her new total endowment vector at the beginning of the next period is $\left(w_{t+1}^i, z_{t+1}^i\right) = \left(\varphi^i\left(y_t^i\right), z_t^i + \Delta z_t^i\right)$.

Consumer-producers maximize their discounted expected lifetime utility:

$$\max_{\left\{\Delta w_t^i, \Delta z_t^i, y_t^i\right\}} \mathbb{E}_0 \sum\nolimits_{t=0}^{\infty} \beta^t u^i(w_t^i + \Delta w_t^{ij} - y_t^i) \tag{1}$$

subject to

$$\begin{aligned}
\left(\Delta w_t^i, \Delta z_t^i, y_t^i\right) &\in \Gamma^i\left(\pi_t^i, w_t^i, z_t^i, P_t\right), \\
\left(w_{t+1}^i, z_{t+1}^i\right) &= \left(\varphi^i\left(y_t^i\right), z_t^i + \Delta z_t^{ij}\right), \\
0 &\leq w^i, z^i, y_t^i, \\
\sum\nolimits_{k=1}^n w_k^i &\leq \bar{w},
\end{aligned}$$

for all $t \in N = \{1, 2, ...\}$, and $w_0^i, z_0^i$ given. The expectation is taken with respect to the joint distribution of $\pi_0^i, ..., \pi_{t-1}^i$. The discount factor is $\beta \in (0, 1)$. The value function of the agents can be written as follows:

$$\begin{aligned}
v^i\left(j, w^i, z^i, P\right) &= \max_{(\Delta w^{ij}, \Delta z^{ij}, y^i) \in \Gamma^i(j, w^i, z^i, P)} \left\{ u^i\left(w^i + \Delta w^{ij} - y^i\right) + \beta \frac{1}{n} \sum_{j=0 \wedge j \neq i}^{n} v^i\left[j, w^{i\prime}, z^{i\prime}, P'\right] \right\} \\
& \quad s.t. \\
w^{i\prime} &= \varphi^i\left(y^i\right) \\
z^{i\prime} &= z^i + \Delta z^{ij}
\end{aligned}$$

Proofs for the existence of an optimal solution are stated in the Appendix.

# 4 Trade with the Intermediary

The intermediary also faces a two-tiered decision process. She first engages in an optimal price setting decision when matched with an agent. This price is set with respect to a given demand function, $D(j, P_t)$ that the intermediary expects from agent $j$. Once the agent traded optimally, she engages in an optimal consumption and savings decision, taking the proceeds from trading as given.

Every trading period the intermediary meets a consumer-producer type. Once the intermediary is matched with an agent, she can observe the agent's endowment bundle $(w_t^i, z_t^i)$ and her type $i$. Furthermore, the intermediary assumes a given demand function of her trading partner, $D(i, P_t)$. The intermediary assumes that this demand function is continuous. Since prices are stated as exchange rates, an increase in one particular price (or exchange-rate), say $p_{ij}$ increases the demand for good $i$ and decreases the demand for good $j$.[10] Based on this information and her own endowment bundle $\left(w_t^0, z_t^0\right)$ she sets her prices $P_t$ optimally.

We give the following intuition for the setting of an optimized price strategy. The intermediary holds a set of maximizing price setting strategies contingent on the agent type she is going to meet. These strategies are formed on the assumption that prior to trade, the intermediary can observe the type and endowment of her trading partner and hence faces a demand function from this agent. Once a trading partner is randomly matched with the intermediary, she just picks the respective price-setting strategy that maximizes her own lifetime discounted utility.

[10] Recall that $p_{ij}$ states the amount of good $i$ the intermediary is willing to give for one unit of good $j$. So if this number increases, the consumer-producer is faced with the situation of a "cheap" good $i$ with respect to good $j$.

### 4.1 The Maximization Problem of the Intermediary

At the beginning of each period $t$ the intermediary enters into a trading round with her total endowment vector $\left(w_t^0, z_t^0\right)$. She is matched with one specific agent $i \in \{1, 2, ..., n\}$. Prior to any trade, the intermediary sets the exchange rates $P_t$ assuming the following net-trade demand function:

$$D(i, P_t) = \left\{ \left(\Delta w_t^{0i}, \Delta z_t^{0i}\right) \in \left[-\bar{w}^0, \bar{w}^0\right] \times \left[-\bar{z}^0, \bar{z}^0\right] \right\} \tag{2}$$

where $\bar{w}^0$ and $\bar{z}^0$ reflect the maximum amounts of goods and IOUs tradeable. The net-trade bundle $\left(\Delta w_t^{0i}, \Delta z_t^{0i}\right)$ is the net-trade bundle she expects from agent $i$ at price $P_t$.

With this demand function the intermediary assumes that any matched agent would react with a net-demand depending on prices. In the next step, she engages in a consumption and savings decision. The quantity of goods invested into the next period is denoted by $y_t^0 \in \mathbb{R}_+^n$. Hence, her new total endowment vector at the beginning of the next period is $\left(w_{t+1}^0, z_{t+1}^0\right) = \left(\varphi^0(y_t^0), z_t^0 + \left(-\Delta z_t^{0i}\right)\right)$. The intermediary maximizes her discounted expected lifetime utility:

$$\max_{\{y_t^0, P_t\}} \mathbb{E}_0 \sum\nolimits_{t=0}^{\infty} \beta^t u^0(w_t^0 + \left(-\Delta w_t^{0i}\right) - y_t^0) \tag{3}$$

such that

$$\begin{aligned}
\left(\Delta w_t^{0i}, \Delta z_t^{0i}\right) &\in D(i, P_t), \\
y_t^0 &\leq w^{\mathbf{0}}{}_t + \left(-\Delta w_t^{0i}\right), \\
\left(w_{t+1}^0, z_{t+1}^0\right) &= \left(\varphi^0(y_t^0), z_t^0 + \left(-\Delta z_t^{0i}\right)\right), \\
0 &\leq y_t^0, w_t^0, \\
0 &\leq P_t, \\
z_t^0 &\leq 0, \\
\sum\nolimits_{j=1}^{n} w_j^0 &\leq \bar{w}^0,
\end{aligned}$$

for all $t \in \{1, 2, ..., \infty\}$. The expectation is taken with respect to $\pi^0 = i$, which is the probability of meeting a specific agent. This probability is the same for all agents. The value function of the intermediary is:

$$\begin{aligned}
v^0\left[\left(i, w^0, z^0\right)\right] &= \max_{0<P, y^0 \leq w^0 + \left(-\Delta w_t^{0i}\right), \left(\Delta w_t^{0i}, \Delta z_t^{0i}\right) = D(i, P_t)} \\
&\left\{ u^0\left[w^0 + \left(-\Delta w_t^{0i}\right) - y^0\right] + \beta \frac{1}{n} \sum\nolimits_{i=1}^{n} v^0\left[i, \varphi^0(y_t^0), z^0 + \left(-\Delta z_t^{0i}\right)\right] \right\}
\end{aligned}$$

The proof for the existence of the maximization problem is similar to the agent maximization problem.

