Homework 3

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Microeconomics-1

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Problem 1. .

a.

$$U(x_1, x_2) = \left[\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right]^{1/\rho}$$

Therefore, if $\rho = 1$

$$U = \left[\alpha_1 x_1 + \alpha_2 x_2\right]$$

$$\iff x_1 = \frac{U}{\alpha_1} - \frac{\alpha_2}{\alpha_1} x_2$$

Therefore, the indifference curves are linear

b.

$$\log(U) = \frac{1}{\rho} \log\left[\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right]^{1/\rho}$$

Therefore, when $\rho \to 0$

$$\log(U) = \lim_{\rho \to 0} \frac{1}{\rho} \log\left[\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right]^{1/\rho}$$

Using L'Hopital rule we get

$$\log(U) = \lim_{\rho \to 0} \frac{\alpha_1 x_1^{\rho} \log(x_1) + \alpha_2 x_2^{\rho} \log(x_1)}{\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}}$$

$$= \frac{\alpha_1 \log(x_1) + \alpha_2 \log(x_1)}{\alpha_1 + \alpha_2} = \log(x_1^{\alpha_1} x_2^{\alpha_2})$$

Therefore, when $\rho \to 0, \log(U) = \log(x_1^{\alpha_1} x_2^{\alpha_2}) \iff U = x_1^{\alpha_1} x_2^{\alpha_2}$.

c. We can prove this by first assuming that $x_1 \geq x_2$ and then showing $\left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]^{1/\rho} = min\{x_1, x_2\} = x_2$ when $\rho \to \infty$

$$x_1 \geq x_2 \iff x_1^\rho \leq x_2^\rho \qquad \text{(Since } \rho < 0\text{)}$$

$$\iff \alpha_1 x_1^\rho \leq \alpha_1 x_2^\rho \qquad \text{(Since } \alpha_1 \geq 0\text{)}$$

$$\iff \alpha_1 x_1^\rho + \alpha_2 x_2^\rho \leq \alpha_1 x_2^\rho + \alpha_2 x_2^\rho$$

$$\iff \left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]^{\frac{1}{p}} \geq \left[\alpha_1 x_2^\rho + \alpha_2 x_2^\rho\right]^{\frac{1}{p}} \qquad \text{(Since } 1/\rho < 0\text{)}$$

$$\iff \left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]^{\frac{1}{p}} \geq \left[(\alpha_1 + \alpha_2) x_2^\rho\right]^{\frac{1}{p}} \qquad (1)$$

Further we also have the below:

$$\alpha_2 x_2^\rho \leq \alpha_1 x_1^\rho + \alpha_2 x_2^\rho \iff \left[\alpha_2 x_2^\rho\right]^{\frac{1}{p}} \geq \left[\alpha_1 x_2^\rho + \alpha_2 x_2^\rho\right]^{\frac{1}{p}} \qquad (2)$$

Therefore combining equation 1 and 2 we get the below:

$$\left[\alpha_2 x_2^\rho\right]^{\frac{1}{p}} \geq \left[\alpha_1 x_2^\rho + \alpha_2 x_2^\rho\right]^{\frac{1}{p}} \geq \left[(\alpha_1 + \alpha_2) x_2^\rho\right]^{\frac{1}{p}}$$

$$\iff \lim_{\rho \to -\infty} \left[\alpha_2 x_2^\rho\right]^{\frac{1}{p}} \geq \lim_{\rho \to -\infty} \left[\alpha_1 x_2^\rho + \alpha_2 x_2^\rho\right]^{\frac{1}{p}} \geq \lim_{\rho \to -\infty} \left[(\alpha_1 + \alpha_2) x_2^\rho\right]^{\frac{1}{p}}$$

$$\iff x_2 \geq \lim_{\rho \to -\infty} \left[\alpha_1 x_2^\rho + \alpha_2 x_2^\rho\right]^{\frac{1}{p}} \geq x_2$$

$$\implies \lim_{\rho \to -\infty} \left[\alpha_1 x_2^\rho + \alpha_2 x_2^\rho\right]^{\frac{1}{p}} = x_2$$

Therefore, $\left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]^{1/\rho} = min\{x_1, x_2\}$ when $\rho \to \infty$ and thus represents two goods that are perfect complements.

