SUFFICIENT CONDITIONS FOR ACHIEVING
MINIMUM DISTORTION IN A QUANTIZER

P. E. Fleischer
Bell Telephone Laboratories, Incorporated
Murray Hill, New Jersey

Abstract

A quantizer is said to be optimum when the intervals and the representative values - one in each interval - are chosen to produce minimum error power. Necessary conditions for a quantizer to be optimum have previously been obtained. The computations required to satisfy these conditions are iterative and necessitate the use of a digital computer for all but the most trivial problems. When certain rapidly convergent methods are used, the result may correspond to a maximum or a saddle point of the error power curve instead of a minimum.

In this paper a very simple sufficient condition for checking the minimality of a quantizer structure is derived. A strengthened form of this condition lends considerable insight into the existence of unique optimum quantizers. Thus, it is shown that signals having Gaussian, Laplacian, or Rayleigh probability densities admit unique optimum quantizers.

1. Introduction

In order to transform an analog signal into a form suitable for transmission by PCM, it is necessary to sample and quantize the signal. When a PCM signal is received, it is decoded and filtered to recover the original signal. A simplified diagram of such an operation is shown in Fig. 1. Assuming perfect coding, transmission, and decoding, the remaining system errors are due to sampling and quantization. It is easy to show that if the signal and the quantizing noise are not correlated the contributions of these two sources to the mean square error add. Furthermore, it has been shown by Lloyd(1) that if a band-limited signal is sampled at the Nyquist rate, quantized, and then reconstructed by means of an ideal low-pass filter, the system error becomes equal to the quantizer error. Thus, a study of quantization noise by itself is of interest.

In this paper only the noise due to quantization will be considered. After reviewing previous work which established necessary conditions for minimum quantizer noise, a sufficient condition will be presented. This condition affords a quick test of minimality once the necessary conditions are satisfied. A strong form of the sufficiency condition can be used to establish the uniqueness of many quantizers.

2. Preliminary Analysis

For purposes of this discussion, quantization is defined as the subdivision of the input signal's amplitude range into N nonoverlapping intervals, R_k, and the assignment of N representative amplitudes, x_k. Whenever the signal sample falls into R_k, the representative value x_k is transmitted. It will be noted that quantization is a memoryless operation and, thus, only the input's amplitude probability density, $p(x)$, is of interest. The diagram of Fig. 2 will be of help in defining the symbols to be used. Note that the 2N-1 variables $x_1, x_2, \ldots, x_N$ and $e_1, e_2, \ldots, e_{N-1}$ completely define the N-level quantizer associated with the probability density $p(x)$ which has $-\infty \leq e_0 \leq x \leq e_N \leq +\infty$ as its support.

The performance of a quantizer will be measured by the error power,

$$P = \sum_{k=1}^{N} \int_{e_{k-1}}^{e_k} (x-x_k)^2 p(x)dx. \tag{1}$$

The use of a mean square error criterion makes sense only if the probability density in question has a finite second moment; i.e., the signal has finite power. This hypothesis will henceforth be made.

The following necessary conditions for minimizing P have been obtained by Lloyd(1) and Max(2);

$$\int_{e_{k-1}}^{e_k} (x-x_k)p(x)dx = 0$$

$$k = 1,2,3,\ldots,N, \tag{2}$$

(i.e., x_k is the centroid of R_k) and

$$e_k = \frac{(x_k+x_{k+1})}{2}$$

$$k = 1,2,3,\ldots,N-1 \tag{3}$$

These equations follow directly when appropriate partial derivatives of P are set equal to zero. An explicit solution for the x_k and e_k has not been obtained except in the trivial case where $p(x)$ = constant. There are two schemes of iteration, however, which lead to a solution. In the first scheme one starts with an assumed set of end-points. Equation (2) then yields a set of centroids whence, from (3), a new set of end-points is obtained, and so on. This method has the advantage that P is decreasing at each step; however, as shown by Goldstein,[3] it converges very slowly, even when N is moderately large, say N = 16. The second scheme starts with an assumed x_1; use of (2) then determines e_1, whence x_2 is found by means of (3) and so on down to x_N. If x_N is indeed the centroid of the last interval, one is finished. Otherwise, a suitable adjustment must be made in x_1, and the whole process repeated. This method has been programmed by Goldstein and Reading;[4] it does converge rapidly. Its disadvantage is that during the iteration one does not really know whether the error power is decreasing. It is quite possible, therefore, to converge to a local maximum or a saddle point of P. The need for a quick test to detect these situations led to the work which is described in the next section.

The problem can also be handled by a dynamic programming approach[5] where any error criterion is usable. Such a program, however, would require a very large computer memory for speed and accuracy.