## 5 Equilibrium

In the following we combine both, agents and the intermediary and define a Nash equilibrium based on the dynamic maximization problems. The results that follow build on this particular notion of equilibrium.

**Definition 10** *We define an equilibrium as a price-sequence* $\{(P_t)\}_{t=0}^{\infty}$, *set by the intermediary, an allocation* $\left\{\left(y_t^0, \Delta w_t^{0i}, \Delta z_t^{0i}, w_t^0, z_t^0\right)\right\}_{t=0}^{\infty}$ *for the intermediary with* $\left(\Delta w_t^{0i}, \Delta z_t^{0i}\right) = D\left(i, P_t\right)$, *and an allocation* $\left\{\left(y_t^i, \Delta w_t^{i0}, \Delta z_t^{i0}, w_t^i, z_t^i\right)\right\}_{t=0}^{\infty}$ *for the typical agent* $i$, *such that*
*(a)* $\{(P_t)\}_{t=0}^{\infty}$ *and* $\left\{\left(y_t^0, \Delta w_t^{0i}, \Delta z_t^{0i}, w_t^0, z_t^0\right)\right\}_{t=0}^{\infty}$ *solve* (3) *with*

$$D\left(i, P_t\right) = \left\{ \begin{array}{c} \left(\Delta w_t^{0i}, \Delta z_t^{0i}\right) \in \mathbb{R}^n \times \mathbb{R}^n : \left(\Delta w_t^{0i}, \Delta z_t^{0i}, y_t^i\right) = \underset{(\Delta w, \Delta z, y) \in \Gamma^i(j,w,z,P)}{\arg\max} \\ \left\{u^i\left(w^i + \Delta w^{i0} - y^i\right) + \beta \frac{1}{n} \sum_{j \neq i=0}^{n} v^i\left[j, \varphi^i\left(y^i\right), z^i + \Delta z^{i0}, P\right]\right\} \end{array} \right\}$$

*(b)* $\left\{\left(y_t^i, \Delta w_t^{i0}, \Delta z_t^{i0}, w_t^i, z_t^i\right)\right\}_{t=0}^{\infty}$ *solves* (1);
*(c) all markets clear:* $\Delta w_t^{i0} = \Delta w_t^{0i}$ *and* $\Delta z_t^{i0} = \Delta z_t^{0i}$ *for a series of agents with* $\pi_t^i = 0$.

## 5.1 The Commodity Investment Threshold

In this section we want to develop a feasible range for the commodity investment vector $y$ of any agent. We do not express the optimal quantity an agent might want to save, since this is part of the dynamic optimization problem. Here we simply intend to introduce a threshold level of investment for a good $i$, above which investment can never be optimal (regardless of any future dynamics). The threshold is unique, the same for every good and depends on the parameter $\theta$ of the depreciation function $\delta$. From the maximization problem of the agent (equation (1)) we see that commodity holdings follow the transition rule:

$$w_{t+1}^i = \varphi^i\left(y_t^i\right),$$

where $w_{t+1}^i$ is the goods endowment vector for period $t+1$. If we write down this equation in extensive form for good $j$ (dropping time subscripts) we get:

$$w_j^i = y_j^i - \frac{(y_j^i)^2}{2\theta} + F_j^i\left(y_j^i - \frac{(y_j^i)^2}{2\theta}\right) \tag{4}$$

Taking the derivative and setting it equal to zero we get:

$$\begin{aligned} \frac{\partial w_j^i}{\partial y_j^i} &= 1 - \frac{y_j^i}{\theta} + F^{i\prime}\left(y_j^i - \frac{(y_j^i)^2}{2\theta}\right)\left(1 - \frac{y_j^i}{\theta}\right) \\ &= \left(1 - \frac{y_j^i}{\theta}\right)\left(1 + F^{i\prime}\left(y_j^i - \frac{(y_j^i)^2}{2\theta}\right)\right) = 0 \end{aligned}$$

We see that the feasible range of investment in good $j$ is $y_j^i \in [0, \theta)$.[11] If an agent would invest $y_j^i > \theta$ she actually ends up with less of the good in the next period than if she had just stored an equally feasible amount $\theta$ or less. The additional quantity, $y_j^i - \theta > 0$, could have been consumed or traded, and both are actions that cannot be worse than just losing the good via depreciation. Note that for all goods $j \neq (i+1)$, so for all non-production goods, the production part of equation (4) is zero. The same argument also holds for the intermediary, i.e. $\hat{y}_j^i \in [0, \hat{\theta})$. For the rest of the paper we will concentrate our analysis on this feasible investment range.

[11] We use the open interval because we want to make sure that the transition function is monotone increasing (see also Lemma 21 in the Appendix).

## 5.2 Price Structure

The next proposition puts more structure on prices and rules out price arbitrage (a "money machine" or "money pump").

**Proposition 11** *A **trade-cycle**, $TC$, is a sequence of consecutive endowment holdings (this includes holdings of IOUs) that ends up with the good we started with: $TC \equiv \left\{ w^{k}_{\tau=1}, w^{k'}_{\tau=2}, w^{k''}_{\tau=3}, .., w^{\bar{k}}_{\tau-1}, w^{k}_{\tau} \right\}$ where $\tau$ marks the $\tau$'th trade "within" a period. The **product of a trade-cycle**, $PTC$, is the product of the respective exchange rates that come into play once the sequence of trades is executed: $PTC = p_{k',k} * p_{k'',k'} * ... * p_{k,\bar{k}}$.*
*With no restrictions on the number of trades between intermediary and consumer-producer there can be no $TC$ such that the $PTC$ is greater than one.*

**Proof.** Suppose not. Then there exists a $PTC > 1$ for some sequence of holdings $\{k\}_{\tau}$. By repeatedly running through such a trade-cycle the consumer-producer can extract the entire quantity of the particular endowment in $k$ or the maximum quantity of IOUs, $\bar{z}^0$, in case of an IOU standing at the beginning of the cycle. This does not maximize the objective function from (3) and is therefore not an equilibrium. ■

## 5.3 Existence of Commodity Trade Equilibria

The following proposition states why it is the case that agents want to engage in commodity trade.

**Proposition 12** *If two agents $i$ and $j$ meet, holding respective commodity endowments $w_i \neq w_j$ they will engage in an optimizing bilateral trade move that makes no trader worse off. Note that this could also be a null trade (autarchy).*

**Proof.** The proof can be found in Feldman (1973) and rests on the assumption of convex preferences. If it is possible for two agents to increase consumption in goods where marginal utility is high and give up goods where marginal utility is lower, we would assume that agents would engage in such a trade at some endogenous price $p$. The consumption and production specialists presented in this paper create a natural environment for such trades to occur. ■

## 5.4 Existence of Equilibria with Investments in Goods

We have already seen that there is a threshold amount $\theta$ of storage, above which storing cannot be optimal. Now we want to show that agents have an incentive to store a positive amount of goods that could go up to this threshold. This storing or investment decision is governed by the intertemporal consumption and savings decision as described in the agent maximization problem. However, since we employ a very specific production structure in the model we need to be careful when we start interpreting the Euler equations.

