Economic application

Production function : gives the relationship b/w inputs of capital K and labor L and resulting output Q of some product:

$$Q = f(K, L)$$

Neoclassical assumptions

(i) Q, L, K are infinetely divisible and $f(K, L)$ is smooth and continuous

(ii) $f(0, L) = 0 = f(K, 0)$

$\Rightarrow f(0,0) = 0$

(iii) For $L > 0$ and $K > 0$

increasing either L or K will increase Q

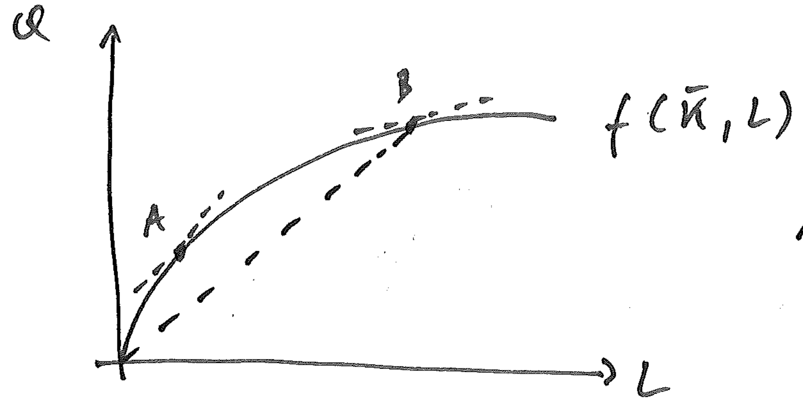
(iv) law of diminishing marginal product applies ($\rightarrow$ later)

Iso-K-section

How does Q vary with labor input L if capital input K is held constant at some level $\bar{K}$.

$$\Rightarrow Q = f(\bar{K}, L)$$

→ short-run production function
b/c firms cannot vary K in the short-run



$$APL = \frac{Q}{L}$$

marginal product of labor

$$MPL = \frac{\partial f(\bar{K}, L)}{\partial L} \geq 0$$ → slope of iso-K-section

by assumption (iii)

diminishing marginal product

successive ~~step~~ increases of output become smaller and smaller as the input L increases

$$\Rightarrow \frac{\partial^2 f(\bar{K}, L)}{\partial L^2} = \frac{\partial MPL}{\partial L} < 0$$

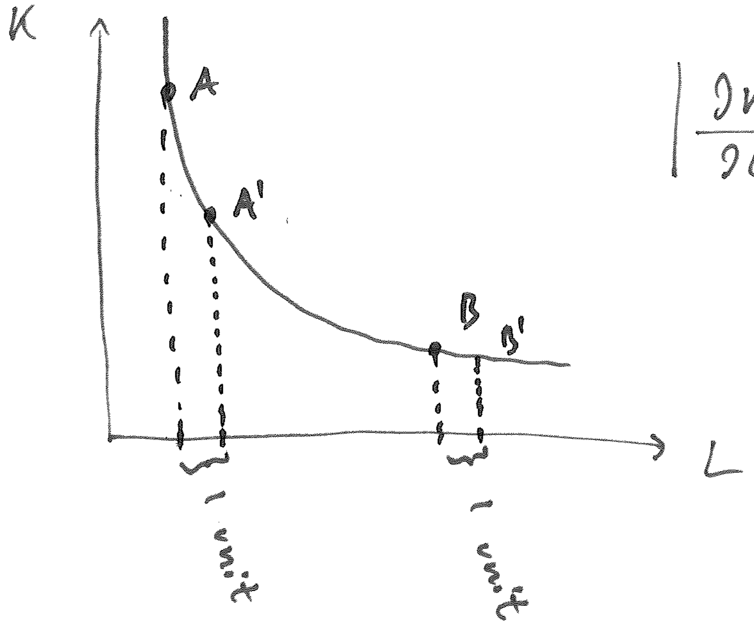
by assumption (iv)

⇝ repeat this for iso-L section

obtain: $MPK = \dfrac{\partial f(K, \bar{L})}{\partial K} \geq 0$

$\dfrac{\partial MPK}{\partial K} < 0$ $\qquad$ $APK = Q/K$

Iso - Q - section = Isoquants



$$\left| \frac{\partial K}{\partial L} \right| = MRS$$

decreasing marginal rate of substitution

- comparing A to B:
  - MPL in A larger than in B
  - MPK in A lower than in B

Extrema for multivariate functions

Local extrema:

Let $f(a,b)$ be a local extremum (min/max) for the function f. If both f_x and f_y exist at the point (a,b) then

$$f_x(a,b) = 0$$

and $f_y(a,b) = 0$.

The converse is not true:

If $f_x(a,b) = f_y(a,b) = 0$ then the point (a,b) may be a local extremum or it may also be a saddle point.

Sufficient conditions for extrema

Let point (a,b) be a critical point of $z = f(x,y)$, meaning $f_x(a,b) = f_y(a,b) = 0$.

Define:
$$A = f_{xx}(a,b)$$
$$B = f_{xy}(a,b)$$
$$C = f_{yy}(a,b)$$

Then, if

(i) $A \cdot C - B^2 > 0$ and $A < 0$
then $f(a,b)$ is a local maximum.

(ii) $A \cdot C - B^2 > 0$ and $A > 0$
then $f(a,b)$ is a local minimum.

(iii) $A \cdot C - B^2 < 0$
then $f(a,b)$ has a saddle point.

(iv) $A \cdot C - B^2 = 0$
then no conclusion can be drawn about extrema.

