EMET1001 Documentation

Release 1.0

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October 03, 2013

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This website hosts all the lecture notes and exercises for the ANU course EMET1001, Foundations of Economic and Financial Models.

CHAPTER

ONE

ANNOUNCEMENTS

Tutorial exercises

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`Tutorial exercises for week 12`

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`Tutorial exercises for week 10`

`Tutorial exercises for week 9`

Tutorial exercises for week 8: Exercise 9 of Differential Calculus and Exercises 1 - 2 of Cost and Profit Functions

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Alternative tutes for students who cannot attend their Monday tute (due to the public holidays)

- Tue 10am Arndt TR5
- Tue 16am Arndt TR5
- Wed 11am Arndt TR4
- Thu 16am Arndt TR5

Helpdesk times (Arndt 1002)

- Wed 12-2
- Thu 4-6

Juergen's consultation times (Arndt 1022)

- Wed 10-12

`Practice midterm` and `answer key`

`pdf-version of website` **(last update: August 29)**

CHAPTER

TWO

COURSE OUTLINE

2.1 Course Outline

Read this entire course outline carefully!

Any items, rules, requirements in this course outline may be subject to changes. When this happens I will announce it during the lecture. Announcements in the lecture supersede any information contained in this course outline.

2.1.1 Course Description

In this course we will study mathematical concepts from real analysis, calculus, and linear algebra and apply them to economics and finance. Particular mathematical concepts and their applications include: sequences and series with application to discounting and present value calculations in finance; differential calculus with application to optimization problems (unconstrained and constrained) such as profit maximization and cost minimization; linear algebra with application to solving systems of equations such as supply and demand models.

Learning Outcome

Ideally, at the end of this course you will

- have a sound understanding of mathematical techniques discussed;
- understand mathematical concepts such as the derivatives from first principles (and not merely from memory)
- be in a position to formulate problems in economics, business, and finance in mathematical terms and apply the tools provided in the course for analyzing them;
- be able to apply the basic principles of maximization and minimization to optimization problems; and to
- apply matrix algebra to simple problems and models;

Textbook

The textbook for the course is *College Mathematics for Business, Economics, Life Sciences and Social Sciences* 12ed, by Barnett, Ziegler, and Byleen. Chiefly library has several copies of the textbook.

Prerequisites

It is strongly recommended that you take EMET1001 during the first year of your studies. Please be aware of the assumed level of mathematical knowledge for EMET1001. As you may recall, the math requirements for the Bachelor of Commerce, the Bachelor of Economics, the Bachelor of Finance, and the Bachelor of International Business is equivalent to the *Maths Methods (Tertiary Major)* in the ACT and *HSC Maths (2 unit) including the study of calculus and algebra* in NSW. (Check the ANU College of Business and Economics 2013 Undergraduate Programs Handbook for other states.)

A standard school curriculum will include (from the Board of Studies NSW):

- Basic arithmetic and algebra,
- Real functions,
- Trigonometric ratios,
- Linear functions,
- The quadratic polynomial and the parabola,
- Plane geometry and geometrical properties,
- Tangent to a curve and derivative of a function,
- Coordinate methods in geometry,
- Applications of geometrical properties,
- Geometrical applications of differentiation,
- Integration,
- Trigonometric functions,
- Logarithmic and exponential functions,
- Applications of calculus to the physical world,
- Probability,
- Series and series applications.

Some of these topics will be treated as a pure prerequisite and will not be covered in the course. Others, such as calculus, will be an elementary part of EMET1001. Sections of the textbook that will be treated as assumed knowledge include chapters 1 and 2 as well as Appendix A. Many other topics will be developed from first principles. For example, in the study of financial mathematics, we begin with a brief refresher of sequences and series before we move on to more complicated formulas (such as the present value of an annuity).

Staff

Administrative

For any *administrative* inquiries or problems (e.g., tutorial enrollment, exam scheduling, supplementary exams, etc.) you should contact Terry Embling (School of Economics Course Administrator) or Finola Wijnberg (School of Economics School Administrator).

Name	Terry Embling	Finola Wijnberg
Job title	Course administrator	School administrator
Office	HW Arndt Building 25a	HW Arndt Building 25a
Location	Room 1013	Room 1014
Hours	9:00-16:00	9:00-16:00
E-mail	terry.embling@anu.edu.au	finola.wijnberg@anu.edu.au

Academic

If you have any *academic* inquiries or problems regarding the course, you should contact your regular tutor first. They should be able to handle most of the problems you might have. If you need to contact the Instructor or Head Tutor:

Name	Juergen Meinecke	Dana Hanna
Job title	Instructor	Head tutor
Office	HW Arndt Building 25a	HW Arndt Building 25a
Location	Room 1022	Room 2002
Hours	TBA	TBA
E-mail	juergen.meinecke@anu.edu.au	dana.hanna@anu.edu.au

Lectures

There will be three lectures each week. You are expected to attend all of them. Lectures will be held in the following venues at the following times:

Day	Tuesday	Wednesday	Thursday
Time	9-10	9-10	10-11
Location	MCC T1	MCC T1	MCC T1

Tutorials

In addition to lectures, you are expected to attend and actively participate in weekly tutorials (weeks 2 through 13). Tutes are an integral part of this course. During each tute, your tutor will develop and present solutions to selected exercises *in cooperation* with students. Solutions will not be made available in any other form.

Allocation to tutorial groups will be made via the ETA (Electronic Tutorial Administration) system which you can find following this link.

Help Desk

The EMET1001 helpdesk is by far the best and most flexible opportunity for students seeking answers to their EMET1001 questions. Tutors will be available to help you with any questions you might have regarding the course material. Each week (starting week 2) the help desk will be open for at least 4 hours. Exact days and times for each week will be posted here soon.

Lecture Attendance and Time Conflicts

You are expected to attend all lectures and your weekly tutorial. You should not enrol in EMET1001 if you have persistent time conflicts with other lectures. Audio recordings may be made available.

2.1.2 Course Requirement and Grade Composition

Assessment in this course will consist of two *compulsory* examinations: a midterm exam and a final exam.

Midterm Examination

The midterm examination is expected to be held during week 8 (23/09/2013 – 27/09/2013) outside the usual lecture times (usually after 6pm). The exam covers material from weeks 1 through 6 of the lecture (and weeks 2 through 7 of the tutorials). Participation in the midterm exam is *compulsory*. The exam will be marked out of 100.

It is your responsibility to make yourselves available for the midterm examination during week 8. Details of the midterm exam will be announced as soon as the date is set and the venue is secured (the ANU examination office usually sets dates and times by the middle of August). As soon as I find out I will announce the exact date and time of the midterm exam in the lecture.

No make-up midterm examination will be offered. Should you miss the midterm exam for a valid reason (see Special Examinations below) then your grade will be based solely on your final exam.

Final Examination

Examinable material covers the whole semester, including material already covered in the midterm exam. Participation in the final exam is *compulsory* The exam will be marked out of 100.

The final exam will be held in the exam period at the end of the semester. Details will be posted on the ANU exam timetable site.

Weighting Scheme

The highest final course mark (FCM) that you can achieve in EMET1001 is 100 which translates to a course grade of HD. You will obtain a FCM of 100 if you score maximal on both the midterm and the final exams. Between these two exams the following weighting scheme applies (midterm / final):

- scheme 1: 50% / 50%
- scheme 2: 25% / 75%

Whichever weighting scheme gives you a better FCM will be applied to you. Weighting scheme 2 makes the midterm exam partially redeemable (relative to scheme 1): 25 percentage points are shifted towards the final exam.

Scaling of Grades

Final scores for the course will be determined by scaling the raw score totals to fit a sensible distribution of grades. Scaling can increase or decrease a mark but does not change the order of marks relative to the other students in the course. If it is decided that scaling is appropriate, then the final mark awarded in a course may differ from the aggregation of the raw marks of each assessment component.

2.1.3 Rules and Policies

Special Examinations

Both the midterm examination and the final examination are *compulsory* However, if you are unable to attend a scheduled examination due to *extraordinary circumstances* you may apply for a special examination. For details, check the ANU Special Examinations Policy

Extraordinary circumstances may include hospitalisation, inability to walk, or being so incapacitated that you cannot attend the examination room. For additional grounds for a special examination please read the Special Examinations Protocol. The Special Examinations Protocol also lists in more detail circumstances that do NOT warrant special examination.

When applying for a special examination you are required to provide appropriate evidence for non–attendance. In the case of illness or accident you must attend a doctor or medical clinic on the day of the examination, unless there are special and documented circumstances, and provide a medical certificate or police report where relevant. Please read the Special Examinations Protocol for more details regarding appropriate evidence and medical documentation.

If you have an ongoing medical condition you are advised to register with the Disability Services Centre. The Special Examinations policy is designed for short-term, unexpected illness. The Disability Services Centre is able to provide more practical solutions to long term medical conditions.

Requests for a Special Examination must be made on the standard University Application for Special Examination Form. The completed Special Examination Form should be submitted to:

- Mid-Semester Examinations: Research School of Economics
- Final Examinations: CBE Student Office

Applications lodged more than 3 working days after the date of the examination will not normally be accepted.

Do not assume that non-attendance at the examination and an application will lead to a special examination being given. The granting of a Special Examination is solely at the College's discretion and provision of a medical certification does not necessarily guarantee the granting of a special examination. The College will judge the severity based on the medical report and taking into account all relevant factors.

Calculator Policy

The only kind of calculator permissible in this course is a *non-programmable* calculator. All other types of calculators are not permitted. Examples of calculators that are not permitted include (but are not restricted to): programmable calculators, iPhones, Android phones, tablet computers.

Communication Policies

The official forum for announcements of any kind are the lectures. If necessary, I will contact students electronically using their official ANU student e-mail address. If you want to contact me send an e-mail to

- juergen.meinecke@anu.edu.au

E-mail addresses are only to be used when you need to contact staff about administrative or academic matters. They are NOT to be used for instructional purposes.

Workload

University study requires at least as much time and effort as a full–time job. You are expected to attend all lectures and tutorials (4 hours per week). You should expect to put in at least 6 hours per week of your own study time for this course in addition to the 4 hours of lectures and tutorials.

Extra Examinations

It is your responsibility to familiarize yourself with the rules and regulations and the policies and procedures that are relevant to your studies at the ANU. The following two links direct you to websites that contain information about proper academic conduct, academic honesty and plagiarism, discrimination, harassment and bullying, as well as examination policies (covering special considerations, supplementary examinations, and special examinations). You are expected to be aware of these policies:

- Policies and Procedures
- Examinations and Assessment

If you seek more information, please feel free to visit the College Office (room 2.01 CBE Building 26C) to talk to a student administrator.

Supplementary Examinations

Should you receive a mark of PX you will be required to take a *written supplementary exam*. No other form of supplementary assessment will be offered (e.g., no oral examinations). None of you should ignore the possibility of having to take the supplementary exam. When you enroll in this course you implicitly agree to be able to take a supplementary exam, should it be required of you, at the *end of February 2014* (it will be your responsibility to contact the course administrator to inquire about the exact date). You are required to factor this into your planning for the coming summer. I do not accept requests for early supplementary exams.

Misconduct

In relation to an examination, misconduct on the part of a student includes:

- cheating;
- plagiarism (including the reproducing in, or submitting for assessment for, any examination, by way of copying, paraphrasing or summarizing, without acknowledgement and with the intention to deceive, any work of another person as the student's own work, with or without the knowledge or consent of that other person);
- submitting for an examination any work previously submitted for examination (except with the approval of the prescribed authority);
- failing to comply with the University's instructions to students at, or in relation to, an examination;
- acting, or assisting another person to act dishonestly, in or in connection with an examination;
- taking a prohibited document into an examination venue.

The administrative procedures regarding misconduct are incorporated in the ANU Discipline Rules.

Academic Honesty

The university has strict rules in relation to Academic Honesty, visit the weblink to learn about the ANU's policies and advice on how you can avoid cheating and plagiarism!

CHAPTER

THREE

COURSE MATERIAL

- Weeks 1 through 6—mathematical foundations:

3.1 Sets, Relations, Functions

3.1.1 Set Theory

We start by defining what sets are

Definition (Set)

A **set** is a collection of distinct objects.

Example

All students sitting in this room.

All students standing in this room. (empty set?)

The numbers $8, 10, -4, 349$.

Two ways of defining sets:

- By description
- By enumeration

Example

All students sleeping during lecture. (description)

All non-negative integers. (description)

$S = \{3, 12, -5, 555\}$. (enumeration)

If two sets A and B contain exactly the same elements, that is, $x \in A$ iff $x \in B$, then it is said that both sets are identical, writing $A = B$. Otherwise we write $A \neq B$. (Reminder: iff means 'if and only if'.)

Example

$$\{x, y\} = \{y, x\}$$
$$\{x, x\} = \{x\}$$
$$\{\{x\}\} \neq \{x\}(\text{why?})$$

Definition (Empty set)

The **empty set**, denoted $\emptyset$ is the set that contains no element; if a set contains at least one element it is called nonempty.

Definition (Subset)

Let A and B be sets. We say that A is a **subset** of B if $x \in A$ implies $x \in B$. We write $A \subset B$.

Intuitively, the set A is a subset of B whenever it is contained in B.

Definition (Natural numbers)

The set of **natural numbers** is denoted by $\mathbb{N}$ and defined by $\mathbb{N} = \{1, 2, 3, \dots\}$. This set is also called the set of positive integer numbers.

Definition (Integer numbers)

The set of **integer numbers** is denoted by $\mathbb{Z}$ and defined by $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

Definition (Non-positive integer numbers)

The set of **non-positive integer numbers** is denoted by $\mathbb{Z}_-$ and defined by $\mathbb{Z}_- = \{\dots, -3, -2, -1, 0\}$.

Definition (Negative integer numbers)

The set of **negative integer numbers** is denoted by $\mathbb{Z}_{--}$ and defined by $\mathbb{Z}_{--} = \{\dots, -3, -2, -1\}$.

Definition (Rational numbers)

The set of **rational numbers** is denoted by $\mathbb{Q}$ and defined by $\mathbb{Q} = \left\{\frac{p}{q} | p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\}\right\}$.

Is the set of rational numbers complete? Intuitively, for $\mathbb{Q}$ to be complete we need to be able to solve any equation within the set of rational numbers.

Example

Imagine a square with side length 1. What is the length of its diagonal r? We know from geometry that $r^2 = 1^2 + 1^2$. The question now is whether $r \overset{?}{\in} \mathbb{Q}$. Suppose that indeed $r^2 = 2$ for some $r \in \mathbb{Q}$. Thus, r can be expressed as the ratio $\frac{p}{q}$ with $q \neq 0$. Here p and q do not have a common factor. (Note: If they did have a common factor we could cancel it out and continue the analysis with p' and q' which now denote the equivalents of p and q after removing the common factor.) Then clearly, $p^2 = 2q^2$ which implies that p^2 must be an even integer. This in turn is only possible if p itself is an even integer. We can therefore write p

as $p = 2k$ for some $k \in \mathbb{Z}$. It follows that $2q^2 = 4k^2$ so that $q^2 = 2k^2$. By a similar argument to before, q^2 must be even and then also q must be an even integer. But then p and q share the common factor 2 which contradicts what we assumed earlier.

In other words, $r \notin Q$. The rational numbers contain 'holes'. In the above example, the number $\sqrt{2}$ is the solution for r which is not part of the rational numbers. In fact, many square roots are not members of the rational numbers (but some are!). Other examples of 'holes' in the rational numbers are: $\pi, e, \log_2 7$. There are in fact infinitely many such 'holes' in the rational numbers. Without digging too deep, we want to come up with a 'complete' set of numbers that fills up all the 'holes'. It turns out that the real numbers accomplish this. If we denote the set of irrational numbers by $\mathbb{I}$ we can afford a formal defintion of the real numbers:

Definition (Real numbers)

The set of **real numbers** is denoted by $\mathbb{R}$ and defined by

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I} \tag{3.1}$$

This definition has a slightly tautological flavor. We have not really defined precisely what the irrational numbers are. (And we will not have the time this semester!) But if we had that definition (**??**) then the preceding definition would succinctly define the real numbers. For now, all we need to understand is that the irrational numbers fill ALL the remaining gaps of the rational number system. It is therefore pleasant to work in the real number system.