Basically, we will end up with two cases. The first is the consumption-investment decision in the production good $i+1$, and the second is the same decision for all other goods $j \in N \backslash \{i+1\}$. The question becomes whether there are equilibria where agents store goods. If not, we may claim that the agent consumes the entire bundle immediately.

To approach this problem we assume that an agent has aquired a bundle of goods in the past. The agent has to decide whether she wants to consume the total amount immediately, or whether she prefers storing parts of her endowment for future consumption and trade.

Furthermore, let $\Phi^i : S \times \mathcal{P} \times A \rightarrow S$ be the transition function that maps the state endowment vector $\left(w_t^i, z_t^i\right) \in S$ into next period's endowment $\left(w_{t+1}^i, z_{t+1}^i\right) \in S$, following the restrictions in the consmer-producer maximization problem in expression $(1)$. The respective expression is

$$\Phi^i\left[\left\{\left(w^i,z^i\right)\right\},\{P\},\left\{\left(\Delta w^{ij},\Delta z^{ij},y^i\right)\right\}\right]=\left\{\begin{array}{c}\left(w^{i\prime},z^{i\prime},P^{\prime}\right)\in S\times\mathcal{P}:\\\left(w^{i\prime},z^{i\prime},P^{\prime}\right)=\left[\varphi^i\left(y^i\right),z^i+\Delta z^{ij},P\right]\end{array}\right\}\tag{5}$$

and the transition function $\Phi^0 : S^0 \times A^0 \rightarrow S$ for the intermediary is:

$$\Phi^0\left[\left\{\left(w^0,z^0\right)\right\},\left\{\left(y^0,P\right)\right\}\right]=\left\{\begin{array}{c}\left(w^{0\prime},z^{0\prime}\right)\in S^0:\\\left(w^{0\prime},z^{0\prime}\right)=\left[\varphi^0(y^0),z^0+\left(-\Delta z_t^{0j}\right)\right]\end{array}\right\}\tag{6}$$

with $\left(\Delta w_t^{0j}, \Delta z_t^{0j}\right) = D\left(j, P_t\right)$. To derive the Euler equations, we write down the problem for two consecutive periods and derive with respect to investment:

$$\beta^t\frac{\partial u^i\left(y_{t-1},y_t\right)}{\partial y_t}+\beta^{t+1}\frac{\partial u^i\left(y_t,y_{t+1}\right)}{\partial y_t}=0$$

For good $i+1$ (where the production function $F$ is not equal to zero) we have instead:

$$\begin{array}{c}\beta^t\frac{-1}{2\sqrt{y_{t-1}-\delta(y_{t-1})+F(y_{t-1}-\delta(y_{t-1}))+\Delta w_t-y_t}}+\beta^{t+1}\frac{\left(1-\frac{y_t}{\theta}\right)\left[1+F'(y_t-\delta(y_t))\right]}{2\sqrt{y_t-\delta(y_t)+F(y_t-\delta(y_t))+\Delta w_{t+1}-y_{t+1}}}=0\text{ for }i+1\\\beta^t\frac{-1}{2\sqrt{y_{t-1}-\delta(y_{t-1})+\Delta w_t-y_t}}+\beta^{t+1}\frac{\left(1-\frac{y_t}{\theta}\right)}{2\sqrt{y_t-\delta(y_t)+\Delta w_{t+1}-y_{t+1}}}=0\text{ for all }j\neq i+1\end{array}$$

Equivalently, we can use the transition function (5) to get the standard form of the Euler equations:

$$\frac{\nabla u^i\left(c_t\right)}{\nabla u^i\left(c_{t+1}\right)}=\beta\nabla_y\Phi^i\left(s,a\right),$$

From the proof of Lemma (8) in the Appendix we know that

$$\begin{array}{rcl}0&\leq&\Phi^i\left(s,a\right)^{\prime}\leq1\text{ for }y_{j\neq(i+1)}\in[0,\theta]\text{ and}\\1&<&\Phi^i\left(s,a\right)^{\prime}\text{ for }y_{i+1}\in[0,\theta].\end{array}$$

Accordingly, for a large enough $\beta$ there will be an incentive to postpone consumption to a future period. Let us summarize and claim that if $c_t > 0$, then in order for the Euler conditions to hold, $c_{t+1} > 0$. No matter how ineffective our "transition technology" is, the Inada condition on the utility function ensures positive consumption in every period for large enough consumption levels.[12]

This is, however, not the end of the story. If we assume that the initial endowment in goods is small, then the marginal utility loss of immediate consumption is very large. A

[12] Of course it must hold that $\beta f'(\cdot) > 0$, otherwise it is always preferable to consumer the entire bundle immediately.

low discount factor combined with a "bad" investment (or production) technology could then result in zero future consumption in all goods $j \neq (i+1)$. The condition $1 < \Phi^{i}(s,a)'$, hence $\beta^{-1} < \Phi^{i'}(s,a)$ for the production good $(i+1)$, ensures that there will always be an incentive to postpone consumption and invest in the production good.

Summarizing the argument, we suggest that the following outcomes are possible:

1. a decreasing optimal consumption path of all goods $j$ that are held in "high quantities", since then $\beta\Phi'(c_{t+1}) < 1$[13] and
2. an increasing consumption path of all goods $j$ that are held in "low" quantities, since then $\beta\Phi'(c_{t+1}) > 1$, which is the typical growth condition in dynamical models.

### 5.5 Existence of an Equilibrium with IOU Trading

For the following proposition we assume that all agents $j \neq i$ and the intermediary trade in IOUs. We will show that if this is the case, agent $i$ has an incentive to accept IOUs in trade, or alternatively, agent $i$ has no incentive to deviate from accepting IOUs.

We assume thatat time $t = 0$ agent $i$ is carrying only her production good $w^{i}_{i+1}$ at a quantity above the threshold level $\theta$. In addition, we assume that this inital endowment is high enough such that her optimal investment level $y^{*}_{i+1}$ into the next period is exactly $\theta$. The agent is then matched with the intermediary who holds zero endowments. The intermediary offers various IOUs at prices stated in the cross-price matrix, $P_{t=0}$.