Example

$z = f(x,y) = x^2 + y^2 + 2$

critical points:

$$\cancel{f_x(x)}$$

$$\left.\begin{array}{ll} f_x(x,y) = 2x & \Rightarrow \quad x = 0 \\ f_y(x,y) = 2y & \Rightarrow \quad y = 0 \end{array}\right\} \text{critical point } (a,b) = (0,0)$$

$$f_{xx} = 2 \qquad f_{xx}(0,0) = 2 = A$$

$$f_{xy} = 0 \qquad f_{xy}(0,0) = 0 = B$$

$$f_{yy} = 2 \qquad f_{yy}(0,0) = 2 = C$$

therefore: $A \cdot C - B^2 = 2 \cdot 2 - 0^2 = 4 > 0$

and $A > 0$

$\Rightarrow$ local minimum

Differentiating an implicit function

$z = f(x,y) = 0 \quad \rightarrow$ gives an implicit function b/w x and y

goal: $\frac{dy}{dx}$

$$dz = \frac{\partial f(x,y)}{\partial x} dx + \frac{\partial f(x,y)}{\partial y} dy$$

$$= 0$$

$$\Rightarrow \frac{dy}{dx} = - \frac{\frac{\partial f(x,y)}{\partial x}}{\frac{\partial f(x,y)}{\partial y}} = - \frac{f_x}{f_y}$$

Constrained Optimization

Objective: max/min some function $z = f(x,y)$ where x and y are constrained, for example: $y = g(x)$

Example: max $f(x,y)$
subject to: $y = 100 - \frac{1}{2}x$

Lagrange Multiplier Method

aim: max/min $w = f(x,y,z)$
subject to: $g(x,y,z) = 0$

Four steps

(1) Define $F = f(x,y,z) + \lambda \cdot g(x,y,z)$

(2) Get all first order partial derivatives of F

$$F_x = f_x(x,y,z) + \lambda \cdot g_x(x,y,z)$$

$$F_y = f_y(x,y,z) + \lambda \cdot g_y(x,y,z)$$

$$F_z = f_z(x,y,z) + \lambda \cdot g_z(x,y,z)$$

$$F_\lambda = g(x,y,z)$$

(3) Find critical point $(x_0, y_0, z_0, \lambda_0)$
such that all four partial derivatives equal zero.

(4) Evaluate function at (x_0, y_0, z_0) :

$$\rightarrow f(x_0, y_0, z_0)$$

<u>Example</u>

min/max $\quad w = f(x,y,z) = 2x + 4y + 4z$

subject to: $x^2 + y^2 + z^2 = 9$

$$\Rightarrow x^2 + y^2 + z^2 - 9 = 0$$

$$\Rightarrow g(x,y,z) = x^2 + y^2 + z^2 - 9$$

<u>Steps</u>

(1) $F = 2x + 4y + 4z + \lambda \cdot (x^2 + y^2 + z^2 - 9)$

(2) $F_x = 2 + 2\lambda x$

$F_y = 4 + 2\lambda y$

$F_z = 4 + 2\lambda z$

$F_\lambda = x^2 + y^2 + z^2 - 9$

(3) <u>critical point:</u>

$$
\begin{aligned}
2 + 2\lambda x &= 0 && \rightarrow x = -\lambda^{-1} \\
4 + 2\lambda y &= 0 && \rightarrow y = -2\lambda^{-1} \\
4 + 2\lambda z &= 0 && \rightarrow z = -2\lambda^{-1} \\
x^2 + y^2 + z^2 - 9 &= 0
\end{aligned}
$$

$\rightarrow y = z$

plug into last equation:

$$(-\lambda^{-1})^2 + (-2\lambda^{-1})^2 + (-2\lambda^{-1})^2 = 9$$

$$\frac{1}{\lambda^2} + \frac{4}{\lambda^2} + \frac{4}{\lambda^2} = 9$$

$$\Leftrightarrow \quad \frac{1}{\lambda^2} = 1$$

$$\boxed{\lambda_0 = \pm 1}$$

$\Rightarrow$ <u>critical point:</u>

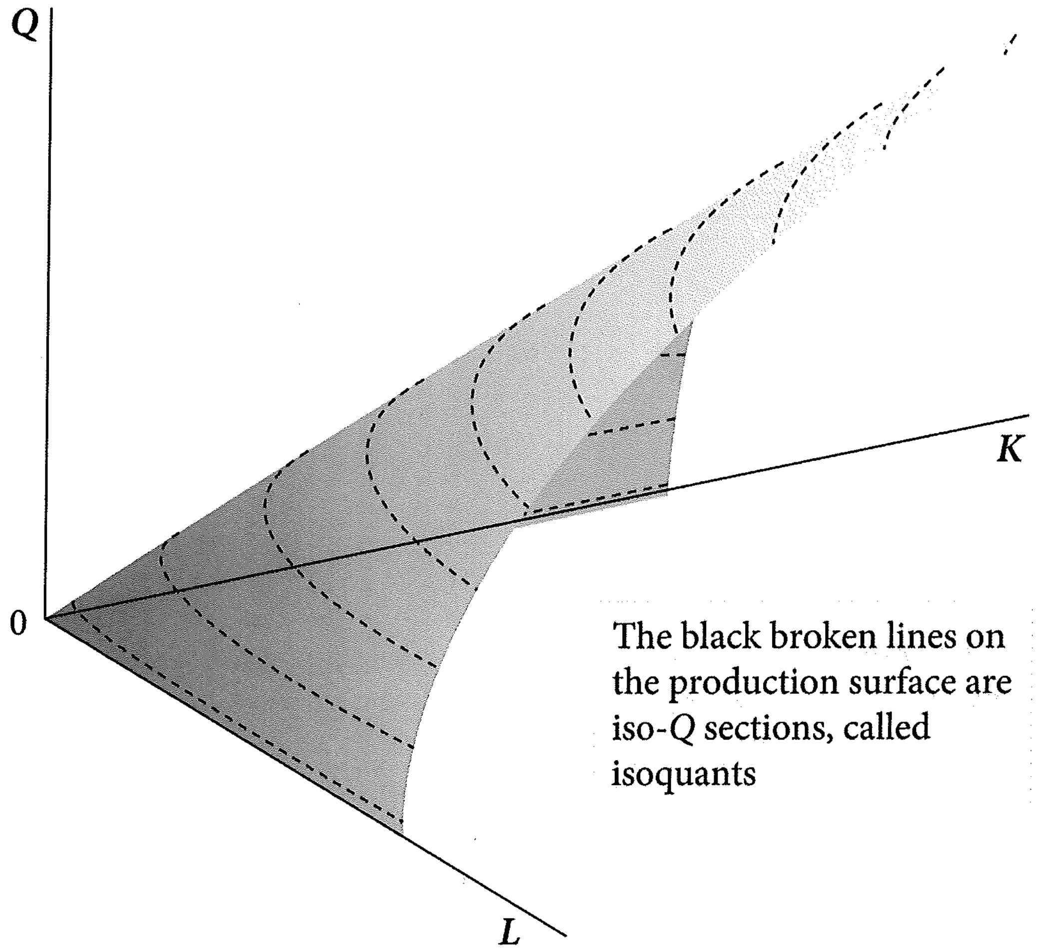
$(x_0, y_0, z_0, \lambda_0)$ is $(-1, -2, -2, 1)$