Definition (Intervals)

We distinguish between the following special sets of real numbers, so-called **intervals**. Let $a, b \in \mathbb{R}$ and suppose that $a \leq b$, then define:

- **Closed interval**: $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$
- **Open interval**: $(a, b) = \{x \in \mathbb{R} | a < x < b\}$
- **Half-open interval**: $[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$
- **Half-open interval**: $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$
- $(-\infty, a] = \{x \in \mathbb{R} | x \leq a\}$
- $(-\infty, a) = \{x \in \mathbb{R} | x < a\}$
- $[a, \infty) = \{x \in \mathbb{R} | x \geq a\}$
- $(a, \infty) = \{x \in \mathbb{R} | x > a\}$

Note, as the symbols $\pm\infty$ do not describe any real number (i.e., $\pm\infty \notin \mathbb{R}$), there is no such thing as $[-\infty, a]$ or $[-\infty, a)$ or $[a, \infty]$ or $(a, \infty]$.

Assumed knowledge: Notions of subset, union, intersection, and complement (see textbook).

3.1.2 Relations and Functions

Definition (Ordered pair)

An **ordered pair** is an ordered list (a, b) consisting of two objects a and b. The ordered pair is ordered in the following sense: For any two ordered pairs (a, b) and (a', b'), we have $(a, b) = (a', b')$ iff $a = a'$ and $b = b'$.

Example

Understand the difference b/w $\{1,3\} = \{3,1\}$ but $(1,3) \neq (3,1)$.

Definition (Cartesian product)

The **Cartesian product** of two nonempty sets A and B, denoted by $A \times B$, is defined as the set of all ordered pairs (a, b) where a comes from A while b comes from B. Formally: $A \times B = \{(a, b) | a \in A, b \in B\}$

Definition (Relation)

Let X and Y be two nonempty sets. A subset R of $X \times Y$ is called a **relation** from X to Y.

Example

The set $\{(x, y) | y \leq 2x\}$ is a relation. (Is it also a function?)

Relations broadly link two sets X and Y. We now want to move on to the more narrow concept of function. Functions transform the elements of one set to those of another. More precisely, this is what we mean by function:

Definition (Function, domain, codomain)

A **function** f that maps the set X into the set Y, denoted by $f : X \to Y$, is a relation $f \subset X \times Y$ such that

- for every $x \in X$ there exists a $y \in Y$ such that $(x, y) \in f$;
- for every $y, z \in Y$ with $(x, y) \in f$ and $(x, z) \in f$ we have $y = z$.

Here X is called the **domain** of f and Y is called the **codomain**.

Sometimes, instead of the word 'function' we use the word 'map'.

Example

The set of all functions that map X into Y is denoted by Y^X. For example, $\{0, 1\}^X$ is the set of all functions on X whose values are either 0 or 1. Likewise, $\mathbb{R}^{[0,1]}$ is the set of all real-valued functions on $[0, 1]$.

Does the above definition of a function look a bit strange? What do we want a function to do? It is supposed to map each member of X to a member of Y. Well, all that f does is completely given by the set $\{(x, f(x)) \in X \times Y | x \in X\}$. The more familiar notation $f(x) = y$ (which we will mostly use throughout the course) is then merely an alternative way of expressing $(x, y) \in f$.

Definition (Image, range, inverse image)

Let $f : X \to Y$ be a function. Let $E \subset X$. Then $f(E)$ is defined to be the set of all elements $f(x)$ such that $x \in E$, formally: $f(E) = \{f(x) | x \in E \subset X\}$. We call $f(E)$ the **image** of E under f. The special case $f(X)$ is called the **range** of f. (The range is the subset of the codomain that the function actually attains.)

Now let $E \subset Y$. Then $f^{-1}(E)$ is defined to be the set of all $x \in X$ such that $f(x) \in E$, formally $f^{-1}(E) = \{x \in X | f(x) \in E \subset Y\}$. We call $f^{-1}(E)$ the **inverse image** of E under f.

$\diamondsuit$

Condition (i) in the definition of a function therefore says that every element in the domain X of f has an image under f in the codomain Y. Condition (ii) states that no element in the domain of f can have more than one image under f.

Definition (Surjection)

A function $f : X \to Y$ is a **surjection** if $f(X) = Y$. In other words, a surjective function has a range that is equal to its codomain. We say that f maps X onto Y. (The function may also be called **surjective** or **onto**.)

Definition (Injection)

A function $f : X \to Y$ is an **injection** if $x \neq y$ implies $f(x) \neq f(y)$ for all $x, y \in X$, in other words, if f maps distinct points in its domain to distinct points in its codomain. (The function may also be called injective or one-to-one.) Equivalently, the function is an injection if $f(x) = f(y)$ implies $x = y$ for all $x, y \in X$.

Definition (Bijection)

A function f is a **bijection** if it is both surjective and injective. (The function may also be called bijective.)

Example

If $X = \{1, \ldots, 10\}$, then $f = \{(1,2), (2,3), \ldots, (10,1)\}$ is a bijection in X^X. However, the function $g \in X^X$ given by $g(x) = 3$ for all $x \in X$ is neither an injection nor a surjection.

Question

Let f from the preceding example be a map in $\left(\{0\} \cup X\right)^X$. Is it an injection? A surjection?

Question

Let the *identity function* be an element of the space X^X (functions in this space are also called *self-maps*), denote it by id_X and define it by $\mathrm{id}_X = x$ for all $x \in X$. Is id_X a bijection?

Definition (Inverse)

For any function $f \in Y^X$ define the set $f^{-1} = \{(y, x) \in Y \times X | (x, y) \in f\}$. If f^{-1} itself is a function then we say that f is **invertible** and f^{-1} is the **inverse** of f.

Example

Let $f : \mathbb{R} \to \mathbb{R}_+$ given by $f(t) = t^2$. Then $(1, 1) \in f^{-1}$ but also $(1, -1) \in f^{-1}$, that is, 1 does not have a unique image under f^{-1} and therefore is not a function.

Example

Take the same f as in the preceding example but restrict the domain to positive real numbers. Thus, $f : \mathbb{R}_+ \to \mathbb{R}_+$. Then $f^{-1}(t) = \sqrt{t}$ for all $t \in \mathbb{R}_+$ is indeed a function.

Proposition

A function $f : X \to Y$ is invertible iff it is bijective.

Question

Does invertibility imply injectivity? Conversely, does injectivity imply invertibility? The first question is answered directly by the preceding proposition: invertibility implies bijectivity and thus injectivity. But does an injective function need to be invertible? Just reconsider the example from above, in which $X = \{1, \ldots, 10\}$ and $f \in \left(\{0\} \cup X\right)^X$ with $f = \{(1,2), (2,3), \ldots, (10,1)\}$.

3.1.3 Exercises

Note: Solutions for exercises will only be given during the tutorials. They will not be posted here.

1. Let $f : (-\infty, 2] \cup [3, \infty) \to \mathbb{R}$ be given by $f(x) = \sqrt{x^2 - 5x + 6}$.
   (a) What are the domain and codomain of f?
   (b) Sketch the function. (Hint: First think about how the graph of $x^2 - 5x + 6$ looks like. Then think about how its square root looks like, especially for large values of $|x|$.)
   (c) Let $E = [4, 5]$. What is the image of E under f?
   (d) Now let $E = [\sqrt{12}, \infty)$. What is the inverse image of E under f?
   (e) What would be the inverse image if instead $E = [\sqrt{-8}, \sqrt{-7}]$?
   (f) What is the range of the function f? Is it equal to the codomain?
2. (Cardinality of sets) The notion of cardinality in set theory concerns the size of sets. Consider the following definition.

   Definition

   We say that two sets X and Y have **equal cardinality** iff there exists a bijection $f : X \to Y$ from X to Y.

   Let n be a natural number. A set X is said to have **cardinality n**, iff it has equal cardinality with the set $\{1, 2, \ldots, n\}$.

   Prove the following:
   (a) The set $\{c, b, a\}$ has cardinality 3.
   (b) The sets $\{0, 1, 2\}$ and $\{d, e, f\}$ have equal cardinality.
   (c) The set of natural numbers and the set of even natural numbers have equal cardinality. (This seems counterintuitive: the set of even natural numbers seems to be contained in the set of natural numbers, yet they have equal cardinality.)
3. (Countability and finiteness of sets). The notions of countability and finiteness should be more or less intuitive. We do not want to rigorously define these concepts here; instead, use your intuition to assign the following sets into the right position in the table below.
   (a) $\{4, \psi, -342\}$
   (b) $\mathbb{N}$

(c) $\mathbb{Z}$

(d) $[-34, 102]$

(e) $\mathbb{R}$

	Countable	Uncountable
Finite	?	?
Infinite	?	?

3.2 Sequences, Series, Limits

3.2.1 Sequences

Definition (Sequence)

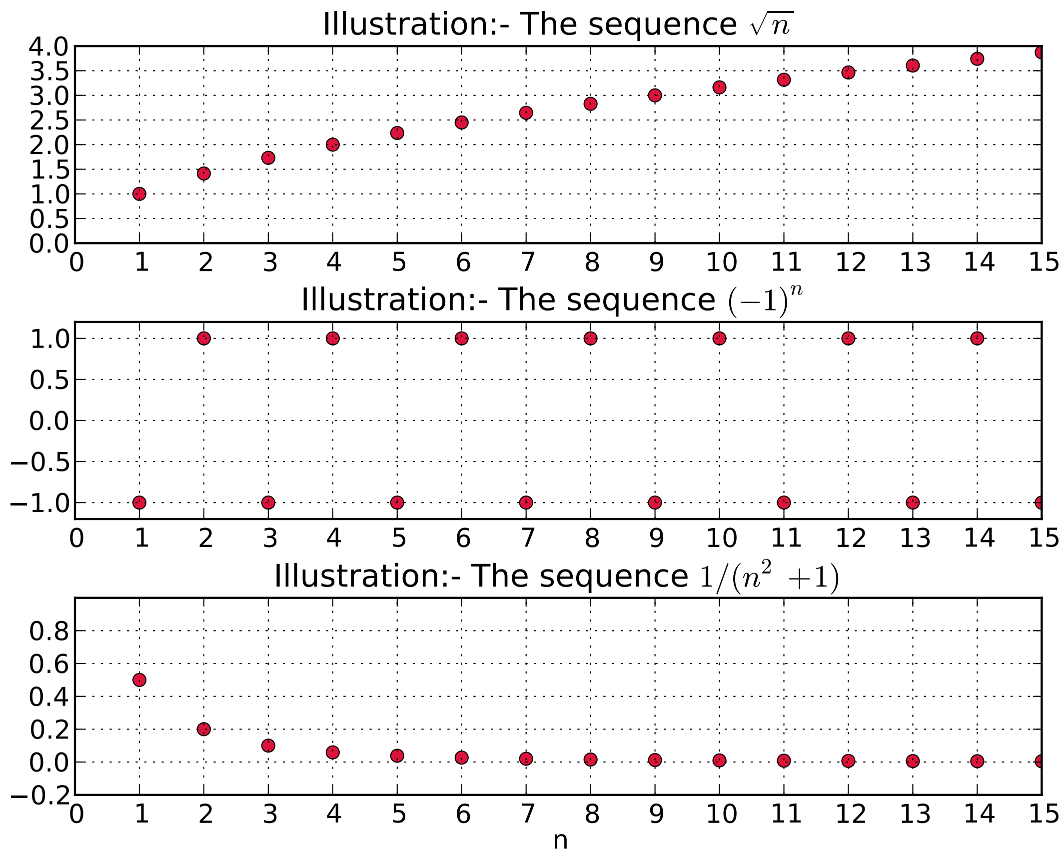
A **sequence** (or **infinite sequence**) is a real-valued function f defined on the set of positive integers, i.e., $f : \mathbb{N} \to \mathbb{R}$. If $f(n) = a_n$, for $n \in \mathbb{N}$, it is convention to write the sequence as $\{a_n\}_{n=1}^{\infty}$ or shorter as $\{a_n\}$ or alternatively as $a_1, a_2, a_3, \ldots$. (The elements $a_1, a_2, a_3, \ldots$ are also called the terms of the sequence.)

Note that the terms $a_1, a_2, a_3, \ldots$ of a sequence need not be distinct.

Example

$$\begin{aligned}\{\sqrt{n}\} &= \{1, 1.4142, 1.7321, \ldots\} \\ \{(-1)^n\} &= \{-1, 1, -1, \ldots\} \\ \left\{\frac{1}{n^2+1}\right\} &= \left\{\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \ldots\right\}\end{aligned}$$

Figure



Sequences can be finite too.

Definition (Finite sequence)

Let $J = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. A **finite sequence** is a real-valued function f defined on J, i.e., $f : J \to \mathbb{R}$.

Definition (Arithmetic sequence)

An **arithmetic sequence** $\{a_n\}$ is a sequence such that $a_{n+1} - a_n = d$ for all $n \in \mathbb{N}$ and some $d \in \mathbb{R}$. The constant d is called the common difference.

Proposition

n-th term of an arithmetic sequence

$a_n = a_1 + (n-1) \cdot d$.

Proof

Omitted.

Definition (Geometric sequence)

A **geometric sequence** $\{a_n\}$ is a sequence such that $\frac{a_{n+1}}{a_n} = i$ for all $n \in \mathbb{N}$ and some $i \in \mathbb{R} \setminus \{0\}$. The constant i is called the common ratio.

Proposition

n-th term of a geometric sequence

$a_n = a_1 \cdot i^{n-1}$.

Proof

Omitted.

It should be clear how the definitions of arithmetic and geometric sequences should be modified to yield *finite* arithmetic and geometric sequences.

3.2.2 Convergence of Sequences

Definition (Convergence, limit)

A sequence $\{a_n\}$ converges to the **limit** a if for every real number $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that $|a_n - a| \leq \epsilon$ for all $n \geq N$.

We say that the sequence $\{a_n\}$ is **convergent** and write $\lim_{n\to\infty} a_n = a$. If a sequence does not converge it is **divergent**.

Example

Let $\{a_n\}$ be such that $a_n = 6$ for every n. This sequence converges to $a = 6$. It is relatively easy to see that for any real number $\epsilon > 0$ there exists a natural number N such that $|a_n - 6| \leq \epsilon$ for all $n \geq N$. To see this, just set $N = 1$ for which $|a_n - 6| = 0 \leq \epsilon$ for all $n \geq N$.

Example

The sequence $\{-3, 7, 8, 5, 6, \pi, 3, 3, 3, \ldots\}$ converges to 3. To see this, write out as follows:

For $a = 3$

$$
\begin{aligned}
|a_1 - a| &= |-3-3| = 6 \\
|a_2 - a| &= |7-3| = 4 \\
|a_3 - a| &= |8-3| = 5 \\
|a_4 - a| &= |5-3| = 2 \\
|a_5 - a| &= |6-3| = 3 \\
|a_6 - a| &= |\pi-3| = \pi - 3 \\
|a_7 - a| &= |3-3| = 0 \leq \epsilon \\
|a_8 - a| &= |3-3| = 0 \leq \epsilon \\
|a_9 - a| &= |3-3| = 0 \leq \epsilon \\
&\vdots
\end{aligned}
$$

which holds for any $\epsilon > 0$ when $N = 7$ and $n \geq N$.