In the following proposition we state that it is advantageous for the agent to accept some IOUs and carry them over into the next period instead of consuming the total amount of her production good (net of optimal commodity investment) immediately. Hence, future holdings in IOUs compensate for the loss in current consumption. Agents believe that holdings in IOUs improve future trade possibilities, since they widen the trade possibility set.[14]

**Proposition 13** *Given that the rest of the world trades in IOUs and agent i only carries (an endowment) in her production good $i+1$, at a high enough quantitiy $w^{i}_{i+1} > \theta$ such that her optimal investment level is $y^{*}_{t+1} = \theta$, then agent i has an incentive to accept IOUs from the intermediary.*
*This implies that the following inequality holds*

$$v^{i*}\left(0, w^{i}, 0, P\right) > v^{i}\left(0, w^{i}, 0, P\right),$$

*where the value functions are defined as*

$$\max_{(\Delta w^{ij}, \Delta z^{ij}, y^{i}) \in \Gamma^{i}(\cdot)} \left\{ u^{i}\left(w^{i} + \Delta w^{*i} - y^{*i}\right) + \beta \tfrac{1}{n} \textstyle\sum_{j \neq i=0}^{n} v^{i}\left(j, y^{*i} - \delta^{i}(y^{*\mathbf{i}}) + F^{i}\left(y^{*i} - \delta^{i}(y^{*\mathbf{i}})\right), \Delta z^{*i}, P'\right) \right\}$$
$$> \max_{(\Delta w^{ij}, \Delta z^{ij}, y^{i}) \in \Gamma^{i}(\cdot)} u^{i}\left(w^{i} - y^{i}\right) + \beta \tfrac{1}{n} \textstyle\sum_{j \neq i=0}^{n} v^{i}\left(j, y^{i} - \delta^{i}(y^{i}) + F^{i}\left(y^{i} - \delta^{i}(y^{i})\right), 0, P'\right)$$

[13] Compare (Blanchard and Fisher, 1989, p.43).

[14] This is similar to "Condition M" put forward in Harris (1979). This condition states that any loss in current utility suffered by maintaining a minimum level of good 1 (the money good) is made up for by the utility of some bundle which one does not expect to be able to obtain without the minimum level of good 1. The assumption means that an agent belives that having a minimum amount of good 1 greatly improves her trading possibilities. Part of "Condition M" is that each agent holds a specific positivie amount of good 1 from the start in order to "tie" the economy together.

*and* $\left(\Delta w^{*i} < 0, \Delta z^{*i} > 0\right)$ *is the net-trade (demand) vector of the agent engaging in trade and* $\left(\Delta w^{i} = 0, \Delta z^{i} = 0\right)$ *is the net-trade vector of agent i not trading IOUs.*[15]

**Proof.** We have to prove that the loss in marginal utility caused by the negative $\Delta w^{*i}$ is sufficiently compensated for by the positive effect of $\Delta z^{*i}$ in the next period. We start our proof by looking at the two cases, non-trade vs. trade separately.
**Case 1:** If the agent does not trade, it is optimal to invest $y^{*}_{i+1} = \theta$ by definition. Accordingly, for the non-trade case we get the following value function:

$$\begin{aligned} v^{1i}\left(j, w^{i}, 0, P\right) &= u^{i}\left(0...0, w^{i}_{i+1} - \theta_{i+1}, 0...0\right) + \beta \frac{1}{n} \sum\nolimits_{j \neq i=0}^{n} v^{1i}\left(j, w^{1i}, 0, P'\right) \\ &= u^{i}\left(0...0, c^{1,i}_{i+1}, 0...0\right) + v^{1i}\left(j, w^{1i}, 0, P'\right) \end{aligned}$$

with $w^{1i} = \left(0...0, \frac{\theta_{i+1}}{2} + F\left(\frac{\theta_{i+1}}{2}\right), 0...0\right)$ being the amount of goods available in the next period after depreciation and new production added.[16]
**Case 2:** We now assume that the agent trades with the intermediary. There are two possible szenarios. *First*, the agent invests the exact same amount as in Case 1 and has therefore the same $w^{1i}$ as next periods endowment plus an endowment $z^{2i} = \Delta z^{2i} > 0$ in IOUs. Since she traded IOUs that must mean that she gave up some consumption in this period. We denote this part of "lost immediate consumption" as $\Delta w^{2i}_{i+1} \in \mathbb{R}_{-}$. The following value function describes this scenario:

$$v^{2i}\left(j, w^{i}, 0, P\right) = u^{i}\left(0...0, w^{i}_{i+1} + \left(-\Delta w^{2,i0}_{i+1}\right) - \theta_{i+1}, 0...0\right) + \beta \frac{1}{n} \sum\nolimits_{j \neq i=0}^{n} v^{2i}\left(j, w^{2i}, \Delta z^{2i}, P'\right)$$

where $c^{1,i}_{i+1} > c^{2,i}_{i+1}$ and $w^{2i} = w^{1i}$ so that

$$v^{2i}\left(j, w^{i}, 0, P\right) = u^{i}\left(0...0, c^{2,i}_{i+1}, 0...0\right) + v^{2i}\left(j, w^{1i}, \Delta z^{2i}, P'\right).$$

*Second*, the agent can consume the exact same amount as in Case 1 which means that her investment into next period must be somewhat lower. Her value function for this scenario is

$$v^{3i}\left(j, w^{i}, 0, P\right) = u^{i}\left(0...0, w^{i}_{i+1} + \left(-\Delta w^{3,i0}_{i+1}\right) - y^{3i}_{i+1}, 0...0\right) + \beta \frac{1}{n} \sum\nolimits_{j \neq i=0}^{n} v^{3i}\left(j, w^{3i}, \Delta z^{3i}, P'\right)$$

where $c^{1,i}_{i+1} = c^{3,i}_{i+1}$ and $w^{3i} < w^{1i}$ so that

$$v^{3i}\left(j, w^{i}, 0, P\right) = u^{i}\left(0...0, c^{3,i}_{i+1}, 0...0\right) + v^{3i}\left(j, w^{3i}, \Delta z^{3i}, P'\right)$$

From Corollary 27 (page 26 in the Appendix) we know that $v^{i}$ is monotone increasing in $(w, z)$. For *szenario one* this implies that

$$v^{2i}\left(j, w^{1i}, \Delta z^{2i}_{1}, P'\right) \geq v^{1i}\left(j, w^{1i}, 0, P'\right)$$

[15] We know that the weaker form of the inequality ($\leq$) holds by definition. Should it be optimal to not trade in IOUs, then the resulting $\Delta w^{*i} = 0$.

[16] We drop the *max* function and mark next period's endowment with a $(^{*})$ since it is the result of an optimizing decision.

and from the utility functions we know that

$$u^i\left(0...0, c_{i+1}^{2,i}, 0...0\right) < u^i\left(0...0, c_{i+1}^{1,i}, 0...0\right).$$

Now we show that the difference gained in the value functions (future wealth) is larger than the utility lost from lower consumption today.

$$\beta\left[v^{2i}\left(j, w^{1i}, \Delta z^{2i}, P'\right) - v^{1i}\left(j, w^{1i}, 0, P'\right)\right] > u^i\left(..., w_{i+1}^i - \theta_{i+1}, ...\right) - u^i\left(..., \overbrace{w_{i+1}^i + \Delta w_{i+1}^{2i} - \theta_{i+1}}^{c_{i+1}^{2,i}}, ...\right)$$