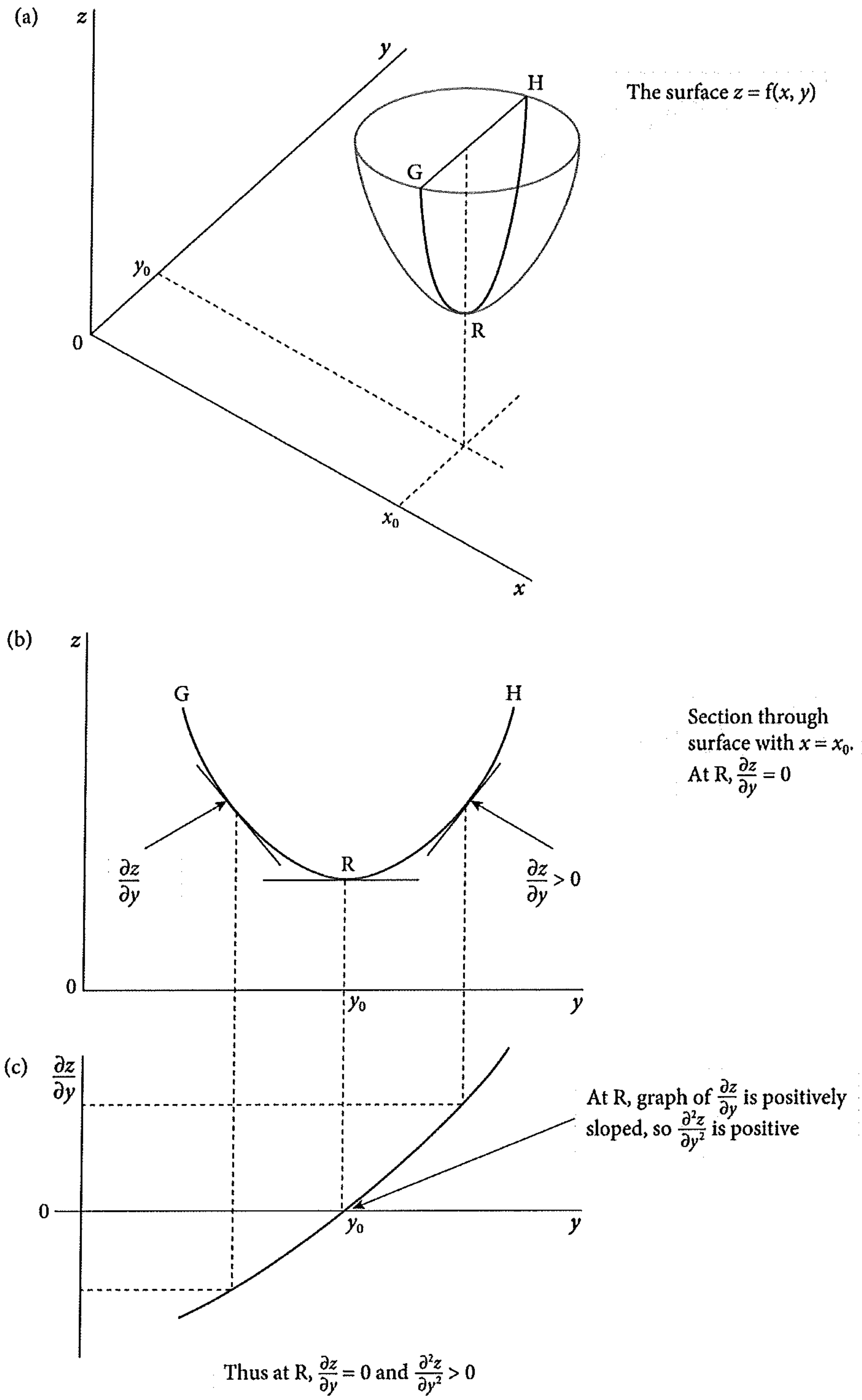
$(1, 2, 2, -1)$

(4) Evaluate function:

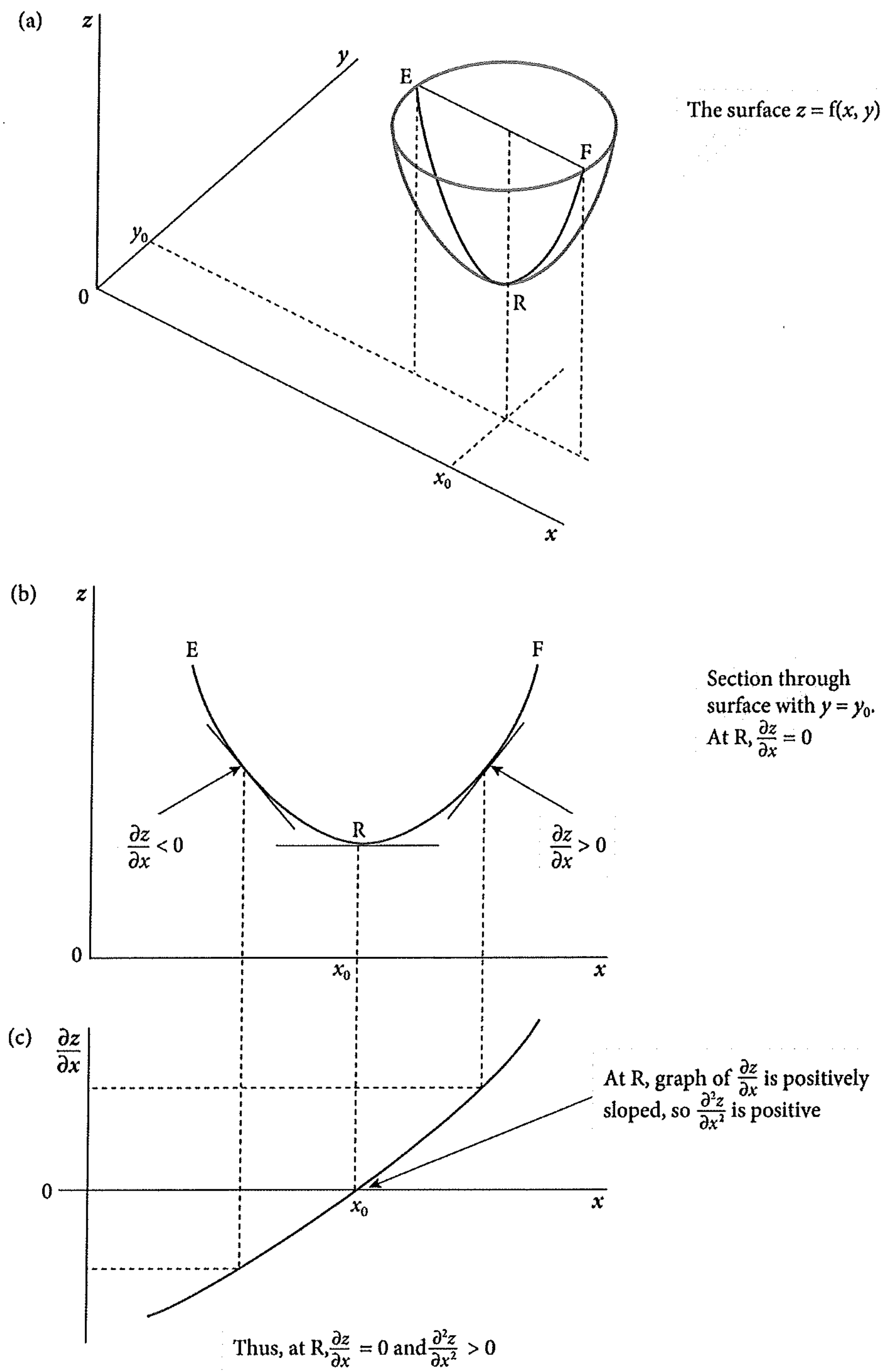
$f(-1, -2, -2) = -18 \rightarrow$ min!

$f(1, 2, 2) = 18 \rightarrow$ max!