Problem 2. .

First we need to find indirect utility, $v(p,w)$, by setting $u = v(p,w)$, in the expenditure function.

$w = g(p) + \big[v(p,w) + f(p)\big] \iff v(p,w) = \dfrac{w - g(p)}{f(p)}$

Next we need to find Walrasian Demand, using Roy's identity $x_i(p,w) = -\dfrac{\dfrac{\partial v(p,w)}{\partial p_i}}{\dfrac{\partial v(p,w)}{\partial w}}$

$$\frac{\partial v(p,w)}{\partial p_i} = \frac{-f(p)g_{p_i}(p) - (w - g(p))f_{p_i}(p)}{f(p)^2} \qquad \text{and} \qquad \frac{\partial v(p,w)}{\partial w} = \frac{1}{f(p)}$$

(Note: $g_{p_i}(p) = \dfrac{\partial g(p)}{\partial p_i}$ and $f_{p_i}(p) = \dfrac{\partial f(p)}{\partial p_i}$).

Next, Using the above we get:

$$x_i = -\frac{\dfrac{\partial v(p,w)}{\partial p_i}}{\dfrac{\partial v(p,w)}{\partial w}} = \frac{f(p)g_{p_i}(p) + (w - g(p))f_{p_i}(p)}{f(p)} = g_{p_i}(p) + \frac{f_{p_i}(p)}{f(p)}(w - g(p))$$

$$= g_{p_i}(p) + \frac{f_{p_i}(p)}{f(p)}w - \frac{f_{p_i}(p)}{f(p)}g(p)$$

Therefore, $\frac{\partial x_i}{\partial w} = \frac{f_{p_i}(p)}{f(p)}$

Next we need to check what happens to $\epsilon_{x_i,w} = \frac{\partial x_i}{\partial w} * \frac{w}{x_i}$, when $w \to \infty$

$$\lim_{w\to\infty} \epsilon_{x_i,w} = \lim_{w\to\infty} \frac{f_{p_i}(p)}{f(p)} * \frac{w}{g_{p_i}(p) + \dfrac{f_{p_i}(p)}{f(p)}w - \dfrac{f_{p_i}(p)}{f(p)}g(p)}$$

Using L'Hopital rule we get

$$\lim_{w\to\infty} \epsilon_{x_i,w} = \frac{f_{p_i}(p)}{f(p)} * \frac{f(p)}{f_{p_i}(p)} = 1$$

Therefore, since $\lim_{w\to\infty} \epsilon_{x_i,w} = 1 \implies$ a 1 % increase in wealth leads to exactly a 1 % increase in consumption.

Problem 3. .

a.

$$L = \sum_{i=1}^{L} \alpha_i \; ln \; x_i + \lambda\big[w - \sum_{i=1}^{L} p_i x_i\big] \tag{1}$$

F.O.C is below:

$$\frac{dL}{dx_i} = \frac{\alpha_i}{x_i} - \lambda p_i = 0 \iff \lambda = \frac{\alpha_i}{p_i x_i}$$

Therefore the shadow price of wealth, $\lambda^* = \sum_{i=1}^{L} \frac{\alpha_i}{p_i x_i} = \frac{1}{w}$.

(Since $\sum_{i=1}^{L} \alpha_i = 1$ and $\sum_{i=1}^{L} p_i x_i = w$)

Next, plugging λ^* into equation we get the Walrasian Demand $x_i(p,w) = \frac{\alpha_i}{p_i} w$

b. The indirect utility function is,

$$v(p,w) = \sum_{i=1}^{L} \alpha_i \; ln\left(\frac{\alpha_i w}{p_i}\right) = ln(w) - \sum_{i=1}^{L} \alpha_i \; ln(p_i) + \sum_{i=1}^{L} \alpha_i \; ln(\alpha_i)$$

Next the shadow price of wealth $= \frac{\partial v(p,w)}{\partial w} = \frac{1}{w}$.

This does coincide with what we found in part (a).

Problem 4. .

