Multivariate Optimization

$z = f(x,y) = 150 - 2x - 3y$

"reduce" function to two dimensions

(i) $\left.\begin{aligned} z = f(x,40) &= 150 - 2x - 3\cdot 40 \\ &= 30 - 2x \end{aligned}\right\}$ iso-y-section

(ii) $\left.\begin{aligned} z = f(60,y) &= 150 - 2\cdot 60 - 3y \\ &= 30 - 3y \end{aligned}\right\}$ iso-x-section

(iii) $\left.\begin{aligned} 30 = f(x,y) &= 150 - 2x - 3y \\ \Rightarrow \quad y &= 40 - \frac{2}{3}x \end{aligned}\right\}$ iso-z-section

Partial Derivatives

slopes of functions in higher-dimensional space

Definition

For any function $z = f(x,y)$ there are two first order partial derivatives:

(i) $$\frac{\partial f(x,y)}{\partial x} = \lim_{x \to x_0} \frac{f(x,y) - f(x_0,y)}{x - x_0} = f_x$$

$$\cancel{\frac{df(x)}{dx}} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

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(ii) $\dfrac{\partial f(x,y)}{\partial y} = \lim\limits_{y \to y_0} \dfrac{f(x,y) - f(x,y_0)}{y - y_0} = f_y$

Example

$f(x,y) = x^3 + 3x^2y^2 + y^3$

$\dfrac{\partial f(x,y)}{\partial x} = 3x^2 + 6xy^2$

$\dfrac{\partial f(x,y)}{\partial y} = 3x^2 2y + 3y^2$

Second order partial derivative

Definition:

Any function $f(x,y)$ has two direct second order partial derivatives:

(i) $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f(x,y)}{\partial x}\right) \Rightarrow \dfrac{\partial^2 f(x,y)}{\partial x \partial x} = f_{xx}$

(ii) $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f(x,y)}{\partial y}\right) \Rightarrow \dfrac{\partial^2 f(x,y)}{\partial y \partial y} = f_{yy}$

Example

$f(x,y) = x^{1/2} \cdot y^{1/2} - 10$

$\dfrac{\partial f(x,y)}{\partial x} = y^{1/2} \, \frac{1}{2} \, x^{-\frac{1}{2}}$

$$\frac{\partial^2 f(x,y)}{\partial x \partial x} = y^{1/2} \frac{1}{2} \left(-\frac{1}{2}\right) x^{-3/2}$$

Definition:

Any function $f(x,y)$ has two second order cross partial derivatives:

(i) $\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) \Rightarrow \frac{\partial^2 f(x,y)}{\partial y \partial x} = f_{xy}$

(ii) $\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right) \Rightarrow \frac{\partial^2 f(x,y)}{\partial x \partial y} = f_{yx}$