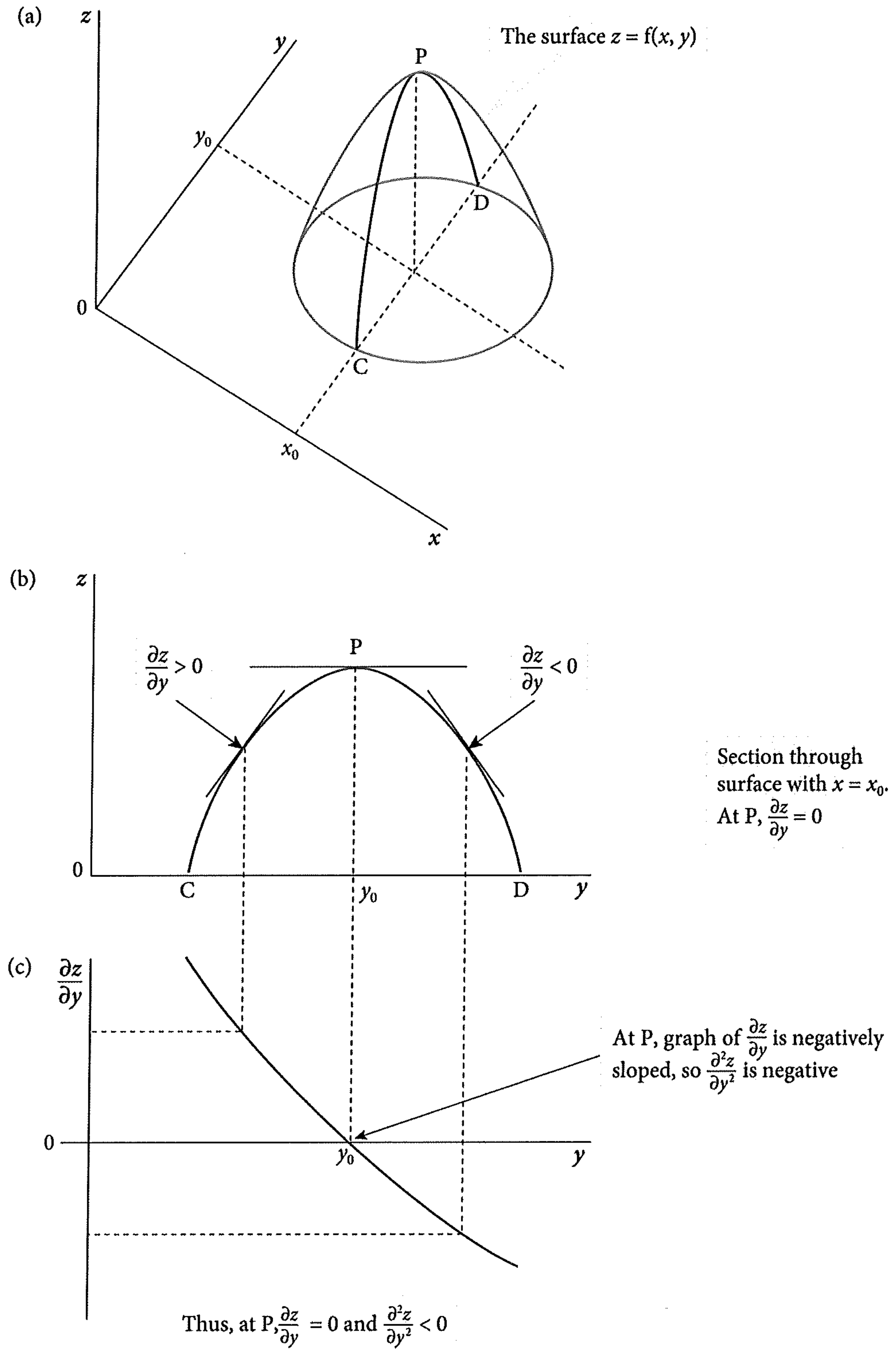

Q
0
K
L
The black broken lines on the production surface are iso-Q sections, called isoquants



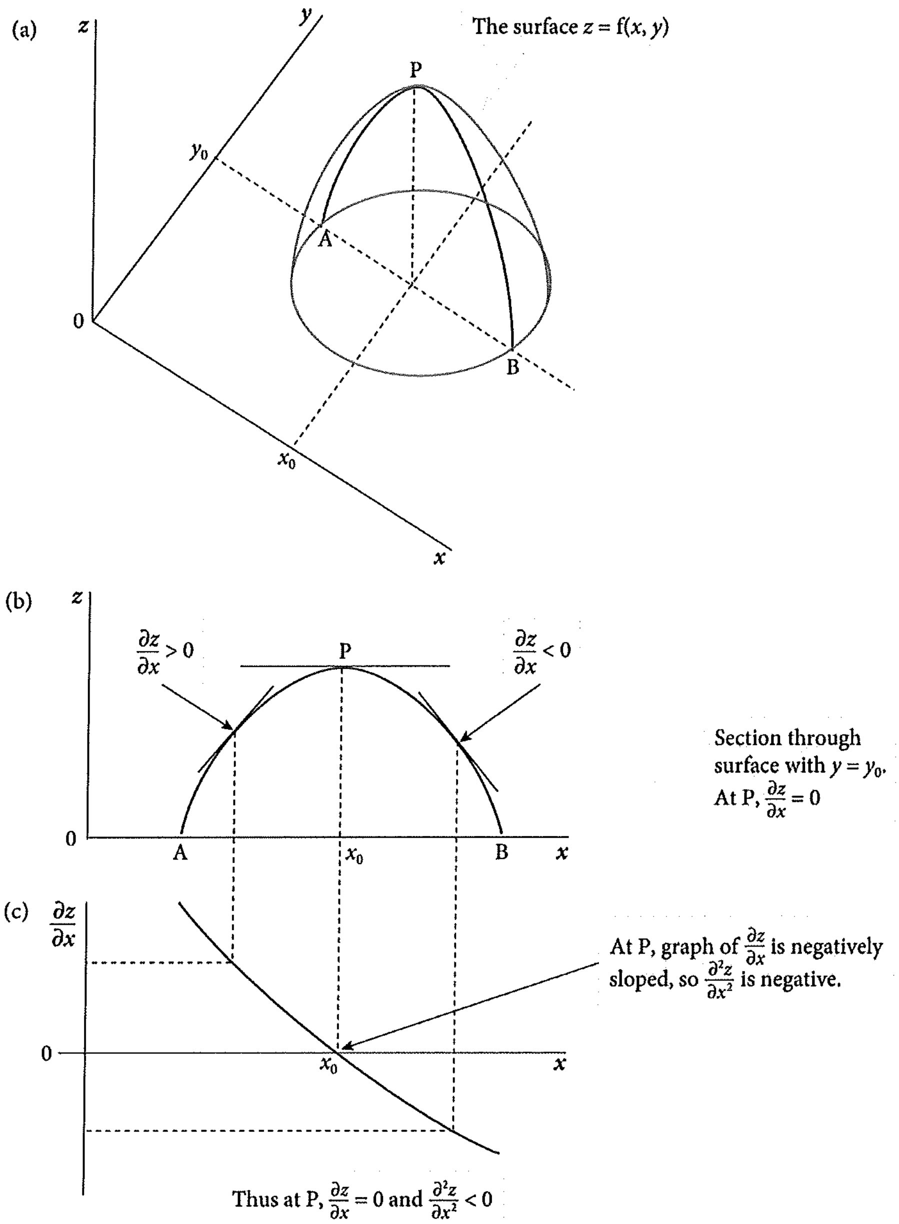
(a)
z
y
H
The surface $z = f(x, y)$
G
y_0
0
R
x_0
x
(b)
z
G
H
Section through surface with $x = x_0$. At R, $\frac{\partial z}{\partial y} = 0$
R
$\frac{\partial z}{\partial y}$
$\frac{\partial z}{\partial y} > 0$
0
y_0
y
(c) $\frac{\partial z}{\partial y}$
At R, graph of $\frac{\partial z}{\partial y}$ is positively sloped, so $\frac{\partial^2 z}{\partial y^2}$ is positive
0
y_0
y
Thus at R, $\frac{\partial z}{\partial y} = 0$ and $\frac{\partial^2 z}{\partial y^2} > 0$