From homework 2 where we had the same utility function (i.e if $\alpha_1 = \alpha_2 = 1$), the Walrasian demand were:

$$x_2^* = \frac{w}{p_1\left(\dfrac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} + p_2} = \frac{w\ p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}$$

And:

$$x_1^* = \frac{w}{p_2\left(\dfrac{p_2}{p_1}\right)^{\frac{1}{\rho-1}} + p_1} = \frac{w\ p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}$$

Thus the indirect utility function is:

$$v(p,w) = w\left[\frac{\left(p_2^{\frac{p}{\rho-1}} +\ p_1^{\frac{p}{\rho-1}}\right)}{\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\rho}}\right]^{\frac{1}{\rho}} = \frac{w}{\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}}$$

Therefore if we set $w = e(u,p)$ we get the expenditure function:

$$e(u,p) = \bar{u}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}.$$

Thus the hicksian demand for good i is $x_i^H(p,u) = \dfrac{\partial e(u,p)}{\partial p_i} = \bar{u}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{-1}{\rho}} p_i^{\frac{1}{\rho-1}}$

Now we will verify the properties of 3.E.2

1. $e(u,\alpha p) = \bar{u}\left((\alpha p_1)^{\frac{\rho}{\rho-1}} + (\alpha p_2)^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} = \alpha\bar{u}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} = \alpha e(u,\alpha p) \implies$ homogenous of degree 1.

2. $\dfrac{\partial e(u,p)}{\partial p_i} = \bar{u}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{-1}{\rho}} p_i^{\frac{1}{\rho-1}} \geq 0 \implies$ non decreasing for all p_i.

$\dfrac{\partial e(u,p)}{\partial u} = \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} > 0 \implies$ increasing in u.

3. $\dfrac{\partial^2 e(u,p)}{\partial p_i^2} = -\frac{1}{\rho}\bar{u}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{-1-\rho}{\rho}} p_i^{\frac{-1}{\rho-1}} + \frac{1}{\rho-1}\bar{u}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{-1}{\rho}} p_i^{\frac{2-\rho}{\rho-1}} < 0 \implies$ concave for all p_i.

4. It is clear from the function that the function is defined and continous for all u and p_i.

Finally we will verify the properties of 3.E.3

1. $x_i^H(\alpha p, u) = \bar{u}\left((\alpha p_1)^{\frac{\rho}{\rho-1}} + (\alpha p_2)^{\frac{\rho}{\rho-1}}\right)^{\frac{-1}{\rho}} (\alpha p_i)^{\frac{1}{\rho-1}}$

$= \alpha^{\frac{\rho-\rho}{\rho-1}} \bar{u}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{-1}{\rho}} p_i^{\frac{1}{\rho-1}} = \bar{u}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{-1}{\rho}} p_i^{\frac{1}{\rho-1}} = x_i^H(p,u) \implies$ homogeneous of degree 0.

2. By plugging in the hicksion demand in the utility function we get:

$u(H(p,u)) = \left[u^\rho \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{-\rho}{\rho}} p_1^{\frac{\rho}{\rho-1}} + u^\rho \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{-\rho}{\rho}} p_2^{\frac{\rho}{\rho-1}}\right]^{\frac{1}{\rho}}$

$= u\left[\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-1} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)\right]^{\frac{1}{\rho}} = u$

Therefore, there is no excess utility i.e $\forall x \in H(p,u), u(x) = u$

3. Corollary to the previous property it is straightforward that there exists a unique element in $H(p,u)$ given a prices and utility level.

Problem 5. .

As mentioned in 3.D.6 we will assume $\alpha+\beta+\gamma=1$

a.

$$L = p_1x_1 + p_2x_2 + p_3x_3 + \lambda\big[\bar{u} - (x_1-b_1)^\alpha(x_2-b_2)^\beta(x_3-b_3)^\gamma\big]$$

F.O.C's are below:

$$\frac{dL}{dx_1} = p_1 - \lambda\alpha(x_1-b_1)^{\alpha-1}(x_2-b_2)^\beta(x_3-b_3)^\gamma = 0 \tag{1}$$

$$\frac{dL}{dx_2} = p_2 - \lambda\beta(x_1-b_1)^{\alpha}(x_2-b_2)^{\beta-1}(x_3-b_3)^\gamma = 0 \tag{2}$$

$$\frac{dL}{dx_3} = p_3 - \lambda\gamma(x_1-b_1)^{\alpha}(x_2-b_2)^{\beta}(x_3-b_3)^\gamma = 0 \tag{3}$$

$$\frac{dL}{d\lambda} = \bar{u} - (x_1-b_1)^{\alpha}(x_2-b_2)^{\beta}(x_3-b_3)^{\gamma-1} = 0 \tag{4}$$

