Homework 1

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Macroeconomics-1

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Problem 1. .

(a) $U'(c_t) = \frac{1}{c_t^\sigma} > 0 \implies$ strictly increasing. (Since $\sigma > 0$)

Next, $U''(c_t) = -\sigma \frac{1}{c_t^{\sigma-1}} < 0 \implies$ strictly concave. (Since $\sigma > 0$)

And finally $\lim_{c_t \to 0} \frac{1}{c_t^\sigma} = +\infty$

(b) If $\sigma \leq 0 \implies \sigma \frac{1}{c_t^{\sigma-1}} \geq 0 \implies$ not strictly concave.

Also if $\sigma \leq 0 \implies \lim_{c_t \to 0} \frac{1}{c_t^\sigma} \neq +\infty$

(c) $\lim_{\sigma \to 1} \frac{c_t^{1-\sigma}-1}{\sigma-1} = \lim_{\sigma \to 1} \frac{e^{ln(c_t)(1-\sigma)}-1}{\sigma-1} = \lim_{\sigma \to 1} \frac{-e^{\ln(c_t)(1-\sigma)}\ln(c_t)}{-1} = \frac{-\ln(c_t)}{-1} = \ln(c_t)$

(d) $U'(c_t) = \frac{1}{c_t^\sigma}$ and $U''(c_t) = -\sigma \frac{1}{c_t^{\sigma-1}}$

Therefore, $-\frac{cU''(c_t)}{U'(c_t)} = \frac{cc^\sigma \sigma}{c^{\sigma+1}} = \sigma$ (a constant).

(e) $U(c_0, c_1, ..., c_T) = \sum_{t=0}^T \beta^t \frac{c_t^{1-\sigma}-1}{1-\sigma} = \sum_{t=0}^T \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} - \sum_{t=0}^T \frac{\beta^t}{1-\sigma}$

Let $g(x) = \left[x + \sum_{t=0}^T \frac{\beta^t}{1-\sigma}\right]^{\frac{1}{1-\sigma}}$ and $g'(x) = \frac{1}{1-\sigma}\left[x + \sum_{t=0}^T \frac{\beta^t}{1-\sigma}\right]^{\frac{1}{1-\sigma}-1} > 0. \implies$ Strictly increasing.

Therefore, $g(U(c_0, c_1, ..., c_T)) = \left[\sum_{t=0}^T \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} - \sum_{t=0}^T \frac{\beta^t}{1-\sigma} + \sum_{t=0}^T \frac{\beta^t}{1-\sigma}\right]^{\frac{1}{1-\sigma}} = \left[\sum_{t=0}^T \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}\right]^{\frac{1}{1-\sigma}}$

And if we set $\alpha^t = \frac{\beta^t}{1-\sigma}$

we get $g(U(c_0, c_1, ..., c_T)) = \left[\sum_{t=0}^T \alpha^t c_t^{1-\sigma}\right]^{\frac{1}{1-\sigma}} = \left[\alpha^0 c_0^{1-\sigma} + \alpha^1 c_1^{1-\sigma} + .. + \alpha^T c_T^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$

Therefore the utility function is a CES function.

(f) Since $\lim_{\sigma\to 1}\frac{c_t^{1-\sigma}-1}{\sigma-1}=\ln(c_t)$

$\implies$ when $\sigma=1$,
$U(c_0,c_1,...,c_T)=\sum_{t=0}^{T}\beta^t\frac{c_t^{1-\sigma}-1}{1-\sigma}=\sum_{t=0}^{T}\beta^t\ln(c_t)=\sum_{t=0}^{T}\ln(c_t^{\beta^t})=\big[\ln(c_0^{\beta^0})+\ln(c_1^{\beta^1})+....+\ln(c_T^{\beta^T})\big]$

Next let $g(x)=e^x$ and $g'(x)=e^x>0 \implies$ (Strictly increasing)

Therefore, $g(U(c_0,c_1,...,c_T))=e^{\big[\ln(c_0^{\beta^0})+\ln(c_1^{\beta^1})+...+\ln(c_T^{\beta^T})\big]}=e^{\big[\ln(c_0^{\beta^0}c_1^{\beta^1}...c_T^{\beta^T})\big]}=c_0^{\beta^0}c_1^{\beta^1}....c_T^{\beta^T}$

Thus the utility function is a Cobb Douglas function when $\sigma=1$.

