→ solving linear eq. systems

$$A X = B \quad | \cdot \text{ pre-multiply by } A^{-1}$$

$$\Rightarrow \quad A^{-1} A X = A^{-1} B$$

$$\Leftrightarrow \quad X = A^{-1} B$$

$$= \frac{1}{-10} \underbrace{\begin{bmatrix} -2 & -4 \\ -1 & 3 \end{bmatrix}}_{2\times2} \cdot \underbrace{\begin{bmatrix} 1 \\ 7 \end{bmatrix}}_{2\times1}$$

$$= -\frac{1}{10} \begin{bmatrix} -30 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Definition

A matrix is called singular if its determinant equals zero. Otherwise it is called non-singular.

Inverse of a 3x3 matrix

Determinant of a 3x3 matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

To compute $|A|$ for a ~~A~~ 3×3 matrix, need 5 steps:

(i) Get determinant of the sub-matrix M_{11}

$$M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \Leftarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|M_{11}| = a_{22}\,a_{33} - a_{32}\,a_{23}$$

(ii) Get determinant of the sub-matrix M_{12}

$$M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$|M_{12}| = a_{21}\,a_{33} - a_{31}\,a_{23}$$

(iii) Get determinant of the sub-matrix M_{13}

$$M_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$|M_{13}| = a_{21}\,a_{32} - a_{31}\,a_{22}$$

$\Rightarrow$ the determinants $|M_{11}|$, $|M_{12}|$, $|M_{13}|$ are called <u>minors</u>.

(iv) calculate the "signed minors" or "cofactors"

- for minor $|M_{11}|$ the signed minor equals
$$C_{11} = |M_{11}|$$
- for minor $|M_{12}|$ the signed minor equals
$$C_{12} = -|M_{12}|$$
- for minor $|M_{13}|$ the signed minor equals
$$C_{13} = |M_{13}|$$

→ trick for figuring out signs:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

(v) Combine all three cofactors to obtain $|A|$

$$\Rightarrow |A| = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

Example

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{bmatrix} \qquad \rightarrow |A| = ?$$

5 steps

(i) $|M_{11}| = 2$

(ii) $|M_{12}| = -2$

(iii) $|M_{13}| = -2$

(iv) cofactors: $C_{11} = 2$

$C_{12} = 2$

$C_{13} = -2$

(v) $|A| = 2 \cdot 2 + 4 \cdot 2 - 1 \cdot 2$

$= 10$

Example

$$A = \begin{bmatrix} 10 & 7 & 5 \\ 0 & 2 & 0 \\ 2 & 7 & 3 \end{bmatrix} \rightarrow |A| = ?$$

→ base calculations on 2nd row,
b/c then need to calculate cofactor C_{22}
(no need to obtain C_{21} or C_{23})

$\Rightarrow |M_{22}| = 20$

& $C_{22} = 20$

$|A| = 2 \cdot 20 = 40$

Inverse of a 3x3 matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Then the inverse A^{-1} of A is

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

where C_{ij} are the corresponding cofactors of A.

Example

$$A = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix} \rightarrow A^{-1} = ?$$

3 steps

(i) Obtain all 9 cofactors

$C_{11} = 13$ $\quad C_{12} = 11$ $\quad C_{13} = -7$

$C_{21} = 1$ $\quad C_{22} = 31$ $\quad C_{23} = 7$

$C_{31} = 16$ $\quad C_{32} = 6$ $\quad C_{33} = 14$

(ii) Obtain determinant : $|A| = 4 \cdot 13 + 1 \cdot 11 - 5 \cdot (-7) = 98$

(iii) <u>Obtain inverse A^{-1} of A:</u>

$$A^{-1} = \frac{1}{98} \cdot \begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}$$

<u>double check:</u>

$$A \cdot A^{-1} = \underset{3 \times 3}{\begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}} \cdot \frac{1}{98} \underset{3 \times 3}{\begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}}$$

$$= \frac{1}{98} \begin{bmatrix} 98 & 0 & 0 \\ 0 & 98 & 0 \\ 0 & 0 & 98 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving systems of equations

Example

$$4x_1 + 1x_2 - 5x_3 = 8$$
$$-2x_1 + 3x_2 + x_3 = 12$$
$$3x_1 - x_2 + 4x_3 = 5$$

compact notation:

$$AX = B \quad | \cdot \text{ pre-multiply by } A^{-1}$$
$$\underbrace{A^{-1}A}_{I} X = A^{-1}B$$
$$I \quad X = A^{-1}B$$
$$X = A^{-1}B$$

here

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad A = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B$$
$$= \frac{1}{98} \underset{3 \times 3}{\begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}} \cdot \underset{3 \times 1}{\begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix}}$$
$$= \frac{1}{98} \begin{bmatrix} 196 \\ 490 \\ 98 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{x_1 = 2 \qquad x_2 = 5 \qquad x_3 = 1}$$

Cramer's rule

In a system $AX = B$ with

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

the solution for x_i is given by

$$x_i = \frac{|A_i|}{|A|} \qquad \text{for } i = 1, \dots, n$$

where A_i is the $n \times n$ matrix formed by replacing the i-th column of matrix A by vector B.

