# Chapter 2
# The Abelian Sandpile Model

## The State of the Art

It has been more than 20 years since Bak, Tang and Wiesenfeld's landmark papers on self-organized criticality (SOC) appeared [1]. The concept of self-organized criticality has been invoked to describe a large variety of different systems. We shall describe the model object of our interest: the *Abelian Sandpile Model* (ASM). The sandpile model was first proposed as a paradigm of SOC and it is certainly the simplest, and best understood, theoretical model of SOC: it is a non-equilibrium system, driven at a slow steady rate, with local threshold relaxation rules, which shows in the steady state relaxation events in bursts of a wide range of sizes, and long-range spatio-temporal correlations. The ASM consists of a special subclass of the sandpile models that exhibits, in the way we will discuss later, the mathematical structure of an abelian group, and its statistics is connected to that of spanning trees on the relative graph. There is a number of review articles on this subject, taking into account the connection of the model with the theme of SOC and its inner mathematical properties: Dhar [2], Priezzhev [3] and Redig [4, 5].

Here we present a review of the Sandpile Model theory based on the material than can be found therein with particular emphasis on mathematical aspects and on its stochastic dynamics; some further development given by us complete the review. This material will be necessary for the comprehension of the studies discussed in the following chapters.

## 2.1 General Properties

The ASM is defined as follows [6, 2]: we consider any (directed) graph $G = (V, E)$ with $|V| = N$ and vertices labeled by integers $i = 1, \ldots, N$, at each site we define a nonnegative integer height variable $z_i$, called the height of the sandpile, and a threshold value $\bar{z}_i \in \mathbb{N}^+$. We define an *allowed* configuration of the sandpile as a set $z \in \mathbb{N}^N$ of integer heights $z = \{z_i\}_{i \in V}$ such that $z_i \geq 0\ \forall i \in V$; an allowed configuration $\{z_i\}$ is said to be *stable* if $z_i < \bar{z}_i \quad \forall i \in V$. Therefore the set $S$ of stable

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configurations is $S = \bigotimes_{i\in V}\{0,\ldots,\bar{z}_i\}$. If we call $S_{\pm} \subset \mathbb{Z}^n$ the sets respectively such that $z_i \geq 0$ for all $i \in V$, and $z_i < \bar{z}_i$ for all $i \in V$. Then $S^{+}$ is the set of *allowed* configurations and is *stable* if it is in $S := S_+ \cap S_-$. The involution $z_i \to \bar{z}_i - z_i - 1$ exchanges $S_+$ and $S_-$.

The stochastic time evolution of the sandpile is defined in term of the *toppling matrix* $\Delta$ according to the following rules:

1. *Adding a particle*: Select one of the sites randomly, the probability that the site $i$ is picked being some given value $p_i$, and add a grain of sand there. Obviously $\sum_i p_i = 1$. On addition of the grain at site $i$, $z_i$ increases by 1, while the height at the other sites remains unchanged.
2. *Toppling*: If for any site $i$ it happens that $z_i \geq \bar{z}_i$, then the site is said to be unstable, it *topples*, and lose some sand grains to other sites. This sand's grains transfer is defined in terms of an $N \times N$ integer valued toppling matrix $\Delta$, which properties will be specified in (2.2a). On toppling at site $i$, the configuration $z$ is updated globally according to the rule:

$$z_j \to z_j - \Delta_{ij} \quad \forall j \in V \tag{2.1}$$

If the toppling results in some other sites becoming unstable, they are also toppled simultaneously (it will be clear in the following that the order of toppling is unimportant). The process continues until all sites become stable[1] (we will see later under which conditions on the set of threshold values and $\Delta$ the final stability is guaranteed)

At each time step of the stochastic evolution, first we add a particle, as specified in rule 1, then we relax the configuration, that means to perform the necessary topplings to reach a stable configuration as stated in rule 2.

The *toppling matrix* $\Delta$ has the following properties:

$$\Delta_{ii} > 0, \ \forall i \in V \tag{2.2a}$$

$$\Delta_{ij} \leq 0, \ \forall i \neq j \tag{2.2b}$$

$$b_i^- := \sum_{j\in V} \Delta_{ij} \geq 0, \ \forall i \in V \tag{2.2c}$$

For future convenience we also define the integers

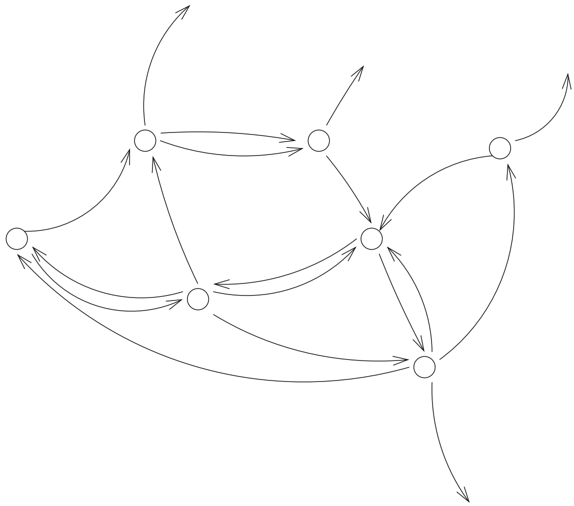
$$b_i^+ := \sum_{j\in V} \Delta_{ij}^T \geq 0 \tag{2.3}$$

for any vertex $i$.

We will adopt a *vector* notation for the collection of elements $\vec{\Delta}_i = \{\Delta_{ij}\}_{j=1,\ldots,N}$. With this notation it is possible to rewrite the *toppling rule* (2.1) for the toppling at site $i$ as

[1] This process is called an *avalanche*

**Fig. 2.1** A graphical representation of the general ASM. Each node denotes a site. On topplings at any site, one particle is transferred along each arrow directed outward from the site, each arrow corresponding to a unit in $-\Delta_{ij}$



$$z \to z - \vec{\Delta}_i \tag{2.4}$$

The conditions (2.2a) just ensure that, on toppling at site $i$, $z_i$ must decrease heights at other sites $j$ can only increase and there is no creation of sand in the toppling process. In some sites could be possible to *lose* some sand during a toppling. If the graph $G$ is undirected, the toppling matrix $\Delta$ is symmetric and $b_i^- = b_i^+ = b_i$.

The graph, in general directed and with multiple edges, is thus identified by the non-diagonal part of $-\Delta$, seen as an adjacency matrix, while the non-zero values $b_i^-$ are regarded as *connections to the border*, and the sites $i$ with nonzero $b_i^-$ are said to be *on the border*, see Fig. (2.1). $b_i^-$ is the total sand lost in the toppling process on $i$, so that, pictorially, we can think of this *lost sand* as dropping out of some boundary. $b_i^+$ is the difference between the amount of sand that can leave the site in a toppling and the amount that is added if all other sites would make a toppling. In the formulation on an arbitrary graph, as presented here, this concept of boundary does not need to correspond to any geometrical structure. We note that no stationary state of the sandpile is possible unless the particles can leave the system.

As an example, the original BTW model [1] is defined on an undirected graph which is a rectangular domain of the $\mathbb{Z}^2$ lattice. We have in this case

$$\Delta_{ij} = \begin{cases} +4 & \text{if } i = j \\ -1 & \text{if } i, j \text{ are nearest-neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

Here the connections with the border $b_i^+ = b_i^- = b_i$ are given, along the sides of the rectangle, by $b_i = 1$, while on the corners by $b_i = 2$. In this framework to be on the boundary (or in a corner) has a direct correspondence with the geometrical structure of the lattice.

We assume, without loss of generality, that $\bar{z}_i = \Delta_{ii}$ (this amounts to a particular choice of the origin of the $z_i$ variables). Then we know that if a site $i$ is stable, and the initial conditions for the heights are $z_i(t = 0) \geq 0 \; \forall i \in V$, at all times the allowed values for $z_i$ are the ones for which holds $0 \leq z_i < \bar{z}_i$.

The procedure rule 1 followed by rule 2 defines a Markov chain on the space of stable configurations, with a given equilibrium measure. Running the stochastic dynamics for long times, that means after a large amount of sand added, the system reaches the stationary state.

As stated in rule 2, if a configuration is *unstable*, there is at least a vertex $i$ where the configuration $z$ has $z_i \geq \bar{z}_i$. The vertex $i$ *topples* and the configuration $z$ is updated following the rule (2.4). The new configuration reached after a toppling at site $i$ is $t_i z = z - \vec{\Delta}_i$, where we call $t_i$ the toppling operator at site $i$.

The collection of topplings needed to produce a stable configuration is called an *avalanche*. We shall assume that an avalanche always stops after a finite number of steps, which is to say that the diffusion is strictly *dissipative*. The size of avalanches can be studied statistically for interesting graphs (e.g. for a partition of $\mathbb{Z}^2$). In many cases of interest it seems to have a power law tail, which is signal of existence of long-range correlations in the system, see [7].