Dividing equation 1 by 2 we get:

$$\frac{\lambda\alpha(x_1-b_1)^{\alpha-1}(x_2-b_2)^\beta(x_3-b_3)^\gamma}{\lambda\beta(x_1-b_1)^{\alpha}(x_2-b_2)^{\beta-1}(x_3-b_3)^\gamma} = \frac{p_1}{p_2} \iff x_2 = (x_1-b_1)\left(\frac{\beta p_1}{\alpha p_2}\right) + b_2$$

And dividing equation 1 by 3 we get:

$$\frac{\lambda\alpha(x_1-b_1)^{\alpha-1}(x_2-b_2)^\beta(x_3-b_3)^\gamma}{\lambda\gamma(x_1-b_1)^{\alpha}(x_2-b_2)^{\beta}(x_3-b_3)^\gamma} = \frac{p_1}{p_3} \iff x_3 = (x_1-b_1)\left(\frac{\gamma p_1}{\alpha p_3}\right) + b_3$$

Next plugging x_2 and x_3 back into the utility constraint we get:

$$\bar{u} = (x_1-b_1)^\alpha\left[(x_1-b_1)\left(\frac{\beta p_1}{\alpha p_2}\right) + b_2 - b_2\right]^\beta\left[(x_1-b_1)\left(\frac{\gamma p_1}{\alpha p_3}\right) + b_3 - b_3\right]^\gamma$$

$$\therefore \bar{u} = (x_1-b_1)\left(\frac{\beta p_1}{\alpha p_2}\right)^\beta\left(\frac{\gamma p_1}{\alpha p_3}\right)^\gamma = (x_1-b_1)\left(\frac{\alpha}{p_1}\right)^\alpha\left(\frac{\beta}{p_2}\right)^\beta\left(\frac{\gamma}{p_3}\right)^\gamma\left(\frac{p_1}{\alpha}\right)$$

Therefore the hicksian demand for good 1 is

$$x_1^H = \bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\left(\frac{\alpha}{p_1}\right) + b_1$$

And by symmetry the hicksian demand for good 2 and 3 are below

$$x_2^H = \bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\left(\frac{\beta}{p_2}\right)+b_2 \qquad \text{and} \qquad x_3^H = \bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\left(\frac{\gamma}{p_3}\right)+b_3$$

Further using the hicksian demands the expenditure function is:

$$e(p,u) = p_1\left[\bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\left(\frac{\alpha}{p_1}\right) + b_1\right] + p_2\left[\bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\left(\frac{\beta}{p_2}\right) + b_2\right] + p_3\left[\bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\left(\frac{\gamma}{p_3}\right) + b_3\right]$$

Therefore $e(p,u) = \sum_{i=1}^{3} p_i b_i + (\alpha+\beta+\gamma)\bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma} = \sum_{i=1}^{3} p_i b_i + \bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}$

Now we will verify the properties of 3.E.2

1. $e(u,tp) = t\sum_{i=1}^{3} p_i b_i + t\bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma} = te(u,p) \implies$ homogenous of degree1

2. $\frac{\partial e(u,p)}{\partial p_1} = \bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\left(\frac{\alpha}{p_1}\right) + b_1 \geq 0 \implies$ non decreasing for p_1.

$\frac{\partial e(u,p)}{\partial p_2} = \bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\left(\frac{\beta}{p_2}\right) + b_2 \geq 0 \implies$ non decreasing for p_2.

$\frac{\partial e(u,p)}{\partial p_3} = \bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\left(\frac{\gamma}{p_3}\right) + b_3 \geq 0 \implies$ non decreasing for p_3.

$\frac{\partial e(u,p)}{\partial u} = \left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma} > 0 \implies$ increasing in u.