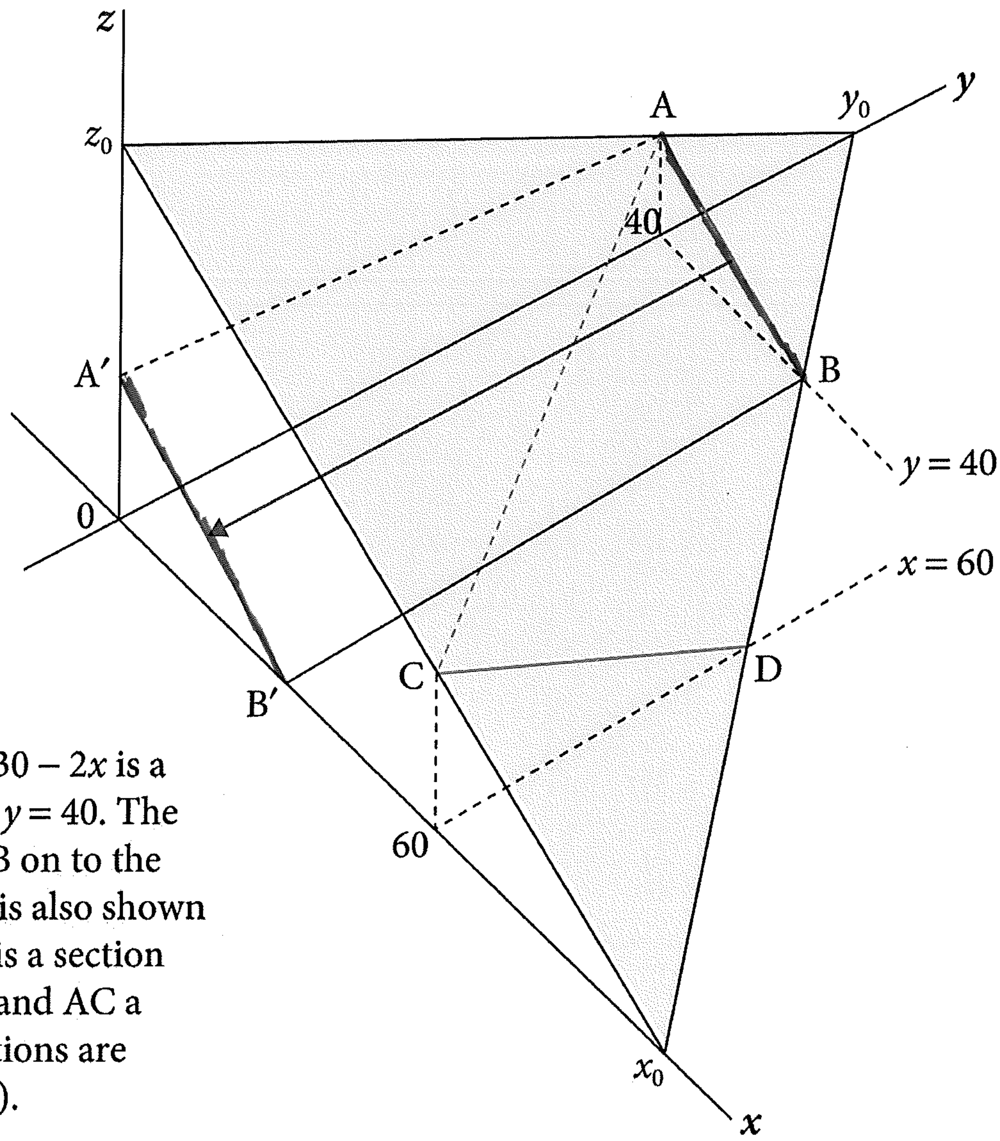
Example

$$f(x,y) = x^3 \cdot y^4$$

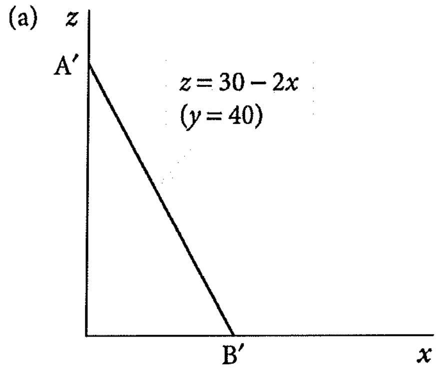
$$\frac{\partial f(x,y)}{\partial x} = y^4 \; 3x^2$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = 3x^2 \; 4y^3$$