We shall denote by $\mathcal{R}(z)$ the stable configuration obtained from the relaxation of the configuration $z$, so $\mathcal{R}(z) \in S$ and

$$z \in S \quad \Leftrightarrow \quad z = \mathcal{R}(z) \,. \tag{2.6}$$

Given two configurations $z$ and $w$ we introduce the configuration $z + w$ which has at each vertex $i$ the height $z_i + w_i$. Call $e^{(i)}$ the configuration which has non-vanishing height only at the site $i$ where it has height 1, that is $e^{(i)}_j = \delta_{ij}$. Of course each configuration $z$ can be obtained by deposing $z_i$ particles at the vertex $i$

$$z_i e^{(i)} = \underbrace{e^{(i)} + e^{(i)} + \cdots + e^{(i)}}_{z_i} \tag{2.7}$$

so that summing on every vertex $i$, this means

$$z = \sum_{i \in V} z_i e^{(i)} \,. \tag{2.8}$$

### *2.1.1 Abelian Structure*

Let $\hat{a}_i$ be the operator which adds a particle at the vertex $i$

$$\hat{a}_i z := z + e^{(i)} \tag{2.9}$$

then if $z$ is not stable at the vertex $j$,

$$t_j \hat{a}_i z = \hat{a}_i t_j z \tag{2.10}$$

is easily verified.

Let now $a_i$ be the addition of a particle at the vertex $i$ followed by a sequence of topplings which makes the configuration stable. The stable configuration

$$a_i z = \mathcal{R}(e^{(i)} + z) \tag{2.11}$$

is independent from the sequence of topplings, because topplings commute. Indeed, the final configuration of a sequence of topplings does not depend on the order of unstable vertices chosen for each intermediate toppling. For this reason the model is said to be *abelian sandpile model* (ASM). More precisely, if the configuration $z$ is such that $z_i > \bar{z}_i$ and $z_j > \bar{z}_j$ then

$$t_i t_j z = t_j t_i z \tag{2.12}$$

can be easily verified. Let us consider an unstable configuration with two unstable sites $\alpha$ and $\beta$, toppling first the site $\alpha$ leaves $\beta$ unstable thanks to (2.2b), and, after the toppling of $\beta$, we get a configuration in which $z \to z - (\vec{\Delta}_\alpha + \vec{\Delta}_\beta)$ this expression is clearly symmetrical under exchange of $\alpha$ and $\beta$. Thus we get the same final configuration irrespective of whether $\alpha$ or $\beta$ is toppled first. By repeated use of this argument we see that, in an avalanche, the same final state is reached irrespective of the sequence chosen for the unstable sites to topple. Similar reasoning apply for toppling of a site $\alpha$ followed by addition of a sand grain in $\beta$, so this gives the same result of the reverse ordered operation.

It is clear now that, applying two operators $a_i$ and $a_j$, the configurations $a_j a_i z$ and $a_i a_j z$ coincide

$$a_i a_j z = a_j a_i z = \mathcal{R}(e^{(i)} + e^{(j)} + z) \tag{2.13}$$

so that $a_i$ and $a_j$ do commute, or in other word

$$[a_i, a_j] = 0 \quad \forall i, j \in V \tag{2.14}$$

Note that, while this property seems very general, it is not shared with most of the other SOC models, even other sandpile models, for example when the toppling condition depends on the gradient, in this case the order of toppling would matter, being the toppling rule not local and dependent on the actual height's values of the whole configuration.

Given two configurations $z$ and $z'$ we define an abelian composition $z \oplus z'$ as the sum of the local height variables, followed by relaxation

$$z \oplus z' = \mathcal{R}(z + z') = \left(\prod_{i \in V} a_i^{z_i}\right) z' = \left(\prod_{i \in V} a_i^{z'_i}\right) z \tag{2.15}$$

and thus, for a configuration $z$, we define multiplication by a positive integer $k \in \mathbb{N}$:

$$k\,z = \underbrace{z \oplus \cdots \oplus z}_{k} \tag{2.16}$$

The operators $a_i$'s have some interesting properties. For example, on a square lattice, when 4 grains are added at a given site, this is forced to topple once and a grain is added to each of his neighbors. Thus:

$$a_j^4 = a_{j_1} a_{j_2} a_{j_3} a_{j_4} \tag{2.17}$$

where $j_1, j_2, j_3, j_4$ are the nearest-neighbors of $j$.

In the general case one has, instead of (2.17),

$$a_i^{\Delta_{ii}} = \prod_{j \neq i} a_j^{-\Delta_{ij}}. \tag{2.18}$$

by definition, because if we add $\bar{z}_i$ particles at the vertex $i$, after its toppling these particles move on the nearest-neighbor vertices, and, since now on, the relaxation to the stable configuration will be identical. Using the abelian property, in any product of operators $a_i$, we can collect together occurrences of the same operator, and using the reduction rule (2.18), it is possible to reduce the power of $a_i$ to be always less than $\Delta_{ii}$. The $a_i$ are therefore the generators of a finite abelian semi-group (in which the associative property follow from their definition) subject to the relation (2.18); these relations define completely the semi-group.

Let now consider the repeated action of some given operator $a_i$ on some configuration $z$. Since the number of possible states is finite, the orbit of $a_i$ must close on itself, at some stage, so that $a_i^{n+p} z = a_i^n z$ for some positive period $p$, and non negative integer $n$. The first configuration that occurs twice in the orbit is not necessary $z$, so that the orbit consists of a sequence of transient configurations, followed by a cycle. If this orbit does not exhaust all configurations, we can take a configuration outside this orbit and repeat the process. So the space of all configurations is broken up into disconnected parts, each one containing one limit cycle.

Under the action of $a_i$ the transient configurations are unattainable once the system has reached one of the periodic configurations. In principle the recurrent configurations might still be reachable as a result of the action of some other operator, say $a_j$, but the abelian property implies that if $z$ is a configuration part of one of the limit cycles of $a_i$, then so is $a_j z$, in fact $a_i^p z = z$ implies that $a_i^p a_j z = a_j a_i^p z = a_j z$. Thus the transient configurations with respect to an operator $a_1$ are also transient with respect to the other operators $a_{j_1}, a_{j_2}, \ldots$, and hence occur with zero probability in the steady state. The abelian property thus implies that $a_i$ maps the cycles of $a_i$

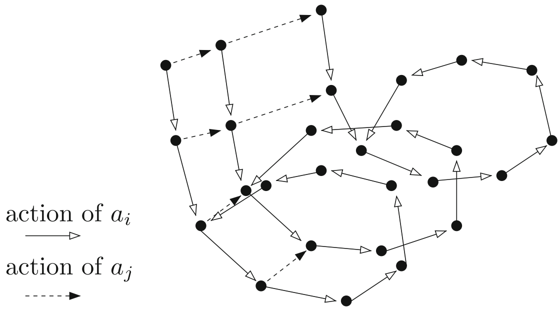



**Fig. 2.2** graphical representation of the combined action of $a_i$ and $a_j$

into cycles of $a_i$, and moreover that all this cycles have the same period Fig. (2.2). Repeating our previous argument we can show that the action of $a_j$ on a cycle is finally closed on itself to yield a torus, possibly with some transient cycles, which may be also discarded. Continuing with the same arguments for the other cycles and other generators leads to the conclusion that the set of all the configurations in the various cycles form a set of multi-dimensional tori under the action of the $a_i$'s.

The configurations that belong to a cycle are said to be *recurrent*, and can be defined, if we allow addition of sand with non zero probability in any site ($p_i > 0$ $\forall i$), as the configurations reachable by any other configuration with addition of sand followed by relaxation. We denote the set of all recurrent configurations as **R** and the set of the transient ones as **T**.

Given the natural partial ordering, $z \preccurlyeq z'$ iff $z_i \leq z'_i$ for all $i$, then **R** is in a sense "higher" than **T**, more precisely

$$\nexists\, (z, z') \in \mathbf{T} \times \mathbf{R} \;: \qquad z \succ z'\,; \tag{2.19}$$

$$\forall z \in \mathbf{T} \quad \exists z' \in \mathbf{R} \;: \qquad z \prec z'\,. \tag{2.20}$$

In particular, the *maximally-filled* configuration $z_{max} = \{\bar{z}_i - 1\}$ is in **R**, and higher than any other stable configuration.

## 2.2 The Abelian Group

The set **R** of recurrent configurations is special. Indeed in **R** it is possible to define the inverse operator $a_i^{-1}$ for all $i$, as each configuration in a cycle has exactly one incoming arrow corresponding to the operator $a_i$. Thus the $a_i$ operators generate a group. The action of the $a_i$'s on the states correspond to translations of the torus. From the symmetry of the torus under translations, it is clear that all recurrent states occur in the steady state with the same probability.

This analysis, which is valid for every finite abelian group, leaves open the possibility that some recurrent configurations are not reachable from each other, in which case there would be some mutually disconnected tori. However, such a situation cannot happen if we allow addition of sand at all sites with non zero probabilities ($p_i > 0\,\forall i$). Let us define $z_{max}$ as the configuration in which all sites have their maximal height, $z_i = \Delta_{ii} - 1 \quad \forall i$. The configuration $z_{max}$ is reachable from every other configuration, is therefore recurrent, and since inverses $a_i$'s exist for configurations in $\mathbf{R}$, every configuration is reachable from $z_{max}$ implying that every configuration lie in the same torus.