(g) We will use the result from e. i.e. $g(U(c_0,c_1,...,c_t))=\big[\alpha^0c_0^{1-\sigma}+\alpha^1c_1^{1-\sigma}+..+\alpha^Tc_T^{1-\sigma}\big]^{\frac{1}{1-\sigma}}$

Therefore,

$$
\begin{aligned}
g(U(\beta c_0,\beta c_1,...,\beta c_T)) &= \big[\alpha^0(\beta c_0)^{1-\sigma}+\alpha^1(\beta c_1)^{1-\sigma}+..+\alpha^T(\beta c_T)^{1-\sigma}\big]^{\frac{1}{1-\sigma}}\\
&= \big[\alpha^0\beta^{1-\sigma}c_0^{1-\sigma}+\alpha^1\beta^{1-\sigma}c_1^{1-\sigma}+..+\alpha^T\beta^{1-\sigma}c_T^{1-\sigma}\big]^{\frac{1}{1-\sigma}}\\
&= \beta^{\frac{1-\sigma}{1-\sigma}}\big[\alpha^0c_0^{1-\sigma}+\alpha^1c_1^{1-\sigma}+..+\alpha^Tc_T^{1-\sigma}\big]^{\frac{1}{1-\sigma}}\\
&= \beta\big[\alpha^0c_0^{1-\sigma}+\alpha^1c_1^{1-\sigma}+..+\alpha^Tc_T^{1-\sigma}\big]^{\frac{1}{1-\sigma}}\\
&= \beta g(U(c_0,c_1,...,c_t))
\end{aligned}
$$

Thus $g(U(\beta c_0,\beta c_1,...,\beta c_T))=\beta g(U(c_0,c_1,...,c_t)) \implies U(c_0,c_1,...,c_T)$ is homothetic.

Problem 2. .

(a)

$$F(tk,tn) = A\big[\alpha(tk)^{1-\sigma} + (1-\alpha)(tn)^{1-\sigma}\big]^{\frac{1}{1-\sigma}}$$

$$= A\big[(t^{1-\sigma})(\alpha k^{1-\sigma} + (1-\alpha)n^{1-\sigma})\big]^{\frac{1}{1-\sigma}}$$

$$= t^{\frac{1-\sigma}{1-\sigma}} A\big[(\alpha k^{1-\sigma} + (1-\alpha)n^{1-\sigma}\big]^{\frac{1}{1-\sigma}}$$

$$= tA\big[(\alpha k^{1-\sigma} + (1-\alpha)n^{1-\sigma}\big]^{\frac{1}{1-\sigma}}$$

$$= tF(k,n)$$

Therefore, $F(tk,tn) = tF(k,n)$

(b) As hinted in the question we will first take the log of the equation in order to find the limit.

$$\log(F(k,n)) = \log(A) + \frac{1}{1-\sigma}\log\big[\alpha(k)^{1-\sigma} + (1-\alpha)(n)^{1-\sigma}\big]$$

$$\therefore \lim_{\sigma\to 1}\log(F(k,n)) = \log(A) + \lim_{\sigma\to 1}\frac{\log\big[\alpha(k)^{1-\sigma} + (1-\alpha)(n)^{1-\sigma}\big]}{1-\sigma}$$

$$= \log(A) + \lim_{\sigma\to 1}\frac{-\big[\alpha\log(k)e^{\log(k)(1-\sigma)} + (1-\alpha)\log(n)e^{\log(n)(1-\sigma)}\big]}{-\big[\alpha(k)^{1-\sigma} + (1-\alpha)(n)^{1-\sigma}\big]} \quad \text{(L'Hopitals rule)}$$

