Homework 7

Jugal Marfatia

Microeconomics-1

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Problem 1. .

a. Expected value $= 0.5 * 4,000 + 0.5 * 104,000 = 54,000$

b. Expected utility $= 0.5 * \sqrt{4,000} + 0.5 * \sqrt{104,000} = 192.87$

c.

d. In order to find risk premium we need to solve for RP in the below equation:

$192.87 = \sqrt{54,000 - RP} \iff (192.87)^2 - 54,000 = -RP$

$\iff RP = 54,000 - (192.87)^2 = 16801.16$

e. Certainty Equivalence $= 54,000 - RP = 54,000 - 16801.16 = 37198.84$

f. In the above graph.

Problem 2. .

$u'(x) = \beta^2(\alpha+\beta x)^{-\frac{1}{\beta}}$ and $u''(x) = -\beta^2(\alpha+\beta x)^{\frac{-(1+\beta)}{\beta}}$

Therefore the Arrow-Pratt coefficient of absolute risk-aversion,

$$r_A(x,u) = \frac{-u''(x)}{u'(x)} = \frac{\beta^2(\alpha+\beta x)^{\frac{-(1+\beta)}{\beta}}}{\beta^2(\alpha+\beta x)^{-\frac{1}{\beta}}} = (\alpha+\beta x)^{-1}$$

Problem 3. .

a. The investor's expected utility maximization problem is below:

$$\max_{x_1,x_2\geq 0} \frac{1}{2}(-e^{-\alpha x_1}) + \frac{1}{2}(-e^{-\alpha x_2}) \text{ s.t. } \pi_1 x_1 + \pi_2 x_2 \leq w$$

b. In order to find the utility-maximizing purchases of assets 1 and 2 we need to solve the below Lagrangian.

$$L = \frac{1}{2}(-e^{-\alpha x_1}) + \frac{1}{2}(-e^{-\alpha x_2}) - \lambda\big[\pi_1 x_1 + \pi_2 x_2 - w\big]$$

F.O.C is below:

$$\frac{dL}{dx_1} : \frac{\alpha}{2}(e^{-\alpha x_1}) = \lambda\pi_1 \tag{1}$$

$$\frac{dL}{dx_2} : \frac{\alpha}{2}(e^{-\alpha x_2}) = \lambda\pi_2 \tag{2}$$

$$\frac{dL}{d\lambda} : w = \pi_1 x_1 + \pi_2 x_2 \tag{3}$$

Dividing equation 1 and 2 we get:

$$\frac{\frac{\alpha}{2}(e^{-\alpha x_1})}{\frac{\alpha}{2}(e^{-\alpha x_2})} = \frac{\lambda\pi_1}{\lambda\pi_2} \iff e^{-\alpha x_1+\alpha x_2} = \frac{\pi_1}{\pi_2}$$

By taking log on both sides we get:

$$-\alpha(x_1 - x_2) = \ln(\pi_1) - \ln(\pi_2) \iff x_1 = x_2 - \frac{1}{\alpha}\Big[\ln(\pi_1) - \ln(\pi_2)\Big]$$

Next plugging x_1 back into the equation 3 we get:

$$w = \pi_1 x_2 - \pi_1 \frac{1}{\alpha}\Bigg[\ln(\pi_1) - \ln(\pi_2)\Bigg] + \pi_2 x_2 \iff x_2^* = w + \frac{\pi_1}{\alpha}\Bigg[\ln(\pi_1) - \ln(\pi_2)\Bigg]$$

Further using symmetry we get:

$$x_1^* = w + \frac{\pi_2}{\alpha}\Bigg[\ln(\pi_2) - \ln(\pi_1)\Bigg]$$

c. $\forall i \in \{1, 2\}, u'(x_i) = \frac{\alpha}{2}(e^{-\alpha x_1})$ and $u''(x_i) = -\frac{\alpha^2}{2}(e^{-\alpha x_i})$

Therefore the Arrow-Pratt coefficient of absolute risk-aversion,

$$r_A(x, u) = \frac{-u''(x)}{u'(x)} = \frac{\frac{\alpha^2}{2}(e^{-\alpha x_i})}{\frac{\alpha}{2}(e^{-\alpha x_i})} = \alpha$$

Further $\dfrac{dx_2^*}{dw} = -\frac{\pi_1}{\alpha^2}\Bigg[\ln(\pi_1) - \ln(\pi_2)\Bigg] \leq 0$

And $\dfrac{dx_1^*}{dw} = -\frac{\pi_2}{\alpha^2}\Bigg[\ln(\pi_2) - \ln(\pi_1)\Bigg] \leq 0$

From the above two derivatives we know that an increase in α reduces the holding of assets 1 and 2. In other words, since α is equal to the Arrow-Pratt coefficient of absolute risk-aversion, an increase in risk aversion reduces the holding of assets 1 and 2.

d. In this case α is equal to the Arrow-Pratt coefficient of absolute risk-aversion. i.e. the risk aversion does not depend on the wealth level. This is entirely due to the form of the given utility function as Arrow-Pratt coefficient of absolute risk-aversion, $r_A(x, u) = \dfrac{-u''(x)}{u'(x)}$. In case of some other functional forms of the utility function the risk aversion could depend on wealth.