Let $\mathcal{G}$ be the group generated by operators $\{a_i\}_{i=1,\dots,N}$. This is a finite group because the operators $a_i$'s, due to (2.18), satisfy the closure relation:

$$\prod_{i=1}^{N} a_i^{\Delta_{ji}} = I \qquad \forall i = 1, \dots, N \tag{2.21}$$

the order of $\mathcal{G}$, denoted as $|\mathcal{G}|$, is equal to the number of recurrent configurations. This is a consequence of the fact that if $z$ and $z'$ are any two recurrent configurations, then there is an element $g \in \mathcal{G}$ such that $z' = gz$. We thus have:

$$|\mathcal{G}| = |\mathbf{R}| \tag{2.22}$$

Given the group structure, the semi-group operation (2.15) is raised to a group operation, in particular the composition of whatever $z$ with the set $\mathbf{R}$ acts as a translation on this toroidal geometry. A further consequence is that, for any recurrent configuration $z$, the inverse configuration $-z$ is defined, and $k\,z$ is defined for $k \in \mathbb{Z}$ too.

The identity of this abelian group, denoted by $Id_{\mathrm{r}}$, is called recurrent identity or *Creutz identity* after Creutz first studies in [8, 9] and is the only stable recurrent configuration such that

$$\forall z \in \mathbf{R} \quad Id_{\mathrm{r}} \oplus z = z \tag{2.23}$$

## 2.3 The Evolution Operator and the Steady State

We consider a vector space $\mathscr{V}$ whose basis vectors are the different configurations of $\mathbf{R}$. The state of the system at time $t$ will be given by a vector

$$|P(t)\rangle = \sum_z \mathsf{Prob}(z, t)\,|z\rangle \ , \tag{2.24}$$

where $\mathsf{Prob}(z, t)$ is the probability that the system is in the configuration $z$ at time $t$. The operators $a_i$ can be defined to act on the vector space $\mathscr{V}$ through their operation on the basis vectors.

The time evolution is Markovian, and governed by the equation

$$|P(t+1)\rangle = \mathscr{W}\, |P(t)\rangle \tag{2.25}$$

where

$$\mathscr{W} = \sum_{i=1}^{N} p_i a_i \tag{2.26}$$

To solve the time evolution in general, we have to diagonalize the evolution operator $\mathscr{W}$. Being mutually commuting, the $a_i$ may be simultaneously diagonalized, and this also diagonalizes $\mathscr{W}$. Let $|\{\phi\}\rangle$ be the simultaneous eigenvector of $\{a_i\}$, with eigenvalues $\{e^{i\phi_i}\}$, for $i = 1, \dots N$. Then

$$a_i \left|\{\phi\}\right\rangle = e^{i\phi_i} \left|\{\phi\}\right\rangle \quad \forall i = 1, \dots, N. \tag{2.27}$$

We recall that the $a_i$ operators now satisfy the relation (2.21). Applying the l.h.s. of this relation to the eigenvector $|\{\phi\}\rangle$ gives $\exp(i \sum_j \Delta_{kj}\phi_j) = 1$, for every $k$, so that $\sum_j \Delta_{kj}\phi_j = 2\pi m_k$, or inverting,

$$\phi_j = 2\pi \sum_k \left[\Delta^{-1}\right]_{jk} m_k \ , \tag{2.28}$$

where $\Delta^{-1}$ is the inverse of $\Delta$, and the $m_k$'s are arbitrary integers.

The particular eigenstate $|\{0\}\rangle$ ($\phi_j = 0$ for all $j$) is invariant under the action of all the $a$'s, $a_i\, |\{0\}\rangle = |\{0\}\rangle$. Thus $|\{0\}\rangle$ must be the stationary state of the system since

$$\sum_i p_i a_i\, |\{0\}\rangle = \sum_i p_i\, |\{0\}\rangle = |\{0\}\rangle \ . \tag{2.29}$$

We now see explicitly that the steady state is independent of the values of the $p_i$'s and that *in the steady state all recurrent configurations occur with equal probability.*

## 2.4 Recurrent and Transient Configurations

Given a stable configuration of the sandpile, how can we distinguish between transient and recurrent configurations? A first observation is that there are some forbidden subconfiguration that can never be created by addiction of sand and relaxation, if not already present in the initial state. The simplest example on the square lattice case is a configuration with two adjacent sites of height 0, $\boxed{0\,|\,0}$. Since $z_i \geq 0$, a site of height 0 can only be created as a result of toppling at one of the two sites (toppling from anywhere else can only increase his height). But a toppling of either of this sites results in a height of at least 1 in the other. Thus any configuration which contains two adjacent 0's is transient. With the same argument it is easy to prove that the following configurations can never appear in a recurrent configuration:

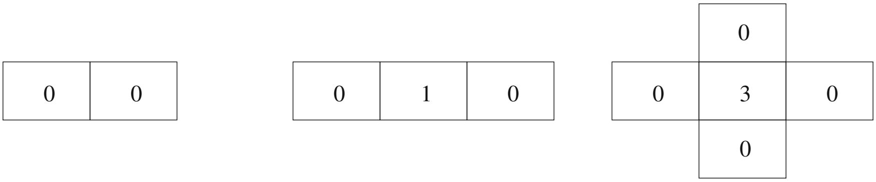



**Fig. 2.3** Examples of forbidden subconfigurations

In general a *forbidden subconfiguration* (FSC) is a set $F$ of $r$ sites ($r \geq 1$), such that the height $z_j$ of each site $j$ in $F$ is less of the number of neighbors than $j$ in $F$, precisely:

$$z_i < \sum_{j \in F \setminus \{i\}} (-\Delta_{ij}) \quad \forall i \in F \tag{2.30}$$

The proof of this assertion is by induction on the number of sites in $F$. For example the creation of the $\boxed{0 \mid 1 \mid 0}$ subconfiguration must involve toppling at one of end sites, but then the subconfiguration must have had a $\boxed{0 \mid 0}$ before the toppling, and this was shown before to be forbidden.

An interesting consequence of the existence of forbidden configurations is the following: consider an ASM on an undirected graph, with $N_b$ bonds between sites, then in any recurrent configuration the number of sand grains is greater or equal to $N_b$. Here we do not count the boundary bonds, corresponding to particles leaving the system. To prove this, we note that if the inequality is not true for any configuration, it must have a FSC in it.

### *2.4.1 The Multiplication by Identity Test*

Consider the product over sites $i \in V$ of equations (2.17)

$$\prod_i a_i^{\Delta_{ii}} = \prod_i \prod_{j \neq i} a_j^{-\Delta_{ij}} = \prod_i a_i^{-\sum_{j \neq i} \Delta_{ji}} \tag{2.31}$$

On the set $\mathbf{R}$, the inverses of the formal operators $a_i$ are defined, so that we can simplify common factors in (2.31), recognize the expression for $b_i^+$, and get

$$\prod_i a_i^{b_i^+} = I \tag{2.32}$$

so that $\prod_i a_i^{b_i^+} z = z$ is a necessary condition for $z$ to be recurrent (but it is also sufficient, as no transient configuration is found twice in the same realization of the

Markov chain), and goes under the name of *identity test*. If we denote by $b^+$ the configuration

$$b^+ = \sum_{i \in V} b_i^+ e^{(i)} \tag{2.33}$$

the identity test means that

$$z \in \mathbf{R} \Leftrightarrow z = z \oplus b^+ \tag{2.34}$$

In the next section we will see how to obtain the same result in an easier and faster way, without actually performing the addition and relaxation.

This relation gives information also on the recurrent identity itself. Indeed it says that if $b^+ \in \mathbf{R}$ then $b^+ = Id_\mathrm{r}$. Otherwise there must be a positive integer $k$ such that

$$\forall \ell \in \mathbb{N} : \ell \geq k \quad \underbrace{b^+ \oplus b^+ \oplus \cdots \oplus b^+}_{\ell} = Id_\mathrm{r} \tag{2.35}$$

then

$$Id_\mathrm{r} = k b^+. \tag{2.36}$$

### *2.4.2 Burning Test*

We define a simple recursive procedure to discover if a configuration is recurrent, based on the mechanical check of presence or absence of FSC in the configuration. We consider at each step a test set, say $T$, of sites. At the beginning $T$ consists of all the sites of the lattice we are considering; we test the hypothesis that $T$ is a FSC using the inequalities (2.30). If these inequalities are satisfied for all sites in $T$, then the hypothesis is true, $T$ has a FSC, and the configuration in exam is transient. Otherwise there are some sites for which the inequalities are violated, these sites cannot be part of any FSC, in fact the inequalities will remain unsatisfied even though $T$ is replaced by a smaller subset of sites. We delete these sites from $T$ and we have a new subset $T'$, we say we *burn* these sites while the remaining are *unburnt*; at this point we repeat the procedure to check whether $T'$ is a FSC. We follow this scheme until we cannot burn anymore site. If we are left with a finite subset $F$ of unburnt sites this is a FSC and the configuration is transient, if the set of unburnt sites eventually becomes empty the configuration in exam is found to be recurrent. We call the procedure just presented the *burning test*.