$$= \log(A) + \big[\alpha\log(k) + (1-\alpha)\log(n)\big]$$

$$\implies \lim_{\sigma\to 1}\log(F(k,n)) = \log(Ak^{\alpha}n^{1-\alpha})$$

Therefore, $\lim_{\sigma\to 1} e^{\log(F(k,n))} = e^{\log(Ak^{\alpha}n^{1-\alpha})} \iff \lim_{\sigma\to 1} F(k,n) = Ak^{\alpha}n^{1-\alpha}$

(c).

$F_k(k,n) = An^{1-\alpha}\alpha k^{\alpha-1} = \frac{An^{1-\alpha}\alpha}{k^{1-\alpha}} > 0$ (Since all terms are > 0)

$F_n(k,n) = \frac{A(1-\alpha)k^{\alpha}}{n^{\alpha}} > 0$ (Since all terms are > 0)

$\lim_{k\to 0}\frac{An^{1-\alpha}\alpha}{k^{1-\alpha}} = \frac{An^{1-\alpha}\alpha}{\lim_{k\to 0}k^{1-\alpha}} = \infty$

$\lim_{k\to \infty}\frac{An^{1-\alpha}\alpha}{k^{1-\alpha}} = \frac{An^{1-\alpha}\alpha}{\lim_{k\to \infty}k^{1-\alpha}} = 0$

(d).

$\frac{F_k(k,n)k}{F(k,n)} = \frac{An^{1-\alpha}\alpha(k^{\alpha-1})k}{An^{1-\alpha}(k^{\alpha})} = \alpha$

$\frac{F_n(k,n)n}{F(k,n)} = \frac{A(1-\alpha)k^{\alpha}n^{1-\alpha}}{An^{1-\alpha}(k^{\alpha})} = (1-\alpha)$

Problem 3. .

(a). We have $k_{t+1} = s(k_t)F(k_t, 1) + (1-\delta)k_t$

Further since $s(k_t) = \bar{s}$, $F(k_t, 1) = k_t^{\alpha}$

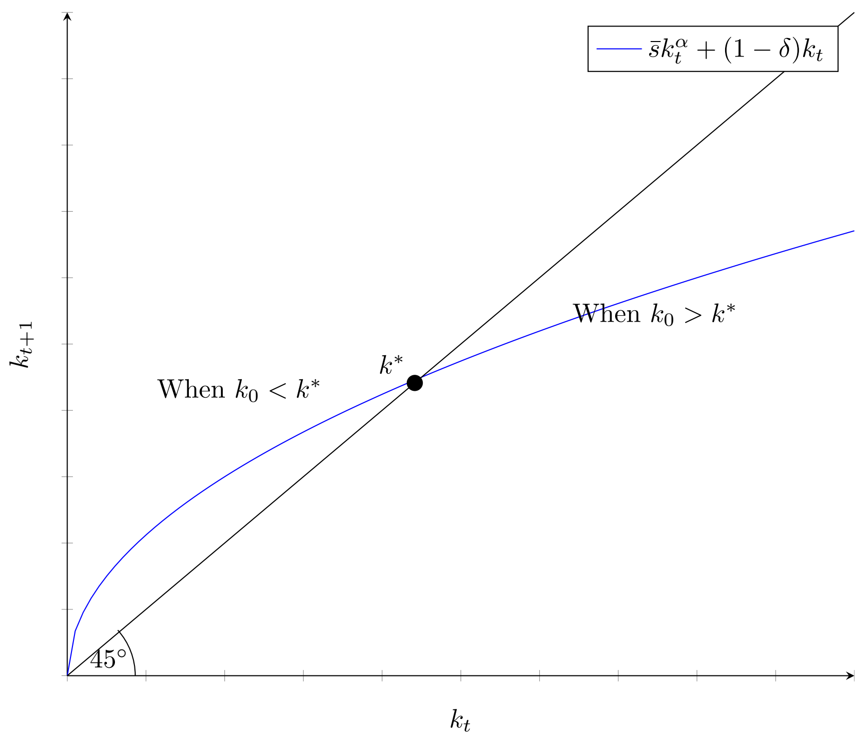
$k_{t+1} = \bar{s}k_t^{\alpha} + (1-\delta)k_t$