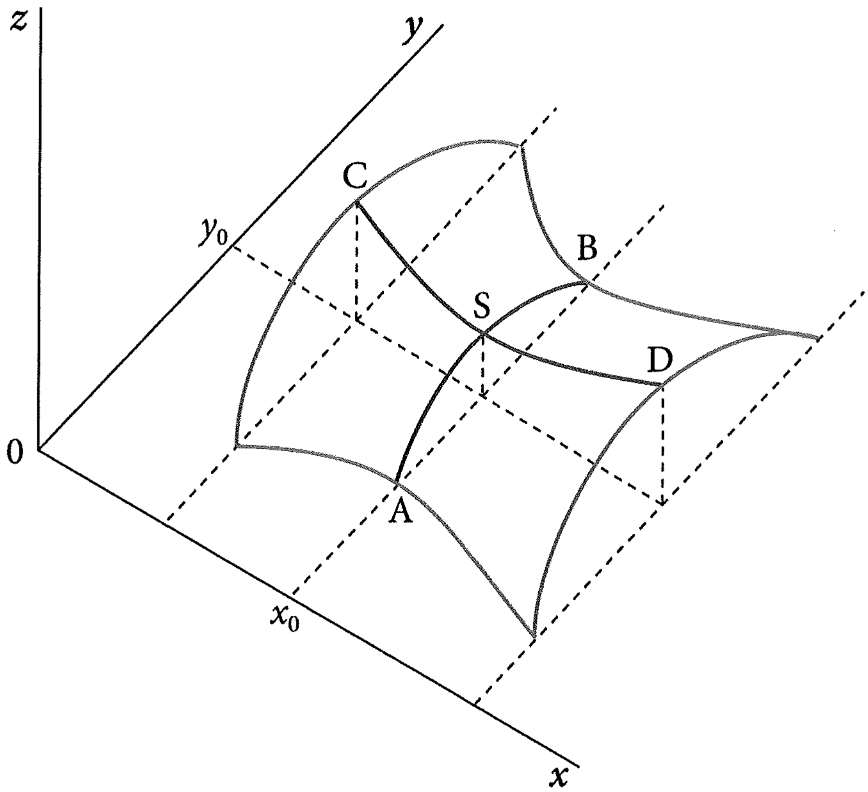
(a)
z
y
E
F
R
y_0
0
x_0
x
The surface $z = f(x, y)$
(b)
z
E
F
R
$\frac{\partial z}{\partial x} < 0$
$\frac{\partial z}{\partial x} > 0$
0
x_0
x
Section through surface with $y = y_0$. At R, $\frac{\partial z}{\partial x} = 0$
(c)
$\frac{\partial z}{\partial x}$
0
x_0
x
At R, graph of $\frac{\partial z}{\partial x}$ is positively sloped, so $\frac{\partial^2 z}{\partial x^2}$ is positive
Thus, at R, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$



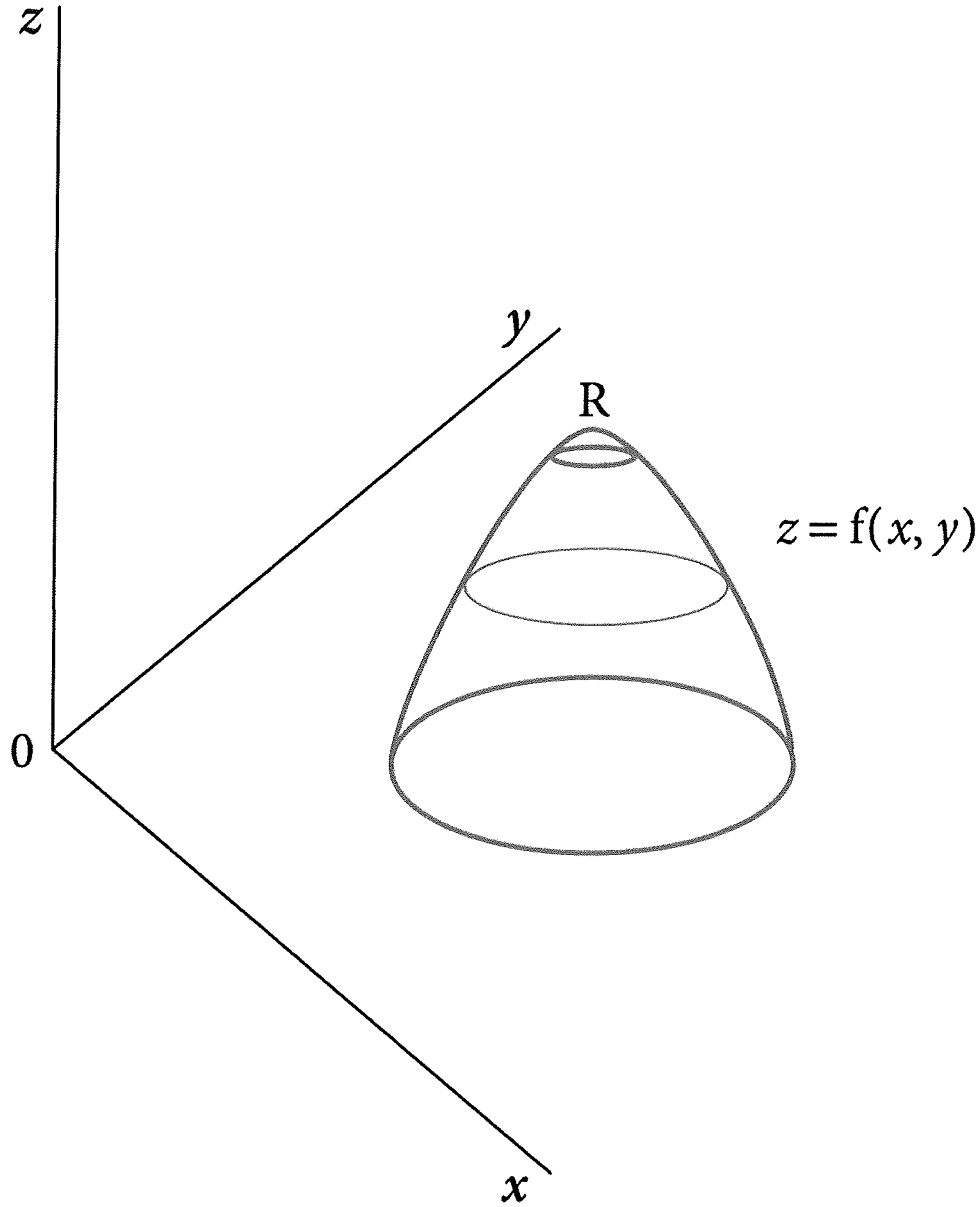
(a)
z
y
y_0
P
The surface $z = f(x, y)$
D
0
C
x_0
x
(b)
z
$\frac{\partial z}{\partial y} > 0$
P
$\frac{\partial z}{\partial y} < 0$
Section through surface with $x = x_0$. At P, $\frac{\partial z}{\partial y} = 0$
0
C
y_0
D
y
(c)
$\frac{\partial z}{\partial y}$
At P, graph of $\frac{\partial z}{\partial y}$ is negatively sloped, so $\frac{\partial^2 z}{\partial y^2}$ is negative
0
y_0
y
Thus, at P, $\frac{\partial z}{\partial y} = 0$ and $\frac{\partial^2 z}{\partial y^2} < 0$