In the *burning test* it does not matter in which order the sites are burnt. It is however useful to introduce the concept of time of burning and to add to the graph a site, named *sink*, which is connected to all the "boundary" sites with as many links as the number of lost particles in a toppling by the boundary sites in exam, it never topples and only collects sand. There is a natural way to choose a time of burning for each site. At time $t = 0$, all the sites in $T^{(0)} = T$ are unburnt except the sink. At any time $t$ a site is called *burnable* iff the inequality (2.30) is unsatisfied with respect to the set $T^{(t)}$ reached at that time, then a burnable site at time $t$ becomes burnt at time $t+1$, and so remains for the successive times. With this prescription we label each site of the graph with a burning time, depending just on the configuration in exam.

We want now to draw a path for the "fire" to propagate to the whole graph, starting from the sink. Take an arbitrary site $i$, except the sink. Let $\tau_i + 1$ be the time step at which this is burnt, then the burning rule implies that at time $\tau_i$ at least one of his neighbor sites has been burnt. Let $r_i$ be the number of such neighbors and let us write:

$$\xi_i = \sum_{j=1}^{r_i} (-\Delta_{ij}) \tag{2.37}$$

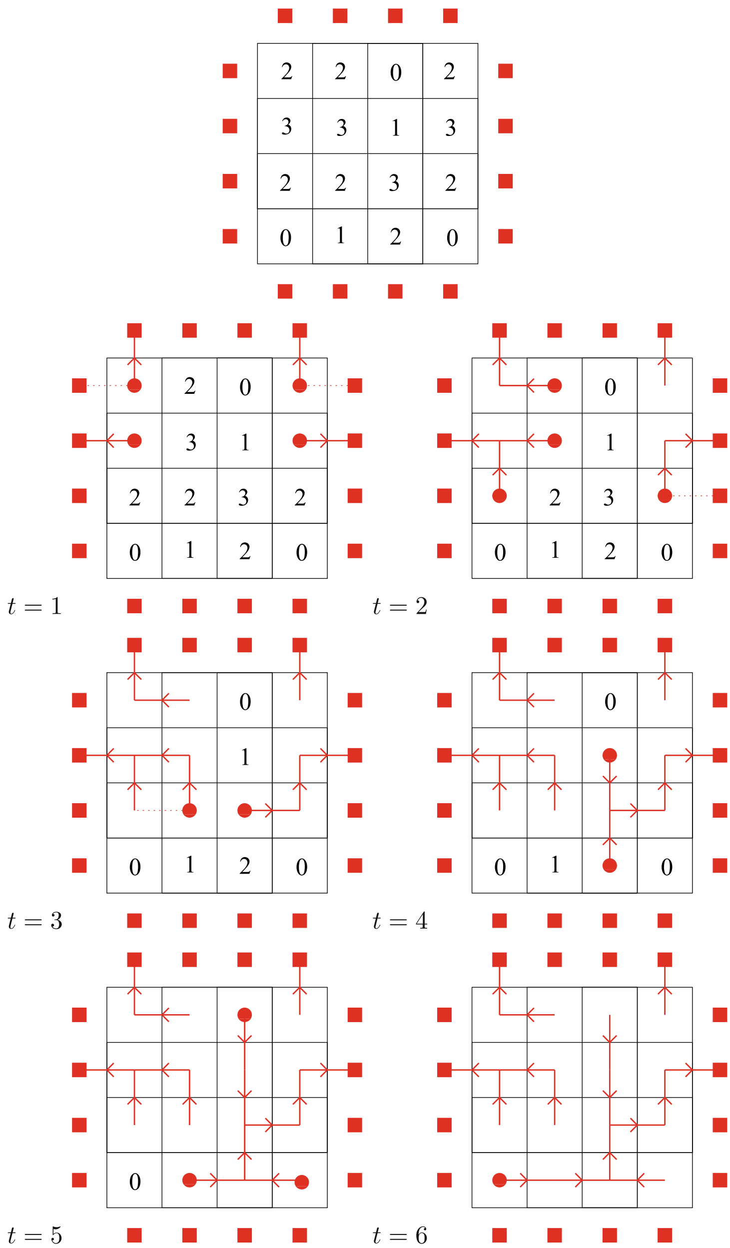
where the primed summation runs over all unburnt neighbors of $i$ at time $\tau_i$. Then we have $z_i \geq \xi_i$ since the site $i$ is burnable at time $\tau_i$; but, since it was not burnable at time $\tau_i - 1$ we must also have:

$$z_i < \xi_i + K \tag{2.38}$$

where $K$ is the number of bonds linking $i$ to his neighbors which were unburnt at time $\tau_i$. During the burning test we say that fire reaches the site $i$ by one of the $K$ bonds. Obviously when $K = 1$ there is just one possibility so there are no problems, and we say that the fire reaches $i$ from the only site possible. If $K > 1$ we have to select one bond through which the fire reaches $i$ depending on his height $z_i$. For this purpose we order the bonds converging on the site $i$ in some sequence (e.g. $\{(i, i_1), (i, i_2), (i, i_3), \ldots\}$), the order is arbitrary and can be chosen independently on each site $i$. Now we can write $z_i = \xi_i + s - 1$ for some $s > 0$, we say that fire reaches $i$ using the $s$-th link in the ordered list of the possible ones. This procedure gives a unique path for the fire to reach each site $i$, given the configuration of heights in the sandpile and the prescription on the order of bonds converging on each site. The set of bonds along which fire propagates, connects the sink with each site in the graph, and there are no loops in each path. Thus the set just obtained is a spanning tree on the graph $G' = G + \{sink\}$.

Choosing a particular prescription for a given graph we can obtain for each recurrent configuration a unique spanning tree. For example on a square lattice we have four bonds for each site, let us call them N-E-S-W, where the cardinal points denotes the direction of incidence, we can choose the prescription N>E>S>W and obtain for a recurrent configuration the corresponding spanning tree. In Fig. (2.4) is shown an example of burning test for a $4 \times 4$ square lattice, with prescription NESW.

Let us note a few facts about the burning test before going on to the next section. Although all recurrent configurations were shown in [6] to pass the burning test; conversely, it was shown in [10] only for sandpiles with symmetric toppling rules –which is if the same amount of sand is transferred to site $j$ when site $i$ topples as it is transferred to $i$ when $j$ topples– that all stable configurations which pass the test are recurrent. The burning test is not therefore valid in general, indeed there exist simple asymmetric sandpiles having stable configurations which pass the test but are




**Legend:** with ■ are denoted the sites such that, connected together, represent the sink, with ● the sites burnt at each step of the algorithm and with ⋯⋯, the bonds through which the fire could have reached the site but were rejected.

**Fig. 2.4** Example of burning test acting on a given configuration. At each time is displayed the progress of the algorithm until at t=6 all sites become "burnt" not recurrent; i.e. in the ASM with toppling matrix $\Delta = \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$, then there are $\det \Delta = 5$ recurrent configurations (2, 0), (3, 0), (1, 1), (2, 1) and (3, 1) and satisfy the burning test, but (1, 0) passes the burning test even if it is not recurrent. A site like 2 which has more incoming arrows than outgoing arrows is called *greedy* or *selfish*. In this case when adding a frame identity to the configuration, then some sites topple twice, and this makes the burning test fail as it assumes that under multiplication by the identity operator (2.32), each site topples only once.

This gap has been filled by Speer that in [11] introduced the *script test*, which is a generalization of the burning test valid in case of asymmetric sandpiles. Sandpile configurations which pass the script test are precisely the recurrent configurations of a sandpile. For sandpiles without greedy sites, the script test reduces to the burning test even for asymmetric sandpiles.

## 2.5 Algebraic Aspects

We report here some features of the abelian group $\mathcal{G}$ associated to the ASM. In particular we determine scalar function, invariant under toppling, and the rank of the group for the square lattice.

First we recall that any finite abelian group $\mathcal{G}$ can be expressed as a product of cyclic groups in the following form:

$$\mathcal{G} \cong \mathbf{Z}_{d_1} \times \mathbf{Z}_{d_2} \times \cdots \times \mathbf{Z}_{d_g} \tag{2.39}$$

That is, the group is isomorphic to the direct product of $g$ cyclic groups of order $d_1, d_2, \ldots, d_g$. Moreover the integers $d_1 \geq d_2 \geq \ldots \geq d_g > 1$ can be chosen such that $d_i$ is an integer multiple of $d_{i+1}$ and, under this condition, the decomposition is unique. In the following we determine the canonical decomposition of the group.