$\frac{dk_{t+1}}{dk_t} = \alpha\bar{s}k_t^{\alpha-1} + (1-\delta) > 0 \implies$ strictly increasing (Since all terms are > 0 and given $\alpha \in (0,1)$.)

$\frac{d^2k_{t+1}}{dk_t^2} = \alpha(\alpha-1)\ \bar{s}\ k_t^{\alpha-2} < 0 \implies$ strictly concave (Given $\alpha \in (0,1)$.)

Further when $k_t = 0 \implies k_{t+1} = 0$

There there $\exists! k^* > 0$ and below is the graphical representation.



(b) Since $s(k_t) = \beta k_t^{1-2\alpha}$ and $F(k_t, 1) = k_t^{\alpha}$

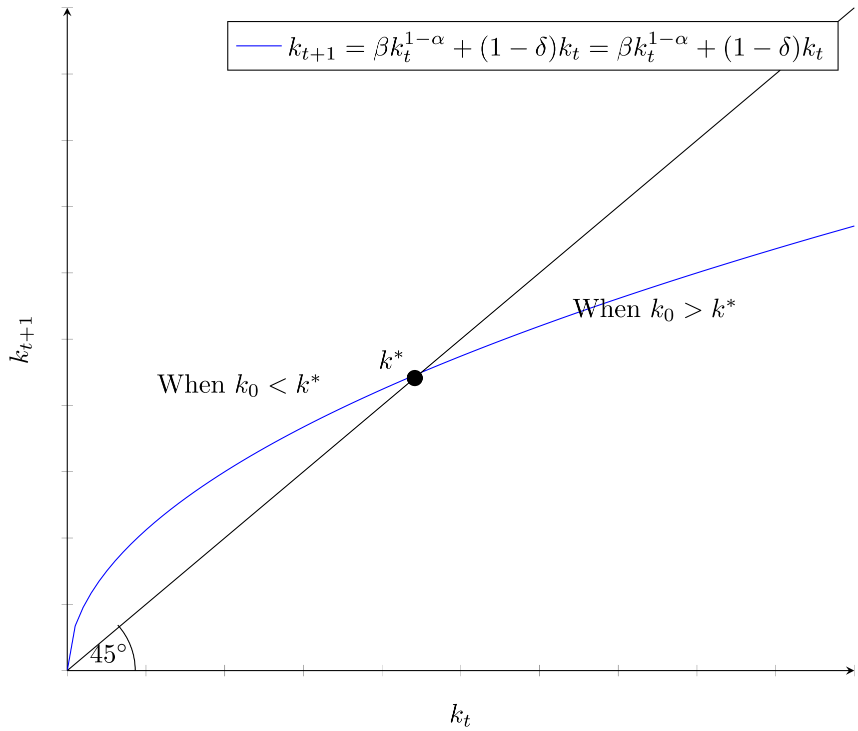
$k_{t+1} = \beta k_t^{1-2\alpha}k_t^{\alpha} + (1-\delta)k_t = \beta k_t^{1-\alpha} + (1-\delta)k_t$

$\frac{dk_{t+1}}{dk_t} = (1-\alpha)\ \beta\ k_t^{-\alpha} + (1-\delta) > 0 \implies$ strictly increasing (Since all terms are > 0 and given $\alpha \in (0,1)$.)

$\frac{d^2k_{t+1}}{dk_t^2} = -\alpha(1-\alpha)\ k_t^{-\alpha-1} < 0 \implies$ strictly concave (Given $\alpha \in (0,1)$.)

