Foundations of Economic and Financial Models — Jürgen Meinecke
EMET 1001/7001 — Research School of Economics
Semester 2, 2013 — Australian National University

Answer Key for Practice Midterm Exam

Note: Some solutions provided below are detailed and comprehensive. Some solutions, on the other hand, only give the final result. In the actual midterm you will have to provide detailed and comprehensive solutions. You obtain no partial credit for merely giving a final result (even if it is correct). May contain errors and typos.

1. There are three cases.

   (i) $x_0 > 0$
   Then $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x \overset{(*)}{=} x_0 = f(x_0)$. Note that (*) holds by the limit laws for polynomial functions (x is a first order polynomial). This shows that the limit as x approaches x_0 exists, is equal to x_0 and therefore coincides with the function value at x_0. Therefore, f is continuous for positive real numbers.

   (ii) $x_0 < 0$
   Following a similar argument to (i), f is continuous for negative real numbers.

   (iii) $x_0 = 0$
   The function behaves differently to the left and right of zero. To figure out what the limit is at zero, need to break this up in a discussion of one-sided limits.

   $$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (-x) = 0.$$
   $$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x = 0.$$

   Thus both one-sided limits exist and are identical which means that $\lim_{x \to 0} f(x) = 0$. We also have that $f(0) = 0$ and it follows that the function is also continuous at zero.

   In summary, the function is continous over the whole real line.

2. For $c \in \mathbb{R}$ let $\delta = \epsilon / |c|$ and x such that $|x - x_0| < \delta$. Therefore, $|f(x) - cx_0| = |cx - cx_0| = |c(x - x_0)| = |c| \cdot |x - x_0| < |c| \cdot \delta = \epsilon$. In other words, there does exist a $\delta > 0$ such that for all x with $|x - x_0| < \delta$ it holds that $|f(x) - cx_0| < \epsilon$ for any $\epsilon > 0$.

3. **Stating the MVT** Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ such that

   $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Continuing with the original solution: We have $f'(z) = a(1+z)^{a-1}$ for all $z > -1$. Now, for $z > 0$, applying the MVT to f on the interval $[0, z]$, there exists c with $0 < c < z$ such that $f(z) - f(0) = f'(c)(z-0)$. Therefore, it follows that

$$(1+z)^a - 1 = a(1+c)^{a-1}z.$$

Since $c > 0$ and $a - 1 > 0$, it follows that $(1+c)^{a-1} > 1$ and thus $(1+z)^a > 1 + az$.

Similarly for $-1 < z < 0$ applying the MVT to f on the interval $[z, 0]$, there exists c with $z < c < 0$ such that $f(0) - f(z) = f'(c)(0-z)$. Therefore, it follows that

$$1 - (1+z)^a = -a(1+c)^{a-1}z.$$

Since $-1 < c < 0$ and $a - 1 > 0$, it follows that $0 < (1+c)^{a-1} < 1$ and thus, again, $(1+z)^a > 1 + az$.

4. (a) Domain and range are both $\mathbb{R}$. Recall that numbers like $c^{2/3}$ are called fractionals and are defined by $c^{2/3} = (c^2)^{1/3} = \sqrt[3]{c^2}$. In particular, c may take on a negative value. For example, $-8^{2/3} = 4$ because $4 \cdot 4 \cdot 4 = (-8)^2$.

   (b) The function and its derivatives are

$$\begin{aligned} h(w) &= w \cdot (4+w)^{2/3} \\ h'(w) &= (4+w)^{2/3} + \tfrac{2}{3}w(4+w)^{-1/3} \\ h''(w) &= \tfrac{2}{3}(4+w)^{-1/3} + \tfrac{2}{3}(4+w)^{-1/3} - \tfrac{2}{9}w(4+w)^{-4/3} \end{aligned}$$

   (c) The necessary condition based on the first derivative is

$$\begin{aligned} 0 &= (4+w)^{2/3} + \tfrac{2}{3}w(4+w)^{-1/3} \\ &= (4+w) + \tfrac{2}{3}w, \end{aligned}$$

   suggesting that there could be local extrema at $w = -12/5$. The instructions on the front page ask to also determine whether this local extremum is a minimum or a maximum. We therefore evaluate the second derivative, $h(-12/5) = 1.42 > 0$ and conclude that at $w = -12/5$ the function has a local minimum.

