1 2-D FTCS

$$T_{a,b}^{\theta+1} = T_{a,b}^{\theta} + C\left(T_{a+1,b}^{\theta} + T_{a-1,b}^{\theta} - 4T_{a,b}^{\theta} + T_{a,b-1}^{\theta} + T_{a,b+1}^{\theta}\right) \tag{1}$$

$$T_{a,b}^{\theta} = G^{n}e^{-ika\Delta x}e^{-ilb\Delta x} \tag{2}$$

Substituting (2) into (1) and simplifying...

$$\frac{G^{n+1}}{G^{n}} = 1 + C\left(e^{ik\Delta x} + e^{-ik\Delta x} + e^{il\Delta x} + e^{-il\Delta x} - 4\right)$$

$$\frac{G^{n+1}}{G^{n}} = 1 + C\left(2cos(k\Delta x) + 2cos(l\Delta x) - 4\right)$$

$$-1 \leq 1 - 8C \leq 1$$

$$-2 \leq -8C \leq 0$$

$$0 \leq C \leq \frac{1}{4}$$

2 3-D FTCS

$$T^{\theta+1}_{a,b,c} = T^{\theta}_{a,b,c} + C\left(T^{\theta}_{a+1,b,c} + T^{\theta}_{a-1,b,c} + T^{\theta}_{a,b+1,c} + T^{\theta}_{a,b-1,c} + T^{\theta}_{a,b,c+1} + T^{\theta}_{a,b,c-1} - 6T^{\theta}_{a,b,c}\right) \tag{3}$$

$$T^{\theta}_{a,b} = G^n e^{-ika\Delta x} e^{-ilb\Delta x} e^{-imc\Delta x} \tag{4}$$

Substituting (4) into (3) and simplifying...

$$\frac{G^{n+1}}{G^n} = 1 + C\left(e^{ik\Delta x} + e^{-ik\Delta x} + e^{il\Delta x} + e^{-il\Delta x} + e^{im\Delta x} + e^{-im\Delta x} - 6\right)$$

$$\frac{G^{n+1}}{G^n} = 1 + C\left(2cos(k\Delta x) + 2cos(l\Delta x) + 2cos(m\Delta x) - 6\right)$$

$$-1 \leq 1 - 12C \leq 1$$

$$-2 \leq -12C \leq 0$$

$$0 \leq C \leq \frac{1}{6}$$

3 1-D BECS

$$T_a^{\theta+1} = T_a^{\theta} + C\left(T_{a+1,b}^{\theta+1} - 2T_a^{\theta+1} + T_{a-1}^{\theta+1}\right) \tag{5}$$

$$T_a^{\theta} = G^n e^{-ika\Delta x} \tag{6}$$

Substituting (6) into (5) and simplifying...

$$G^{n+1} = G^n + CG^{n+1}\left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right)$$

$$\frac{G^n}{G^{n+1}} = 1 - C\left(2cos(k\Delta x) - 2\right)$$

$$\frac{G^{n+1}}{G^n} = \frac{1}{1 - C\left(2cos(k\Delta x) - 2\right)}$$

$$-1 \leq \frac{1}{1+4C} \leq 1$$

$$-1 - 4C \leq 1 \leq 1 + 4C$$

$$-1 \leq 2C \leq 4C$$

BECS is unconditionally stable, since C is unbounded.

4 2-D BECS

$$T^{\theta+1}_{a,b} = T^{\theta}_{a,b} + C\left(T^{\theta+1}_{a+1,b}T^{\theta+1}_{a,b+1} - 4T^{\theta+1}_{a,b} + T^{\theta+1}_{a-1,b}T^{\theta+1}_{a,b-1}\right) \tag{7}$$

$$T^{\theta}_{a,b} = G^{n}e^{-ika\Delta x}e^{-ilb\Delta x} \tag{8}$$

Substituting (8) into (7) and simplifying...

$$G^{n+1} = G^{n} + CG^{n+1}\left(e^{ik\Delta x} + e^{-ik\Delta x} + e^{il\Delta x} + e^{-il\Delta x} - 4\right)$$

$$\frac{G^{n}}{G^{n+1}} = 1 + C\left(2cos(k\Delta x) + 2cos(l\Delta x) - 4\right)$$

$$\frac{G^{n+1}}{G^{n}} = \frac{1}{1 + C\left(2cos(k\Delta x) + 2cos(l\Delta x) - 4\right)}$$

$$-1 \leq \frac{1}{1+8C} \leq 1$$

$$-1 - 8C \leq 1 \leq 1 + 8C$$

$$-1 \leq 4C \leq 8C$$

BECS is unconditionally stable, since C is unbounded.