3. A Sufficient Condition

Inspection of (1) reveals immediately that the noise power P is a function of the 2N-1 variables $x_1, x_2, \ldots, x_N$ and $e_1, e_2, \ldots, e_{N-1}$. The simplest test for minimality would formulate the matrix of second derivatives of the noise power as given in (1). The method to be presented here is much neater and simpler to use. The following corollary of (2) and (3) is crucial:

> If the representative values, x_k, are arbitrarily preassigned, the necessary and sufficient condition for P to be a minimum is that the end-points of the regions be chosen according to (3).*

Consider now

$$P_o(x_1, x_2, \ldots, x_N) = \sum_{k=1}^{N} \int_{\varepsilon_{k-1}}^{\varepsilon_k} (x-x_k)^2 p(x)dx \quad (4)$$

where the endpoints are constrained:

$$\varepsilon_o = e_o$$

$$\varepsilon_k = \frac{(x_k+x_{k+1})}{2} \qquad k = 1,2,\ldots,N-1 \qquad (5)$$

$$\varepsilon_N = e_N.$$

As defined above, P_o is a function only of the N representative levels. In view of the corollary, for an arbitrary set $\{x_k\}$,

$$P_o(x_1,\ldots,x_N) \leq P(x_1,\ldots,x_N,e_1,\ldots,e_{N-1}), \quad (6)$$

with equality if and only if the e_k satisfy (3), i.e., $e_k = \varepsilon_k$. Thus, in looking for a minimum of P, it is sufficient to minimize P_o. This reduces the number of variables to N.

Now, a (necessary and) sufficient condition for P_o to have a minimum at a point in the N-dimensional space $(x_1, x_2, \ldots, x_N)$ is that

$$\frac{\partial P_o}{\partial x_i} = 0, \qquad i = 1,2,\ldots,N \qquad (7)$$

and that the N×N matrix of second derivatives

$$G = [g_{ij}] = \left[\frac{\partial^2 P_o}{\partial x_i \partial x_j}\right] \qquad (8)$$

be positive (semi)definite at that point.[6]

The differentiation embodied in (7) is easy to perform. Thus, differentiating (4) subject to the constraints (5) yields the following after cancellations:

$$\frac{\partial P_o}{\partial x_i} = -2 \int_{\varepsilon_{i-1}}^{\varepsilon_i} (x-x_i)p(x)dx \qquad (9)$$

$$i = 1,2,\ldots,N.$$

---

*In fact, this statement is true for any symmetric error criterion which is monotonically increasing for $c > 0$. See Max.[2]

It will be noted that setting $\frac{\partial P_o}{\partial x_i} = 0$ is equivalent to satisfying the necessary condition (2). This contributes no new information. The sufficiency condition for a minimum resides in the positive definiteness of the second derivative matrix. Straightforward manipulation leads to the following matrix. Assuming that p(x) is a continuous function, one obtains

$$\begin{bmatrix} 2a_1-b_1 & -b_1 & 0 & 0 & & \\ -b_1 & 2a_2-b_1-b_2 & -b_2 & 0 & & \\ 0 & -b_2 & 2a_3-b_2-b_3 & -b_3 & & \\ 0 & 0 & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & -b_{N-1} & \\ & & & -b_{N-1} & 2a_N-b_{N-1} & \end{bmatrix} \tag{10}$$

where

$$a_i = \int_{\varepsilon_{i-1}}^{\varepsilon_i} p(x)dx \qquad i = 1,2,\ldots,N \tag{11}$$

and

$$b_i = \frac{x_{i+1}-x_i}{2}\, p(\varepsilon_i) \qquad i = 1,2,\ldots,N-1 \tag{12}$$

The matrix of second derivatives, G, is symmetric and has nonzero entries only on the main diagonal and the two contiguous diagonals. The entries in the matrix are simple to calculate and the special form of the matrix allows a very rapid numerical check of its positive definiteness. To do this, the following theorem[7] is used:

Theorem: A necessary and sufficient condition for the N×N matrix G to be positive definite is that the determinant of G, as well as its N-1 principal minors* be positive.

Consider now the matrix $G' = [g'_{ij}]$ which is obtained as follows:

$$g'_{1j} = g_{1j}$$

$$g'_{2j} = g_{2j} + \frac{b_1}{g'_{11}} g'_{1j}$$

$$\vdots$$

$$g'_{kj} = g_{kj} + \frac{b_{k-1}}{g'_{k-1,k-1}} g'_{k-1,j}$$

$$\vdots$$

$$g'_{Nj} = g_{Nj} + \frac{b_{N-1}}{g'_{N-1,N-1}} g'_{N-1,j}$$

It can be readily confirmed that the new matrix is of the superdiagonal form. Furthermore, it follows from elementary determinant theory that the kth principal minor of G is given by:

$$M_k = g'_{11}g'_{22}\cdots g'_{kk} \qquad k = 1,\ldots,N$$

Thus, a necessary and sufficient condition for G to be positive definite is that $g'_{kk} > 0$, $k = 1,\ldots,N$.