Lemma (Archimedian principle)

Let r be a real number. Then there exists a natural number n greater than r. Formally,

$$\forall r \in \mathbb{R} : \exists n \in \mathbb{N} : n > r$$

Proof

Suppose the lemma is false and $n < r$ for all natural numbers n. Then the set of natural numbers $\mathbb{N}$ is bounded above by r and, by the Completeness Axiom (not stated here), it has a least upper bound $M \in \mathbb{R}$. Since M is the least upper bound, $M - 1$ cannot be an upper bound. Thus, there is a natural number $x \in \mathbb{N}$ such that $M - 1 < x$. Adding 1 gives $M < x + 1$ but $x + 1$ must also be a natural number which contradicts the fact that M is an upper bound of $\mathbb{N}$.

Example

Let $a_n = \left\{ \frac{2n+1}{n+1} \right\}$. This sequence converges to 2. How do you prove this?

We need to show that there exists a natural number N such that $|a_n - 2| \leq \epsilon$ for all positive real numbers ϵ and $n \geq N$. Here, we can simplify and obtain $|a_n - 2| = \left| \frac{2n+1}{n+1} - \frac{2n+2}{n+1} \right| = \frac{1}{n+1} < \frac{1}{n}$. Now pick $N \in \mathbb{N}$ such that $1/N < \epsilon$. For any $n \geq N$ it also holds that $1/n < \epsilon$ and thus $|a_n - 2| < \epsilon$. Therefore the sequence $\{a_n\}$ converges to 2.

Note: How do we know that there always exists an $N \in \mathbb{N}$ such that $1/N < \epsilon$? This follows from the Archimedean Principle.

Example

Does the sequence $\{n/2 + 1/n\}$ converge? Let's assume that it indeed does converge to the real number a. In that case we should be able show that the difference $\left| \frac{n^2+2}{2n} - a \right|$ can be made arbitrarily small by choosing

n large enough. Does this work here? Rewrite

$$\begin{aligned}\left|\frac{n^2+2}{2n}-a\right| &= \left|\frac{n^2+2}{2n}-\frac{2an}{2n}\right| = \left|\frac{n^2+2-2an}{2n}\right| \\ &> \left|\frac{n^2-2an}{2n}\right| = \left|\frac{n-2a}{2}\right|,\end{aligned}$$

where the last expression will be equal to or greater than zero for $n \geq N$ if $N = 2a$ (assuming, without loss of generality, that $2a$ is a natural number). It thus follows that $\left|\frac{n^2+2}{2n}-a\right|$ will be strictly greater than zero and can therefore not be made arbitrarily small by choosing large values for n. In fact, for large values of n the difference $\left|\frac{n^2+2}{2n}-a\right|$ becomes quite large.

Proposition (Limit laws for sequences)

Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences, and let a and b be their limits, i.e., $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Then,

1. $\lim_{n\to\infty} c = c$, for any real constant c
2. $\lim_{n\to\infty}(a_n \pm b_n) = a \pm b$
3. $\lim_{n\to\infty}(c \cdot a_n) = c \cdot a$ for any real constant c
4. $\lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b$
5. $\lim_{n\to\infty}(b_n^{-1}) = b^{-1}$, assuming that $b \neq 0$ and $b_n \neq 0$ for all n
6. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$, assuming that $b \neq 0$ and $b_n \neq 0$ for all n
7. $\lim_{n\to\infty} a_n^p = a^p$, for a real constant $p > 0$ and assuming that $a_n > 0$ for all n

---

Proof

Omitted.

◊

3.2.3 Finite Series

Definition (Summation operator)

Given a sequence $\{a_n\}$, we use the **summation operator**

$$\sum_{n=p}^{q} a_n \qquad (p, q \in \mathbb{Z} \text{ with } p \leq q)$$

to denote the sum $a_p + a_{p+1} + \cdots + a_q$.

Definition (Finite series)

Given the sequence $\{a_n\}$, we define the **finite series** S_n by $S_n = \sum_{k=1}^{n} a_k$.

The finite series S_n takes the first n terms of the sequence $\{a_n\}$ and adds them together. Note that we can construct a sequence $\{S_n\}$ out of the series S_n. The value of the finite series S_n depends on the choice of n.

It should be obvious how one would define the concepts of finite arithmetic and finite geometric series.

Definition (Finite arithmetic series)

A **finite arithmetic series** S_n is defined by the first term a_1 of an arithmetic sequence, the common difference d, and the number of elements n in the sequence: $S_n = \sum_{k=1}^{n} a_1 + (k-1) \cdot d$.

It is easy to verify that $S_n = a_1 + \cdots + a_n$, where $a_1, \ldots, a_n$ are the terms of the underlying arithmetic sequence. Some textbooks let the sum index start at zero, $S_n = \sum_{k=0}^{n-1} a_1 + k \cdot d$. This, of course, is equivalent notation.

Proposition

The n terms of a finite arithmetic series S_n add up to

$$S_n = n(2a_1 + (n-1)d)/2. \tag{3.2}$$

Proof

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ &= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-2)d) + (a_1 + (n-1)d), \end{aligned}$$

and then rewriting in reverse order

$$S_n = (a_1 + (n-1)d) + (a_1 + (n-2)d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1.$$

Thus for $2S_n$ we get

$$\begin{aligned} 2S_n &= (2a_1 + (n-1)d) + (2a_1 + (n-1)d) + \cdots + (2a_1 + (n-1)d) + (2a_1 + (n-1)d) \\ &= n(2a_1 + (n-1)d), \end{aligned}$$

which yields the result.

Corollary

Alternatively, the n terms of a finite arithmetic series S_n add up to

$$S_n = n \cdot \frac{a_1 + a_n}{2}.$$

Proof

Replace $a_1 + (n-1)d$ in equation (**??**) by a_n. (Why is this justified?)

Definition (Finite geometric series)

A **finite geometric series** S_n is defined by the first term a_1 of a geometric sequence, the common ratio i, and the number of elements n in the sequence: $S_n = \sum_{k=1}^{n} a_1 \cdot i^{k-1}$.

Proposition

The n terms of a finite geometric series S_n add up to

$$S_n = \frac{a_1(i^n - 1)}{i - 1}, \qquad \text{for } i \neq 1. \tag{3.3}$$

Proof

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ &= a_1 + a_1 i + a_1 i^2 + \cdots + a_1 i^{n-2} + a_1 i^{n-1}, \end{aligned}$$

and then multiplying both sides by i

$$iS_n = a_1 i + a_1 i^2 + a_1 i^3 + \cdots + a_1 i^{n-1} + a_1 i^n.$$

Subtracting gives

$$iS_n - S_n = a_1 i^n - a_1,$$

which yields the result.

Corollary

Alternatively, the n terms of a finite geometric series S_n add up to

$$S_n = \frac{ia_n - a_1}{i - 1}, \qquad \text{for } i \neq 1.$$

Proof

Since $a_n = a_1 i^{n-1}$, or equivalently $a_1 i^n = ia_n$, plugging into equation (**??**) delivers the result.

3.2.4 Infinite Series

So far we defined finite series. But how would we handle objects such as $\sum_{k=1}^{\infty} a_k$? It seems reasonable to think of this as an infite series. But how would we determine its 'value'? For example, what is $\sum_{k=1}^{\infty} 1 = 1 + 1 + 1 + \cdots$ (a never-ending sum of ones) equal to?

To address this problem, we need to define the concept of convergence for infinite series. Before we do that, let's properly define the term.

Definition (Series)

An **infinite series** (or just a series) is any expression of the form

$$\sum_{n=1}^{\infty} a_n,$$

where a_n is a real number. We sometimes write this series as $a_1 + a_2 + a_3 + \cdots$.

Out of convention, we changed the index variable to n. This is just a label, it does not matter whether we use k, n, or any other symbol. Also, for convenience, we sometimes label an infinite series $\sum_{n=0}^{\infty} a_n$ instead (this will become clear later in context).

Definition (Convergence of series)

The infinite series $\sum_{n=1}^{\infty} a_n$ is said to **converge** to the real number S if the sequence $\{S_n\}$ of partial sums $S_n = \sum_{k=1}^{n} a_k$ converges to S. We write

$$\sum_{n=1}^{\infty} a_n = S.$$

If the sequence of partial sums $\{S_n\}$ diverges then we say that the infinite series **diverges** (or is divergent).

The definition of convergence of an infinite series is thus based on the definition of convergence of the underlying sequence. In that sense we are not defining a new concept of convergence. Convergence of a series is understood as convergence of a sequence.

This suggests that the convergence behavior of a sequence may influence whether or not the related series converges. There exists a handy result that links the two.

Proposition (Zero test)

Let $\{a_n\}$ be a sequence. If $\lim_{n\to\infty} a_n$ is non-zero, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof

Omitted.

Example

The sequence $\{1, 1, 1, \ldots\}$ clearly does not converge to zero. By the zero test, we know that $\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \cdots$ is a divergent infinite series. Note though that the underlying sequence $\{1, 1, 1, \ldots\}$ itself is actually convergent (to 1, obviously).

On the other hand, if a sequence $\{a_n\}$ does converge to zero, then the series $\sum_{n=1}^{\infty} a_n$ may or may not converge. It depends. Take, for example, the sequence $\{1/n\}$. The series $\sum_{n=1}^{\infty} 1/n$ diverges even though $1/n$ converges to zero (see exercises below).

Now we state and proof a result that is used in finance, accounting and financial econometrics all the time.

Proposition

The series $\sum_{n=1}^{\infty} i^{n-1}$ converges to $1/(1-i)$ if $-1 < i < 1$.

Proof

The finite sums $S_n = \sum_{k=1}^{n} i^{k-1}$ are straightforward geometric series. By Proposition ref{prop: geometric series} we have $S_n = (1 - i^n)/(1 - i)$. To get the limit of the series, we need to get the limit of the sequence

$\{S_n\}$. This requires us to show that there exists a natural number N such that

$$\left|\frac{(1-i^n)}{1-i} - \frac{1}{1-i}\right| = \left|\frac{-i^n}{1-i}\right| < \epsilon$$

for any real number $\epsilon > 0$ and $n \geq N$. We will not prove this rigorously here but appeal to the intuition that indeed $|-i^n|$ is positive and can be made arbitrarily small by picking n large enough. (Note that the denominator is positive and finite.)

♢

3.2.5 Exercises

Note: Solutions for exercises will only be given during the tutorials. They will not be posted here.

1. You deposit \$2,000 in a bank account which pays 6% interest each year. You do not withdraw money and keep it in the bank for 12 years. What is the amount of money A saved over that time span? Provide a formula for A in terms of the principal P, the interest rate r and the time span T. (Apply the n-th term formula for geometric sequences. How do A, P, r and T relate to the elements of the n-th term formula?)
2. Redo the previous exercise assuming that the bank pays a *nominal* annual interest rate of 6% compounded half-yearly. What is the amount of money A saved over that time span? Provide a formula for A in terms of the principal P, the interest rate r and the time span T as well as the number of compounding periods per year m. (How do A, P, r, m and T relate to the elements of the n-th term formula?)
3. Redo the previous exercise with monthly and daily (assume that the year has 365 days) and continuous compounding. For all five cases of compounding, what is the *effective annual rate* (also called the annual percentage yield)? (See textbook for definition.)
4. Given your initial deposit of \$2,000 and the 6% nominal interest rate, how long will it take you to save up \$5,000? (Calculate this for monthly as well as continuous compounding.)
5. What is the limit of the sequence $\left\{\frac{3n+2}{n+1}\right\}$? Prove. (*Note: Depending on time constraints, part or all of this question will also be answered in next week's tutorial.*)

You can find related exercises in the textbook under:

- Exercises 3-1 and 3-2.

Make sure to use these to practice with! The tutors at the EMET1001 help desk are happy to help, if you have any questions.

6. On her 25th birthday, Maria has nothing better to do than worry about her financial situation when she retires. She wants to save up enough money for a comfortable retirement. On the website of the Association of Superannuation Funds of Australia (ASFA) Maria discovers that she needs \$3,350 a month for a comfortable retirement lifestyle in NSW (as a single female retiree). She would like to retire on her 65th birthday.

   The current life expectancy at age 65 for females is about 22 years. This means, that a 65 year old woman is currently expected to live until age 87. Maria is conservative and factors in that technological progress in medical treatment will raise her life expectancy (once she reaches 65) to 30 years. She therefore wants to make sure that she can guarantee a secure monthly income stream of \$3,350 for 30 years starting at her 65th birthday. (Note: For simplicity, we ignore inflation.)

   Maria's bank promises a 5.5% nominal annual interest rate compounded monthly forever.

(a) On her 65th birthday, how much money needs to be in Maria's bank account to guarantee a comfortable retirement? (She does not plan to leave an inheritance.)

(b) How much money would Maria need to have now (at her 25th birthday) to guarantee that she can retire at age 65 with the amount of savings determined under part (i)? (This question asks for the *present value*.)

(c) Maria currently has no savings but luckily she has a steady job. In order to come up with the necessary retirement savings, she decides to put aside a constant dollar amount (an *annuity*) at the beginning of every month until she retires. How much money should Maria set aside each month?

7. Prove that the series $\sum_{n=1}^{\infty} 1/n$ diverges. Recall from the lecture that the underlying sequence $\{1/n\}$ does converge to zero. To answer this question, you need to study the convergence behavior of the sequence of *partial sums* $\{S_n\}$ where each partial sum is defined by $S_n = \sum_{k=1}^{n} 1/k$.

You can find related exercises in the textbook under:

- Exercises 3-3 and 3-4.

Make sure to use these to practice with! The tutors at the EMET1001 help desk are happy to help, if you have any questions.

3.3 Limits of Functions

3.3.1 Two-sided limits

Definition

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function. We say that $f(x)$ approaches the limit L as x approaches x_0 and write

$$\lim_{x \to x_0} f(x) = L$$

if f is defined on some neighborhood of x_0 and, for every $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all x with $0 < |x - x_0| < \delta$.

We say that L is the **limit of the function at** x_0.

Example

Let $f(x) = c$ for some $c \in \mathbb{R}$ and for all $x \in \mathbb{R}$. We want to show that $\lim_{x \to x_0} f(x) = c$. For all $\epsilon > 0$ just let $\delta = 1$ (in fact any positive δ will do). Then whenever x is such that $0 < |x - x_0| < 1$ we get that $|f(x) - c| = |c - c| = 0 < \epsilon$. This holds for arbitrary ϵ and thus it shows that the limit is indeed equal to c.

Example

Let $f(x) = x + 1$. Then $\lim_{x \to 1} f(x) = 2$. While it seems obvious, how do you prove this? We need $|f(x) - 2| = |x - 1| < \epsilon$ in a neighborhood $0 < |x - 1| < \delta$ for some $\delta > 0$ of our choice. Well, just set $\delta = \epsilon$ and the result follows.

This example shows that sometimes, in order to find the limit of a function $f(x)$ at x_0 you can simply just evaluate the function at the limit, i.e. $f(x_0)$.

Question

Let $f(x) = c \cdot x$ which is just a linear function through the origin (for some real number c). Then $\lim_{x \to x_0} f(x) = c \cdot x_0$. How do you prove that?

The function does not need to be defined at x_0. The limit looks at the function value in a neighborhood of x_0, what happens at x_0 is irrelevant. Often times determining the limit of a function at x_0 is akin to just plugging in x_0 and getting the function value $f(x_0)$. But this appraoch may not always work. The next example illustrates this.