3. $\dfrac{\partial^2 e(u,p)}{\partial p_1^2} = -(1-\alpha)\bar{u}\left(\dfrac{p_1}{\alpha}\right)^{\alpha}\left(\dfrac{p_2}{\beta}\right)^{\beta}\left(\dfrac{p_3}{\gamma}\right)^{\gamma}\left(\dfrac{\alpha}{p_1^2}\right) < 0 \implies$ concave for p_1 (Since $\alpha < 1$).

$\dfrac{\partial^2 e(u,p)}{\partial p_2^2} = -(1-\beta)\bar{u}\left(\dfrac{p_1}{\alpha}\right)^{\alpha}\left(\dfrac{p_2}{\beta}\right)^{\beta}\left(\dfrac{p_3}{\gamma}\right)^{\gamma}\left(\dfrac{\alpha}{p_2^2}\right) + b_1 < 0 \implies$ concave for p_2 (Since $\beta < 1$).

$\dfrac{\partial^2 e(u,p)}{\partial p_3^2} = -(1-\gamma)\bar{u}\left(\dfrac{p_1}{\alpha}\right)^{\alpha}\left(\dfrac{p_2}{\beta}\right)^{\beta}\left(\dfrac{p_3}{\gamma}\right)^{\gamma}\left(\dfrac{\gamma}{p_3^2}\right) + b_1 < 0 \implies$ concave for p_3 (Since $\gamma < 1$).

4. It is clear from the functional form that the it is defined and continous for all u and p_i.

Finally we will verify the properties of 3.E.3, we will check for x_1^H since if the properties hold for x_1^H it will hold for x_2^H and x_3^H because of symmetry :

1. $x_1^H(tp,u) = \bar{u}\left(\dfrac{tp_1}{\alpha}\right)^{\alpha}\left(\dfrac{tp_2}{\beta}\right)^{\beta}\left(\dfrac{tp_3}{\gamma}\right)^{\gamma}\left(\dfrac{\alpha}{tp_1}\right) + b_1$

$= t^{\alpha+\beta+\gamma-1}\ \bar{u}\left(\dfrac{p_1}{\alpha}\right)^{\alpha}\left(\dfrac{p_2}{\beta}\right)^{\beta}\left(\dfrac{p_3}{\gamma}\right)^{\gamma}\left(\dfrac{\alpha}{p_1}\right) + b_1 = x_i^H(p,u) \implies$ homogeneous of degree 0.

2. By plugging in the hicksion demand in the utility function we get:

$u(H(p,u)) = \left[u\left(\dfrac{p_1}{\alpha}\right)^{\alpha}\left(\dfrac{p_2}{\beta}\right)^{\beta}\left(\dfrac{p_3}{\gamma}\right)^{\gamma}\left(\dfrac{\alpha}{p_1}\right) + b_1 - b_1\right]^{\alpha}\left[u\left(\dfrac{p_1}{\alpha}\right)^{\alpha}\left(\dfrac{p_2}{\beta}\right)^{\beta}\left(\dfrac{p_3}{\gamma}\right)^{\gamma}\left(\dfrac{\beta}{p_2}\right) + b_2 - b_2\right]^{\beta}\left[u\left(\dfrac{p_1}{\alpha}\right)^{\alpha}\left(\dfrac{p_2}{\beta}\right)^{\beta}\left(\dfrac{p_3}{\gamma}\right)^{\gamma}\left(\dfrac{\gamma}{p_3}\right) + b_3 - b_3\right]^{\gamma}$

$$= u\left(\frac{p_1}{\alpha}\right)^{\alpha(\alpha+\beta+\gamma-1)}\left(\frac{p_2}{\beta}\right)^{\beta(\alpha+\beta+\gamma-1)}\left(\frac{p_3}{\gamma}\right)^{\gamma(\alpha+\beta+\gamma-1)} = u$$

Therefore, there is no excess utility i.e $\forall x \in H(p,u), u(x) = u$

3. Corollary to the previous property it is straightforward that there exists a unique element in $H(p,u)$ given a prices and utility level.

b.