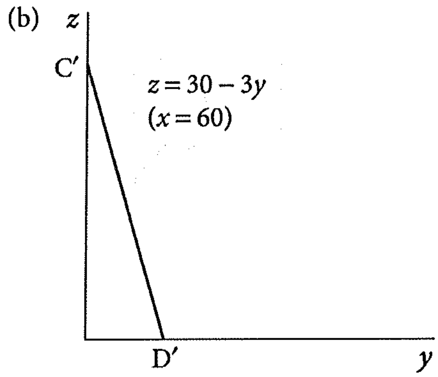
$z = 150 - 2x - 3y$



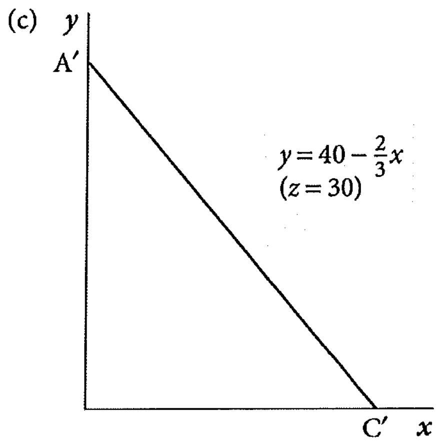
The line AB with equation $z = 30 - 2x$ is a section through the surface for $y = 40$. The line A′B′ is the projection of AB on to the $0xz$ plane. The projection A′B′ is also shown in figure 14.4(a). Similarly CD is a section through the surface for $x = 60$, and AC a section for $z = 30$. Their projections are shown in figures 14.4(b) and (c).



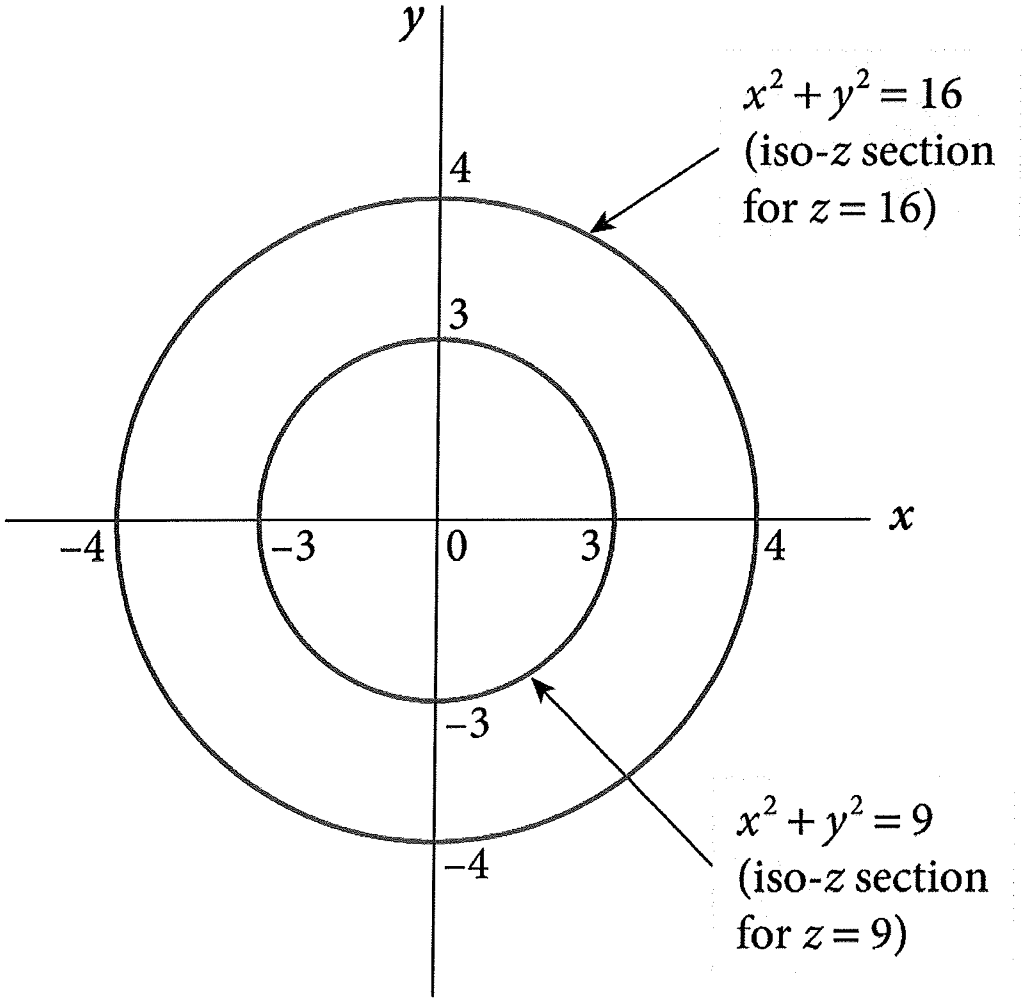
Iso-y section AB through the surface in figure 14.3, for $y = 40$, projected on to the $0xz$ plane