Example

Let $g : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ be defined as $g(x) = \frac{x^2-1}{x-1}$. Then $\lim_{x \to 1} g(x) = 2$. To see this note first that we cannot simply guess that $g(1)$ is the correct limit. The function is not defined at 1. Instead we have to make sure that $|g(x) - 2| < \epsilon$ for some x in the neighborhood of $x_0 = 1$. Writing out gives

$$|g(x) - 2| = \left| \frac{x^2 - 1}{x - 1} - 2 \right| = \left| \frac{(x+1)(x-1)}{(x-1)} - 2 \right| = |(x+1) - 2| = |f(x) - 2|,$$

where $f(x) = x + 1$. What this shows is that the functions $g(x) = \frac{x^2-1}{x-1}$ and $f(x) = x + 1$ are identical for the purpose of determining the limit at $x_0 = 1$. While this is not trivial for $g(x)$, we have already shown in one of the previous examples that the limit of $f(x)$ is 2 at $x_0 = 1$.

It is important to understand that the functions $f(x)$ and $g(x)$ are not the same. The domain of f is $\mathbb{R}$ while the domain of g is $\mathbb{R} \setminus \{1\}$. However, what this last example shows is that when we restrict the function f to the set $\mathbb{R} \setminus \{1\}$ then we can treat f as being identical to the function g and exploit this to find the limit for g at x_0.

Theorem (Limit laws for functions)

If $\lim_{x \to x_0} f(x) = L_1$ and $\lim_{x \to x_0} g(x) = L_2$ for $L_1, L_2 \in \mathbb{R}$ then

1. Sum and difference rule

   $\lim_{x \to x_0} \big(f(x) \pm g(x)\big) = L_1 \pm L_2$

2. Multiplicative constant rule

   $\lim_{x \to x_0} \big(c \cdot f(x)\big) = c \cdot L_1$ for some $c \in \mathbb{R}$

3. Product rule

   $\lim_{x \to x_0} \big(f(x) \cdot g(x)\big) = L_1 \cdot L_2$

4. Quotient rule

   $\lim_{x \to x_0} \left(\frac{f(x)}{g(x)}\right) = \frac{L_1}{L_2}$, if $g(x) \neq 0$ and $L_2 \neq 0$

5. Rule for polynomial functions

   If $f(x)$ is a polynomial function, i.e. $f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ for some $c_0, \ldots, c_n \in \mathbb{R}$, then $\lim_{x \to x_0} f(x) = f(x_0)$

6. Rule for rational functions

   If $f(x)$ and $g(x)$ are both polynomial functions then the rational function constructed as $r(x) = f(x)/g(x)$ has the limit $\lim_{x \to x_0} r(x) = r(x_0)$, provided that $g(x_0) \neq 0$

Proof

Omitted.

Example

Let $f(x) = 2x^2 + x - 3$ and $g(x) = x^3 + 4$ and $r(x) = f(x)/g(x)$. Then

$$\lim_{x \to 1} r(x) = \lim_{x \to 1} \frac{2x^2 + x - 3}{x^3 + 4} = \frac{\lim_{x \to 1}(2x^2 + x - 3)}{\lim_{x \to 1}(x^3 + 4)} = \frac{2 \cdot 1^2 + 1 - 3}{1^3 + 4} = 0$$

But since $r(x)$ is a rational function we could have know from the beginning that the limit coincides with the function value $r(1) = 0$.

3.3.2 One-sided limits

For some functions, the above definition of a limit does not seem to apply. Consider the function

$$f(x) = \begin{cases} x + 1 & \text{if } x > 0 \\ x - 1 & \text{if } x < 0. \end{cases}$$

What is the limit of this function at zero? First, the function is not defined at zero. This need not be a problem as we discussed earlier. The problem here is that the function behaves in 'different ways' to the left and to the right of zero. (Can you sketch this function?)

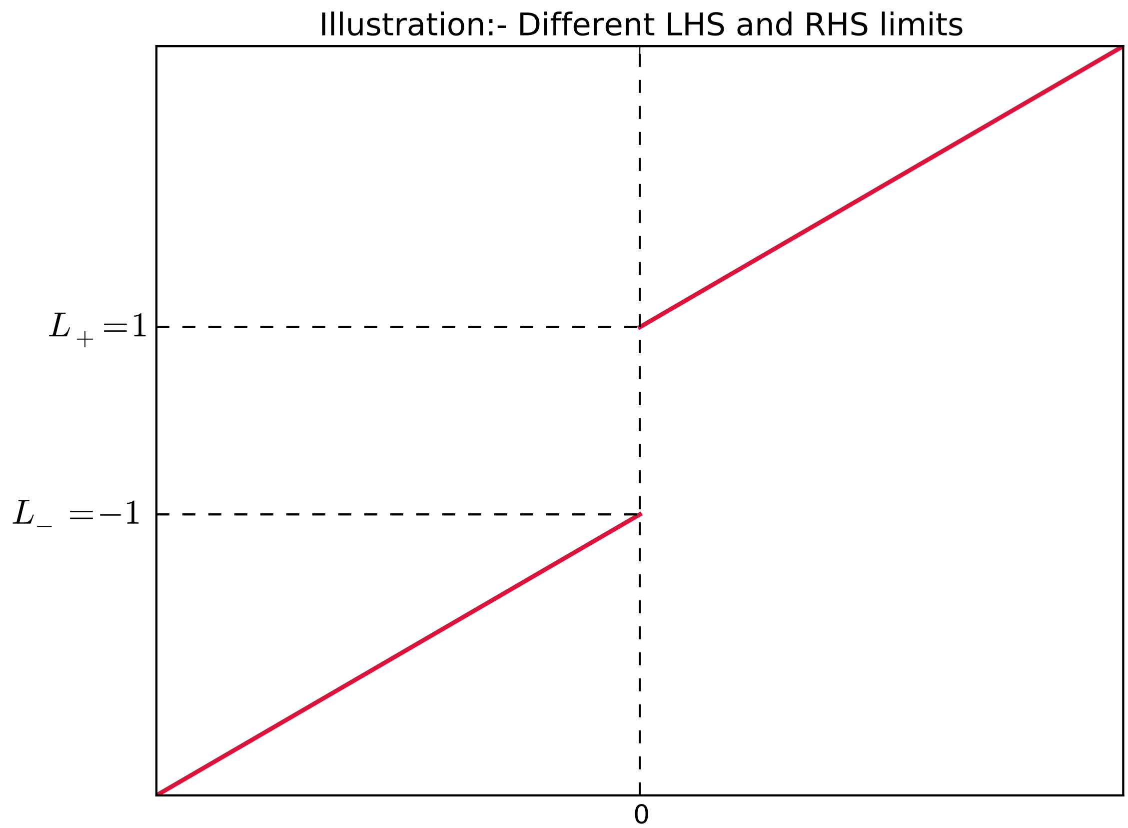
The following versions of limit may be more applicable in such cases.

Definition (One-sided limit)

We say that $f(x)$ approaches the left-hand limit L_- as x approaches x_0 from the left and write $\lim_{x \to x_0 -} f(x) = L_-$ if f is defined on some open interval (a, x_0) and, for each $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L_-| < \epsilon$ if $x_0 - \delta < x < x_0$.

We say that $f(x)$ approaches the right-hand limit L_+ as x approaches x_0 from the right and write $\lim_{x \to x_0 +} f(x) = L_+$ if f is defined on some open interval (x_0, b) and, for each $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L_+| < \epsilon$ if $x_0 < x < x_0 + \delta$.

Figure



Proposition

The limit laws for functions also apply to one-sided limits.

Proof

Omitted.

Example

Let $f(x) = \frac{x}{|x|}$ for $x \in \mathbb{R} \setminus \{0\}$. Then $\lim_{x \to 0+} f(x) = 1$ and $\lim_{x \to 0-} f(x) = -1$.

But does this function have a limit at zero? The next theorem clarifies.

Theorem

A function f has a limit at x_0 iff it has a left- and a right-hand limit at x_0 and they are equal. Thus

$$\lim_{x \to x_0} f(x) = L \qquad \text{iff} \lim_{x \to x_0-} f(x) = \lim_{x \to x_0+} f(x) = L.$$

It follows that the function from the previous example does not have a limit at zero because their one-sided limits do not coincide (although they both exist).

3.3.3 Limits at infinity

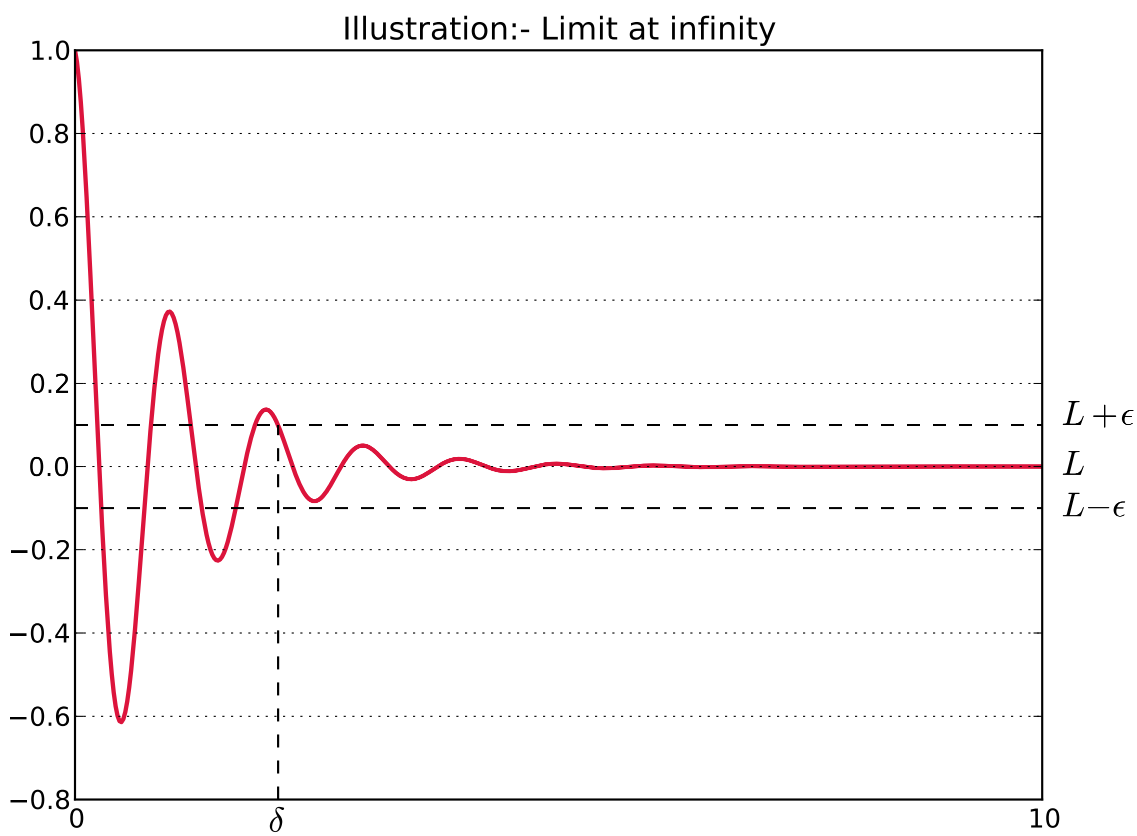
How do functions behave when x approaches a very, very large (small) number?

Definition (Limits at infinity)

We say that $f(x)$ approaches the limit L as x approaches ∞ and write $\lim_{x\to\infty} f(x) = L$ if f is defined on an interval (a, ∞) and, for each $\epsilon > 0$, there is a number δ such that $|f(x) - L| < \epsilon$ if $x > \delta$.

We say that $f(x)$ approaches the limit L as x approaches $-\infty$ and write $\lim_{x\to-\infty} f(x) = L$ if f is defined on an interval $(-\infty, b)$ and, for each $\epsilon > 0$, there is a number δ such that $|f(x) - L| < \epsilon$ if $x < \delta$.

Figure



Example

Let $f(x) = 1 - 1/x^2$. Then $\lim_{x\to\infty} f(x) = 1$ because $|f(x) - 1| = 1/x^2 < \epsilon$ if $x > 1/\sqrt{\epsilon} = \delta$.

Example

Let $f(x) = \frac{2\cdot|x|}{1+x}$. Then $\lim_{x\to\infty} f(x) = 2$ because $|f(x)-2| = \left|\frac{2x}{1+x} - 2\right| = \frac{2}{1+x} < \frac{2}{x} < \epsilon$ if $x > 2/\epsilon = \delta$.

3.3.4 Infinite limits

When do limits not exist?

Definition (Infinite limits)

We say that $f(x)$ approaches ∞ as x approaches x_0 from the left and write

$$\lim_{x\to x_0-} f(x) = \infty,$$

if f is defined on some open interval (a, x_0) and, for each real number $M \in \mathbb{R}_+$, there is a $\delta > 0$ such that

$$f(x) > M \qquad \text{if } x_0 - \delta < x < x_0.$$

We say that $f(x)$ approaches $-\infty$ as x approaches x_0 from the left and write

$$\lim_{x\to x_0-} f(x) = -\infty,$$

if f is defined on some open interval (a, x_0) and, for each real number $M \in \mathbb{R}_-$, there is a $\delta > 0$ such that

$$f(x) < M \qquad \text{if } x_0 - \delta < x < x_0.$$

Likewise for right-sided limits:

We say that $f(x)$ approaches ∞ as x approaches x_0 from the right and write

$$\lim_{x\to x_0+} f(x) = \infty,$$

if f is defined on some open interval (x_0, b) and, for each real number $M \in \mathbb{R}_+$, there is a $\delta > 0$ such that

$$f(x) > M \qquad \text{if } x_0 < x < x_0 + \delta.$$

We say that $f(x)$ approaches $-\infty$ as x approaches x_0 from the right and write

$$\lim_{x\to x_0+} f(x) = -\infty,$$

if f is defined on some open interval (x_0, b) and, for each real number $M \in \mathbb{R}_-$, there is a $\delta > 0$ such that

$$f(x) < M \qquad \text{if } x_0 < x < x_0 + \delta.$$

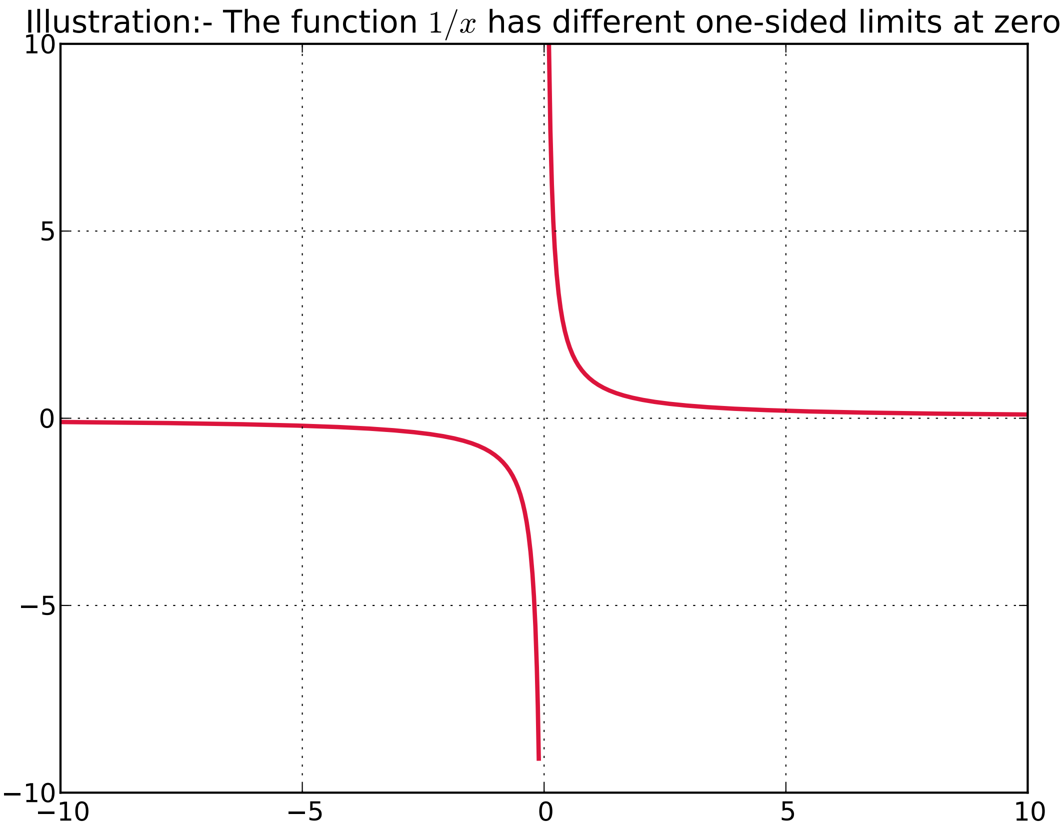
Example

Let $f(x) = 1/x$. Then $\lim_{x\to 0+} f(x) = \infty$ and $\lim_{x\to 0-} f(x) = -\infty$. Even though this may seem obvious, how do you prove this?