   The necessary condition based on the first derivative can only help with finding local extrema for such points in the domain of h for which h' exists. At $w = -4$ the function h' is undefined while the function h is well defined with $h(-4) = 0$. Is there a local extremum at $w = -4$? To address this, we need to recall the definition of a local extremum. In this example, the function h would attain a local maximum at $w = -4$ iff there exists a $\delta > 0$ such that $h(x) < h(-4)$ for all $x \in (-4-\delta, 4+\delta)$. Does this happen here? Studying the function, $h(w) = w \cdot (4+w)^{2/3}$, we see that the second factor, $(4+w)^{2/3}$, is always non-negative (i.e., for all $w \in \mathbb{R}$). The second factor, around $w = -4$, is negative. More precisely, pick $\delta = 0.5$ and see that $h(w) < 0 = h(-4)$ for $w \in (-4.5, -3.5)$. Therefore, this function does attain a local maximum at $w = -4$. (This would work for many different choices of δ. For example, $\delta = -0.2$ would also have worked.)

5. 2.4

6. (a) Careful with the timing; it's the end of each month.
   $1000000 = P(1 + 0.075/12)^{479} + P(1 + 0.075/12)^{478} + P(1 + 0.075/12)^{477} + \cdots + P(1 + 0.075/12)^2 + P(1 + 0.075/12) + P$.
   The last term, P, represents that last payment which happens just one day before retirement.

   (b) $a_1 = P$, $i = 1 + 0.075/12$, $n = 480$.

   (c) From geometric series: $A = P(1 - i^n)/(1 - i)$. Solving for P yields:

$$P = \frac{A(1-i)}{1-i^n} = \frac{1000000(1-1.00625)}{1-1.00625^{480}}$$

7. (a)

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 40 & \text{if } 0 < x \leq 1 \\ 70 & \text{if } 1 < x \leq 2 \\ 100 & \text{if } 2 < x \leq 3 \\ 130 & \text{if } 3 < x \leq 4 \\ 160 & \text{if } 4 < x \leq 5 \end{cases}$$

   (b) Don't exist

8. Looking at the limit of $g(x)$:

$$\lim_{x \to a} g(x) = \lim_{x \to a} \frac{x \cdot (x+1)}{x^2 - 1} = \lim_{x \to a} \frac{x \cdot (x+1)}{(x+1) \cdot (x-1)} = \lim_{x \to a} \frac{x}{x-1} = \lim_{x \to a} f(x).$$

   where we defined $f(x) = x/(x-1)$. We learned that for a rational function (such as $f(x)$ here) $\lim_{x \to a} f(x) = f(a)$ as long as the denominator of $f(x)$ is not equal to zero. Therefore, limit exists for all $x \in \mathbb{R} \setminus \{1\}$, this includes $x = -1$!

   We can rule out $x = 1$ as a point of continuity because limit does not exist. What about $x = -1$? Although the limit exists there, the function is not defined for that value. Therefore, the function is not continuous at -1 either.

   Is the function $g(x)$ continuous everywhere else? Yes, because for x everywhere else we have $-\infty < g(x) < \infty$ and the limit exists and $\lim_{x \to a} g(x) = g(a)$.

9. $-4x^{-3}$

10. (a) too easy
    (b) increasing: $(-\infty, 0) \cup (2, \infty)$, decreasing $(0, 2)$
    (c) $(0, 0)$ is a max (check soc!) and $(2, -4)$ is a min (check soc!)
    (d) convex: $(1, \infty)$ concave: $(-\infty, 1)$
11. (a) 2 (b) 21 (c) 1/3 (check MC" for minimum!)
12. Two sets X and Y have equal cardinality iff there exists a bijection $f : X \to Y$ from X to Y.
    Let n be a natural number. A set X is said to have cardinality n, iff it has equal cardinality with the set $\{1, 2, \ldots, n\}$.
13. $1000 \cdot e^{0.045 \cdot 3.5}$
14. $1/5$
15. See lecture notes