### 2.5.1 Toppling Invariants

The space of all configurations $S_+$ constitutes a commutative semigroup over the vertex-set of the ambient graph, with the addition between configurations defined as a sitewise addition of heights with, if necessary, relaxation as in (2.15). We define an equivalence relation on this semigroup by saying that two configurations $z$ and $z'$ are equivalent iff there exists $N = |V|$ integers $n_j$, $j = 1, \ldots N$, such that:

$$z'_i = z_i - \sum_{j=1}^{N} \Delta_{ij} n_j \qquad \forall i \in V \tag{2.40}$$

This equivalence is said *equivalence under toppling*, and each equivalence class with respect to (2.40) contains one and only one recurrent configuration. One can associate to each configuration $z \in S^+$ a recurrent configuration $C[z]$ defining:

$$C[z] = \prod_i a_i^{z_i} z^* \tag{2.41}$$

where $z^*$ is a given recurrent configuration. If $z$ and $z'$ are in the same equivalence class, then $C[z] = C[z']$, indeed we have that:

$$\begin{aligned} C[z'] &= \prod_i a_i^{z_i - \sum_j \Delta_{ij} n_j} z^* = \Big(\prod_i a^{z_i}\Big)\Big(\prod_{ij} a^{\Delta_{ij} n_j}\Big) z^* \\ &= \Big(\prod_i a^{z_i}\Big)\Big(\prod_j \big(\prod_i a^{\Delta_{ij}}\big)^{n_j}\Big) z^* = \prod_i a^{z_i} z^* = C[z_i]. \end{aligned} \tag{2.42}$$

Using the relation (2.40) two stable configurations can be equivalent under toppling. As a consequence of this equivalence relation and the existence of a unique recurrent representative for each equivalence class we will denote a class by $[z]$, being $[z] = [w]$ if $z$ and $w$ are equivalent under toppling. Furthermore the set of all configurations is a *superlattice* whose fundamental cell is the set **R**, the rows of $\Delta$ are the principal vectors of the superlattice and det $\Delta$ is the volume of the fundamental cell, that is the number of stable recurrent configurations $|\mathbf{R}|$.

We define *toppling invariants* as scalar functions over the space $S_+$ of all the configurations of the sandpile, such that their value is the same for configurations equivalent under toppling. Given the toppling matrix $\Delta$ for an $N$ sites sandpile, we define $N$ rational functions $Q_i$, $i \in \{1, \ldots, N\}$, as follows

$$Q_i(z) = \sum_j \Delta_{ij}^{-1} z_j \mod 1 \tag{2.43}$$

It is straightforward to prove that the functions $Q_i$ are toppling invariants, indeed a toppling at site $k$ changes $z \equiv \{z_i\}$ into $z' \equiv \{z_i - \Delta_{ik}\}$, and the linearity of the $Q_i$'s in the height variables allows to write:

$$Q_i(z') = Q_i(z) - \sum_j \Delta_{ij}^{-1} \Delta_{jk} = Q_i(z) \mod 1 \tag{2.44}$$

These functions are rational-valued but they can be made integer-valued by multiplication upon an adequate integer. So these functions can be used to label the recurrent configurations. Thus the space of recurrent configurations **R** can be replaced by the set of $N$-uples $(Q_1, Q_2, \ldots, Q_N)$, but this labeling is generally overcomplete, they being not all independent.

It is desirable to isolate a minimal set of invariants, and this can be done for an arbitrary ASM using the classical theory of Smith normal form for integer matrices [12].

Any nonsingular $N \times N$ integer matrix $\Delta$ can be written in the form:

$$\Delta = ADB \tag{2.45}$$

where $A$ and $B$ are $N \times N$ integer matrices with determinant $\pm 1$, and $D$ is a diagonal matrix

$$D_{ij} = d_i \delta_{ij} \tag{2.46}$$

where the eigenvalues $d_i$ are defined as follows:

1. $d_i$ is a multiple of $d_{i+1}$ for all $i = 1$ to $N - 1$
2. $d_i = e_{i-1}/e_i$ where $e_i$ stands for the greatest common divisor of the determinants of all the $(N - i) \times (N - i)$ submatrices of $\Delta$ (note that $e_N = 1$)

The matrix $D$ is uniquely determined by $\Delta$ but the matrices $A$ and $B$ are far from unique. The $d_i$ are called the *elementary divisors* of $\Delta$.

In terms of the decomposition (2.45), we define the set of scalar functions $I_i(z)$ by

$$I_i(z) = \sum_j (A^{-1})_{ij} z_j \mod d_i \tag{2.47}$$

Due to the unimodularity of $A$ (fact that guarantees the existence of an integer inverse matrix for $A$), these functions are integer-valued, and are toppling invariant, explicitly, given the equivalence under toppling relation (2.40), we have for $z \sim z'$:

$$I_i[z'] = \sum_j (A^{-1})_{ij} z_j - \sum_{jk} (A^{-1})_{ij} \Delta_{jk} n_k \tag{2.48}$$

$$= I_i[z] - \sum_{jk\ell m} (A^{-1})_{ij} A_{j\ell} D_{\ell m} B_{mk} n_k \tag{2.49}$$

$$= I_i[z] - \sum_{jk} D_{ij} B_{jk} n_k \tag{2.50}$$

$$= I_i[z] - d_i \sum_k B_{ik} n_k = I_i[z] \mod d_i \tag{2.51}$$

Only the $I_i$ for which $d_i \neq 1$ are nontrivial, and we note that this invariants are far from unique, because they are defined in the term of $A$ which is not unique itself. The $I_i$'s can also be written in term of the $Q_i$'s as follows:

$$I_i[z] = \sum_j d_i B_{ij} Q_j[z] \tag{2.52}$$

We now show that the set of nontrivial invariants is always minimal and complete. Let $g$ be the number of $d_i > 1$, we associate at each recurrent configuration a $g$-uple $(I_1, I_2, \dots, I_g)$ where $0 \le I_i < d_i$. The total number of distinct $g$-uple is $\prod_{i=1}^{g} d_i = |\mathcal{G}|$.

We first show that this mapping from the set of recurrent configurations to $g$-uples is one-to-one. Let us define operators $e_i$ by the equation

$$e_i = \prod_{j=1}^{N} a_j^{A_{ji}} \qquad 1 \leq i \leq g \tag{2.53}$$

Acting on a fixed configuration $z^* = \{z_j\}$, $e_i$ yields a new configuration, equivalent under toppling to the configuration $\{z_j + A_{ji}\}$. If the $g$-uple corresponding to $z^*$ is $(I_1^*, I_2^*, \ldots, I_g^*)$, from (2.47) follows that $e_i z^*$ has toppling invariants $I_k = I_k^* + \delta_{ik}$. By operating with this operators $\{e_i\}$ sufficiently many times on $z^*$, all $|\mathcal{G}|$ values for the $g$-uple $(I_1, I_2, \ldots, I_g)$ are obtainable. Thus there is at least one recurrent configuration corresponding to any $g$-uple $(I_1, I_2, \ldots, I_g)$. As the total number of recurrent configurations equals the number of $g$-uples (2.22), we see that there is a one to one correspondence between the $g$-uples $(I_1, I_2, \ldots, I_g)$ and the recurrent configurations of the ASM.

To express the operators $a_j$ in terms of $e_i$, we need to invert the transformation (2.53). This is easily seen to be:

$$a_j = \prod_{i=1}^{g} e_i^{(A^{-1})_{ij}} \qquad 1 \leq j \leq N \tag{2.54}$$

Thus the operators $e_i$ generate the whole of $\mathcal{G}$. Since $e_i$ acting on a configuration increases $I_i$ by one, leaving the other invariants unchanged, and since $I_i$ is only defined modulo $d_i$, we see that

$$e_i^{d_i} = I \quad \text{for } i \ 1 \ \text{ to } \ g \tag{2.55}$$

Note that the definition (2.53) makes sense for $i$ between $g + 1$ and $N$, and implies relations among the $a_j$ operators.

This shows that $\mathcal{G}$ has a canonical decomposition as a product of cyclic groups as in (2.39), with $d_i$'s defined in (2.46). We thus have shown that the generators and the group structure of $\mathcal{G}$ can be entirely determined from its toppling matrix $\Delta$, through its normal decomposition (2.45).

The invariants $\{I_i\}$ also provide a simple additive representation of the group $\mathcal{G}$. We define a binary operation of "addition" (denoted by $\oplus$) on the space of recurrent configurations by adding heights sitewise, and then allowing the resulting configuration to relax see (2.15). From the linearity of the $I_i$'s in the height variables, and their invariance under toppling, it is clear that under this addition of configurations, the $I_i$'s also simply add. Thus for any recurrent configurations $z$ and $w$ one has

$$I_i(z \oplus w) = I_i(z) + I_i(w) \mod d_i \tag{2.56}$$

The $I_i$'s provide a complete labeling of **R**. There is a unique recurrent configuration, denoted by $Id_\mathrm{r}$, for which all $I_i(Id_\mathrm{r})$ are zero. Also, each recurrent $z$ has a unique inverse $-z$, also recurrent, and determined by $I_i(-z) = -I_i(z) \bmod d_i$. Therefore the addition $\oplus$ is a group law on **R**, with identity given by $Id_\mathrm{r}$. M. Creutz first gave an algorithm to compute this configuration in [8, 9].