First, for the right-hand limit choose any real number $M \in \mathbb{R}_+$ and set $\delta = 1/M$. Then $0 < x < \delta = 1/M$ for which $1/x > 1/\delta$ and thus indeed $f(x) > M$. For instance, let $M = 1000$; then x has to stay in a $\frac{1}{1000}$-th neighborhood to the right of $x_0 = 0$ in order to make sure that $f(x)$ is larger than $M = 1000$.

Second, for the left-hand limit choose any real number $M \in \mathbb{R}_-$ and set $\delta = -1/M$. Then $1/M = -\delta < x < 0$ for which $-1/\delta > 1/x$ (careful here!) and thus indeed $1/x = f(x) < M$.

Figure



3.3.5 Continuity

We now learn about one fundamental concept in the theory of functions.

Definition (Continuity)

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function. Let $x_0 \in X$. We say that f is continuous at x_0 iff we have

$$\lim_{x \to x_0} f(x) = f(x_0).$$

We say that f is continuous on X (or simply continuous) iff f is continuous at x_0 for every $x_0 \in X$. We say that f is discontinuous at x_0 iff it is not continuous at x_0.

Therefore, for a function to be continuous at a point x_0 in the domain we need three things: the limit as x approaches x_0 needs to exist, the function value at x_0 needs to exist, and both the limit and the function value need to be identical.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = c$ for some real number c. Then for any $x_0 \in \mathbb{R}$ we have $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} c = c = f(x_0)$ and thus f is continuous at every point $x_0 \in \mathbb{R}$. This in fact makes f continuous on the whole real line $\mathbb{R}$.

Question

Let $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

For which values of x is this function continuous?

♢

3.3.6 Exercises

Note: Solutions for exercises will only be given during the tutorials. They will not be posted here.

1. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the function $f(x) = 1/x$. Prove that

$$\lim_{x\to\infty} f(x) = 0.$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = e^{-x}$. Prove that

$$\lim_{x\to\infty} f(x) = 0.$$

3. Use the limit laws and what you learned about limits at infinity and infinite limits to determine

$$\lim_{x\to\infty} \frac{e^{-x}}{x}.$$

4. Use the limit laws and what you learned about limits at infinity and infinite limits to determine

$$\lim_{x\to\infty} \frac{2x^2 - x + 1}{3x^2 + 2x - 1}.$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = |x|$. Is this function continuous? Show your work!
6. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with $f(x) = 1/x$. For which values of x is this function continuous? Show your work!

Related exercises in the textbook you should study, include (but are not limited to):

- Exercises 10-1 — Problems 21-38, 43-74
- Exercises 10-2 — Problems 9-52
- Exercises 10-3 — Problems 23-34, 57-62

The tutors at the EMET1001 help desk are happy to help, if you have any questions.

3.4 Differential Calculus

3.4.1 Difference quotient and derivatives

Definition (Difference quotient)

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function. For $x, x_0 \in X$, we call

$$\frac{f(x) - f(x_0)}{x - x_0}, \qquad \text{with } x \neq x_0,$$

the **difference quotient**.

Figure

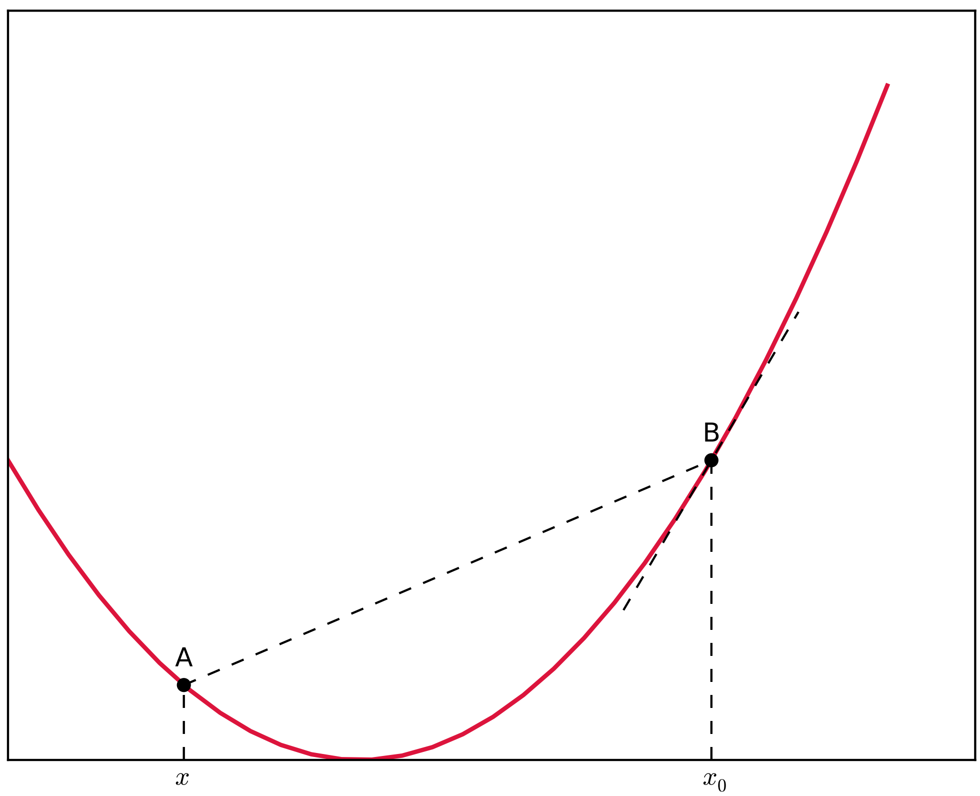


Illustration of the difference quotient. The difference quotient is the slope of the line segment AB. Alternatively, it is the average slope on the curve as one walks from A to B. If instead we are interested in the slope at a point, we need to study the difference quotient as x_0 and x get closer and closer to each other. For example, the slope of the curve at point B is represented by the limit of the difference quotient as x approaches x_0.

The next definition makes this formal.

Definition (Differentiability, derivative)

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function. For $x_0 \in X$, if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = L \in \mathbb{R}, \qquad \text{with } x \neq x_0,$$

then we say that f is **differentiable** at x_0 with **derivative** L and write

$$f'(x_0) = L.$$

If the limit does not exist then we say that the function is **not differentiable**.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$. Then

$$\begin{aligned}\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} \\ &= \lim_{x \to x_0} (x + x_0) = 2x_0.\end{aligned}$$

Note that cancelling the $(x - x_0)$ factors is legitimate because $x \neq x_0$ (otherwise the derivative would not be properly defined).

Some basic derivatives that should be familiar (assumed knowledge) are given by the next two propositions:

Proposition (Basic derivatives)

$$\begin{aligned}&\text{(Constant function)} \\ &f(x) = c \\ \Rightarrow\ &f'(x) = 0 \\ &\text{(Power rule)} \\ &f(x) = x^n \\ \Rightarrow\ &f'(x) = n \cdot x^{n-1}\end{aligned}$$

Proof

Omitted.

Proposition (Derivatives of exponential and logarithmic functions)

Let $f : \mathbb{R} \to \mathbb{R}_+$ with $f(x) = b^x$ for some $b \in \mathbb{R}_+ \setminus \{1\}$. Then f is called an exponential function. The derivative is $f'(x) = b^x \ln b$. For the special case $b = e$ we get $f'(x) = e^x \ln e = e^x$.

Let $f : \mathbb{R}_+ \to \mathbb{R}$ with $f(x) = \log_b x$ for some $b \in \mathbb{R}_+ \setminus \{1\}$. Then f is called a logarithmic function. The derivative is $f'(x) = \frac{1}{\ln b} \cdot \frac{1}{x}$. For the special case $b = e$ we get $f'(x) = \frac{1}{\ln e} \frac{1}{x} = \frac{1}{x}$.

Proof

Omitted.

The following standard derivative laws can be useful.

Theorem (Derivative laws)

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ both be functions that are differentiable at x_0. Let c be some real constant. Then

1. Sum and difference rule $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$
2. Multiplicative constant rule $(c \cdot f)'(x_0) = c \cdot f'(x_0)$
3. Product rule $(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$
4. Quotient rule $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)}{g(x_0)^2}$, for $g(x_0) \neq 0$
5. Chain rule $g\big(f(x_0)\big)' = g'(y_0) \cdot f'(x_0)$, where $y_0 = f(x_0)$, and g is differentiable at y_0

Proof

Omitted.

3.4.2 Determining local and global extrema

Derivatives are often used for curve sketching of functions. Loosely speaking, if the derivatives equals zero then the function attains an extremum point (a maximum or a minimum). But we need to be careful what exactly we mean by that.

Definition (Global extrema)

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function and let $x_0 \in X$. We say that f attains a *global maximum* at x_0 if $f(x_0) \geq f(x)$ for all $x \in X$. We say that f attains a *global minimum* at x_0 if $f(x_0) \leq f(x)$ for all $x \in X$.

Example

The constant function $f(x) = c$ with $c \in \mathbb{R}$ has infinitely many global maxima and minima.

Proposition EP (Extremum principle)

Let $a < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$. Then f attains its global maximum at some point $x_{max} \in [a, b]$ and it also attains its global minimum at some point $x_{min} \in [a, b]$.

Proof

Omitted.

Example

Let $f : [-5, -2] \to \mathbb{R}$ be the function $f(x) = |x|$. We have learnt previously that this function is continuous on $\mathbb{R}$. Thus, Proposition EP tells us that the function must attain its maximum and minimum on the closed interval $[-5, -2]$. The corresponding x-values are, of course, $x_{max} = -5$ and $x_{min} = -2$.

Definition (Local extrema)

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function and let $x_0 \in X$. Then $f(x_0)$ is a *local maximum* of f at x_0 if there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in X \cap (x_0 - \delta, x_0 + \delta)$. Analogously, $f(x_0)$ is a *local minimum* of f at x_0 if there exists $\delta > 0$ such that $f(x) \geq f(x_0)$ for all $x \in X \cap (x_0 - \delta, x_0 + \delta)$.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2 - x^4$. What kind of extrema does this function have at zero? It cannot be a global minimum because $f(2) = -12 < 0 = f(0)$. But it is local, because if we choose, for example, $\delta = 1$ and study f on the open interval $(-1, 1)$ then we get $x^4 \leq x^2$ and therefore $f(x) = x^2 - x^4 \geq 0 = f(0)$.

Proposition NC (Necessary condition for local extrema)

Let $a < b$ be real numbers and let $f : (a, b) \to \mathbb{R}$ be a function. If $x_0 \in (a, b)$, f is differentiable at x_0, and f attains either a local maximum or a local minimum at x_0 then $f'(x_0) = 0$.

Proof

Omitted.

Example

Take $f(x) = x^2 - x^4$ from the previous example. We learnt that this function has a local minimum at $x = 0$. Proposition NC requires that the derivative $f'(0)$ must be zero. This indeed is true, as we can readily see.

Example

Let $f : (-1, 1) \to \mathbb{R}$ be the function $f(x) = x^2$. Then f attains a local minimum at $x_0 = 0$ for which indeed we have $f'(0) = 0$.

Example

It is crucial that the domain inside Proposition NC be an open set. If the function from the previous example instead had been defined as $(-1, 1] \to \mathbb{R}$ with $f(x) = x^2$ then, by construction, it would have attained another local extremum, namely a local maximum at $x_0 = 1$ for which the first derivative is not equal to zero, i.e. $f'(1) \neq 0$.

$\diamondsuit$

Combining Proposition EP with Proposition NC results in

Proposition (Rolle's theorem)

Let $a < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Suppose also that $f(a) = f(b)$. Then there exists an $x \in (a, b)$ such that $f'(x) = 0$.

Proof

Omitted.

Figure

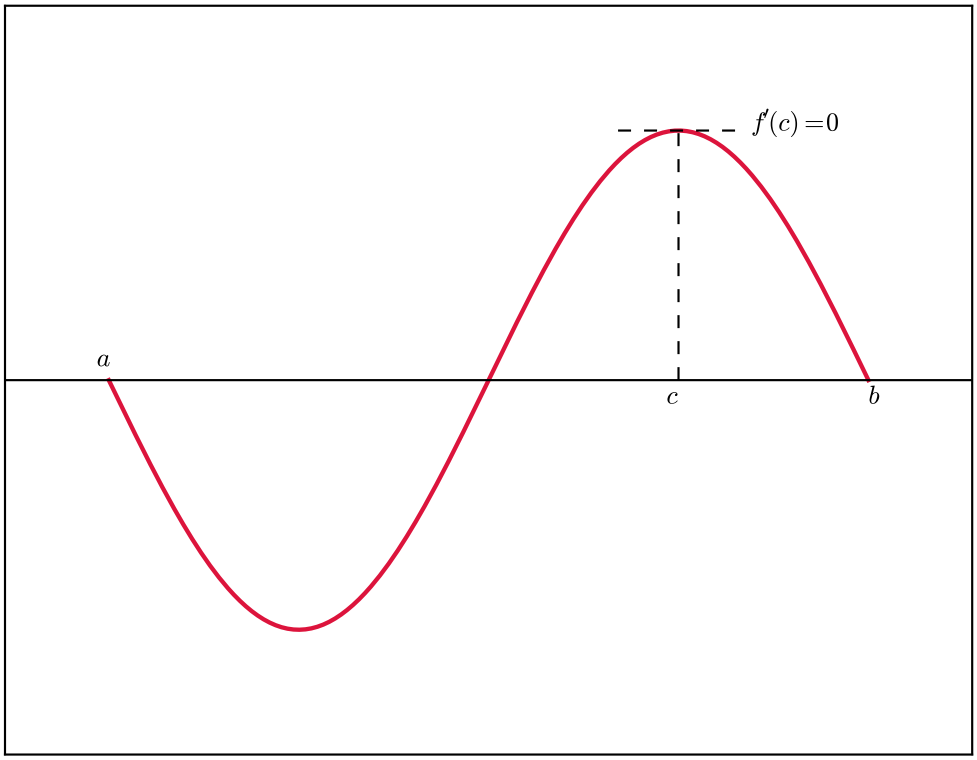


Illustration of Rolle's Theorem.

Rolle's Theorem has an important corollary.