Example

$$A = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix}$$

$\rightarrow$ solve for x_1 only!

by Cramer's rule: $\quad x_1 = \dfrac{|A_1|}{|A|}$

from yesterday: $|A| = 98$

for the numerator:

$$A_1 = \begin{bmatrix} 8 & 1 & -5 \\ 12 & 3 & 1 \\ 5 & -1 & 4 \end{bmatrix}$$

$$\Rightarrow |A_1| = 8 \cdot 13 + 1 \cdot (-43) - 5 \cdot (-27)$$

$$= 196$$

$$\Rightarrow x_1 = \frac{196}{98} = 2$$

Simple macro model

- income: $Y = C + G$
- consumption: $C = a + b(Y - T)$
- tax revenue: $T = d + t \cdot Y$

endogeneous variables: Y, C, T

parameters: a, b, d, t

exogoneous variables: G

→ write compactly as $AX = B$

where $X = \begin{bmatrix} Y \\ C \\ T \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} G \\ a \\ d \end{bmatrix}$$

→ <u>calculate sol'n for Y</u>

$$Y = \frac{|A_1|}{|A|}$$

<u>where:</u>

$$|A| = 1 \cdot 1 \quad + 1 \cdot (-b + tb) + 0 \cdot$$
$$= 1 + b(t-1)$$

$$A_1 = \begin{bmatrix} G & -1 & 0 \\ a & 1 & b \\ d & 0 & 1 \end{bmatrix}$$

$$|A_1| = G \cdot 1 \quad + 1 \cdot (a - bd)$$

$$= G + a - bd$$

$$\Rightarrow \boxed{Y = \frac{G + a - bd}{1 + b(t-1)}}$$

Practice final

Q3 Prove that the limit of a fct is unique.

$$\lim_{x \to x_0} f(x) = L_1 \leftarrow$$

$$\lim_{x \to x_0} f(x) = L_2$$

Recall:

$$|f(x) - L_1| < \varepsilon$$

for all $0 < |x - x_0| < \delta_1$

at same time

$$|f(x) - L_2| < \varepsilon$$

for all $0 < |x - x_0| < \delta_2$

aside

$|a+b| \leq |a| + |b|$

Therefore

$$2\varepsilon > |f(x) - L_1| + |f(x) - L_2|$$

$$= |-(L_1 - f(x))| + |f(x) - L_2|$$

$$= |L_1 - f(x)| + |f(x) - L_2|$$

$$\geq |L_1 \cancel{- f(x)} \cancel{+ f(x)} - L_2|$$

$$= |L_1 - L_2|$$

$$\Rightarrow |L_1 - L_2| = 0 \Rightarrow \boxed{L_1 = L_2}$$

$\boxed{Q4}$

$$\frac{G_0}{Y_0} = 0.4$$

G grows at 3%

Y 2% per year

$$\rightarrow \boxed{\frac{G_T}{Y_T} = 0.5 \quad ?}$$

(a) <u>con't growth</u>

$$A = P e^{rT}$$

$$\left.\begin{aligned} G_T &= G_0 e^{0.03T} \\ Y_T &= Y_0 e^{0.02T} \end{aligned}\right\} \quad 0.5 = \frac{G_0 e^{0.03T}}{Y_0 e^{0.02T}}$$

$$= \underbrace{0.4 \frac{e^{0.03T}}{e^{0.02T}}}_{\text{solve for } T}$$

(b) $A = P(1+r)^T$

Lagrange multiplier method

$Q = K^{0.2} L^{0.6}$

$w = 2$

$r = 8$

$p = 20$

$\rightarrow$ max profit st: production fct

4 steps

(1) $F = 20 \cdot Q - 2L - 8K + \lambda (K^{0.2} L^{0.6} - Q)$

(2) $F_Q = 20 - \lambda$

$F_K = -8 + 0.2 \lambda K^{-0.8} L^{0.6}$

$F_L = -2 + 0.6 \lambda K^{0.2} L^{-0.4}$

$F_\lambda = K^{0.2} L^{0.6} - Q$

(3) $\lambda = 20$

from F_K: $\quad 8 = 0.2 \cancel{\lambda} K^{-0.8} L^{0.6}$

F_L: $\quad 2 = 0.6 \cancel{\lambda} K^{0.2} L^{-0.4}$

$$\Rightarrow \quad 4 = \frac{\cancel{0}.2}{\cancel{0}.6} \frac{L}{K}$$

$$\Rightarrow \quad \underline{\underline{L = 12 K}}$$

.... → final result

$L = 648$

$K = 54$

$Q = 108$

$\text{profit} = 432$