As $\Delta z^{2i} \rightarrow 0$, the left hand side becomes the derivative of the value function with respect to $\Delta z^{2i}$, whereas for $\Delta w_{i+1}^{2i} \rightarrow 0$, the right hand side becomes the derivative of the utility function with respect to the net trade vector $\Delta w_{i+1}^{2i}$, that is

$$\beta \nabla_z \left[v^{2i}\left(j, w^i, z^i, P'\right)\right] |_{(w^i = w^{2i}, z^i = \Delta z^{2i})} > \left| \frac{\partial u^i\left(..., w_{i+1}^i + \Delta w_{i+1}^{2i} - \theta_{i+1}, ...\right)}{\partial c_{i+1}^{2,i}} \right|$$

where $\nabla_z \left[v^{2i}\left(j, w^i, z^i, P'\right)\right]$ is the gradient vector of function $v^{2i}$. Some of the transferred IOUs as represented in the $\Delta z$ vector can be exchanged into goods in future periods. Using the envelope condition we can replace the left hand side with the respective utility function times the marginal transition function:

$$\begin{aligned}
\beta \nabla_w u^i(w) \nabla_z \Phi^i \left[\left(w^{2i}, 0\right), P, \left(\Delta w^{2,i0}, \Delta z^{2,i0}, y^i\right)\right] &> \left| \frac{\partial u^i\left(..., w_{i+1}^i + \Delta w_{i+1}^{2i} - \theta_{i+1}, ...\right)}{\partial c_{i+1}^{2,i}} \right| \\
\beta \nabla_w u^i(w) \nabla_z \Phi^i [\cdot] &> \left| u^{i\prime}\left(..., w_{i+1}^i + \Delta w_{i+1}^{2i} - \theta_{i+1}, ...\right) \right|
\end{aligned}$$

Since we already proved that $\Phi^{i\prime}$ is positive in the $y \in [0, \theta]$ region, $\beta$ is a non-zero discount factor and some of the entries in $w$ are positive and possibly close to zero, the left hand side is larger than the right hand side for large enough values of initial endowments in $w_{i+1}$. Then the marginal loss in forgone consumption is very small and easily compensated for by the relatively large marginal gains of the left hand side expression. The left hand side expression is a vector product that sums up the marginal utility (potentially very large due to Inada conditions) gains through additional trade facilitated by the IOUs. This proves that an equilibrium where IOUs are held exists.

In the second szenario the first period consumption level is identical and the only difference is in the next period where we compare value functions

$$v^{1i}\left(j, w^{1i}, 0, P'\right) \text{ and } v^{3i}\left(j, w^{3i}, \Delta z^{3i}, P'\right) \text{ for } w^{1i} > w^{3i}.$$

Proof pending. ■

### 5.6 Existence of an Equilibrium without IOU Trading

**Proposition 14** *If the rest of the world does not trade in IOUs, agent $i$ has no incentive to accept an IOU in a trade.*

**Proof.** The agent would give up immediate or future consumption of a good and accept something intrinsically worthless. Without further trade possibilities the IOU does have no trade value. Agent $i$ would be clearly worse off swapping a commodtiy for an IOU. ■

## 6 Additional Results

### 6.1 Intermediary Default Rules

In the next proposition we put a behavioral assumption on all agents. Given that the intermediary defaults on an IOU, all agents are informed about this and refuse to trade with the intermediary for all future periods. The intermediary drops out of the market. We show that it is not in the intermediary's interest to default on her IOUs under such circumstances.

**Proposition 15** *The Intermediary does not default on IOUs issued by her.*

**Proof.** The intermediary compares two alternatives. First, after defaulting she consumes the rest of her leftover endowments in an optimal way. The problem then reduces to an optimal consumption-savings decision with no more interaction with the rest of the economy. The second alternative is to never default and stay in the economy forever.
Let us assume that the intermediary holds an initial endowment $w^0 \neq 0$. If she drops out of the market she engages in the following consumption-savings decision.

$$v^0\left[\left(0,w^0,0\right)\right] = \max_{y^0}\left\{u^0\left[w^0 - y^0\right] + \beta v^0\left[0, y^0 - \delta(y^0), 0\right]\right\}$$

If she stays in the market she gets:

$$v^0\left[\left(j,w^0,z^0\right)\right] = \max_{y^0}\left\{u^0\left[w^0 + \Delta w_t^{0i} - y^0\right] + \beta\frac{1}{n}\sum\nolimits_{j=1}^{n} v^0\left[j, y^0 - \delta(y^0), z^0 + \left(-\Delta z_t^{0i}\right)\right]\right\}$$

Proof pending. ■

### 6.2 Intermediary Commitment

Let us assume the rest of the world does not trade in IOUs.

**Proposition 16** *If the intermediary cannot credibly commit to repay its IOU, then an agent $i$ has no incentive to accept an IOU. All other consumer-producers have the same incentive. Hence, the intermediary stays out of the market.*

**Proof.** In the case of an equilibrium where IOUs are not accepted by consumer-producers, no agent has an incentive to deviate from this behavior and accept IOUs once she is matched with the intermediary. Would an agent do so, she would give up a commodity which has consumption utility for something that has no utility at all. She would be worse off. Since this equilibrium exists, we clearly see that the value function can only be weakly monotone in $z$, the vector of IOU holdings. The intermediary cannot be part of this market. ■

**Proposition 17** *If the intermediary can credibly commit to repay its IOU, then an agent $i$ has an incentive to accept an IOU. All other consumer-producers have the same incentive. The intermediary stays in the market.*

**Proof.** The small chance of meeting the intermediary again (it is a positive probability in the finite agent case) is enough to postpone some consumption into a future period. ■

# 7 Conclusion

In this paper, we take a small step towards reducing the gap between models of money that are most useful in applied work and those models that describe the institution of money as an endogenous or spontaneous process. We are working with an equilibrium that evolves out of an economy with decentralized trade, where agents are production and consumption specialists. Agents are facing an absence of double coincidence of wants and are forced to trade indirectly if they want to consume their consumption good.

Trade is motivated by the agents' desire to level out marginal utility levels in immediate consumption and, speaking losely, by an incentive to have the right trading basket ready for future trade. An increasing depreciation rate creates a further motive for trade. Once an agent has accumulated a critical amount of a commodity, it is advantageous for the agent to reduce respective holdings in that good by way of three alternatives.

First, she can consume the surplus amount. Second, she can trade against another good that she is holding in a lower quantity, and third, she can trade against an IOU. The IOU presents an alternative that is costless to store. Intrinsically worthless IOUs are injected into the economy by an intermediary.

We show that at least one equilibrium exists that describes a situation where it is advantageous for agents to hold and use IOUs. In this equilibrium IOUs are used as a store of value and as means of exchange in bilateral trading. Money, in the form of intrinsically worthless IOUs, emerges endogenously. It is issued by a trade specialist and accepted by consumer-producers.

# 8 Appendix

This appendix contains definitions and proofs that establish the existence of an optimal solution for the agent and intermediary maximization problem. The following assumptions are fulfilled by previous model definitions and guarantee a solution to the value function.