Corollary (Mean Value Theorem)

Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Figure

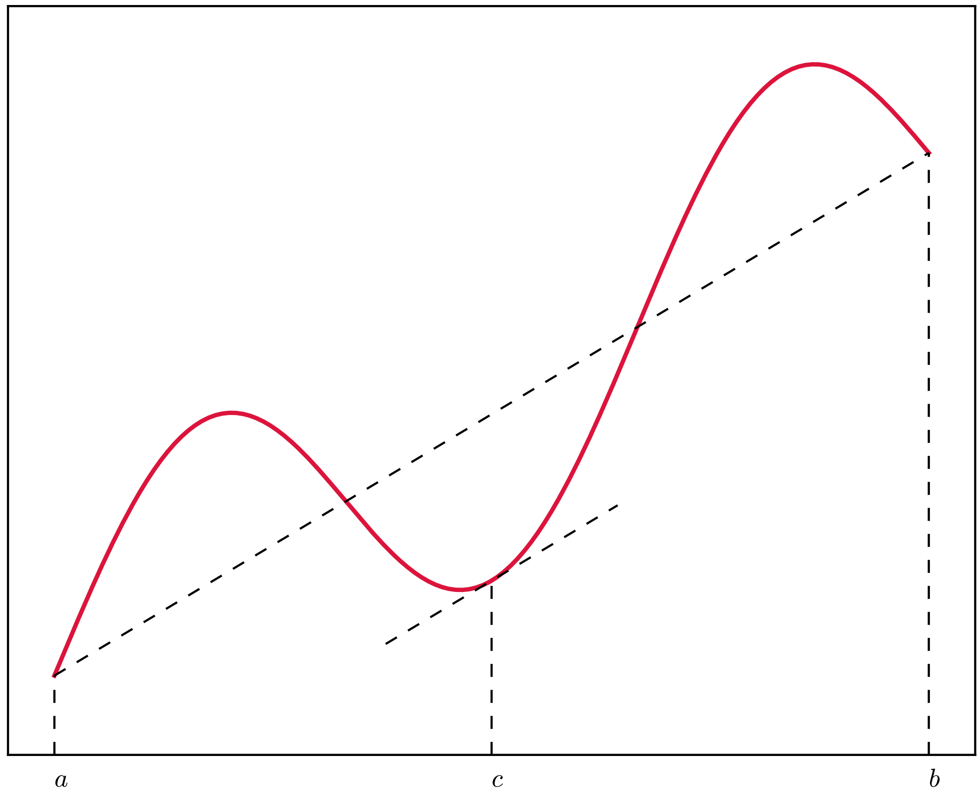


Illustration of the Mean Value Theorem. Intuitively, the mean value theorem states that there exists some point $(c, f(c))$ on the graph of f at which a tangent line is parallel to the line segment connecting the points $(a, f(a))$ and $(b, f(b))$.

The main use of the mean value theorem, however, is to prove the following results.

Proposition FDT (First derivative test for local extrema)

Let $f : [a, b] \to \mathbb{R}$ be a continuous and differentiable function. Let $x_0 \in (a, b)$. Then

1. If there is a neighborhood $(x_0 - \delta, x_0 + \delta)$ such that $f'(x) \geq 0$ for $x \in (x_0 - \delta, x_0)$ and $f'(x) \leq 0$ for $x \in (x_0, x_0 + \delta)$, then f attains a local maximum at x_0.
2. If there is a neighborhood $(x_0 - \delta, x_0 + \delta)$ such that $f'(x) \leq 0$ for $x \in (x_0 - \delta, x_0)$ and $f'(x) \geq 0$ for $x \in (x_0, x_0 + \delta)$, then f attains a local minimum at x_0.

Proof

We just prove (i) here, the proof for (ii) is similar. For $x \in (x_0 - \delta, x_0)$ it follows, from the mean value theorem, that there exists a point $c \in (x, x_0)$ such that $f(x_0) - f(x) = (x_0 - x) \cdot f'(c)$. But $f'(c) \geq 0$ and $(x_0 - x) > 0$ so that $f(x_0) \geq f(x)$ for $x \in (x_0 - \delta, x_0)$. Similarly, for $x \in (x_0, x_0 + \delta)$ it follows, from the mean value theorem, that there exists a point $c \in (x_0, x)$ such that $f(x) - f(x_0) = (x - x_0) \cdot f'(c)$. But $f'(c) \leq 0$ and $(x - x_0) > 0$ so that $f(x_0) \geq f(x)$ for $x \in (x_0, x_0 + \delta)$. In summary, $f(x_0) \geq f(x)$ for all $x \in (x_0 - \delta, x_0 + \delta)$ which is what is required for a local maximum.

The First derivative test for local extrema helps determine whether a point x_0 with $f'(x_0) = 0$ is a local minimum or a local maximum. Checking the first derivative to the left and to the right of a x_0 is not always easy to do. There exists a more practical way, based on the second derivative, of determining the nature of extrema.

Definition (Second derivative)

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function with derivative $f'(x_0)$ at $x_0 \in X$. Then we call the derivative of the function $f'(x)$ at the point x_0 the **second derivative of** f, denoted by $f''(x_0)$.

A similar definition holds for higher order derivatives.

We now use the second derivative to provide a more practical sufficient condition for extrema.

Proposition SC (Second derivative test for local extrema)

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function with second derivative $f''(x_0)$ at $x_0 \in X$. Then, if $f'(x_0) = 0$ and $f''(x_0) < 0$ the function attains a local maximum. If, instead, $f'(x_0) = 0$ and $f''(x_0) > 0$ the function attains a local minimum. Last, if both $f'(x_0) = f''(x_0) = 0$ then the function attains either a local maximum, a local minimum, or it has an inflection point at x_0.

Figure

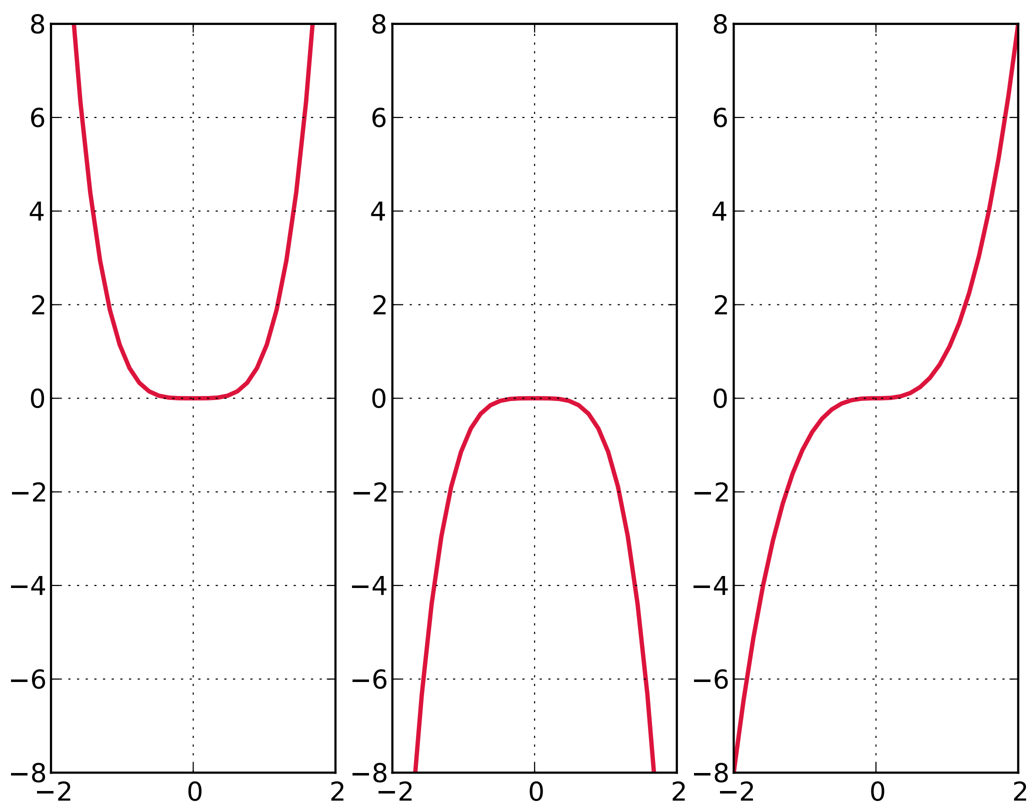


Illustration of the second derivative test. The functions $f_1(x) = x^4$, $f_2(x) = -x^4$, and $f_3(x) = x^3$ are all examples of functions that have a zero first and second derivative.

Proposition SC is much easier to use than Proposition FDT. While the latter requires knowledge of the behavior of $f'(x)$ at points to the left and right of x_0, the former only requires knowing the first two derivatives of the function f but only at the point x_0 itself.

3.4.3 Monotonicity, curvature and inflection points

We now move on to techniques that can be helpful in determining the shape and look of functions.

Definition (Increasing, decreasing, monotone functions)

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function. Pick an interval $J \subset X$.

The function f is said to be *increasing* on J if for all $x_1, x_2 \in J$ such that $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.

The function f is said to be *strictly increasing* on J if for all $x_1, x_2 \in J$ such that $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

The function f is said to be *decreasing* on J if for all $x_1, x_2 \in J$ such that $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.

The function f is said to be *strictly decreasing* on J if for all $x_1, x_2 \in J$ such that $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

If a function is either increasing or decreasing on X, we say that it is *monotone* on X. If it is either strictly increasing or either strictly decreasing on X, we say that it is *strictly monotone* on X.

It can be quite difficult to apply this definition for the purpose of checking whether a particular function is increasing or decreasing on a certain subset of the domain. The following proposition makes that job easier by linking the monotonicity of a function to the first derivative of the function.

Proposition DM

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function that is differentiable on X. Let $J \subset X$. Then

1. f is increasing on J iff $f'(x) \geq 0$ for all $x \in J$
2. f is decreasing on J iff $f'(x) \leq 0$ for all $x \in J$.
3. f is strictly increasing on J if $f'(x) > 0$ for all $x \in J$
4. f is strictly decreasing on J if $f'(x) < 0$ for all $x \in J$

Proof

Omitted.

This proposition is helpful because it allows you to use the derivative in order to learn about the monotonicity of a function.

It is therefore safe to use the weak monotonicity property (increasing/decreasing) interchangeably with the property of weakly positive/negative derivatives. You need to be careful though when dealing with strictly increasing/decreasing functions, note that the *if and only if* statement holds only for cases (i) and (ii) but not for cases (iii) and (iv). The following example illustrates.

Example

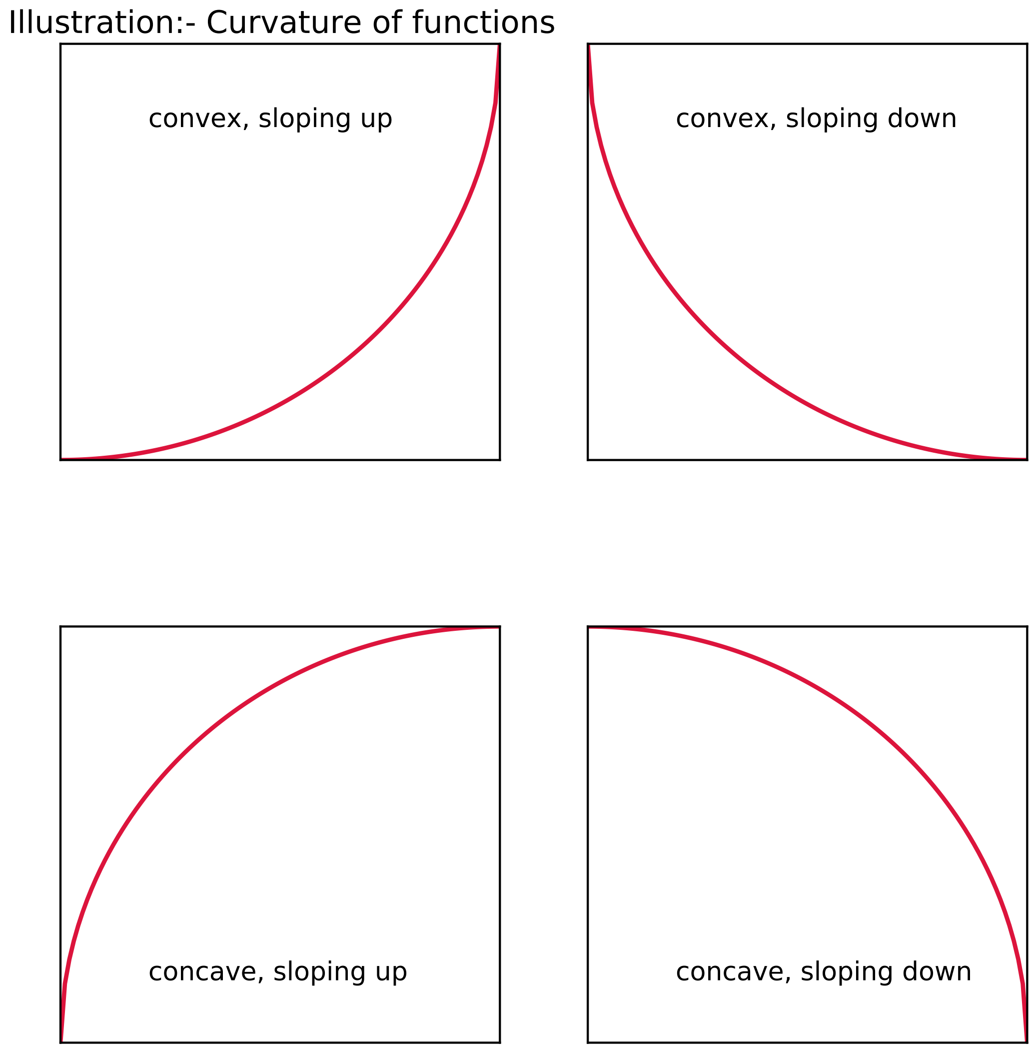
Let $f(x) = x^3$ (the previous figure shows the graph of this function). Then, clearly, $f'(x) = 3x^2$. Now, let J from the proposition be the real line, i.e. $J = \mathbb{R}$. It should be obvious that $f'(x) \ngtr 0$ on $x \in J$ (why?) but the function is in fact *strictly* increasing (you will prove this in next week's tute). On the other hand, for weakly increasing functions Proposition DM works in both directions. Note that f is also weakly increasing (strict increasing implies weak increasing). The proposition therefore states that its derivative must be weakly positive on $\mathbb{R}$. This indeed holds, because $f'(x) \geq 0$ for $x \in J$.

Definition (Curvature of functions)

Let $a < b$ be real numbers and let $f : (a, b) \to \mathbb{R}$ be a continuous and twice differentiable function. Then f is said to be (weakly) convex (textbook: *concave up*) on (a, b) if $f''(x) \geq 0$. The function is said to be (weakly) concave (textbook: *concave down*) on (a, b) if $f''(x) \leq 0$.

Note: this definition of curvature is a little non-standard. There exists a more general definition of convexity that does not invoke the second derivative and therefore is valid even for functions that are not differentiable.

Figure



Combining this definition with Proposition DM results in the following Corollary.

