THE AUSTRALIAN NATIONAL UNIVERSITY

Second Semester Midsemester Examination – September, 2013

Foundations of Economic and Financial Models

(EMET 1001/7001)

Reading Time: 10 Minutes
Writing Time: 90 Minutes
Permitted Materials: Non-programmable Calculator

Instructions:

- This handout of exam questions contains 4 pages (including cover page) with 15 exam questions. Make sure you are not missing any pages!
- Answer **ALL** questions of this handout in the script book provided to you.
- Show your work, provide full and mathematically rigorous derivations! No partial credit will be given for merely stating final results (even if they are correct).
- Unless otherwise stated, when determining local extrema, make sure to also show which ones are minima and which ones are maxima.
- You may not use L'Hospital's rule in solving questions.
- Cheat sheets are not permitted.
- Anything that is not a non-programmable calculator is not permitted. These include, but are not limited to: mobile phones, tablet computers, abacuses.
- Return this handout to the invigilators at the end of the exam.
- Total marks: 100. Good luck!

1. [8 marks] Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = |x|$. Is this function continuous? Provide a full proof!

2. [6 marks] State the zero test. Give a brief discussion of how (or if) the zero test can be helpful in establishing the convergence behaviour of the following two series:

   (a) $\sum_{n=1}^{\infty} 1/n$

   (b) $\sum_{n=1}^{\infty} 7 - 1/n$

3. [10 marks] State the Mean Value Theorem and use it to prove the inequality

$$e^x \geq 1 + x, \qquad \text{for } x \in \mathbb{R}.$$

4. [8 marks] Is the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with $f(x) = 6/x$ surjective? Is it injective? Is it bijective? Is it invertible? Provide full proofs!

5. [5 marks] Find the 300th term and the sum of the first 300 terms for the arithmetic sequence $\{8, 11, 14, \ldots\}$.

6. [6 marks] You deposit \$$P$ in a bank account which pays nominal interest of r% each year compounded monthly. You do not withdraw money and keep it in the bank for T years.

   (a) What is the amount of money A saved over that time span? Provide a formula for A in terms of the principal P, the interest rate r and time span T.

   (b) What is the effective annual rate?

7. [6 marks] Give a definition of an **inflection point** of a function.

8. [8 marks] For what values of x is the following function continuous?

$$g(x) = \frac{x^2 - 9}{x^2 + 2x + 1}$$

9. [7 marks] For the following function, determine the difference quotient and its limit as x approaches x_0.

$$f(x) = \frac{x}{6 - x}$$

10. [6 marks] Let $f(x) = 3x^2 \cdot e^{2x}$

    (a) Find $f'(x)$ and $f''(x)$.

    (b) Determine all intervals on which $f'(x) \geq 0$.

    (c) Determine the location (x-values) of all local extrema.
    (Note: You do not need to check whether the extrema are minima or maxima.)

    Note: Do not use graphs or sign charts to answer the question.

11. [6 marks] The greatest integer function, $f(x) = \lfloor x \rfloor$, is defined to be the greatest integer less than or equal to x, where x is any real number.

    (a) Determine the following function values: $f(3)$, $f(1.999)$, $f\left(\frac{1}{4}\right)$, and $f(-4.5)$.

    (b) Find $\lim_{x \to 3} f(x)$.

12. [6 marks] For the function $f(x) = \sqrt{\ln x}$ determine $f'(e)$, where e is Euler's number.

13. [6 marks] At an annual rate of 10% compounded continuously, how many years does it take for the principal (the initial investment) to triple?

14. [6 marks] Determine the limit:

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 + 2x - 8}$$

15. [6 marks] Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = x^2$.

    (a) Let $E = [-2, 2]$. What is the image of E under h?

    (b) Determine $h^{-1}\big(h([0, 3])\big)$.

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Answers

THIS IS MERELY AN ANSWER KEY. IN SOME CASES FULL ANSWERS ARE PROVIDED. IN SOME CASES, ONLY BRIEF ANSWERS ARE PROVIDED. IN THE ACTUAL EXAM, YOU ARE ALWAYS REQUIRED TO PROVIDE A COMPREHENSIVE AND MATHEMATICALLY RIGOROUS ANSWER (UNLESS STATED OTHERWISE).

1. See tutorial answer key.

2. Stating the zero test: Let $\{a_n\}$ be a sequence. If $\lim_{n\to\infty} a_n$ is non-zero, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

   (a) This series is based on the sequence $\{1/n\}$ which converges to zero (as we know from the lecture), therefore the zero test does not apply.

   (b) This series is based oon the underlying sequence $\{7 - 1/n\}$ which converges to 7 (no prrof necessary, as only a brief argument was required). The zero test thus states that the series diverges.