There is a one-to-one correspondence between recurrent configurations of ASM and elements of the group $\mathcal{G}$: we associate to the group element $g \in \mathcal{G}$, the recurrent configuration $gId_\mathrm{r}$, and from (2.56) follows that for all $g, g' \in \mathcal{G}$

$$gId_\mathrm{r} \oplus g'Id_\mathrm{r} = (gg')Id_\mathrm{r} \tag{2.57}$$

Thus the set of recurrent configurations with the operation $\oplus$ form a group which is isomorphic to the multiplicative group $\mathcal{G}$, result first proved in [8, 9]. The invariants $\{I_i\}$ provide a simple labeling of the recurrent configurations. Since a recurrent configuration can also be uniquely determined by its height variables $\{z_i\}$, the existence of forbidden configuration (2.30) in ASM's implies that this heights satisfy many inequality constraints.

### *2.5.2 Rank of $\mathcal{G}$ for a Rectangular Lattice*

For a general toppling matrix $\Delta$, it is difficult to say much more about the group structure of $\mathcal{G}$. To obtain some useful results we now consider the toppling matrix $\Delta$ of a finite $L_1 \times L_2$ bi-dimensional square lattice. In this framework is more convenient to label the sites not by a single index $i$ running from 1 to $N = L_1 L_2$, but by two Cartesian coordinates $(x, y)$, with $1 \le x \le L_1$ and $1 \le y \le L_2$. The toppling matrix is the discrete Laplacian as defined in (2.5), given by $\Delta(x, y; x, y) = 4$, $\Delta(x, y; x', y') = -1$ if the sites are nearest-neighbors (i.e. $|x - x'| + |y - y'| = 1$), and zero otherwise. We assume, without loss of generality, that $L_1 \ge L_2$. The relations (2.17) satisfied by operators $a(x, y)$, using the fact that the operators has an inverse on **R**, can be rewritten in the form

$$a(x+1, y) = a^4(x, y)a^{-1}(x, y+1)a^{-1}(x, y-1)a^{-1}(x-1, y) \tag{2.58}$$

where we adopt the convention that

$$a(x, 0) = a(x, L_2+1) = a(0, y) = a(L_1+1, y) = I \quad \forall x, y \tag{2.59}$$

The Eq. (2.58) can be recursively solved to express any operator $a(x, y)$ as a product of powers of $a(1, y)$. Therefore the group $\mathcal{G}$ can be generated by the $L_2$ operators $a(1, y)$. Denoting the rank of $\mathcal{G}$ (minimal number of generators) by $g$, this implies that

$$g \le L_2 \tag{2.60}$$

In the special case of a linear chain, $L_2 = 1$, we see that $g = 1$, and thus $\mathcal{G}$ is cyclic. Equation (2.58) permits also to express $a(L_1+1, y)$ in term of powers of $a(1, y)$ say

$$a(L_1 + 1, y) = \prod_{y'} a(1, y')^{n_{yy'}} \tag{2.61}$$

where the $n_{yy'}$ are integers which depend on $L_1$ and $L_2$ and which can be eventually determined by solving the linear recurrence relation (2.58). The condition (2.59), $a(L_1 + 1, y) = I$ then leads to the closure relations

$$\prod_{y'=1}^{L_2} a(1, y')^{n_{yy'}} = I \quad \forall y = 1, \ldots, L_2 \tag{2.62}$$

The Eq. (2.62) give a presentation of $\mathcal{G}$, the structure of which can be determined from the normal form decomposition of the $L_2 \times L_2$ integer matrix $n_{yy'}$. This is considerably easier to handle that the normal form decomposition of the much larger matrix $\Delta$ needed for an arbitrary ASM. Even though this is a real computational improvement, the calculation for arbitrary $L_1$ is not trivial.

In the particular case of square-shaped lattice, where $L_1 = L_2 = L$, using the above algorithm is possible to find the structure of $\mathcal{G}$, and to prove that for an $L \times L$ square lattice we have

$$g = L \quad \text{for} \quad L_1 = L_2 = L \tag{2.63}$$

## 2.6 Generalized Toppling Rules

The two basic rules of the ASM are the *addition rule* and the *toppling rule*. The addition rule has a general formulation, and, in the identification of a Markov Chain, it is flexible because of the possibility to make different choices for the rates $p_i$, at which particles are added on each site. On the other hand the toppling rule is not in the most general formulation. Indeed the toppling rule has the form of a single-variable check, labeled by the site index $i$, $z_i < \bar{z}_i$. When the inequality fails causes an "instability" in the height profile. The heights are then relaxed with a constant shift $z \to z - \vec{\Delta}_i$ such that the total mass can only decrease. Some conditions on $\Delta$ ensure both the finiteness of the relaxation process, and the fact that there is no ambiguity in the "who topple first" in case of multiple violated disequalities at some intermediate steps.

**Theorem 6.1** *Given an ASM on a graph $G = (E, V)$ and a toppling matrix $\Delta$, if $\tilde{z}$ is unstable, consider the set $S$ of sequences $(i_1, \ldots, i_{N(s)})$ such that $t_{i_{N(s)}} \ldots t_{i_2} t_{i_1} \tilde{z}$ is a valid sequence of topplings, and produces a stable configuration $z(s)$, some facts are true:*

*(0) $S$ is non-empty;*

*(1)* $z(s) = z(s')$ *for each* $s, s' \in S$*, i.e. the final stable configuration does not depend upon possible choices of who topples when;*
*(2)* $N(s) = N(s') = N(\tilde{z}) \quad \forall s, s' \in S$
*(3)* $\forall s, s' \in S \quad \exists \pi \in \mathcal{S}_{N(s)} : \; i_\alpha^{(s)} = i_{\pi(\alpha)}^{(s')}$ *for* $\alpha = 1, \ldots, N(\tilde{z})$*, i.e. the toppling sequences differ only by a permutation.*

*Proof* Here we prove *(3)*, given *(1)* and *(0)* As a restatement of *(3)*, we have that one can define a vector $\vec{n}(\tilde{z}) \in \mathbb{N}^{|V|}$ as the number of occurrence of each site in any of the sequence of $S$. Then, we have that the final configuration is

$$z = \tilde{z} + \Delta \cdot \vec{n} \tag{2.64}$$

The fact that $\vec{n}$ is unique is trivially proven. Indeed, as $S$ is non-empty, we have a first candidate $\vec{n}_0$, and thus a solution of the non-homogeneous linear system (in $\vec{n}$)

$$\Delta \cdot \vec{n} = z - \tilde{z}.$$

If there was another solution $\vec{n}_1$, then we would have that $\vec{n}' = \vec{n}_1 - \vec{n}_0$ is a solution of the homogeneous system

$$\Delta \cdot \vec{n}' = 0.$$

But $\Delta$ is a square matrix of the form Laplacian+mass, such that the spectrum is all positive (we saw how it is a strictly-dissipative diffusion kernel), so it can not have non-zero vectors in its kernel. This proves the uniqueness of $\vec{n}$, i.e. *(3)*. But *(3)* is stronger than *(2)*, and the fact that $z = \tilde{z} + \Delta \cdot \vec{n}$ also implies *(1)*. So the theorem is proven.

The standard toppling rule can be shortly rewritten as:

$$\text{if } \exists i \in V \mid z_i \geq \bar{z}_i = \Delta_{ii} \quad \Longrightarrow \quad z \to z + \vec{\Delta}_i \tag{2.65}$$

Pictorially, on a square lattice, we can draw the heights at a given site $i$ and at its nearest-neighbors $i_1, i_2, i_3, i_4$ as

$$\begin{array}{c|c|c} & z_{i_1} & \\ \hline z_{i_4} & z_i & z_{i_2} \\ \hline & z_{i_3} & \end{array} \tag{2.66}$$

and an example of typical toppling is

$$\begin{array}{c|c|c} & 0 & \\ \hline 0 & 4 & 0 \\ \hline & 0 & \end{array} \longrightarrow \begin{array}{c|c|c} & 1 & \\ \hline 1 & 0 & 1 \\ \hline & 1 & \end{array} \tag{2.67}$$

where the initial value of $z_i$ is equal to $\bar{z}_i = 4$, being the only site unstable, it requires the toppling shown in figure.