**Assumption 18** *The consumption space $X$ is convex and bounded.*

**Assumption 19** $u^i(c) : X \to \mathbb{R}_+$ *is bounded for all* $c \in X$, $u'(0) = \infty$ *and* $u'(\infty) = 0$. *Moreover,* $u^i(c)$ *is continous, strictly increasing, and strictly concave on* $X$. *The same applies to* $u^0(c)$.

The boundedness (above) comes from the finite number of agents and their respective upper bounds on production, respectively total storage capacity of $\bar{w}$. The maximum amount of commodities within the economy is $n\bar{w}+\bar{w}^0$. Hence, every agent and the intermediary is carrying her maximum storage.

**Definition 20** *We define the one period reward function of agent $i$ as $r^i : S \times \mathcal{P} \times A \to \mathbb{R}$ such that $r^i\left(s^i, p, a^i\right) = u^i\left(w^i + \Delta w^{ij} - y^i\right)$. For the intermediary $r^0 : S^0 \times A^0 \to \mathbb{R}$ is the reward function such that $r^0\left(s^0, a^0\right) = u^0\left(w^0 + \left(-\Delta w_t^{0i}\right) - y^0\right)$. Both functions are bounded, continuous, strictly increasing, and strictly concave.*

**Lemma 21** *The transition function $\Phi^i$ and $\Phi^0$ satisfy the following conditions:*
*$(a)$ $\Phi^i(0) = 0$ on $S \times \mathcal{P} \times A$.*
*$(b)$ $\Phi^i$ is continuous and concave on $S \times \mathcal{P} \times A$.*
*$(c)$ $\Phi^i$ is strictly increasing on $S \times \mathcal{P} \times A$ in the range of $y_i \in [0,\theta]$.*
*$(d)$ $\lim_{y^i \to 0} \partial\Phi^i / \partial y^i > \frac{1}{\beta}$.*
*and for the intermediary transition function:*
*$(a)$ $\Phi^0(0) = 0$ on $S^0 \times A^0$.*
*$(b)$ $\Phi^0$ is continuous and concave on $S^0 \times A^0$.*
*$(c)$ $\Phi^0$ is strictly increasing on $S^0 \times A^0$ in the range of $y^0 \in [0,\theta]$.*
*$(d)$ $\lim_{y^0 \to 0} \partial\Phi^0 / \partial y^0 > \frac{1}{\beta}$.*

Point $(d)$ puts an Inada condition on $\Phi^i(\cdot)$ and $\Phi^0(\cdot)$. Together with the Inada condition on $u(\cdot)$, this ensures that agents will try to keep consumption levels strictly positive in all periods.

**Proof.** In the following we will drop the $i-$ superscript for variables and call the endowment state $(w, z) = s$ to simplify the notation.
$(a)$ is trivial.
$(b)$ Without loss of generality we assume two state action pairs $(s, P, a) < (s^*, P^*, a^*)$. We define the following convex combination:

$$\left(s^\lambda, P^\lambda, a^\lambda\right) = \lambda(s, P, a) + (1-\lambda)(s^*, P^*, a^*) \text{ for } \lambda \in (0,1).$$

Now for concavity we need:

$$\Phi^i\left(s^\lambda, P^\lambda, a^\lambda\right) \geq \lambda\Phi^i(s, P, a) + (1-\lambda)\Phi^i(s^*, P^*, a^*)$$

From the function output $(w',z',P') = \left(y - \delta^i(y) + F^i\left(y - \delta^i(y)\right), z + \Delta z, P\right)$ we see that:

$$\begin{aligned} & \left[y^\lambda - \delta^i(y^\lambda) + F^i\left(y^\lambda\right), z^\lambda + \Delta z^\lambda, P^\lambda\right] \\ \geq \;& \lambda\left[y - \delta^i(y) + F^i(y - \delta(y)), z + \Delta z, P\right] + (1-\lambda)\left[y^* - \delta^i(y^*) + F^i\left(y^* - \delta^i(y^*)\right), z^* + \Delta z^*, P^*\right] \end{aligned}$$

We know that $\delta^i(\cdot)$ is a continous and strictly convex function and $F^i(\cdot)$ is a continuous and strictly concave function. From this we conclude that for the three components it holds that:

$$\begin{aligned} y^\lambda - \delta^i(y^\lambda) + F^i\left(y^\lambda\right) &> \lambda\left[y - \delta^i(y) + F^i\left(y - \delta^i(y)\right)\right] + (1-\lambda)\left[y^* - \delta^i(y^*) + F^i\left(y^* - \delta^i(y^*)\right)\right] \\ z^\lambda + \Delta z^\lambda &= \lambda[z + \Delta z] + (1-\lambda)[z^* + \Delta z^*] \\ P^\lambda &= \lambda P + (1-\lambda)P^* \end{aligned}$$

and the function $\Phi^i$ is continuous and concave.
$(c)$ We know the new state is $\left[y - \delta^i(y) + F^i\left(y - \delta^i(y)\right), z + \Delta z, P\right]$. It is easy to see that the

second and third component is increasing. However, the commodity component is somewhat more complicated. We check the gradient and find that for entries other than $i+1$ entries in $F^i(\cdot)$ are zero. For all $j \neq (i+1)$ entries in the state commodity vectors is therefore $y - \delta^i(y) + 0$. This is clearly increasing in $y_j \in [0, \theta)$ once we take into account the specific form of the depreciation function $\delta^i(y)$, with entries $j$ defined by: $\delta^i(y_j) = \min\left[\frac{(y_j)^2}{2\theta}, y_j\right]$. A similar result holds for entries at position $i+1$. We summarize and see that for $y_{i+1} \in [0, \theta)$ the first derivative is positive, so that

$$\begin{aligned}
\frac{\partial\left(y_j - \frac{(y_j)^2}{2\theta} + 0\right)}{\partial y_j} &= \left(1 - \frac{y_j}{\theta}\right) > 0 \text{ if } y_j < \theta \text{ for all } j \neq i+1 \\
\frac{\partial\left(y_{i+1} - \frac{(y_{i+1})^2}{2\theta} + F^i\left(y_{i+1} - \frac{(y_{i+1})^2}{2\theta}\right)\right)}{\partial y_{i+1}} &= \left(1 - \frac{y_{i+1}}{\theta}\right)\left(1 + F^{i\prime}\left(y_{i+1} - \frac{(y_{i+1})^2}{2\theta}\right)\right) > 0 \\
\text{if } y_{i+1} &< \theta
\end{aligned}$$

Above the threshold the derivative becomes negative. This establishes the result that as long as the agent invests below the threshold level, the function is increasing in $y$.
$(d)$ Taking derivatives and using the Inada condition on production function $F^i$ we get:

$$\begin{aligned}
\lim_{y \to 0} \partial\Phi^i / \partial y_j &= \lim_{y \to 0}\left(1 - \frac{y_j}{\theta}\right) = 1 \text{ for all } j \neq i+1 \\
\lim_{y \to 0} \partial\Phi^i / \partial y_{i+1} &= \lim_{y \to 0}\left(\left(1 - \frac{y_{i+1}}{\theta}\right) + F^{i\prime}\left(y_{i+1} - \frac{y_{i+1}}{\theta}\right) * \left(1 - \frac{y_{i+1}}{\theta}\right)\right) \\
&= 1 + F^{i\prime}(0)
\end{aligned}$$

Since $F^{i\prime}(0) = \infty$ we can establish the result that $1 + F^{i\prime}(0) > \beta^{-1}$ for any non-zero discount factor for entries in $i+1$. For all other entries $j \neq i+1$ the condition is not satisfied since $1 \leq \beta^{-1}$.