3. Three cases:

   (i) $x = 0$
   It is easy to see that $e^x = 1 + x$ in this case.

   (ii) $x > 0$
   Apply the MVT on the interval $[0, x]$. Then, for some c with $0 < c < x$ we have

   $$e^x - e^0 = e^c(x - 0).$$

   Since $e^0 = 1$ and $e^c > 1$ (Note: $e^c > 1$ for $c > 0$. This is a property of the exponential function!), this turns to $e^x - 1 > x$ or, equivalently, $e^x > 1 + x$.

   (iii) $x < 0$
   Apply the MVT on the interval $[x, 0]$. Then, for some c with $x < c < 0$ we have

   $$e^0 - e^x = e^c(0 - x).$$

   Since $e^0 = 1$ and $e^c < 1$ (Note: $e^c < 1$ for $c < 0$. This is a property of the exponential function!), this turns to $1 - e^x < -x$ or, equivalently, $e^x > 1 + x$.

4. Injective, not surjective, therefore not bijective and thus not invertible. (This is just the short answer, need to be more formal in actual exam.)

5. For arithmetic sequences: $a_n = a_1 + (n-1) \cdot d$. Thus, $a_{300} = 8 + 299 \cdot 3 = 905$. Furthermore, $S_n = n/2(a_1 + a_{300})$ which gives $S_{300} = 150(8 + 905) = 136,950$.

6. See tutorial answer key.

7. See lecture notes.

8. Looking at the limit of $g(x)$:

$$\lim_{x\to a} g(x) = \lim_{x\to a} \frac{x^2-9}{x^2+2x+1} = \lim_{x\to a} \frac{(x+3)\cdot(x-3)}{(x+1)^2} = \lim_{x\to a} f(x).$$

where we defined $f(x) = \frac{(x+3)\cdot(x-3)}{(x+1)^2}$. We learned that $\lim_{x\to a} f(x) = f(a)$ as long as the denominator of $f(x)$ is not equal to zero. Therefore, limit exists for all $a \in \mathbb{R} \setminus \{-1\}$.

We can rule out $x = -1$ as a point of continuity because limit does not exist.

Is the function $g(x)$ continuous everywhere else? Yes, because for x everywhere else we have $-\infty < g(x) < \infty$ and the limit exists and $\lim_{x\to a} g(x) = g(a)$.

9. The difference quotient is

$$\frac{\frac{x}{6-x} - \frac{x_0}{6-x_0}}{x-x_0} = \frac{\frac{x(6-x_0)-x_0(6-x)}{(6-x)(6-x_0)}}{x-x_0} = \frac{6x - xx_0 - 6x_0 + xx_0}{(x-x_0)(36-6x-6x_0+xx_0)} = \frac{6}{36-6x-6x_0+xx_0}.$$

Looking at this as a rational function in x, applying the limit law for rational functions (the limit as x approaches x_0 is equal to the function value at x_0), we get: $6/(36-12x_0+x_0^2) = 6/(6-x_0)^2$. (Note: simply taking the derivative is not a sufficient answer to the question.)

10. (a) $f'(x) = 6x\cdot e^{2x} + 6x^2\cdot e^{2x} = 6e^{2x}\cdot(x+x^2)$
    $f''(x) = 6e^{2x} + 24x\cdot e^{2x} + 12x^2\cdot e^{2x} = 6e^{2x}\cdot(2x^2+4x+1)$

    (b) Need $x + x^2 = x\cdot(1+x) \geq 0$ which implies the set $(-\infty,-1]\cup[0,\infty)$ as the solution.

    (c) Need $x + x^2 = 0$ and therefore $x \in \{-1, 0\}$ are the two x-locations of the local extrema.

11. (a) $f(3) = 3$, $f(1.999) = 1$, $f\left(\frac{1}{4}\right) = 0$, and $f(-4.5) = -5$.

    (b) Does not exist. The left limit is 2 while the right limit is 3, therefore the limit does not exist.

12. $f'(x) = (\ln x)^{-1/2}/(2x)$ and therefore $f'(e) = 1/(2e)$.

13. General formula: $A = Pe^{rt}$ with $A = 3P$. Thus, $t = \frac{\ln 3}{0.10}$.

14. Factoring:

$$\lim_{x\to 2} \frac{(x+3)\cdot(x-2)}{(x+4)\cdot(x-2)} = \lim_{x\to 2} \frac{(x+3)}{(x+4)} = 5/6.$$

15. (a) $h([-2,2]) = [0,4]$.

    (b) First, $h([0,3]) = [0,9]$. Thus, $h^{-1}(h([0,3])) = h^{-1}([0,9]) = [-3,3]$.
    (Note: It may be a common mistake to think that $h^{-1}(h(E)) = E$. But that is not correct.)