Further when $k_t = 0 \implies k_{t+1} = 0$

There there $\exists! k^* > 0$ and below is the graphical representation.

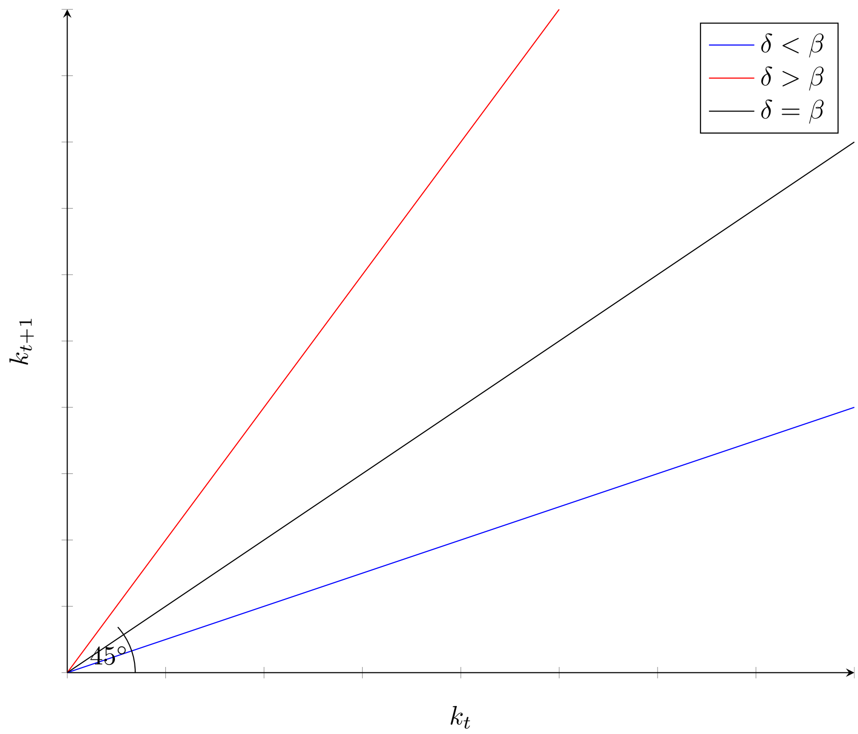


(c) . Since $s(k_t) = \beta k_t^{1-\alpha}$ and $F(k_t, 1) = k_t^{\alpha}$

$k_{t+1} = \beta k_t^{1-\alpha} k_t^{\alpha} + (1-\delta)k_t = \beta k_t^{+}(1-\delta)k_t$

$\frac{dk_{t+1}}{dk_t} = \beta\ +1-\delta > 0 \implies$ Strictly increasing.

$\frac{d^2k_{t+1}}{dk_t^2} = 0 \implies$ linear and below is the graphical representation.



(d) Since $s(k_t) = \beta k_t^{1-\alpha/2}$ and $F(k_t, 1) = k^\alpha$

$k_{t+1} = \beta k_t^{1-\alpha/2} k_t^\alpha + (1-\delta)k_t = \beta k_t^{1+\alpha/2} + (1-\delta)k_t$