A straightforward generalization of this prescription is considering a site stable or unstable not just for "ultra-local" (i.e. single-site) properties -the overcoming of the site threshold- but also for local properties that depend on the heights at more than one sites. Similarly, we would have toppling rules $\vec{\Delta}_\alpha = \{\Delta_{\alpha j}\}_{j \in V}$ with more than a single positive entry. Still we want to preserve the *exchange* properties of toppling procedures which led to the abelianity of the *addition* operators $a_i$, and this could be in principle such a severe constraint that we could not find essentially any new possibility. As we show now, by direct construction, this is *not* the case. We can define some *cluster-toppling* rules, labeled by the subsets $I \subseteq V$ of the set of sites, instead that by a single site. For any subset $I$, we introduce the toppling rule

$$\text{if} \quad \forall i \in I \quad z_i \geq \sum_{\alpha \in I} \Delta_{i\alpha} \quad \Longrightarrow \quad z_k \to z_k - \sum_{\alpha \in I} \Delta_{\alpha k} \quad \forall k \in V \tag{2.68}$$

These rules clearly define a new sandpile model, that, under some constraints on the choice of *toppling clusters set* $\mathcal{L} = \{I\}$, we will prove later to be abelian. Let us first address the simpler issue of checking for the finiteness of the space of configurations. It is trivial to see that if, for any site $i$, there is no single site set $\{i\} \in \mathcal{L}$, but only an arbitrary number of "large" clusters $I \in \mathcal{L}$, $|I| \geq 2, i \in I$, then all the configurations of height

$$z_i = n \in \mathbb{N}, \quad z_j = 0 \;\; j \neq i \tag{2.69}$$

are allowed and stable, thus a necessary condition for having a finite set of stable configurations is

$$\{i\} \in \mathcal{L} \quad \forall i \in V \tag{2.70}$$

this is also sufficient, as even in the standard ASM we have a number $\prod \bar{z}_i$ of stable configurations, and this number can only decrease when adding new toppling rules.

We define $\mathcal{L}_{std}$ the set of toppling cluster for the standard toppling rules, that is

$$\mathcal{L}_{std} = \big\{\{i\}_{i \in V}\big\} \tag{2.71}$$

We ask now whether a given set $\mathcal{L}$ of cluster-toppling subsets give rise to an abelian sandpile. In the following we will denote with $I \setminus J$ the set of all the elements members of $I$ but not members of $J$.

**Theorem 6.2** *Given an ASM on a graph $G = (E, V)$, with a symmetric toppling matrix $\Delta$, $\{\vec{\Delta}_I\}_{I \in \mathcal{L}}$ and $\mathcal{L} \supseteq \mathcal{L}_{std}$. A necessary and sufficient condition for the sandpile to be abelian is that*

$$\textit{each component of } I_{(1)} \setminus I_{(2)} \in \mathcal{L} \qquad \forall I_{(1)}, I_{(2)} \in \mathcal{L} \tag{2.72}$$

*Proof* Let us call $J = I_{(1)} \cap I_{(2)}$, then there are two cases:

(a) $J = \varnothing$
(b) $J \neq \varnothing$

In case **(a)** the compatibility is obvious, indeed if we make the toppling for $I_{(1)}$ then the heights in $I_{(2)}$ can only increase for the properties of the toppling matrix $\Delta$. After the topplings also of the sites in $I_{(2)}$ have been done, the final height configuration will be

$$z'_k = z_k - \sum_{i \in I_{(1)}} \Delta_{ik} - \sum_{j \in I_{(2)}} \Delta_{jk} \tag{2.73}$$

this expression is clearly symmetric under the exchange of $I_{(1)}$ and $I_{(2)}$.

In case **(b)** we shortly recall the toppling rule for $I_{(1)}$ and $I_{(2)}$ (2.68)

$$\text{if } \forall i \in I_{(1)} \quad z_i \geq \sum_{\alpha \in I_{(1)}} \Delta_{i\alpha} \quad \Longrightarrow \quad z_k \to z_k - \sum_{\alpha \in I_{(1)}} \Delta_{\alpha k} \quad \forall k \in V \tag{2.74a}$$

$$\text{if } \forall i \in I_{(2)} \quad z_i \geq \sum_{\alpha \in I_{(2)}} \Delta_{i\alpha} \quad \Longrightarrow \quad z_k \to z_k - \sum_{\alpha \in I_{(2)}} \Delta_{\alpha k} \quad \forall k \in V \tag{2.74b}$$

now we note that we can split the sums in the contribution from $J$ and the one from the remaining sites of each subset

$$\sum_{\alpha \in I_{(1)}} = \sum_{\alpha \in I_{(1)} \setminus J} + \sum_{\alpha \in J}$$
$$\sum_{\alpha \in I_{(2)}} = \sum_{\alpha \in I_{(2)} \setminus J} + \sum_{\alpha \in J}$$

Now we can topple first $I_{(1)}$ and so update the configuration $z \to z'$ as follows

$$z'_i = z_i + \sum_{\alpha \in I_{(1)} \setminus J} \Delta_{\alpha i} + \sum_{\alpha \in J} \Delta_{\alpha i} \quad \forall i \in V$$

At this point, for the updated configuration the following relations are valid

$$\forall i \in I_{(2)} \setminus J \qquad z'_i = z_i - \sum_{\alpha \in I_{(1)} \setminus J} \Delta_{\alpha i} - \sum_{\alpha \in J} \Delta_{\alpha i} \tag{2.75a}$$

$$\geq \sum_{\alpha \in I_{(2)} \setminus J} \Delta_{i\alpha} + \sum_{\alpha \in J} \Delta_{i\alpha} - \sum_{\alpha \in I_{(1)} \setminus J} \Delta_{\alpha i} - \sum_{\alpha \in J} \Delta_{\alpha i} \geq \tag{2.75b}$$

$$\geq \sum_{\alpha \in I_{(2)} \setminus J} \Delta_{i\alpha} - \sum_{\alpha \in I_{(1)} \setminus J} \Delta_{\alpha i} \geq \tag{2.75c}$$

$$\geq \sum_{\alpha \in I_{(2)} \setminus J} \Delta_{i\alpha} \tag{2.75d}$$

in line (2.75b) we used the symmetry property of the toppling matrix to cancel out the second and the fourth terms, in line (2.75c) we used the property the off-diagonal elements $\Delta_{ij}$ to be negative or equal to zero to obtain the inequality in the last line, indeed if $A > B$ and $c_i \geq 0$, then $A + \sum c_i > B$ a *fortiori*.

In case it does not exist the toppling rule for $I_{(2)} \setminus J$, there exists a configuration of heights (the minimal heights such that both $I_{(1)}$ and $I_{(2)}$ are unstable) such that toppling first $I_{(1)}$ or $I_{(2)}$ leads immediately after a single toppling to two distinct stable configurations, so we see that necessary part of the theorem holds. As a consequence, as $\mathcal{L} \supseteq \mathcal{L}_{std}$, given $I \in \mathcal{L}$ we have that all the $I' \subseteq I$ are in $\mathcal{L}$, and thus all of its components. In particular disconnected $I$'s are redundant, and we can restrict $\mathcal{L}$ to contain only connected clusters without loss of generality.

Conversely, if $I_{(2)} \setminus J \in \mathcal{L}$ (and $I_{(1)} \setminus J \in \mathcal{L}$ by symmetry), in the two "histories" in which we toppled $I_{(1)}$ or $I_{(2)}$, we can still topple $I_{(2)} \setminus J \in \mathcal{L}$ and $I_{(1)} \setminus J \in \mathcal{L}$ respectively and put them back on the same track, i.e.

$$t_{I_{(1)} \setminus J} t_{I_{(2)}} \equiv t_{I_{(2)} \setminus J} t_{I_{(1)}} \tag{2.76}$$

as operators when applied to configurations $z$ such that both $I_{(1)}$ and $I_{(2)}$ are unstable.

We want also to produce a proof similar to that for standard toppling rule in theorem (6.1) for the cluster-toppling ASM.

Let suppose we have $G = (E, V)$, and the induced toppling matrix $\Delta$, and a set $\mathcal{L}$ of connected subsets of $V$, with $\mathcal{L} \supseteq \mathcal{L}_{std} = \big\{\{i\}_{i \in V}\big\}$. Call $\vec{\Delta}_i = \{\Delta_{ij}\}_{j \in V}$, and $\vec{\Delta}_I = \sum_{i \in I} \vec{\Delta}_i$. A toppling $t_I$ changes $z$ into $z - \vec{\Delta}_I$.