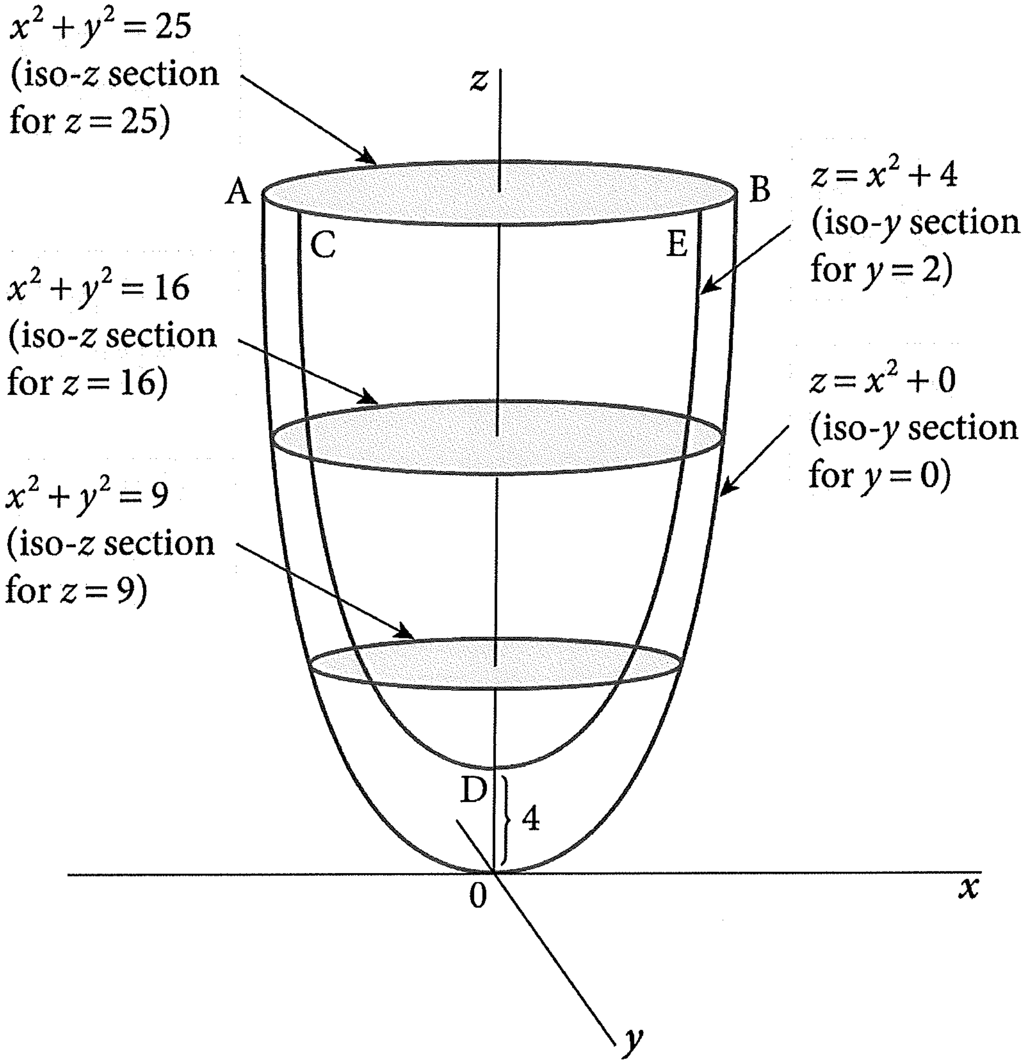
Iso-x section CD through the surface in figure 14.3, for $x = 60$, projected on to the $0yz$ plane