Problem 4. .

a. Max's expected utility maximization problem is below:

$$\max \left\{ u(c_0) + \delta u(c_1) \right\} \text{ s.t. } c_0 = w_0 - s, c_1 = w_1 + \rho s$$

Or similarly:

$$\max \left\{ u(w_0 - s^*) + \delta u(w_1 + \rho s^*) \right\}$$

Therefore the FOC with respect to s is:

$-u'(w_0 - s^*) + \delta \rho u'(w_1 + \rho s^*) = 0$

Therefore in order for $s^* > 0$, we need $-u'(w_0) + \delta \rho u'(w_1) > 0$

b. If we set $w_1 = 0 \implies u'(w_0 - s^*) = \delta \rho u'(\rho s^*)$

Next differentiating both the sides with respect to ρ we get

$$-u''(w_0 - s^*)\frac{ds^*}{d\rho} = \delta \rho u'(\rho s^*) + \delta \rho u''(\rho s^*)\left[s^* + \rho \frac{ds^*}{d\rho}\right]$$

$$\iff \left[-\delta \rho^2 u''(\rho s^*) - u''(w_0 - s^*)\right]\frac{ds^*}{d\rho} = \delta \rho u'(\rho s^*) + \delta \rho u''(\rho s^*) s^*$$

Next dividing both the sides by $u'(w_0 - s^*)$ we get

$$\iff \left[-\frac{\delta \rho^2 u''(ps^*)}{u'(w_0 - s^*)} - \frac{u''(w_0 - s^*)}{u'(w_0 - s^*)}\right]\frac{ds^*}{d\rho} = \frac{\delta \rho u'(\rho s^*)}{u'(w_0 - s^*)} + \frac{\delta \rho s^* u''(\rho s^*)}{u'(w_0 - s^*)}$$

$$\iff \left[-\frac{\delta \rho^2 u''(ps^*)}{\delta \rho u'(\rho s^*)} - \frac{u''(w_0 - s^*)}{u'(w_0 - s^*)}\right]\frac{ds^*}{d\rho} = \frac{\delta \rho u'(\rho s^*)}{\delta \rho u'(\rho s^*))} + \frac{\delta \rho s^* u''(\rho s^*)}{\delta \rho u'(\rho s^*)} \qquad \text{(Since } u'(w_0 - s^*) = \delta \rho u'(\rho s^*)\text{)}$$

$$\iff \left[-\frac{\delta \rho^2 u''(ps^*)}{\delta \rho u'(\rho s^*)} - \frac{u''(w_0 - s^*)}{u'(w_0 - s^*)}\right]\frac{ds^*}{d\rho} = \frac{1}{\rho}\left[1 + \frac{\rho s^* u''(\rho s^*)}{u'(\rho s^*)}\right]$$

Further in the question we are supposed to assume interior solution which implies the utility function is increasing and concave i.e $u' > 0$ and $u'' < 0$

$$\implies \frac{\delta \rho^2 u''(ps^*)}{\delta \rho u'(\rho s^*)} < 0 \text{ and } \frac{u''(w_0 - s^*)}{u'(w_0 - s^*)} < 0 \implies \left[-\frac{\delta \rho^2 u''(ps^*)}{\delta \rho u'(\rho s^*)} - \frac{u''(w_0 - s^*)}{u'(w_0 - s^*)}\right] > 0$$

Therefore, the necessary and sufficient condition for s to be (locally) increasing in ρ i.e $\frac{ds^*}{d\rho} > 0$ is $\left[1 + \frac{\rho s^* u''(\rho s^*)}{u'(\rho s^*)}\right] > 0 \iff -\left[\frac{\rho s^* u''(\rho s^*)}{u'(\rho s^*)}\right] < 1$

c. Under uncertainty the original FOC is as the following:

$$-u'(w_0 - s^{**}) + \delta\rho E\left[u'(w_1 + \tilde{x} + \rho s^{**})\right] = 0$$

Further using Jensen's inequality we get:

$$-u'(w_0 - s^*) + \delta\rho E\left[u'(w_1 + \tilde{x} + \rho s^*)\right] > -u'(w_0 - s^*) + \delta\rho u'(w_1 + E(\tilde{x}) + \rho s^*)$$

Further from the question we know $E(\tilde{x}) = 0$, therefore we get:

$$-u'(w_0 - s^*) + \delta\rho E\left[u'(w_1 + \tilde{x} + \rho s)\right] > -u'(w_0 - s) + \delta\rho u'(w_1 \rho s) = 0 \text{ (From part a)}$$

$$\iff -u'(w_0 - s) + \delta\rho E\left[u'(w_1 + \tilde{x} + \rho s)\right] > 0$$

Therefore, at any level s^{**} is positive and as shown in the below graph $s^{**} > s^*$