$$\frac{\partial e(u,p)}{\partial p_1} = \alpha\bar{u}\left(\frac{p_1^{\alpha-1}}{\alpha^\alpha}\right)\left(\frac{p_2}{\beta}\right)^\beta\left(\frac{p_3}{\gamma}\right)^\gamma + b_1 = \bar{u}\left(\frac{p_1}{\alpha}\right)^\alpha\left(\frac{p_2}{\beta}\right)^\beta\left(\frac{p_3}{\gamma}\right)^\gamma\left(\frac{\alpha}{p_1}\right) + b_1 = x_1^H.$$

$$\frac{\partial e(u,p)}{\partial p_2} = \beta\bar{u}\left(\frac{p_2^{\beta-1}}{\beta^\beta}\right)\left(\frac{p_1}{\alpha}\right)^\alpha\left(\frac{p_3}{\gamma}\right)^\gamma + b_2 = \bar{u}\left(\frac{p_1}{\alpha}\right)^\alpha\left(\frac{p_2}{\beta}\right)^\beta\left(\frac{p_3}{\gamma}\right)^\gamma\left(\frac{\beta}{p_2}\right) + b_2 = x_2^H.$$

$$\frac{\partial e(u,p)}{\partial p_3} = \gamma\bar{u}\left(\frac{p_3^{\gamma-1}}{\gamma^\gamma}\right)\left(\frac{p_2}{\beta}\right)^\beta\left(\frac{p_1}{\alpha}\right)^\alpha + b_3 = \bar{u}\left(\frac{p_1}{\alpha}\right)^\alpha\left(\frac{p_2}{\beta}\right)^\beta\left(\frac{p_3}{\gamma}\right)^\gamma\left(\frac{\gamma}{p_3}\right) + b_3 = x_3^H.$$

c.

First we need to find indirect utility, $v(p,w)$, by setting $u = v(p,w)$, in the expenditure function.

$$w = \sum_{i=1}^{3} p_i b_i + \bar{u}\left(\frac{p_1}{\alpha}\right)^\alpha\left(\frac{p_2}{\beta}\right)^\beta\left(\frac{p_3}{\gamma}\right)^\gamma \iff v(p,w) = \left[w - \sum_{i=1}^{3} p_i b_i\right]\left(\frac{\alpha}{p_1}\right)^\alpha\left(\frac{\beta}{p_2}\right)^\beta\left(\frac{\gamma}{p_3}\right)^\gamma$$

Next we need to find Walrasian Demand, using Roy's identity $x_1(p,w) = -\dfrac{\frac{\partial v(p,w)}{\partial p_1}}{\frac{\partial v(p,w)}{\partial w}}$

$$\frac{\partial v(p,w)}{\partial p_1} = -b_1\left(\frac{\alpha}{p_1}\right)^\alpha\left(\frac{\beta}{p_2}\right)^\beta\left(\frac{\gamma}{p_3}\right)^\gamma - \left[w - \sum_{i=1}^{3} p_i b_i\right]\frac{\alpha}{p_1}\left(\frac{\alpha}{p_1}\right)^\alpha\left(\frac{\beta}{p_2}\right)^\beta\left(\frac{\gamma}{p_3}\right)^\gamma$$

and $$\frac{\partial v(p,w)}{\partial w} = \left(\frac{\alpha}{p_1}\right)^\alpha\left(\frac{\beta}{p_2}\right)^\beta\left(\frac{\gamma}{p_3}\right)^\gamma$$

Next, Using the above we get:

$$x_1 = -\frac{\dfrac{\partial v(p,w)}{\partial p_1}}{\dfrac{\partial v(p,w)}{\partial w}} = \frac{b_1\left(\dfrac{\alpha}{p_1}\right)^{\alpha}\left(\dfrac{\beta}{p_2}\right)^{\beta}\left(\dfrac{\gamma}{p_3}\right)^{\gamma} + \left[w - \displaystyle\sum_{i=1}^{3} p_i b_i\right]\dfrac{\alpha}{p_1}\left(\dfrac{\alpha}{p_1}\right)^{\alpha}\left(\dfrac{\beta}{p_2}\right)^{\beta}\left(\dfrac{\gamma}{p_3}\right)^{\gamma}}{\left(\dfrac{\alpha}{p_1}\right)^{\alpha}\left(\dfrac{\beta}{p_2}\right)^{\beta}\left(\dfrac{\gamma}{p_3}\right)^{\gamma}}$$

$$= b_1 + \left[w - \sum_{i=1}^{3} p_i b_i\right]\frac{\alpha}{p_1}$$

Therefore, $\frac{\partial x_1}{\partial w}x_1 = \left(\frac{\alpha}{p_1}\right)b_1 + \left[w - \sum_{i=1}^{3} p_i b_i\right]\frac{\alpha^2}{p_1^2}$