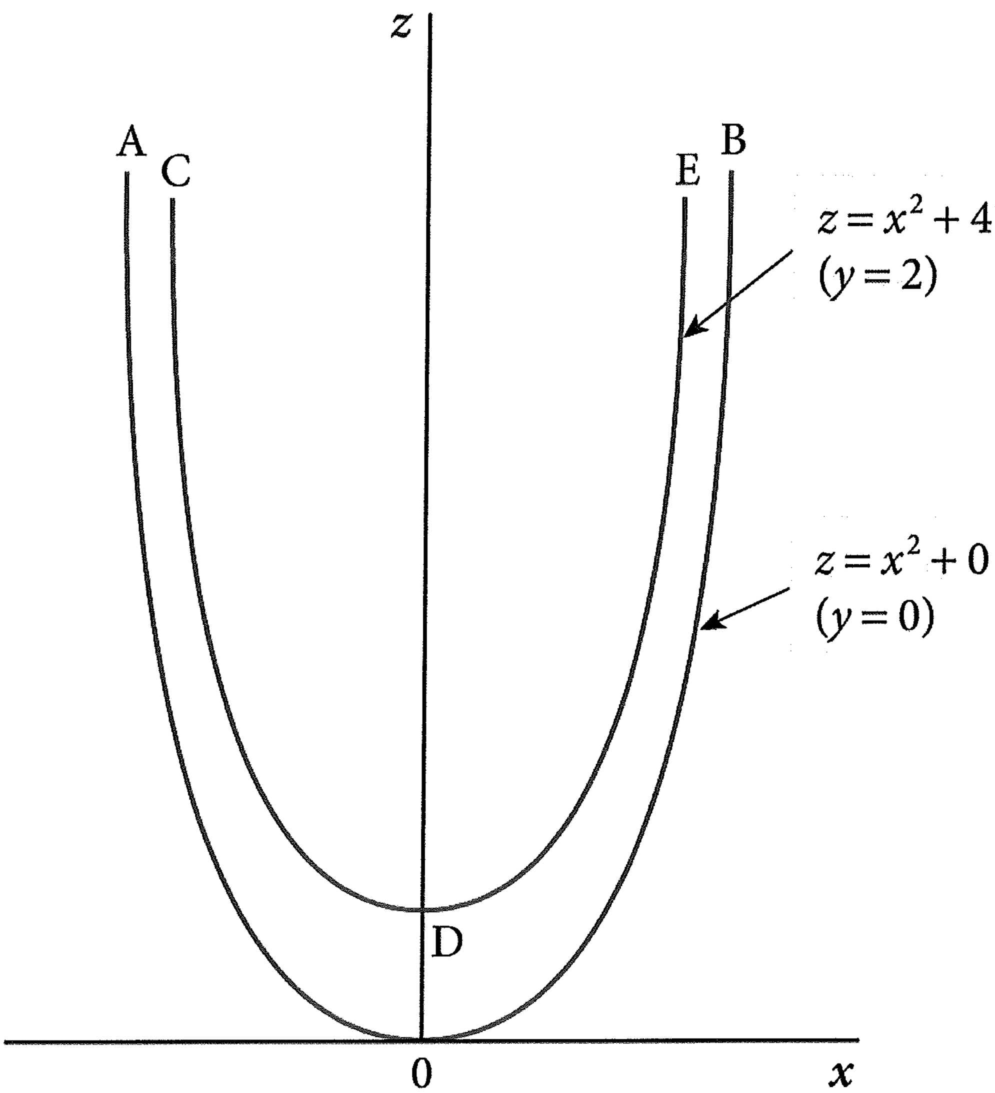
Iso-z section AC through the surface in figure 14.3, for $z = 30$, projected on to the $0xy$ plane