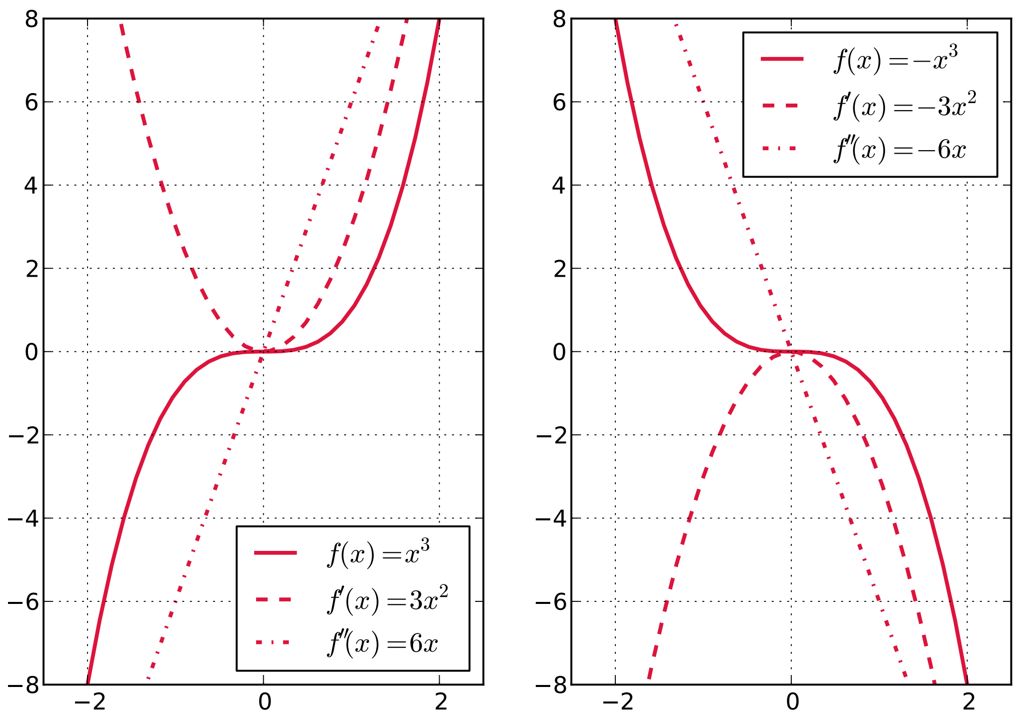
Corollary

Let $a < b$ be real numbers and let $f : (a, b) \to \mathbb{R}$ be a continuous and twice differentiable function. Then f is said to be (weakly) convex on (a, b) iff $f'(x)$ is increasing on (a, b). The function is said to be (weakly) concave on (a, b) iff $f'(x)$ is decreasing on (a, b).

Now we are ready to define inflection points. Loosely speaking, an inflection point is a point where the curvature of a function changes from convex to concave or from concave to convex. A non-conventional way to define inflection point is via local extrema of first derivatives. The following two graphs illustrate that inflection points are points where the first derivative attains a local extremum. Depending on whether the extremum is a minimum or a maximum, the curvature changes from concave to convex (left figure) or the curvature changes from convex to concave (right figure). (Note that it does not matter whether the graph changes from concave to convex sloping upwards (as shown in the figure) or sloping downwards (not shown). Likewise for when the graph changes from convex to concave sloping downwards (shown) or sloping upwards (not shown).)

In the following definition of an inflection point, however, we do not distinguish what direction the curvature changes to.

Figure



Inflection points can be interpreted as local extrema of first derivatives. Here, for example, the function $f(x) = x^3$ has a first derivative that attains a local minimum at zero. The original function $f(x)$ has an inflection point there.

Definition IP (Inflection point)

Let $a < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be a continuous and twice differentiable function. Then the point $x_0 \in (a, b)$ is an inflection point of $f(x)$ if $f'(x)$ attains a local extremum at x_0.

Note: END OF EXAMINABLE MATERIAL (only material up to this point is relevant for the mid-semester exam)

♢

Now we can apply the same machinery from before to figure out whether a point x_0 is an inflection point. The only difference here is that we are looking for an extremum of f' and not f. The necessary condition follows straightforwardly.

Proposition (Necessary condition for inflection point)

Let $a < b$ be real numbers and let $f : (a, b) \to \mathbb{R}$ be a function. If f has an inflection point at $x_0 \in (a, b)$ and is twice differentiable at x_0 then $f''(x_0) = 0$.

Proof

This is the same necessary condition as before, just that now f' (instead of f) attains a local extremum at x_0 (which is equivalent to having an inflection point there).

Thus, determining the roots of $f''(x)$ gives us a list of candidates for inflection points. Not all of these roots will be inflection points. We need to study sufficient conditions.

Corollary IP (Second derivative test for inflection points)

Let $f : [a, b] \to \mathbb{R}$ be a continuous and twice differentiable function. Let $x_0 \in (a, b)$. Then

1. If there is a neighborhood $(x_0 - \delta, x_0 + \delta)$ such that $f''(x) \geq 0$ for $x \in (x_0 - \delta, x_0)$ and $f''(x) \leq 0$ for $x \in (x_0, x_0 + \delta)$, then f' attains a local maximum at x_0 and therefore f has an inflection point (convex to concave) at x_0.
2. If there is a neighborhood $(x_0 - \delta, x_0 + \delta)$ such that $f''(x) \leq 0$ for $x \in (x_0 - \delta, x_0)$ and $f''(x) \geq 0$ for $x \in (x_0, x_0 + \delta)$, then f' attains a local minimum at x_0 and therefore f has an inflection point (concave to convex) at x_0.

Proof

Follows again by the mean value theorem.

Under the conditions given (twice differentiable and x_0 an interior point), both Definition IP and Corollary IP can be treated as being synonymous. Some textbooks reinterpret Corollary IP as the actual *definition* of an inflection point.

Parallel to before, Corollary IP may not be practical because it could be difficult to check the derivative in a neighborhood around x_0. Instead we propose the sufficient condition based on the higher-order derivative—here the third derivative.

Proposition (Third derivative test for inflection point)

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function with third derivative $f'''(x_0)$ at $x_0 \in X$. Then, if $f''(x_0) = 0$ and $f'''(x_0) \neq 0$ the function attains an inflection point.

Last, if both $f''(x_0) = f'''(x_0) = 0$ then the function may or may not have an inflection point at x_0.

3.4.4 Exercises

Note: Solutions for exercises will only be given during the tutorials. They will not be posted here.

1. Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = |x|$. Is this function differentiable? Show your work!
2. True or False? Argue.

   Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function.

   (a) If f is continuous at a point then it is also differentiable there.

   (b) If f is differentiable at a point then it is also continuous there.

3. In the lecture you learnt about a necessary condition for local extrema. Loosely speaking, if a function attains a local extremum, then its derivative there is equal to zero. How does this argument work (or not!) for the function $f(x) = |x|$ at $x = 0$?
4. Proof the power rule (i.e, if $f(x) = x^n$ then $f'(x) = n \cdot x^{n-1}$).
5. Let $f : \mathbb{Z} \to \mathbb{R}$ be the function $f(x) = x$. Determine all local and global extrema.

Related exercises in the textbook you should study, include (but are not limited to):

- Exercises 10-4 — Problems 1-26, 45-60
- Exercises 10-5 — Problems 1-18, 25-52, 69-80
- Exercises 11-2 — Problems 1-22, 27-42
- Exercises 11-3 — Problems 1-82
- Exercises 11-4 — Problems 1-76, 79-90

The tutors at the EMET1001 help desk are happy to help, if you have any questions.

6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$.
   (a) Prove that f is strictly increasing on $\mathbb{R}$.
   (b) Prove that f is weakly increasing on $\mathbb{R}$.
7. The total cost of producing q units of a good is

$$TC(q) = aq^2 + bq + c, \qquad q > 0,$$

   where a, b, and c are positive constants.
   (a) Find the local extrema of the average total cost function

$$ATC(q) = TC(q)/q.$$

   Determine the nature of the local extrema (minimum/maximum) and also discuss curvature of the ATC curve.
   (b) Sketch the graph of $ATC(q)$ together with the graphs of the functions

$$\begin{aligned} f(q) &= aq + b \\ g(q) &= 2aq + b. \end{aligned}$$

   What is the interpretation of the function f? The function g?
8. A function f is given by $f(x) = (1 + 2/x) \cdot \sqrt{x + 6}$.
   (a) What is the domain of f?
   (b) On which intervals is $f(x)$ positive?
   (c) Find the local extrema of f. Are they minima or maxima?

Related exercises in the textbook you should study, include (but are not limited to):

- Exercises 12-1 — Problems 19-40, 47-54, 79-88
- Exercises 12-2 — Problems 7-34

The tutors at the EMET1001 help desk are happy to help, if you have any questions.

9. Evaluate the first three derivatives of the functions

$$f(x) = x^4, \qquad g(x) = -x^4, \qquad h(x) = x^5$$

at zero. What do you conclude about possible inflection points there? Draw the function h and its first three derivatives in one graph.

- Weeks 7 through 8—cost and profit functions, optimisation:

3.5 Cost and Profit Functions

3.5.1 Total and Average Cost

Example of a total cost function:

$$TC(q) = aq^2 + bq + c,$$

with $c > 0$ and $q \geq 0$. This gives the total cost as a function of output q. This is a generic *quadratic* cost function. Of course, cost functions do not need to be quadratic.

Decomposition of TC:

$$\begin{aligned} TC(q) = & \underbrace{TVC(q)}_{\text{total variable cost}} \\ & + \underbrace{TFC(q)}_{\text{total fixed cost}} \\ & = (aq^2 + bq) \\ & + c. \end{aligned}$$

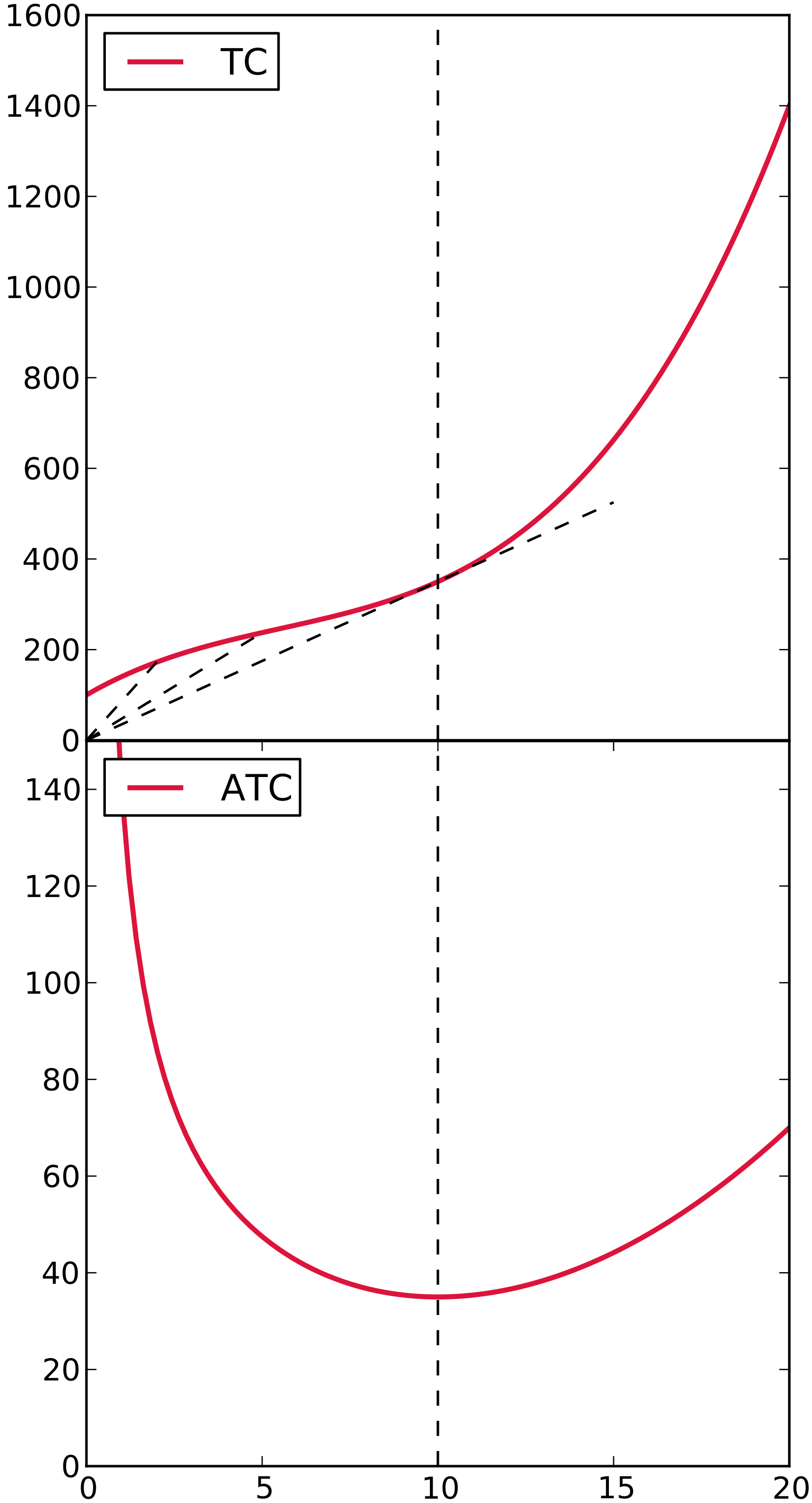
Average total cost

$$ATC(q) := TC(q)/q = \frac{aq^2+bq+c}{q} = aq + b + c/q.$$

Example

$a = 5$, $b = 5$, $c = 2,000$. Then $TC(0) = 2,000$, $TC(1000) = 5,007,000$ and $ATC(1000) = 5,007$. The last result means that each of the $1,000$ units cost, *on average* $ 5,007 to produce. Note: This does not mean that each unit cost $ 5,007 to produce. Some units may cost more, some less. But on average each cost $ 5,007. This is the important distinction between *marginal* cost and *average* cost, as will become clearer later.

Figure


TC
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Relationship between $TC(q)$ and $ATC(q)$-curves. The $ATC(q)$-curve (bottom) can be derived from the slopes of the secant lines (top, dashed lines). These secant lines connect the origin to points on the $TC(q)$-curve. Three such secant lines are shown here; there are, of course, infinitely many more, eventually tracing out the entire $ATC(q)$-curve.

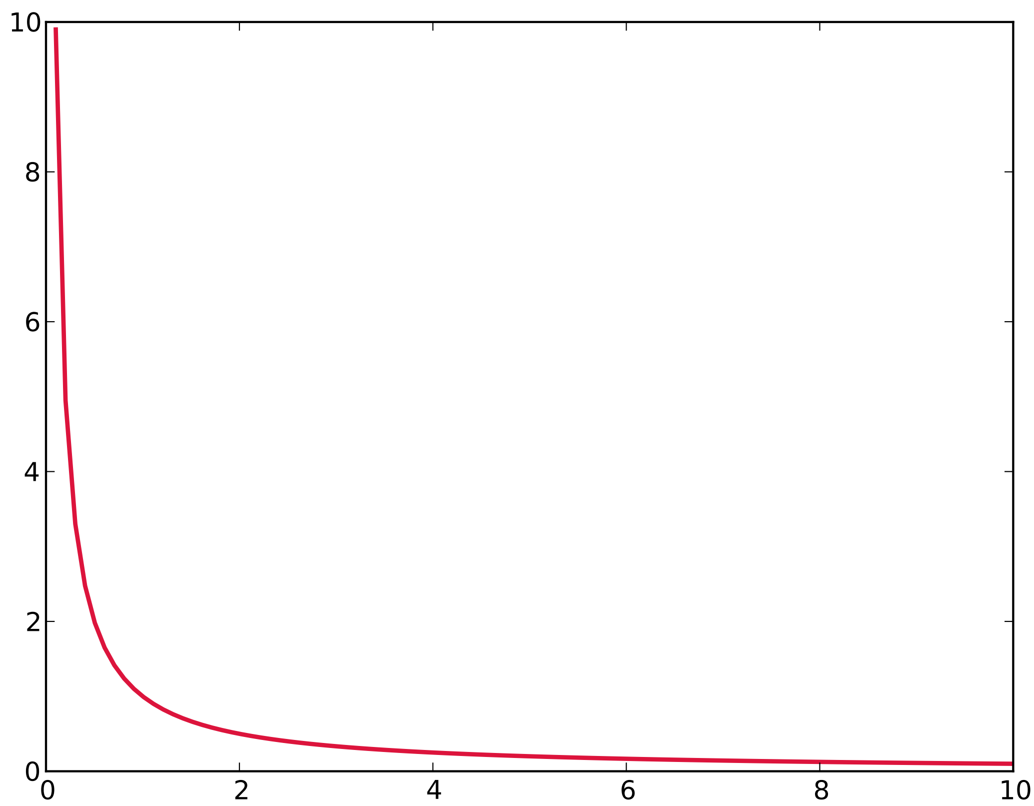
Asymptotic behavior of ATC

$$
\begin{aligned}
ATC(q) :=& \frac{TC(q)}{q} = \frac{TVC(q) + TFC(q)}{q} \\
=& \frac{TVC(q)}{q} + \frac{TFC(q)}{q} \\
=& AVC(q) + AFC(q) \\
=& aq + b + c/q.
\end{aligned}
$$

What happens to $AFC(q)$ as $q \to 0$? Well, $\lim_{q \to 0+} AFC(q) = \lim_{q \to 0+} c/q = \infty$.