And $\frac{\partial x_1}{\partial p_1} = -\frac{\alpha}{p_1}b_1 - \left[w - \displaystyle\sum_{i=1}^{3} p_i b_i\right]\frac{\alpha}{p_1^2}$

Therefore $\frac{\partial x_1}{\partial w}x_1 + \frac{\partial x_1}{\partial p_1} = \left[w - \displaystyle\sum_{i=1}^{3} p_i b_i\right]\frac{\alpha^2}{p_1^2} - \left[w - \displaystyle\sum_{i=1}^{3} p_i b_i\right]\frac{\alpha}{p_1^2} = -\frac{\alpha(1-\alpha)}{p_1^2}\left[w - \displaystyle\sum_{i=1}^{3} p_i b_i\right]$

And from above we know that $\left[w - \displaystyle\sum_{i=1}^{3} p_i b_i\right] = \bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}$

Therefore, $\frac{\partial x_1}{\partial w}x_1 + \frac{\partial x_1}{\partial p_1} = -\frac{\alpha(1-\alpha)}{p_1^2}\left[\bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\right] = \frac{\partial x_1^H}{\partial p_1}$

And therefore the slutsky's equation is satisfied for x_1 and by symmetry it will be satisfied for other goods as well.

d.

The slutsky's Matrix is

$$\begin{bmatrix} \frac{dx_1^H}{dp_1} & \frac{dx_1^H}{dp_2} & \frac{dx_1^H}{dp_3} \\ \\ \frac{dx_2^H}{dp_1} & \frac{dx_2^H}{dp_2} & \frac{dx_2^H}{dp_3} \\ \\ \frac{dx_3^H}{dp_1} & \frac{dx_3^H}{dp_2} & \frac{dx_3^H}{dp_3} \\ \end{bmatrix} = \bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma} \begin{bmatrix} -\frac{\alpha(1-\alpha)}{p_1^2} & \frac{\alpha\beta}{p_1p_2} & \frac{\alpha\gamma}{p_1p_3} \\ \\ \frac{\alpha\beta}{p_1p_2} & -\frac{\beta(1-\beta)}{p_2^2} & \frac{\beta\gamma}{p_2p_3} \\ \\ \frac{\alpha\gamma}{p_1p_3} & \frac{\beta\gamma}{p_2p_3} & -\frac{\gamma(1-\gamma)}{p_3^2} \\ \end{bmatrix}$$

From the above matrix we can conclude that the own substitution term are negative. Further the matrix is symmetric which implies the cross price effects are symmetric.

e.

The slutsky's Matrix is

$$\begin{bmatrix} \frac{dx_1^H}{dp_1} & \frac{dx_1^H}{dp_2} & \frac{dx_1^H}{dp_3} \\ \\ \frac{dx_2^H}{dp_1} & \frac{dx_2^H}{dp_2} & \frac{dx_2^H}{dp_3} \\ \\ \frac{dx_3^H}{dp_1} & \frac{dx_3^H}{dp_2} & \frac{dx_3^H}{dp_3} \\ \end{bmatrix} = \bar{u}\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma} \begin{bmatrix} -\frac{\alpha(1-\alpha)}{p_1^2} & \frac{\alpha\beta}{p_1p_2} & \frac{\alpha\gamma}{p_1p_3} \\ \\ \frac{\alpha\beta}{p_1p_2} & -\frac{\beta(1-\beta)}{p_2^2} & \frac{\beta\gamma}{p_2p_3} \\ \\ \frac{\alpha\gamma}{p_1p_3} & \frac{\beta\gamma}{p_2p_3} & -\frac{\gamma(1-\gamma)}{p_3^2} \\ \end{bmatrix}$$

Since the slutsky's matrix is a symmetric matrix and the non diagonal values are positive and the diagonal values are negative, it is indeed a negative semi-definite matrix. Lastly due to concavity of the expenditure function the rank of the matrix is 2.