However, once we would like to establish the property for the entire range of goods we basically face a multidimensional Inada condition. For this case we do not look at the single entries and check whether the (then) one-dimensional Inada condition holds, we rather check the Inada condition on the norm of the gradient-vector of the transition function. Since the norm is nothing more than square root of the sum of squares of the vector components and one vector component goes to infinity, namely the $(i+1)^{th}$ entry, the whole norm goes to infinity. Therefore the property for the entire problem is established. ■

This allows us to rewrite the maximization problem as a stationary discounted dynamic programming problem.

**Definition 22** *The stationary discounted dynamic programming problem, SPD, is defined as a tuple* $\left\{S, , P, A, \Gamma, \Phi^i, r, \beta\right\}$.[17]

For establishing the required properties of the action-correspondence we introduce the following lemma.

[17] See also (Sundaram, 1996, p. 281) for a similar definition.

**Lemma 23** $\Gamma^i : M \times S \times \mathcal{P} \to A$ *is a compact-valued and continous correspondence on* $M \times S \times \mathcal{P}$.
$a \subset \Gamma^i\left(\pi^i, s, P\right)$ *or in more extensive notation,* $(\Delta w, \Delta z, y) \subset \Gamma^i(j, w,z,P)$.

**Proof.** $(a)$ First we prove upper-hemicontinuity and compactness.
We define the graph $G_{\Gamma^i}$ of the correspondence $\Gamma^i$ as:

$$G_{\Gamma^i} = \left\{\left(\pi^i, s, P, a\right) \in M \times S \times \mathcal{P} \times A : a \in \Gamma^i\left(\pi^i, s, P\right)\right\} \text{ for all } i \in M.$$

or in extended notation

$$G_{\Gamma^i} = \left\{ \begin{array}{c} [j, (w,z), P, (\Delta w, \Delta z, y)] \in M \times S \times \mathcal{P} \times A : \\ (\Delta w, \Delta z, y) \in \Gamma^i(j, w,z,P) \end{array} \right\} \text{ for all } j.$$

Next we demonstrate that $G_{\Gamma^i}$ is a closed graph. Given $S \times \mathcal{P}$ and the closed set $A$, the correspondence $\Gamma^i : M \times S \times \mathcal{P} \to A$ has a closed graph if for any two sequences $(s^m, P^m) \to (s^q, P^q) \in S \times \mathcal{P}$ and $a^m \to a^0$, with $(s^m, P^m) \in S \times \mathcal{P}$ and $a^m \in \Gamma^i\left(\pi^i, s^m, P^m\right)$ for every $m$, we have $a^q \in \Gamma^i\left(\pi^i, s^q, P^q\right)$.
We construct the sequence of states $\{w^m,z^m,P^m\} \to (w^q,z^q,P^q) \in S \times \mathcal{P}$ and a sequence of actions $\{\Delta w^m, \Delta z^m, y^m\} \to (\Delta w^q, \Delta z^q, y^q)$. We know that $\{\Delta w^m, \Delta z^m, y^m\} \in \Gamma^i\left(\pi^i, w^m,z^m,P^m\right)$ by definition. We have to show now that $(\Delta w^q, \Delta z^q, y^q) \in \Gamma^i\left(\pi^i, w^q,z^q,P^q\right)$ or

$$\left\{ \begin{array}{c} -w^q \leq \Delta w^q \leq w_t^j; \\ y^q \leq w^q + \Delta w^q \\ \left\{ \begin{array}{c} -z^q \leq \Delta z^q \leq \bar{z}^0; \\ \left[ \begin{array}{c} \left[P^{*T}\right]_k^q \left[ \begin{array}{c} \Delta w^q \\ \Delta z^q \end{array} \right]_+ + P_k^q \left[ \begin{array}{c} \Delta w^q \\ \Delta z^q \end{array} \right]_- \leq 0 \\ \text{for some } k \in \{1, ..., n\}; \end{array} \right] \quad \text{if } \pi^i = 0 \end{array} \right. \\ \left\{ \begin{array}{c} -z^q \leq \Delta z^q \leq z_t^j; \\ \left[ \begin{array}{c} \exists P \text{ s.t. } \left[p^{*T}\right]_k^q \left[ \begin{array}{c} \Delta w^q \\ \Delta z^q \end{array} \right]_+ + p_k^q \left[ \begin{array}{c} \Delta w^q \\ \Delta z^q \end{array} \right]_- \leq 0 \\ \text{for some } k \in \{1, ..., n\}. \end{array} \right] \quad \forall \pi_t^i \neq 0 \end{array} \right. \end{array} \right\}$$

We know by assumption that

$$\left\{ \begin{array}{c} -w^m \leq \Delta w^m \leq w_t^j; \\ y^m \leq w^m + \Delta w^m; \\ \left\{ \begin{array}{c} -z^m \leq \Delta z^m \leq \bar{z}^0; \\ \left[ \begin{array}{c} \left[P^{*T}\right]_k^m \left[ \begin{array}{c} \Delta w^m \\ \Delta z^m \end{array} \right]_+ + P_k^m \left[ \begin{array}{c} \Delta w^m \\ \Delta z^m \end{array} \right]_- \leq 0 \\ \text{for some } k \in \{1, ..., n\}; \end{array} \right] \quad \text{if } \pi^i = 0 \end{array} \right. \\ \left\{ \begin{array}{c} -z^m \leq \Delta z^m \leq z_t^j; \\ \left[ \begin{array}{c} \exists P \text{ s.t. } \left[P^{*T}\right]_k^m \left[ \begin{array}{c} \Delta w^m \\ \Delta z^m \end{array} \right]_+ + P_k^m \left[ \begin{array}{c} \Delta w^m \\ \Delta z^m \end{array} \right]_- \leq 0 \\ \text{for some } k \in \{1, ..., n\}. \end{array} \right] \quad \forall \pi_t^i \neq 0 \end{array} \right. \end{array} \right\}$$

