Matrix algebra

Definition

A rectangular array of numbers/symbols consisting of m rows and n columns is called an $m \times n$ matrix.

Special case: if either $m = 1$ or $n = 1$ then we call the resulting matrix a vector.

Transposes of matrices

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ 2×3 matrix

The transpose A' of A is defined as

$$A' = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix} \quad 3 \times 2 \text{ matrix}$$

side note: it should be clear that $A \neq A'$.

Addition / subtraction of matrices

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Then $A \pm B = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} \end{bmatrix}$

→ only defined if A and B have same size (same # rows and # columns)

Scalar multiplication

Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$

Then $2 \cdot A = \begin{bmatrix} 2 \cdot 3 & 2 \cdot 1 \\ 2 \cdot 4 & 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 8 & 4 \end{bmatrix}$

Matrix multiplication

Let $A = \begin{bmatrix} 2 & 1 & -6 \\ 1 & -3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 4 & 2 \\ -2 & 1 & 1 \end{bmatrix}$

$\Rightarrow A \cdot B = ?$

Definition

Let A be an $m \times \boxed{n}$ matrix

and let B be an $\boxed{n} \times p$ matrix. Then the product

$C = A \cdot B$ is the $m \times p$ matrix whose elements

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \, b_{kj}$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$$

→ each element in row i of matrix A is multiplied by the corresponding element in column j of matrix B.

→ matrix multiplication only defined if # columns of A = # rows of B

Example

$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 & -3 \\ 0 & 4 & 2 \\ -2 & 1 & 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 & -6 \\ 1 & -3 & 2 \end{bmatrix} & \begin{bmatrix} 14 & -2 & -10 \\ -3 & -10 & -7 \end{bmatrix} \end{array}$$

What about $B \cdot A$?

size of matrix $B = 3 \times 3$

$A = 2 \times 3$

$\Rightarrow$ $B \cdot A$ not defined b/c # columns of B is not equal # rows of A

$\Rightarrow$ $A \cdot B \neq B \cdot A$

Example

$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$\Rightarrow$ $A \cdot B = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 32 \end{bmatrix}$

$$B \cdot A = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \\ \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix} \end{array}$$

again: $A \cdot B \neq B \cdot A$

order of matrix multiplication matters!

Terminology

in the expression $A \cdot B$

we say that matrix A pre-multiplies matrix B

B post-multiplies matrix A

Example

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

aim: (i) $A \cdot B$

(ii) $(A \cdot B)'$

(iii) $B' \cdot A'$

solve: (i) $\underset{2\times 2}{A} \cdot \underset{2\times 2}{B} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

need to be same

(ii) $(A \cdot B)' = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

(iii) $B' \cdot A' = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

$\Rightarrow$ comparing (ii) and (iii):

$$\boxed{(A \cdot B)' = B' \cdot A'} \quad \text{holds in general}$$

it is <u>not</u> true that

$$(A \cdot B)' = A' \cdot B'$$

Solving systems of linear equations by matrix inversion

<u>Example</u>

$$\left.\begin{aligned} 3x_1 + 4x_2 &= 1 \\ x_1 - 2x_2 &= 7 \end{aligned}\right\} \text{ compact notation:}$$

$$AX = B$$

<u>where:</u> $A = \begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ $\quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

<u>Goal:</u>

solve $AX = B$ for X

How would you solve this if A, B were real numbers?

$$\Rightarrow AX = B \quad \Big| \cdot \frac{1}{A}$$

$$1 \cdot X = \frac{B}{A}$$

But A, B are not real numbers

- cannot simply divide both sides by A
- looking for a matrix-equivalent of division by A

$\Rightarrow$ we need a matrix A^{-1} such that

$$A^{-1} \cdot A = I$$

because

$$I \cdot X = X$$

recall:

$$AX = B \qquad | \cdot A^{-1}$$

$$\underbrace{A^{-1} A}_{I} X = A^{-1} B$$

$$I X = A^{-1} B$$

$$X = A^{-1} B$$

Inverse of a 2×2 matrix

Definition:

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Then the inverse A^{-1} of A is

$$\boxed{A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}$$

determinant of A

Definition:

The determinant of matrix A above is defined as: $|A| = a_{11}\, a_{22} - a_{12} \cdot a_{21}$

→ applies only to 2×2 matrices

Where does the definition of an inverse come from?

→ the inverse A^{-1} is defined that way b/c

(i) $A^{-1} A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(ii) $A \cdot A^{-1} = I$

Example

$A = \begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix}$ → get inverse!

two steps

(i) determinant:

$$|A| = 3 \cdot (-2) - 4 \cdot 1 = -10$$

(ii) obtain inverse: $A^{-1} = \frac{1}{-10} \begin{bmatrix} -2 & -4 \\ -1 & 3 \end{bmatrix}$

→ solving linear eq. systems

$$A X = B \quad | \cdot \text{ pre-multiply by } A^{-1}$$

$$\Rightarrow \quad A^{-1} A X = A^{-1} B$$

$$\Leftrightarrow \quad X = A^{-1} B$$

$$= \frac{1}{-10} \underbrace{\begin{bmatrix} -2 & -4 \\ -1 & 3 \end{bmatrix}}_{2\times 2} \cdot \underbrace{\begin{bmatrix} 1 \\ 7 \end{bmatrix}}_{2\times 1}$$

$$= -\frac{1}{10} \begin{bmatrix} -30 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Definition

A matrix is called singular if its determinant equals zero. Otherwise it is called non-singular.

Inverse of a 3×3 matrix

Determinant of a 3×3 matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

To compute $|A|$ for a 3×3 matrix, need 5 steps:

(i) Get determinant of the sub-matrix M_{11}

$$M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \Leftarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|M_{11}| = a_{22}\, a_{33} - a_{32}\, a_{23}$$

(ii) Get determinant of the sub-matrix M_{12}

$$M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$|M_{12}| = a_{21}\, a_{33} - a_{31}\, a_{23}$$

(iii) Get determinant of the sub-matrix M_{13}

$$M_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$|M_{13}| = a_{21}\, a_{32} - a_{31}\, a_{22}$$

$\Rightarrow$ the determinants $|M_{11}|$, $|M_{12}|$, $|M_{13}|$ are called <u>minors</u>.

(iv) calculate the "signed minors" or "cofactors"

- for minor $|M_{11}|$ the signed minor equals

$$C_{11} = |M_{11}|$$

- for minor $|M_{12}|$ the signed minor equals

$$C_{12} = -|M_{12}|$$

- for minor $|M_{13}|$ the signed minor equals

$$C_{13} = |M_{13}|$$