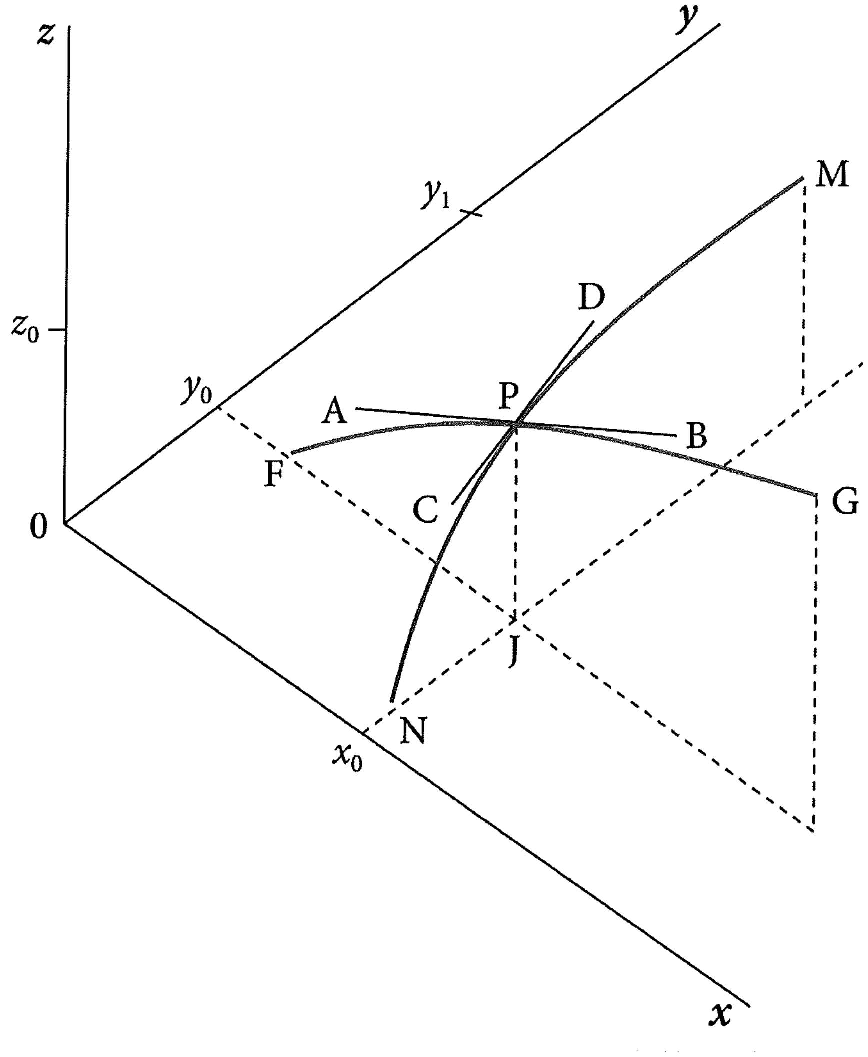


y
x
4
3
-4
-3
0
3
4
-3
-4
$x^2 + y^2 = 16$
(iso-z section
for $z = 16$)
$x^2 + y^2 = 9$
(iso-z section
for $z = 9$)