Taking limits as $m \to \infty$ and bearing in mind that the continuity of the scalar product enables us to write the limit of the product as the product of the limits, one gets:

$$\begin{aligned}
\lim_{m\to\infty} \left[P^{*T}\right]_k^m \begin{bmatrix} \Delta w^m \\ \Delta z^m \end{bmatrix}_+ &= \left[P^{*T}\right]_k^q \begin{bmatrix} \Delta w^q \\ \Delta z^q \end{bmatrix}_+ ; \\
\lim_{m\to\infty} P_k^m \begin{bmatrix} \Delta w^m \\ \Delta z^m \end{bmatrix}_- &= P_k^q \begin{bmatrix} \Delta w^q \\ \Delta z^q \end{bmatrix}_- \\
\lim_{m\to\infty} w^m, z^m, P^m, \Delta w^m, \Delta z^m, y^m &\to w^q, z^q, P^q, \Delta w^q, \Delta z^q, y^q \text{ for every single element}
\end{aligned}$$

Suppose, by way of contradiction, that say $y^q > w^q + \Delta w^q$. Then there will exist some $m'$ such that $y^m > w^m + \Delta w^m$, for all $m > m'$. But this contradicts the hypothesis that $\{\Delta w^m, \Delta z^m, y^m\} \in \Gamma^i\left(\pi^i, w^m, z^m, P^m\right)$ for all $m$. Hence $(\Delta w^q, \Delta z^q, y^q) \in \Gamma^i\left(\pi^i, w^q, z^q, P^q\right)$ ,that is, $G_{\Gamma^i}$ is closed.

In addition we know by assumption that the correspondence maps from $M \times S \times \mathcal{P}$ into the compact set $A$. We therefore fulfill the requirements of Lemma 24 (on page 25) and have shown that the correspondence $\Gamma^i$ is compact-valued and u.h.c.

$(b)$ Next we proof lower hemicontinuity of $\Gamma^i$ at $\left(\pi^i, w^q, z^q, P^q\right)$.

Consider a sequence $\{w^m, z^m, P^m\}$ in $S \times \mathcal{P}$ converging to $(w^q, z^q, P^q)$, and let $(\Delta w^q, \Delta z^q, y^q)$ be a point in $\Gamma^i\left(\pi^i, w^q, z^q, P^q\right)$. We have to show that there is a sequence $\{\Delta w^m, \Delta z^m, y^m\}$ in $A$ such that $\{\Delta w^m, \Delta z^m, y^m\} \to (\Delta w^q, \Delta z^q, y^q)$ and $\{\Delta w^m, \Delta z^m, y^m\} \in \Gamma^i\left(\pi^i, w^m, z^m, P^m\right)$ for all $m$.

To establish this property we use Theorem 3.5 in (Stokey, Lucas and Prescott, 1989, p. 61) and show that $\Gamma$ is nonempty, its graph $G_\Gamma$ is convex; for any bounded set $(M \widetilde{\times S \times} \mathcal{P}) \subseteq M \times S \times \mathcal{P}$, there is a bounded set $\tilde{A} \subseteq A$ s.t. $\Gamma(\cdot) \cap \tilde{A} \neq \emptyset$ for all $\left(\pi^i, s, P\right) \in (M \widetilde{\times S \times} \mathcal{P})$ and that for every state $\left(\pi^i, s, P\right)$ there exists a closed neighborhood $B\left(\left(\pi^i, s, P\right), \varepsilon\right) \cap M \times S \times \mathcal{P}$. ■

**Lemma 24** *Let $\Gamma^i : M \times S \times \mathcal{P} \to A$ be a nonempty-valued correspondence, and let $G_{\Gamma^i}$ be the graph of $\Gamma^i$. Suppose that $G_\Gamma$ is closed, and for any bounded set $\bar{M} \times \bar{S} \times \bar{\mathcal{P}} \subseteq M \times S \times \mathcal{P}$, the set $\Gamma\left(\bar{M} \times \bar{S} \times \bar{\mathcal{P}}\right)$ is bounded, then $\Gamma^i$ is compact-valued and upper hemi continuous (u.h.c.).*

**Proof.** The proof is stated in (Stokey, Lucas and Prescott, 1989, p. 60). ■

**Theorem 25** *Suppose the SDP$^i$ $\left\{S, P, A, \Gamma, \Phi^i, r^i, \beta\right\}$ satisfies the following conditions*[18]*:*
(1) $r^i : S \times \mathcal{P} \times A \to \mathbb{R}$ *is continuous and bounded on* $S \times \mathcal{P} \times A$.
(2) $\Phi^i : S \times \mathcal{P} \times A \to \mathbb{R}$ *is continuous on* $S \times \mathcal{P} \times A$.
(3) $\Gamma^i : M \times S \times \mathcal{P} \to A$ *is a compact-valued, continuous correspondence.*
*Then, there exists a stationary and continuous optimal policy function $g^* : S \times \mathcal{P} \to A$. Furthermore, the value function $v = W(g^*)$ is continuous on $S \times \mathcal{P}$, and is the unique bounded function that satisfies the Bellman Equation at each $(s, P) \in S \times \mathcal{P}$:*

$$\begin{aligned}
W^i(g^*)(j, s, P) &= \underset{a \in \Gamma^i(j,s,P)}{\arg\max} \left\{ r^i(s, P, a) + \beta \frac{1}{n} \sum\nolimits_{j \neq i = 0}^{n} W^i(g^*)\left(j, \Phi^i(s, P, a)\right) \right\} \\
&= r^i(s, P, g(j, s, P)) + \beta \frac{1}{n} \sum\nolimits_{j \neq i = 0}^{n} W^i(g^*)\left[j, \Phi^i(s, P, g^*(j, s, P))\right]
\end{aligned}$$

[18] Given the properties stated above we can see that the assumptions for the existence of an optimal strategy for the intermediary $g^{0*} : S^0 \to A^0$ are sufficient as well.

**Proof.** The proof can be found in (Sundaram, 1996, p. 295-298). ■

Given the properties of the maximization problem we can make more statements about the value function, the proofs of which can also be found in (Sundaram, 1996, p. 301 ff).

**Theorem 26** *Given the properties of the maximization problem stated above and the following additional properties we can state that:*
$(a)$ *Since* $\Phi^i$ *is concave and non-decreasing in* $y \in [0, \theta)$ *and* $\Gamma$ *is convex valued,* $v^i : M \times S \times \mathcal{P} \rightarrow \mathbb{R}$ *is concave, non-decreasing and continuously differentiable on* $(s, P_0) \in int\,(S \times \mathcal{P})$*.*
$(b)$ *Adding the Inada condition on the utility function and the transition function, plus strict concavity on the transition function over* $y \in [0, \theta)$ *ensures an interior solution to the model.*
$(c)$ *Under the same assumptions (concavity may replace strict concavity), the optimal policy function* $g^*$ *is increasing in* $S \times \mathcal{P}$*. That is* $(s, P) > (s', P')$ *implies* $g^*(s, P) > g^*(s', P')$

From this theorem we derive the following Corollary.

**Corollary 27** $v^i(j, w, z, P)$ *is strictly monotone increasing in* $w$ *and weakly monotone in* $z$*.*

**Proof.** Monotonicity in $w$ is trivial. Monotonicity in $z$ can be derived from the budget condition in the feasible net-trade (demand) opportunity set (11). From there we see that a larger $z$ "widens" the budget constraint and can never make an agent worse off. ■

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