THE AUSTRALIAN NATIONAL UNIVERSITY

Second Semester PRACTICE MIDTERM Examination – September, 2013

Foundations of Economic and Financial Models

(EMET 1001/7001)

Reading Time: 10 Minutes
Writing Time: 90 Minutes
Permitted Materials: Non-programmable Calculator

Instructions:

- This handout of exam questions contains 4 pages (including cover page) with 15 exam questions. Make sure you are not missing any pages!
- Answer **ALL** questions of this handout in the script book provided to you.
- Show your work! No partial credit will be given for just providing final results.
- All your derivations should be as mathematically rigorous as possible.
- You may not use L'Hospital's rule in solving questions.
- Unless otherwise stated, when determining local extrema, make sure to also show which ones are minima and which ones are maxima.
- Cheat sheets are not permitted.
- Anything that is not a non-programmable calculator is not permitted. These include, but are not limited to: mobile phones, tablet computers, abacuses.
- Return this handout to the invigilators at the end of the exam.
- Total marks: 100
- Good luck!

Note: The following set of questions covers material from weeks 1 through 6 of the lecture (weeks 2 through 7 of the tutorials). No particular emphasis is put on any particular question/topic for any particular purpose. No claim on comprehensiveness and exhaustiveness of questions/topics covered is made. You should be aware that the actual exam will contain different questions. The priority given to different questions/topics in the exam may be different to the priority these same questions/topics receive here.

1. [6 marks] Let $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0. \end{cases}$$

   Is the function f continuous?

2. [6 marks] Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = c \cdot x$ for some real number c. Prove that

$$\lim_{x \to x_0} f(x) = c \cdot x_0.$$

3. [10 marks] State the Mean Value Theorem and use it to prove the inequality

$$(1 + z)^a \geq 1 + az,$$

   for $-1 < z$ and some constant $a > 1$. (Hint: Let $f(z) = (1 + z)^a$.)

4. [8 marks] A function h is given by $h(w) = w \cdot (4 + w)^{2/3}$.
   (a) What are the domain and range of h?
   (b) Obtain $h'(w)$ and $h''(w)$!
   (c) Find all local extrema of h!

5. [5 marks] Find the sum of the infinite geometric sequence:

$$4, -\frac{8}{3}, \frac{16}{9}, -\frac{32}{27}, \ldots$$

6. [8 marks] You wish to be a millionaire when you retire. To accomplish this you decide to open a savings account and make equal monthly payments at the end of each month. If the account earns a nominal rate of 7.5% compounded monthly and you know you will retire in 40 years, what must be the monthly payment to obtain your $1,000,000?

   To answer this question, do the following:

   (a) Denote the monthly payment by P. Denote by A your savings goal (i.e., 1,000,000). Write down explicitly the payment series. (Only focus on the first three and the last three terms.)

   (b) Map your payment series from part (a) into the generic notation for geometric series. What is the starting term a_1 equal to? What is the common ratio i equal to? What is n?

   (c) Solve for P in terms of A, a_1, i, and n only. In the end, plug in specific numbers and provide the solution.

7. [6 marks] A math tutor charges $40 for the first hour of work at your house and $30 for every hour (or fraction thereof) afterwards.

   (a) Write down the function f that maps hours x into total charges by your tutor. (Assume that $x \leq 5$.)

   (b) Find $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$.

8. [6 marks] For what values of x is the following function continuous?

$$g(x) = \frac{x \cdot (x+1)}{x^2 - 1}$$

9. [6 marks] For the following function, determine the difference quotient and its limit as x approaches x_0:

$$f(x) = \frac{2}{x^2}$$

10. [8 marks] Let $f(x) = x^3 - 3x^2$.

   (a) Find $f'(x)$ and $f''(x)$

   (b) Determine all intervals on which the function is increasing; determine all intervals on which it is decreasing.

   (c) Determine the coordinates of all local extrema.

   (d) Determine all intervals on which the function is convex; determine all intervals on which the function is concave.

   Note: Do not use graphs or sign charts to answer the question.

11. [6 marks] A firm has total cost function

$$TC(q) = 2q^3 - 2q^2 + 5q + 24 \qquad \text{with } \{q \in \mathbb{N} \mid 0 \leq q \leq 8\}.$$

   (a) For what value of q are average total cost minimal?

   (b) Determine the marginal cost $TC'(q)$ at the minimum of the average total cost.

   (c) For what value of q are the marginal cost minimal?

12. [6 marks] What does it mean for a set X to have cardinality n? Give a definition!

13. [5 marks] A person deposits \$1,000 in a savings account that pays an interest rate of 4.75% compounded continuously. Find the balance in the account at the end of 3.5 years.

14. [6 marks] Determine the limit:

$$\lim_{p \to 4} \frac{p^2 - 7p + 12}{p^2 - 3p - 4}$$

15. [8 marks] Define by S_n the arithmetic series with common difference d that starts at a_1 and ends at a_n. Prove, based on the lecture, that

$$S_n = \frac{n}{2}\left[a_1 + a_n\right]$$