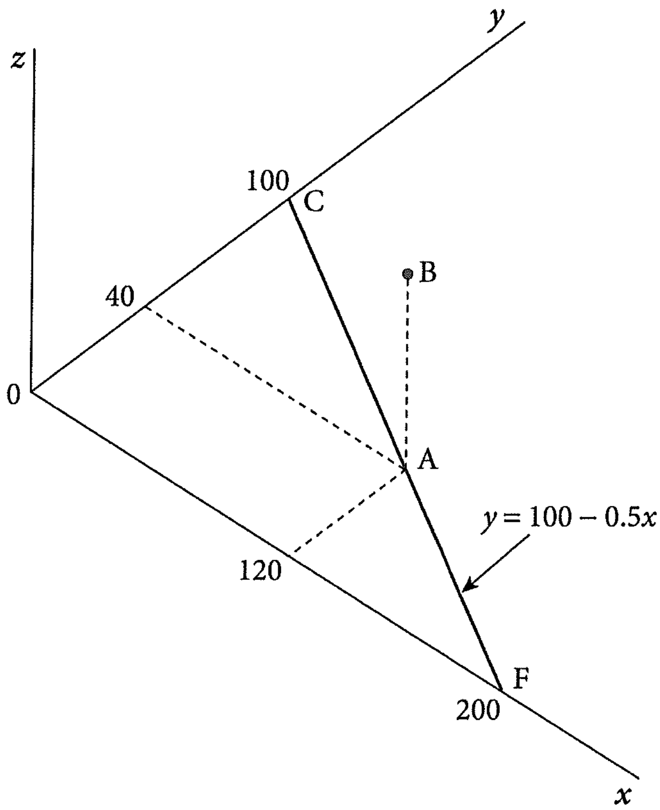
(a)
The surface $z = f(x, y)$
z
y
P
y_0
A
0
B
x_0
x
(b)
z
$\frac{\partial z}{\partial x} > 0$
P
$\frac{\partial z}{\partial x} < 0$
Section through surface with $y = y_0$. At P, $\frac{\partial z}{\partial x} = 0$
0
A
x_0
B
x
(c)
$\frac{\partial z}{\partial x}$
At P, graph of $\frac{\partial z}{\partial x}$ is negatively sloped, so $\frac{\partial^2 z}{\partial x^2}$ is negative.
0
x_0
x
Thus at P, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial^2 z}{\partial x^2} < 0$



