Lagrange multiplier method

Example: cost minimization

Firms produce output by combining K, L

- cost function:
  - labor: per unit cost $w = \$4$
  - capital: $r = \$2$

  $\Rightarrow TC(K, L) = w \cdot L + r \cdot K$

  $= 4L + 2K$

- production function:

  $Q(K, L) = 5K^{1/3} L^{2/3}$

- objective:

  produce $Q = 100$ at minimal cost!

  $\Rightarrow \min TC(K, L)$ st: $Q(K, L) = 100$

Four steps

(1) $F = 4L + 2K + \lambda \cdot (5K^{1/3} L^{2/3} - 100)$

(2) $F_L = 4 + \lambda \cdot \frac{2}{3} 5K^{1/3} L^{-1/3}$

$F_K = 2 + \lambda \frac{1}{3} 5K^{-2/3} L^{2/3}$

$F_\lambda = 5K^{1/3} L^{2/3} - 100$

(3) $$\lambda \cdot \frac{2}{3} 5 K^{1/3} L^{-1/3} = -4 \qquad (1)$$

$$\lambda \cdot \frac{1}{3} 5 K^{-2/3} L^{2/3} = -2 \qquad (2)$$

divide eq (1) by eq(2):

$$\Rightarrow \frac{\cancel{\lambda} \cdot \frac{2}{\cancel{3}} \cancel{5} K^{1/3} L^{-1/3}}{\cancel{\lambda} \cdot \frac{1}{\cancel{3}} \cancel{5} K^{-2/3} L^{2/3}} = \frac{\cancel{-}\cancel{4}\,2}{\cancel{-}\cancel{2}}$$

$$\Leftrightarrow \quad 2\frac{K}{L} = 2$$

$$\Leftrightarrow \quad \boxed{K = L}$$

plug $K = L$ into F_λ and set equal to zero:

$$5K^{1/3} K^{2/3} - 100 = 0$$

$$\Rightarrow \boxed{\begin{aligned} K &= 20 \\ L &= 20 \end{aligned}}$$

(4) $$TC(20, 20) = 4 \cdot 20 + 2 \cdot 20 = \underline{\underline{120}}$$

Systems of linear equations / matrices

The linear system

$$a x_1 + b x_2 = h$$
$$c x_1 + d x_2 = k$$

can be written compactly:

$$AX = B \quad \rightsquigarrow \quad X = \frac{B}{A}$$

where:

$$A = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{coefficient matrix}} \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad B = \underbrace{\begin{bmatrix} h \\ k \end{bmatrix}}_{\text{constant matrix}}$$

Solving linear equation systems using augmented matrices

solve: $3x_1 + 4x_2 = 1$
$x_1 - 2x_2 = 7$

Define: augmented (coefficient) matrix

$$\left[\begin{array}{cc:c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right]$$

Elementary row operations

An augmented coefficient matrix can be transformed into a row-equivalent matrix by performing any of the following 3 operations:

(i) exchange two rows $(R_i \leftrightarrow R_j)$

(ii) multiply a row by non-zero constant $(cR_i \leftrightarrow R_i)$

(iii) add a constant multiple of one row to another row $(cR_j + R_i \leftrightarrow R_i)$

Objective:

$$\underbrace{\left[\begin{array}{cc:c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right]}$$

$$\left.\begin{array}{l} 1 \cdot x_1 + 0 \cdot x_2 = m \\ 0 \cdot x_1 + 1 \cdot x_2 = n \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = m \\ x_2 = n \end{array}$$

back to example

$$\left[\begin{array}{cc:c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \Rightarrow \left[\begin{array}{cc:c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right]$$

$$-3R_1 + R_2 \leftrightarrow R_2 \Rightarrow \left[\begin{array}{cc:c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right]$$

$$\frac{1}{10} R_2 \leftrightarrow R_2 \Rightarrow \left[\begin{array}{cc:c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

$$2R_2 + R_1 \leftrightarrow R_1 \Rightarrow \left[\begin{array}{cc:c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

$$\Rightarrow \boxed{\begin{array}{l} x_1 = 3 \\ x_2 = -2 \end{array}}$$

Example

$$6x_2 + 2x_1 = -3$$
$$x_1 + 3x_2 = 2$$

$$A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc:c} 2 & 6 & -3 \\ 1 & 3 & 2 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \Rightarrow \left[\begin{array}{cc:c} 1 & 3 & 2 \\ 2 & 6 & -3 \end{array}\right]$$

$-2R_1 + R_2 \longleftrightarrow R_2 \Rightarrow \underbrace{\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & -7 \end{array}\right]}$

last row:

$0 \cdot x_1 + 0 \cdot x_2 = -7$

$\rightarrow$ contradiction

$\Rightarrow$ inconsistent, no solution

Example

$2x_1 - x_2 = 4$

$-6x_1 + 3x_2 = -12$

$\Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 4 \\ -6 & 3 & -12 \end{array}\right]$

$\frac{1}{2} R_1 \leftrightarrow R_1 \Rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 2 \\ -6 & 3 & -12 \end{array}\right]$

$6R_1 + R_2 \leftrightarrow R_2 \Rightarrow \underbrace{\left[\begin{array}{cc|c} 1 & -1/2 & 2 \\ 0 & 0 & 0 \end{array}\right]}$

last row:

$0 \cdot x_1 + 0 \cdot x_2 = 0$

$\rightarrow$ not a contradiction

from first row:

$$1 \cdot x_1 - \frac{1}{2} x_2 = 2$$

for example: if $x_2 = 4$ then $x_1 = 4$

more general: if $x_2 = t$ then $x_1 = 2 + \frac{1}{2} t$

$\Rightarrow$ set of solutions:

$$\left\{ \left(2 + \frac{1}{2} t, t\right) \mid t \in \mathbb{R} \right\}$$

$\rightarrow$ infinitely many solutions

in last 3 examples

$$\left[\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & p \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & 0 \end{array}\right]$$

consistent independent | inconsistent no solution | consistent dependent

$\rightarrow$ what about more general / bigger equation systems?

Gauss-Jordan Elimination

Example:

$$x_3 + 2x_1 - 2x_2 = 3$$
$$3x_1 - x_3 + x_2 = 7$$
$$-3x_2 + x_1 + 2x_3 = 0$$

write as: $AX = B$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$$

→ <u>augm. coeff matrix:</u>

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 3 \\ 3 & 1 & -1 & 7 \\ 1 & -3 & 2 & 0 \end{array}\right]$$

$$R_1 \leftrightarrow R_3 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 3 & 1 & -1 & 7 \\ 2 & -2 & 1 & 3 \end{array}\right]$$

$$\begin{array}{l} -3R_1 + R_2 \leftrightarrow R_2 \Rightarrow \\ -2R_1 + R_3 \leftrightarrow R_3 \Rightarrow \end{array} \quad \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 10 & -7 & 7 \\ 0 & 4 & -3 & 3 \end{array}\right]$$

$$\tfrac{1}{10} R_2 \leftrightarrow R_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & -7/10 & 7/10 \\ 0 & 4 & -3 & 3 \end{array}\right]$$

$$\begin{array}{l} 3R_2 + R_1 \leftrightarrow R_1 \Rightarrow \\ -4R_2 + R_3 \leftrightarrow R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -1/10 & 21/10 \\ 0 & 1 & -7/10 & 7/10 \\ 0 & 0 & -2/10 & 2/10 \end{array}\right]$$

$$-5R_3 \leftrightarrow R_3 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & -1/10 & 21/10 \\ 0 & 1 & -7/10 & 7/10 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$\frac{1}{10} R_3 + R_1 \leftrightarrow R_1$

$\frac{7}{10} R_3 + R_2 \leftrightarrow R_2$

$$\Rightarrow \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]}$$

reduced form matrix

$$x_1 = 2$$
$$x_2 = 0$$
$$x_3 = -1$$

Definition:

A matrix is said to be in reduced form if

(i) each row consisting entirely of zeros is below any row having at least one nonzero element.

(ii) the leftmost nonzero element in each row is 1

(iii) all other elements in the column containing the leftmost 1 of a given row are zero.

(iv) the leftmost 1 in any row is to the right of the leftmost 1 in the row above.

Example

$$x_1 + 2x_2 + 4x_3 + x_4 - x_5 - 1 = 0$$

$$2x_1 + 4x_2 + 8x_3 + 3x_4 - 4x_5 = 2$$

$$x_1 + 3x_2 + 7x_3 + 3x_5 + 2 = 0$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

aug. coeff. matrix:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 1 & -1 & 1 \\ 2 & 4 & 8 & 3 & -4 & 2 \\ 1 & 3 & 7 & 0 & 3 & -2 \end{array}\right]$$

$-2R_1 + R_2 \leftrightarrow R_2$

$-1R_1 + R_3 \leftrightarrow R_3$ $\Rightarrow$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 3 & -1 & 4 & -3 \end{array}\right]$$

$R_2 \leftrightarrow R_3 \Rightarrow$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 1 & -1 & 1 \\ 0 & 1 & 3 & -1 & 4 & -3 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

$-2R_2 + R_1 \leftrightarrow R_1 \Rightarrow$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & -9 & 7 \\ 0 & 1 & 3 & -1 & 4 & -3 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

$-3R_3 + R_1 \leftrightarrow R_1$

$1R_3 + R_2 \leftrightarrow R_2$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & -3 & 7 \\ 0 & 1 & 3 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

$\rightarrow$ reduced form

$$\Rightarrow \begin{aligned} 1x_1 + 0x_2 - 2x_3 + 0x_4 - 3x_5 &= 7 \\ 0x_1 + x_2 + 3x_3 + 0x_4 + 2x_5 &= -3 \\ 0x_1 + 0x_2 + 0x_3 + x_4 - 2x_5 &= 0 \end{aligned}$$

$$\Leftrightarrow \begin{aligned} x_1 &= 7 + 2x_3 + 3x_5 \\ x_2 &= -3 - 3x_3 - 2x_5 \\ x_4 &= 2x_5 \end{aligned}$$

If we let $x_3 = s$ and $x_5 = t$

then for any $s, t \in \mathbb{R}$

$$\boxed{\begin{aligned} x_1 &= 7 + 2s + 3t \\ x_2 &= -3 - 3s - 2t \\ x_4 &= 2t \\ x_3 &= s \\ x_5 &= t \end{aligned}}$$