What happens to $AFC(q)$ as $q \to \infty$? Well, $\lim_{q \to \infty} AFC(q) = \lim_{q \to \infty} c/q = 0$.

Figure



Typical shape of an average fixed cost curve. In words: Fixed cost are spread over large output q, each output unit carries an infinitesimally small fixed cost. Asymptotically, the fixed cost become irrelavant. In practice, firms with large fixed cost tend to produce a very large amount of output to spread fixed cost by as much as possible (example: power companies, utilities, car producers).

3.5.2 Marginal cost

Definition (Marginal cost)

For any total cost function $TC(q)$, the *marginal cost* are defined by

$$MC(q) := TC'(q) = \frac{dTC(q)}{dq}.$$

By this definition, the marginal cost are merely the slope of the TC.

Independence between MC and TFC

Recall that

$$\begin{aligned} TC(q) &= TVC(q) + TFC(q) \\ MC(q) &:= \frac{dTC(q)}{dq} \\ &= \frac{dTVC(q)}{dq} + \frac{dTFC(q)}{dq}, \end{aligned}$$

but the last term is equal to zero because TFC is just a constant function. Thus,

$$MC(q) = \frac{TC(q)}{dq} = \frac{dTVC(q)}{dq}$$

and it follows that marginal cost are equal to the slope of the TC (which we knew already) and also are equal to the slope of the TVC.

3.5.3 Relationship between MC and ATC

Recall that $ATC(q) := TC(q)/q$. We want to study the relationship between various cost functions at the minimum of the ATC. Let's start by deriving that minimum. To find the minimum, we look at the first order necessary condition.

$$\frac{dATC(q)}{dq} = \frac{TC'(q) \cdot q - TC(q) \cdot 1}{q^2}.$$

The necessary condition for ATC to attain a local minimum is $dATC(q)/dq = 0$. Therefore we need

$$0 = \frac{TC'(q) \cdot q - TC(q) \cdot 1}{q^2},$$

which is equivalent to having $0 = TC'(q) \cdot q - TC(q)$. Rearranging yields the necessary condition

$$\frac{TC(q)}{q} = TC'(q) \Leftrightarrow ATC(q) = MC(q).$$

So far, all we can say is that at local extrema of ATC we have that $ATC = MC$. We set out to study the behavior of various cost functions at the local minimum of the ATC. We therefore still need to make sure that we restrict our attetion to minima only. To do so, we study the second order condition next.

Recall

$$\frac{ATC(q)}{dq} = \frac{TC'(q) \cdot q - TC(q)}{q^2},$$

then

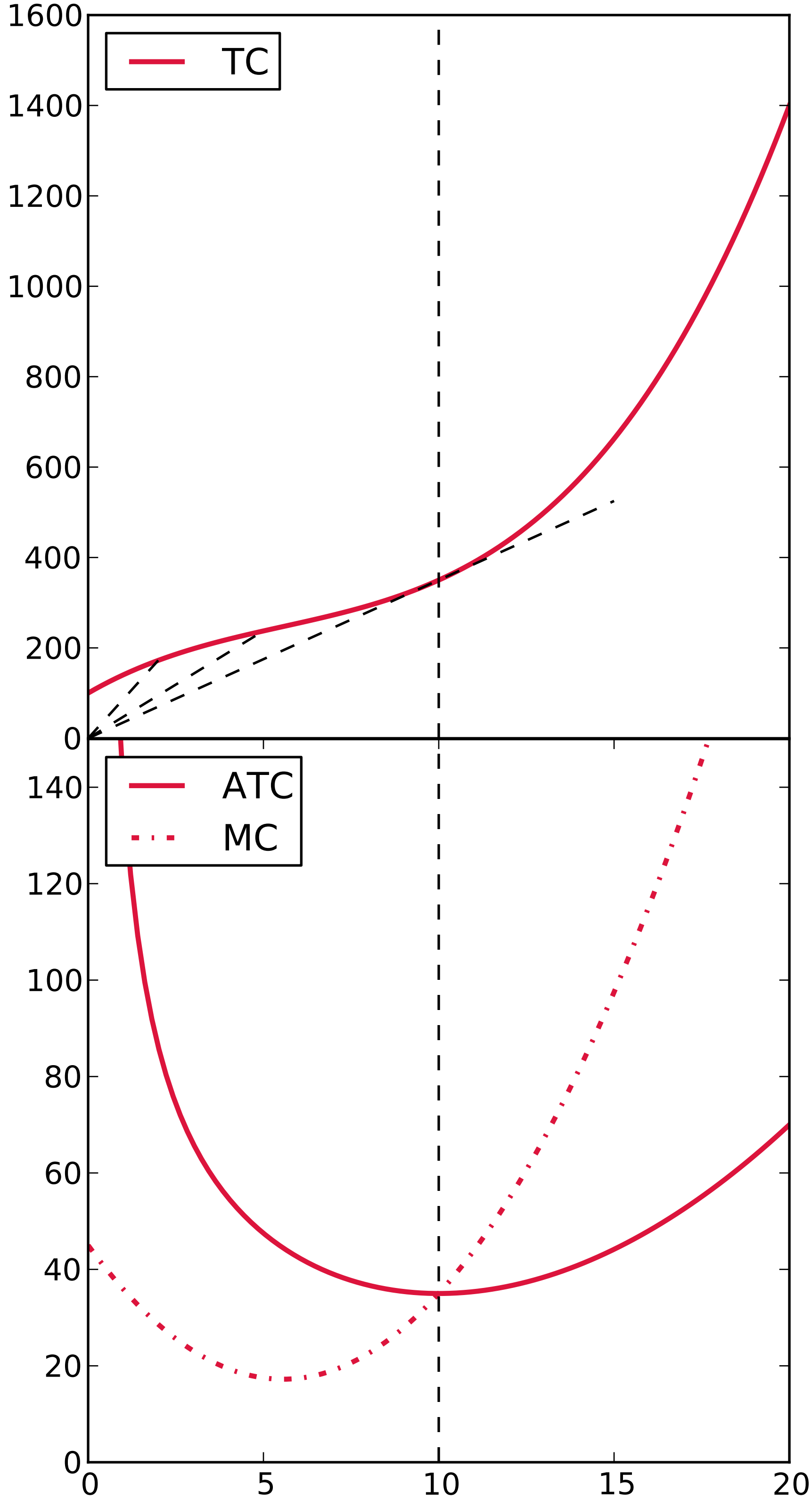
$$
\begin{aligned}
\frac{d^2ATC}{dq^2} &= \frac{\left(TC''(q)\cdot q + TC'(q) - TC'(q)\right)\cdot q^2 - \overbrace{\left(TC'(q)\cdot q - TC(q)\right)}^{=0 \text{ b/c of necessary condition}} \cdot 2q}{q^4} \\
&= \frac{\left(TC''(q)\cdot q\right)\cdot q^2}{q^4} = \frac{TC''(q)}{q} = \frac{dMC(q)}{dq}\cdot\frac{1}{q}.
\end{aligned}
$$

For a local minimum of ATC we therefore need that

$$
\frac{d^2ATC(q)}{dq^2} = \frac{dMC(q)}{dq}\cdot\frac{1}{q} > 0.
$$

Because we always assume that $q > 0$ this condition holds whenever $dMC(q)/dq > 0$. In words, at the minimum of the ATC, the MC curve cuts the ATC curve from below. When MC are below the ATC then the ATC are falling. (One could interpret this as the MC pulling down the ATC to the minimum. After that, the MC are pushing the ATC up away from the minimum.)

Figure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The $MC(q)$-curve cuts the $ATC(q)$-curve from below at the minimum of the $ATC(q)$-curve.

♢

3.5.4 Exercises

1. A firm has total cost function $TC(q) = 0.5q + 2$ for $q \in [0, 10]$.
   (a) Sketch the graph of the total cost function.
   (b) Find the marginal and average total cost functions. Sketch their graphs.
   (c) Explain in words why average total cost is greater than marginal cost at all levels of output.
   (d) If the TC function is linear, on what assumption can $MC = ATC$?
2. A firm has total cost function $TC(q) = q^2 - 3q + 500$. The firm sells in a perfectly competitive market at ruling market price $p = 67$.
   (a) Find the most profitable level of output and the profits at that output.
   (b) Does the firm produce at minimum average total cost? Explain.
   (c) Sketch the graphs of total cost and total revenue with the same axes, and do the same with marginal cost and marginal revenue.
   (d) Sketch the graph of the profit function.
   (e) Assume the market price rises first to 68, then 69, then 70. What's the firm's response to these price increases? Can you deduce from this the firm's supply function (i.e., the relationship between market price and the quantity the firm chooses to supply)? Illustrate with a sketch the graph of the supply function.
   (f) Use the supply function calculated in part (v) to find the price that would induce the firm to produce at minimum average total cost.
   (g) Suppose the fixed costs rise from 500 to 1000 (market price is back to 67). What effect will this have on the firm's chosen level of output? Profits?

3.5.5 Profit Maximization

Firms do not minimize cost, they maximize profits. Profit is defined as the difference between total revenues and total cost.

$$\Pi(q) = TR(q) - TC(q)$$

Example

A firm sells mobile phones for $124 a piece. The firm cannot set it's own market price; instead it accepts the price that is dictated by the market (i.e., this firm is a price-taker). Its total cost of producing q units are given by the function $2q^2 + 4q + 600$. How much should the firm produce to maximize profits?

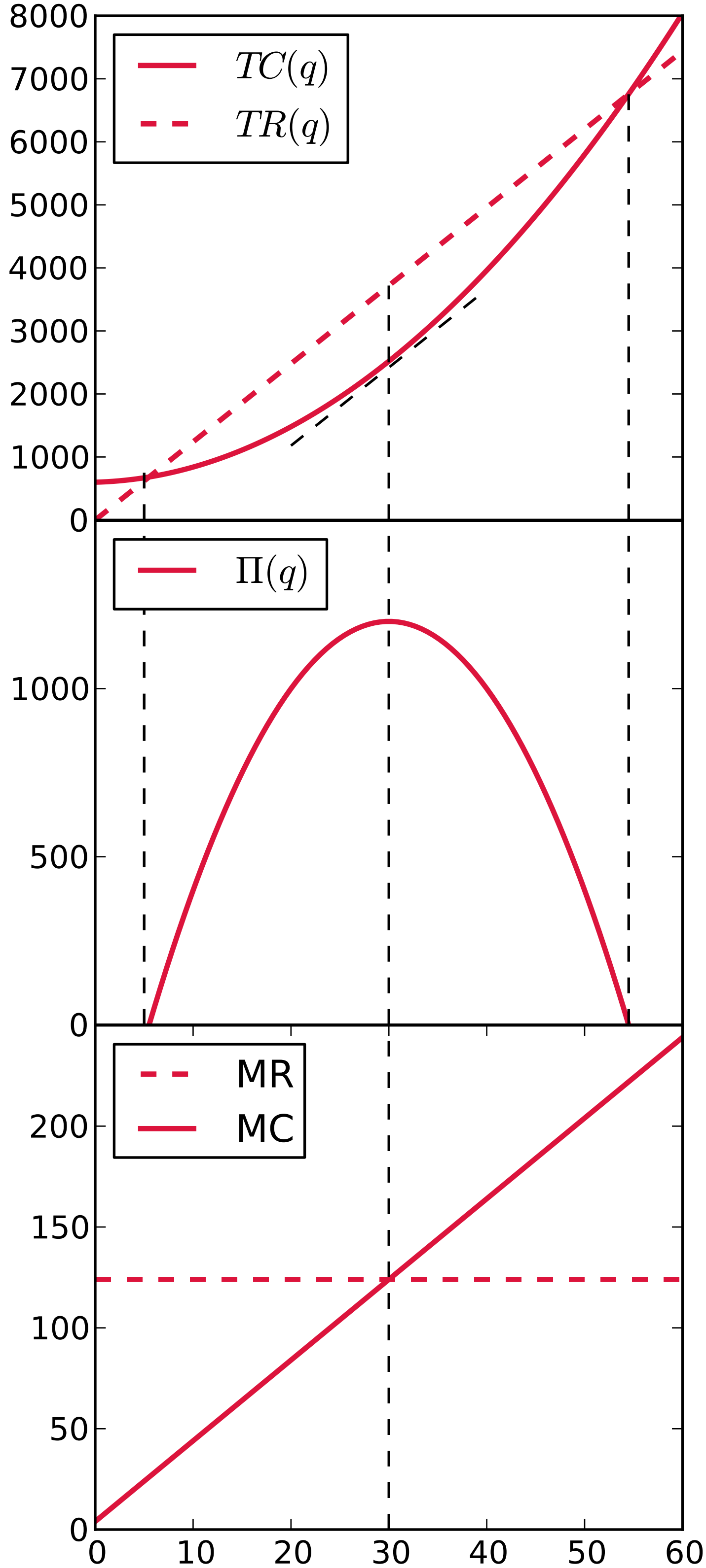
Intuitively, the total revenue function is just a linear function starting in the origin with slope equal to the market price. Therefore the profit function is

$$\begin{aligned}\Pi(q) &= 124q - 2q^2 - 4q - 600 \\ &= -2q^2 + 120q - 600\end{aligned}$$

To maximize profit, we only need to study the first order condition

$$\frac{d\Pi(q)}{dq} = -4q + 120 = 0$$

Figure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$TC(q)$
$TR(q)$
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$\Pi(q)$
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50
0
MR
MC
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50
60

We infer that this firm *may* maximize its profit by producing 30 units. Of course, as we have learnt earlier in this course, setting the first derivative equal to zero is not sufficient for finding a local maximum. We could just as well have detected a profit minimum. To rule this out, we need to check the second derivative. We can easily convince ourselves that the second derivative evaluates to -4 at $q = 30$ and hence we find that indeed the firm is maximizing its profits by producing 30 units.

Now we would like to derive a more general result that relates revenue and cost to the profit maximizing behavior of the firm. Instead of using a particular price and a particular total cost function, we keep things generic. Recall, the general profit function of the firm was

$$\Pi(q) = TR(q) - TC(q)$$

The necessary condition for profit maximization is

$$\frac{d\Pi(q)}{dq} = 0$$

Breaking up the left-hand side, the necessary condition is equivalent to

$$\frac{dTR(q)}{dq} - \frac{dTC(q)}{dq} = 0$$

Therefore, a necessary condition for a profit maximum is simply

$$\frac{dTR(q)}{dq} = \frac{dTC(q)}{dq}$$

or

$$MR(q) = MC(q)$$

This last little equation is quite important in microeconomics. It says that in order to maximize profits, price-taking firms equalize marginal revenue and marginal cost. And since the marginal revenue for price-taking firms is dictated by the market (it is just the market price), all firms need to do is find the output amount q at which their marginal cost are equal to the market price.

Again, we know from earlier in the course that equating marginal revenue and marginal cost may just as well result in a profit *minimum* rather than a maximum. To be sure that we have detected a local maximum we always also need to check the second order condition. That's why we need to be careful and understand that the condition $MR(q) = MC(q)$ is merely a necessary condition and not a sufficient one for profit maximization.

- Weeks 9 through 13—lecture material presented using document camera instead of website

CHAPTER

FOUR

SEARCH SITE

- *search*