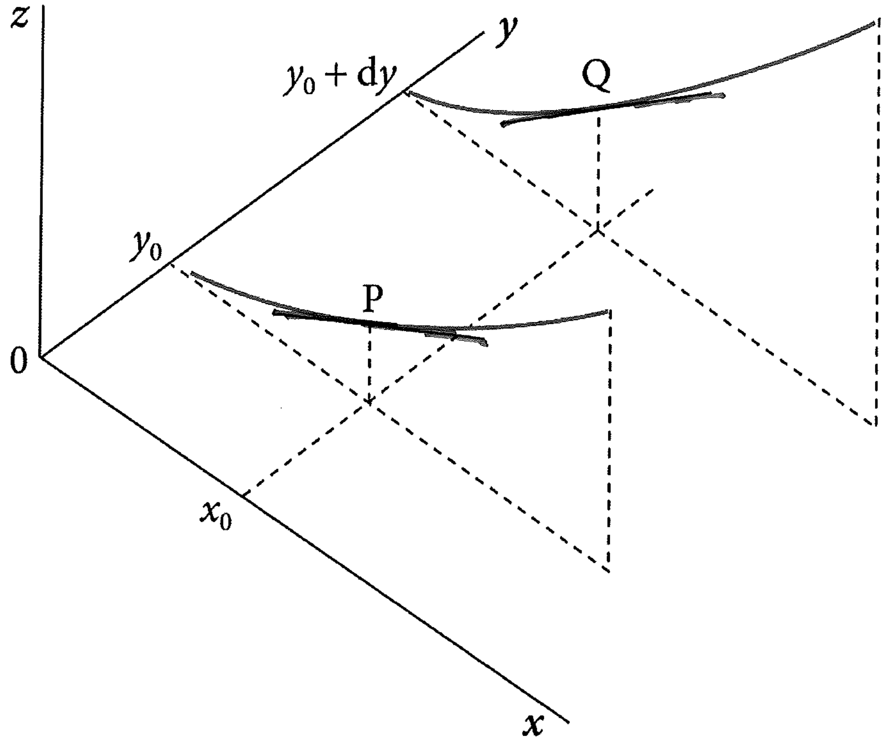


$z = x^2 + y^2$ is a cone with its vertex (point) at the origin. Iso-z sections through it are circles with their centres on the z-axis. Two of these are projected on to the $0xy$ plane in figure 14.6. Iso-y sections are quadratic functions, with the shape of a parabola. Two of these are projected on to the $0xz$ plane in figure 14.7.

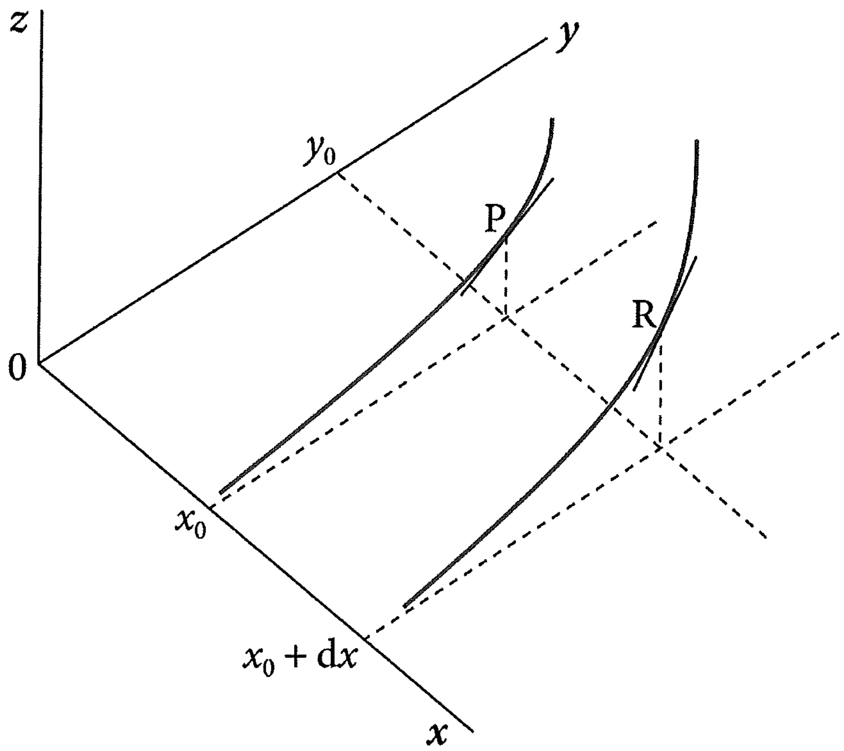

z
A
C
E
B
$z = x^2 + 4$
$(y = 2)$
$z = x^2 + 0$
$(y = 0)$
D
0
x



The tangent APB measures the slope of the surface at P in the x direction (y constant). The tangent CPD measures the slope of the surface at P in the y direction (x constant).



Slope in x direction is steeper at Q than at P, so $\dfrac{\partial^2 z}{\partial y\,\partial x}$ is positive.



Slope in y direction is steeper at R than at P, so $\frac{\partial^2 z}{\partial x\, \partial y}$ is positive.