5 3-D BECS

$$T_{a,b,c}^{\theta+1} = T_{a,b,c}^{\theta} + C\left(T_{a+1,b,c}^{\theta+1}T_{a,b+1,c}^{\theta+1}T_{a,b,c+1}^{\theta+1} - 6T_{a,b,c}^{\theta+1} + T_{a-1,b,c}^{\theta+1}T_{a,b-1,c}^{\theta+1}T_{a,b,c-1}^{\theta+1}\right) \tag{9}$$

$$T_{a,b,c}^{\theta} = G^n e^{-ika\Delta x} e^{-ilb\Delta x} e^{-imb\Delta x} \tag{10}$$

Substituting (10) into (9) and simplifying...

$$G^{n+1} = G^n + CG^{n+1}\left(e^{ik\Delta x} + e^{-ik\Delta x} + e^{il\Delta x} + e^{-il\Delta x} + e^{im\Delta x} + e^{-im\Delta x} - 6\right)$$

$$\frac{G^n}{G^{n+1}} = 1 + C\left(2cos(k\Delta x) + 2cos(l\Delta x) + 2cos(m\Delta x) - 6\right)$$

$$\frac{G^{n+1}}{G^n} = \frac{1}{1 + C\left(2cos(k\Delta x) + 2cos(l\Delta x) + 2cos(m\Delta x) - 6\right)}$$

$$-1 \leq \frac{1}{1 + 12C} \leq 1$$

$$-1 - 12C \leq 1 \leq 1 + 12C$$

$$-1 \leq 6C \leq 12C$$

BECS is unconditionally stable, since C is unbounded.

6 1-D Crank-Nicolson

$$T_a^{\theta+1} = T_a^{\theta} + \frac{C}{2}\left(T_{a+1}^{\theta+1} - 2T_a^{\theta+1} + T_{a-1}^{\theta+1} + T_{a+1}^{\theta} - 2T_a^{\theta} + T_{a-1}^{\theta}\right) \tag{11}$$

$$T_a^{\theta} = G^n e^{-ika\Delta x} \tag{12}$$

Substituting (12) into (11) and simplifying...

$$G^{n+1} = G^n + \frac{C}{2}\left(G^{n+1} + G^n\right)\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$

$$\frac{G^{n+1} - G^n}{G^{n+1} + G^n} = \frac{C}{2}\left(2cos(k\Delta x) - 2\right)$$

C-N 1D is unconditionally stable, since C is unbounded.

7 2-D Crank-Nicolson

$$T_{a,b}^{\theta+1} = T_{a,b}^{\theta} + \frac{C}{2}\left(T_{a+1,b}^{\theta+1} + T_{a,b+1}^{\theta+1} - 2T_{a,b}^{\theta+1} + T_{a-1,b}^{\theta+1} + T_{a,b-1}^{\theta+1} + T_{a+1,b}^{\theta} + T_{a,b+1}^{\theta} - 2T_{a,b}^{\theta} + T_{a-1,b}^{\theta} + T_{a,b-1}^{\theta}\right) \tag{13}$$

$$T_{a,b}^{\theta} = G^n e^{-ika\Delta x} e^{-ilb\Delta x} \tag{14}$$

Substituting (14) into (13) and simplifying...

$$G^{n+1} = G^n + \frac{C}{2}\left(G^{n+1} + G^n\right)\left(e^{ik\Delta x} + e^{il\Delta x} - 4 + e^{-ik\Delta x} + e^{-il\Delta x}\right)$$

$$\frac{G^{n+1} - G^n}{G^{n+1} + G^n} = \frac{C}{2}\left(2cos(k\Delta x) + 2cos(l\Delta x) - 4\right)$$

C-N 1D is unconditionally stable, since C is unbounded.

8 3-D Crank-Nicolson

$$T^{\theta+1}_{a,b,c} = T^{\theta}_{a,b,c} + \frac{C}{2}\left(T^{\theta+1}_{a+1,b,c} + T^{\theta+1}_{a,b+1,c} + T^{\theta+1}_{a,b,c+1} - 2T^{\theta+1}_{a,b} + T^{\theta+1}_{a-1,b,c} + T^{\theta+1}_{a,b-1,c} + T^{\theta+1}_{a,b,c-1} + T^{\theta}_{a+1,b,c} + T^{\theta}_{a,b+1,c} + T^{\theta}_{a,b,c+1} - 2T^{\theta}_{a,b} + T^{\theta}_{a-1,b,c} + T^{\theta}_{a,b-1,c} + \right. \tag{15}$$

$$T^{\theta}_{a,b} = G^n e^{-ika\Delta x} e^{-ilb\Delta x} e^{-imc\Delta x} \tag{16}$$

Substituting (16) into (15) and simplifying...

$$G^{n+1} = G^n + \frac{C}{2}\left(G^{n+1} + G^n\right)\left(e^{ik\Delta x} + e^{il\Delta x} + e^{im\Delta x} - 6 + e^{-ik\Delta x} + e^{-il\Delta x} + e^{-im\Delta x}\right)$$

$$\frac{G^{n+1} - G^n}{G^{n+1} + G^n} = \frac{C}{2}\left(2cos(k\Delta x) + 2cos(l\Delta x) + 2cos(m\Delta x) - 6\right)$$

C-N 1D is unconditionally stable, since C is unbounded.