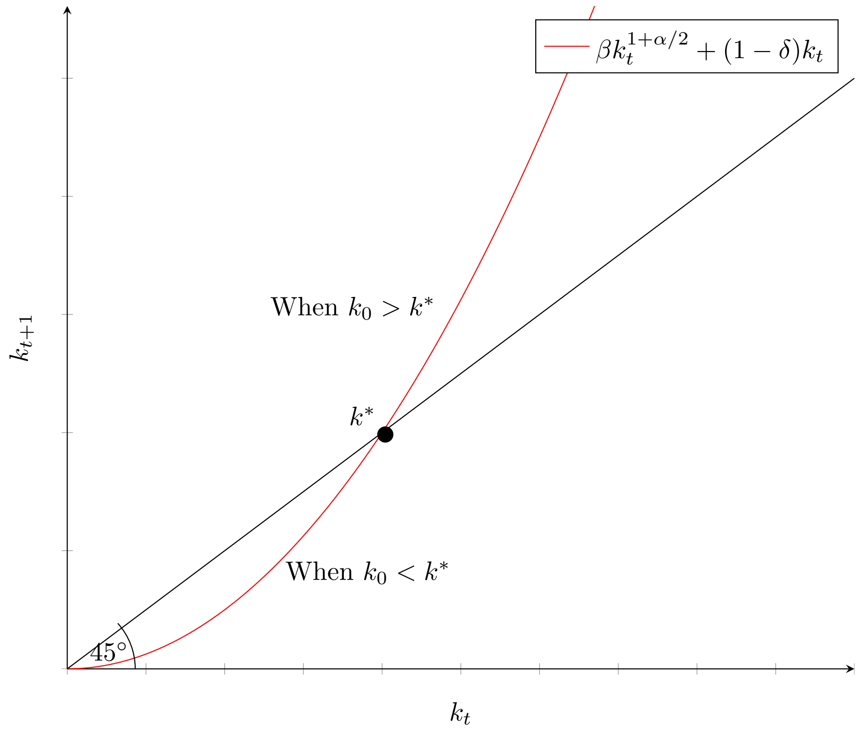
$\frac{dk_{t+1}}{dk_t} = (1-\alpha/2)\beta\ k_t^{\alpha/2} + (1-\delta) > 0 \implies$ strictly increasing (Since all terms are > 0 and given $\alpha \in (0,1)$.)

$\frac{d^2 k_{t+1}}{dk_t^2} = (1-\alpha/2)(\alpha/2)\ k_t^{\alpha/2-1} > 0 \implies$ strictly convex (Given $\alpha \in (0,1)$.)

Further when $k_t = 0 \implies k_{t+1} = 0$

There there $\exists! k^* > 0$ and below is the graphical representation.

Below is the graphical representation.



(e) In order to have a unique $k^* > 0$ we need $g(k_t)$ strictly concave or strictly convex and $g(0) = 0$.

Further the saving function $s(k_t) > 0$ and $s'(k_t) \geq 0$

Problem 4. .

(a) From the question we have $(1+g)k_{t+1} = k_t - \delta k_t + i_t$ and since $i_t - \delta k_t = s_\alpha y_t \implies (1+g)k_{t+1} = k_t + s_\alpha y_t$

We get $(1+g)k^* = k^* + s_\alpha y_t \iff gk^* = s_\alpha y_t \iff k^* = \frac{s_\alpha y_t}{g}$

Therefore, $\frac{k^*}{y^*} = \frac{s_\alpha}{g} \implies$ if growth rate (g) decreases to 0 $\frac{k^*}{y^*}$ will approach infinity given $s_\alpha > 0$. The economic intuition is that if the economy stops growing wages stay constant and the value of the existing capital rises.

(b) $\frac{r^* k^*}{y^*} = \frac{r s_\alpha}{g} \implies$ if growth rate (g) decreases to 0 the capital income share $\frac{r^* k^*}{y^*}$ will approach infinity. The economic intuition follows the one in part a and since capital is more valuable it will yield a much higher return compared to wages.

(c) We have $(1+g)k_{t+1} = (1-\delta)k_t + i_t$ and since $i_t = s_b y_t \implies (1+g)k_{t+1} = (1-\delta)k_t + s_b y_t$.

$\therefore (1+g)k^* = (1-\delta)k^* + s_b y_t \iff (g+\delta)k^* = s_b y_t \iff k^* = \frac{s_\alpha y_t}{(g+\delta)}$

Therefore, $\frac{k^*}{y^*} = \frac{s_b}{(g+\delta)} \implies$ if growth rate (g) decreases to 0 $\frac{k^*}{y^*}$, unlike the case in part a, will not approach infinity given $\delta \neq 0$. Furthermore, the capital income share $\frac{r^* k^*}{y^*}$ will not approach infinity.