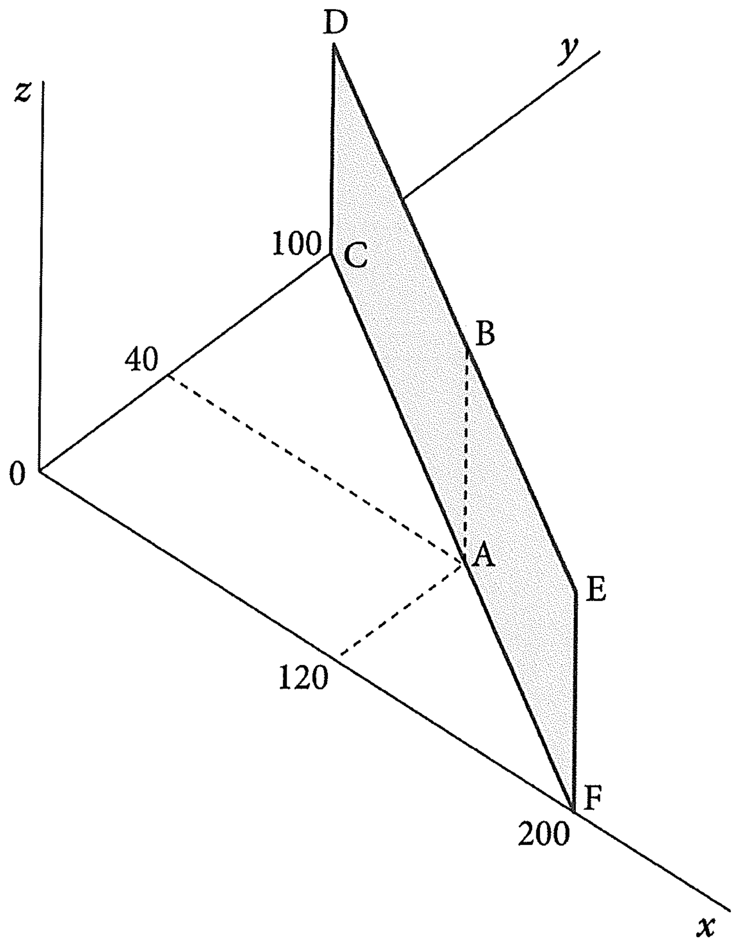
z
y
C
y_0
B
S
D
0
A
x_0
x



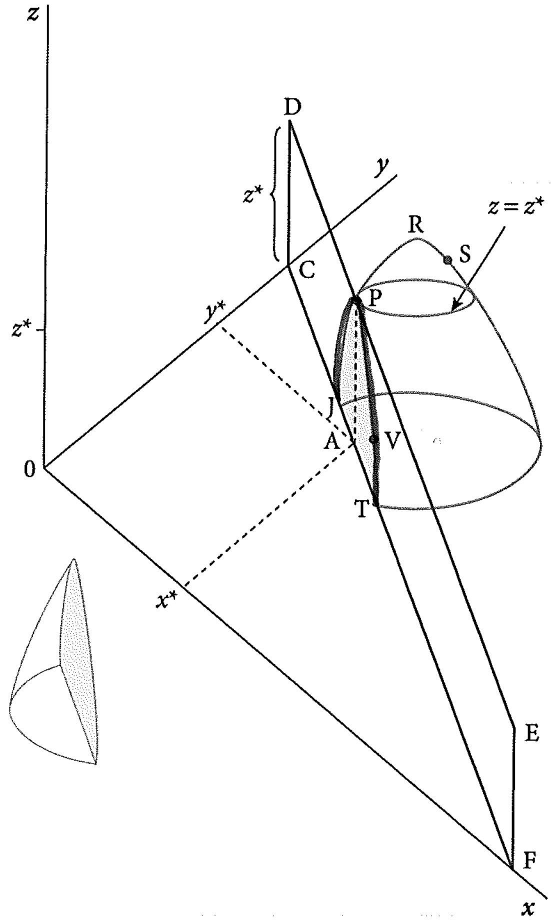
z
y
R
z = f(x, y)
0
x



All points such as A on the line $y = 100 - 0.5x$ satisfy the constraint, but so do all points that lie vertically above it. Thus B satisfies the constraint because it has the same x and y coordinates as A.

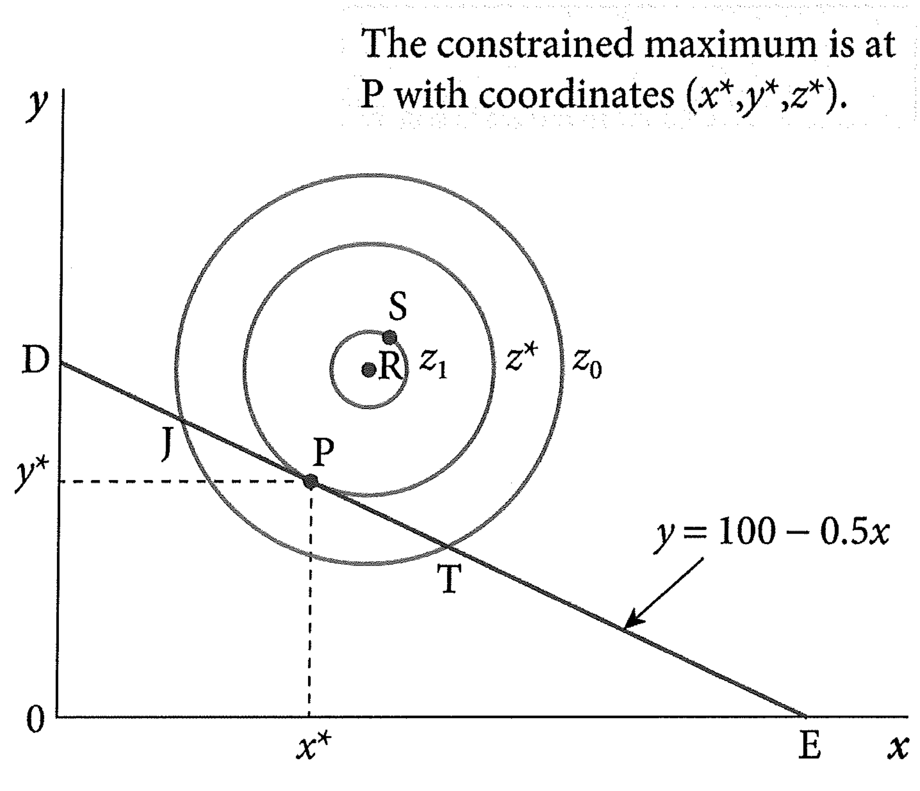


All points on the vertical plane CDEF satisfy the constraint $y = 100 - 0.5x$. The plane extends indefinitely upwards, as satisfying the constraint depends only on the values of y and x. The value of z is immaterial.



The line JPT comprises all the points that lie on the objective function and also satisfy the constraint. Of these points, P has the highest value of z and is therefore the constrained maximum value z.

A point such as V satisfies the constraint but the value of z is lower than at P. A point such as S has a higher value of z than at P but does not satisfy the constraint.


The constrained maximum is at P with coordinates (x^*,y^*,z^*).
y
D
S
R
z_1
z^*
z_0
J
P
y^*
$y = 100 - 0.5x$
T
0
x^*
E
x