**Theorem 6.3** *If $\tilde{z}$ is unstable w.r.t.* (2.68) *given the framework above, consider the set $S$ of sequences $s = (I_1, \dots, I_{N(s)})$ such that $t_{I_{N(s)}} \dots t_{I_2} t_{I_1} \tilde{z}$ is a valid sequence of topplings and produces a stable configuration $z(\tilde{z}; s)$. Some facts are true:*

*(0)* *$S$ is non-empty;*
*(1)* $z(\tilde{z}; s) = z(\tilde{z}; s') \quad \forall s, s' \in S;$
*(2)* $\sum_{\alpha=1}^{N(s)} |I_\alpha^{(s)}| = \sum_{\alpha=1}^{N(s')} |I_\alpha^{(s')}| \quad \forall s, s' \in S;$
*(3)* *defining* $\vec{\chi}_I = \begin{cases} 1 & i \in I \\ 0 & i \notin I \end{cases}$, $\sum_{\alpha=1}^{N(s)} \vec{\chi}_{I_\alpha^{(s)}}$ *is the same for all the sequences and is some vector* $\vec{n}(\tilde{z})$

*Proof* (of *(3)* and *(2)* given *(1)* and *(0)*) Again, the final stable configuration is $z = \tilde{z} + \Delta \cdot \vec{n}$, and the uniqueness of $\vec{n}$ is proven along the same line as the proof for standard ASM. Then, as *(3)* is a strengthening of *(2)*, the theorem is proven. Remark however some qualitative difference with the simpler case of ordinary ASM: it can be that $N(s) \neq N(s')$, and $s$ and $s'$ do not differ simply by a permutation (e.g., in the relaxation of $\boxed{3}\boxed{4}\boxed{3}$ by $t_3 t_{12}$ or by $t_1 t_3 t_2$), and analogously the kernel

$$\sum_j n_I \vec{\Delta}_{Ij} = 0 \quad \forall I \in \mathcal{L}$$

is non-empty for $\mathcal{L} \supsetneq \mathcal{L}_{std}$, as $\Delta$ is rectangular (e.g. $n_{12} = a$, $n_1 = n_2 = -a$, $n_I = 0$ otherwise is a null vector of $\Delta$). Only in the *basis* of $\mathcal{L}_{std}$ we have a unique solution, and of course the versor $\hat{e}_I$, in $\mathbb{Z}^{|\mathcal{L}|}$, reads $\vec{\chi}_I$ in this basis.

We present for clarity the example case in which the rule is defined for all the 2-clusters, *dimers*. In this case, given $G = (E, V)$ we have the set of toppling clusters

$$\mathcal{L} = \Big\{\{i, j\}_{ij \in E}\Big\} \cup \Big\{\{i\}_{i \in V}\Big\}, \tag{2.77}$$

and the general rule (2.68) becomes:

$$\text{if} \begin{cases} ij \in E \\ z_i \geq \Delta_{ii} + \Delta_{ij} \\ z_j \geq \Delta_{jj} + \Delta_{ij} \end{cases} \implies \quad z_k \to z_k - \Delta_{ik} - \Delta_{jk} \quad \forall k \in V, \tag{2.78}$$

we can now pictorially draw on a square lattice the heights for a given cluster, formed by the sites $i$ and $j$, and its nearest-neighbors $i_1, i_2, i_3, j_1, j_2, j_3$ as

$$\begin{array}{c|c|c|c} & z_{i_1} & z_{j_1} & \\ \hline z_{i_3} & z_i & z_j & z_{j_2} \\ \hline & z_{i_2} & z_{j_3} & \end{array} \tag{2.79}$$

so a typical 2-cluster toppling is:

$$\begin{array}{c|c|c|c} & 0 & 0 & \\ \hline 0 & 3 & 3 & 0 \\ \hline & 0 & 0 & \end{array} \longrightarrow \begin{array}{c|c|c|c} & 1 & 1 & \\ \hline 1 & 0 & 0 & 1 \\ \hline & 1 & 1 & \end{array} \tag{2.80}$$

in which in the initial state both the sites $i$ and $j$ have height equal to $\bar{z}_i - 1 = 3$ and become unstable with respect to the (2.78), it is therefore necessary to topple the sites obtaining the final configuration. We note that doing a single cluster-toppling is the same as making two consecutive normal topplings, at condition that we permit negative height in the intermediate steps having forced the toppling in the case $z_i = 3$ when it is not necessary, in fact:

$$\begin{array}{c|c|c|c} & 0 & 0 & \\ \hline 0 & 3 & 3 & 0 \\ \hline & 0 & 0 & \end{array} \longrightarrow \begin{array}{c|c|c|c} & 1 & 0 & \\ \hline 1 & -1 & 4 & 0 \\ \hline & 1 & 0 & \end{array} \longrightarrow \begin{array}{c|c|c|c} & 1 & 1 & \\ \hline 1 & 0 & 0 & 1 \\ \hline & 1 & 1 & \end{array} \tag{2.81}$$

and in the case $z_i \geq 4$ or $z_j \geq 4$ or both, the same result would have been obtained, any possible rule we choose to use, as proved in (6.2). This fact can be better understood recalling the relations (2.17) satisfied by the operators $a_i$ and $a_j$, with $i$ and $j$ corresponding to the ones in (2.79):

$$a_i^4 = a_{i_1} a_{i_2} a_{i_3} a_j \tag{2.82a}$$

$$a_j^4 = a_{j_1} a_{j_2} a_{j_3} a_i \tag{2.82b}$$

If we restrict the attention on recurrent configurations where inverses of $a_i$ exist it is possible to multiply side by side the two equalities obtaining:

$$a_i^4 a_j^4 = a_{i_1} a_{i_2} a_{i_3} a_j a_{j_1} a_{j_2} a_{j_3} a_i \tag{2.83}$$

and dividing (in group sense) each side by $a_i$ and $a_j$ we have the following equality:

$$a_i^3 a_j^3 = a_{i_1} a_{i_2} a_{i_3} a_{j_1} a_{j_2} a_{j_3} \tag{2.84}$$

that is the same of (2.82) for the cluster toppling rule. This rule easily generalizes for arbitrary subsets of $V$. This permit us to state that the different toppling rules we have introduced bring to the same group presentation (and then to the same group structure) for the abelian group associated to the recurrent configurations of the ASM.

We recall now that for the model with the standard toppling rule we have an easy characterization for the subsets $F$ of the graph that are forbidden subconfiguration (FSC), that is:

$$\text{if } \forall i \in F \qquad z_i < \sum_{\substack{j \neq i \\ j \in F}} \left(-\Delta_{ij}\right) \qquad \Longrightarrow \qquad F \ \text{ is a FSC} \tag{2.85}$$

As obvious with the new rules just introduced, some forbidden subconfigurations of the standard ASM can become reachable by adding sand and toppling, e.g. the simplest forbidden configuration in the case of standard toppling rules, $\boxed{0\,|\,0}$, is easily reachable if we allow the dimer-cluster toppling rule, in fact it can simply turn out as result of the basic dimer-cluster toppling $\boxed{3\,|\,3} \to \boxed{0\,|\,0}$. It is also possible to characterize the forbidden subconfigurations with respect to a given $I$-toppling rule, we have:

$$\text{if } \forall I \subseteq F \quad \sum_{j=1}^{k} z_{i_j} < \sum_{\substack{j \in F \setminus I \\ \ell \in I}} \left(-\Delta_{\ell j}\right) \qquad \Longrightarrow \qquad F \text{ is a FSC} \tag{2.86}$$

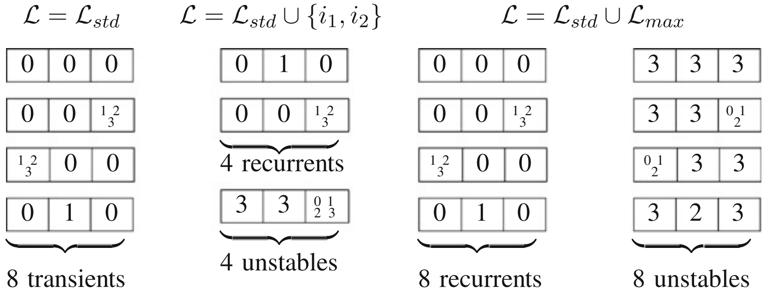
this yielding to the possibility that a transient configuration with respect to a $\mathcal{L}'$ toppling rule becomes recurrent for a $\mathcal{L}''$ toppling rule, with $\mathcal{L}'' \supset \mathcal{L}'$.

On the other hand some configurations stable with respect to a $\mathcal{L}'$ toppling rule become unstable if we allow $\mathcal{L}'$ to increase up to $\mathcal{L}''$, e.g. the basic unstable configuration $\boxed{3\,|\,3}$ for the dimer-toppling rule is perfectly stable in the framework of toppling only for $z_i \geq \bar{z}_i$. Moreover the fact that the group structure of the associated group remains unchanged under the addition of the new toppling rules, yields the number of group elements $g \in \mathcal{G}$ to be the same in the two cases, this forces the number of recurrent configurations to be the same, as the order of group associated, i.e. as many stable recurrent becomes unstable, as transient become recurrent, for each enlargement of $\mathcal{L}$.

In this framework, we see how, remaining unchanged the number of recurrent configurations and growing up the number of unstable configurations, since the set of $\mathcal{L}$-stable configurations becomes a subset of the original set of stable configurations $S = \{0, 1, 2, 3\}^{|V|}$, in some sense the transient configurations that become allowed must correspond to some newly unstable configurations. This kind of symmetry between the two sets, yields to suppose the existence of a bijection between unstable configurations for $\mathcal{L}''$ toppling rules and transient configurations for $\mathcal{L}'$ toppling rules, with $\mathcal{L}'' \supset \mathcal{L}'$. This procedure of enlarging the set of unstable configuration, and at the same time to shrink the set of transient configuration yields to the possibility to completely suppress the set of the transient configurations and to have that the recurrent configurations become all the stable configurations. This situation is reached by letting

$$\mathcal{L} = \{\text{all the subsets of } V\} \tag{2.87}$$

An interesting example is the lattice $3 \times 1$ in which the number of configurations is not huge and we can directly check this statement.

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