4. Consequences of the Sufficient Condition

In the previous section, it was shown that a quantizer is optimal if and only if the relations (5) and (7) are satisfied and the matrix (8) is positive definite.** Expressed in words, (5) and (7) require that the representative values be the centroids of their regions while the endpoints of the regions bisect the adjacent representative values. Unfortunately, these relationships do not provide much insight into the structure of optimum quantizers. The complicated functional relationships imposed by the requirement that the matrix (8) be positive definite further becloud the issue. While these problems cannot be entirely surmounted, it is possible to derive conditions under which the quantizer noise power has a unique stationary point which is a minimum. This also provides considerable insight into the general question. Before deriving this sufficient condition, an example*** is given where the stationary point is not unique. Consider a two-level quantizer for the probability density given in Fig. 3, where both lobes are symmetrical.

*The kth order principal of the square matrix G is the determinant obtained by deleting the last N-k rows and columns of G. The determinant itself may be considered to be its own Nth order principal minor.

**The case where (8) is positive semi-definite is not considered, since this would necessitate the use of higher order derivatives. As a matter of practical interest, this situation is uninteresting.

***This example is a modification of one given by Lloyd[1] and attributed to J. L. Kelly, Jr.

The quantizer drawn satisfies the necessary conditions (5) and (7); yet it is obviously not an optimal solution when $A \gg B$. The optimal solution is indicated at the top of the figure.

The following will be used in the derivation of the sufficient condition:

Lemma: A sufficient condition for a matrix, $G = [g_{ij}]$, of the form (10) to be positive definite is that

$$\sigma_i = \sum_j g_{ij} > 0, \qquad i = 1,2,3,\ldots,N.$$

A proof and slight generalization of this version of a well-known theorem[8] is given in the Appendix.

Let us now consider a typical row-sum σ_i. Referring to (10), (11), and (12), we have

$$\sigma_i = \Sigma\, g_{ij} = 2[a_i - b_i - b_{i-1}]$$

$$= 2\int_{\varepsilon_{i-1}}^{\varepsilon_i} p(x)dx - \frac{x_{i+1}-x_i}{2}\, p(\varepsilon_i) - \frac{x_i - x_{i-1}}{2}\, p(\varepsilon_{i-1}) \tag{13}$$

The above holds for $i = 1,2,\ldots,N$, provided we make the interpretation:

$$b_o = b_N = 0 \tag{14}$$

This is a natural choice, since it is equivalent to $p(\varepsilon_o) = p(\varepsilon_N) = 0$, which follows from the hypothesized continuity of $p(x)$.

The condition $\sigma_i > 0$, $i = 1,2,\ldots,N$, is easily interpreted in terms of Fig. 4 which shows a typical interval. It requires that in each interval the cross-hatched area be less than the area under the curve between corresponding limits. Intuitively, it seems rather likely that $\sigma_i > 0$ for a well-behaved $p(x)$. Since x_i is the centroid of the ith region, it is likely to be closer to that end of the curve where the probability is higher. In that case it will give a lower estimate of the area than the trapezoidal rule. The following precise statement can be made:

Theorem: A sufficient condition for $\sigma_i > 0$ is that, in the region defined, the probability density obey the relation:

$$\frac{d^2}{dx^2} \ln p(x) < 0. \tag{15}$$

Proof: In order to eliminate writing subscripts, the following equivalent symbols will be used:

$\sigma = \sigma_i$ = row-sum of ith interval

$\alpha = \varepsilon_i$ = lower limit of ith interval

$\beta = \varepsilon_{i+1}$ = upper limit of ith interval

$g = x_i$ = centroid of ith interval.

In terms of these variables, the relation for σ becomes:

$$\frac{1}{2}\sigma = \int_\alpha^\beta p(x)dx - (\beta - g)p(\beta) - (g-\alpha)p(\alpha) \tag{16}$$

Integration by parts yields the following:

$$\frac{1}{2}\sigma = \beta p(\beta) - \alpha p(\alpha) - \int_\alpha^\beta xp'(x)dx - (\beta-g)p(\beta) - (g-\alpha)p(\alpha)$$

$$= g[p(\beta) - p(\alpha)] - \int_\alpha^\beta xp'(x)dx$$

$$= \frac{\int_\alpha^\beta xp(x)dx}{\int_\alpha^\beta p(x)dx}\int_\alpha^\beta p'(x)dx - \int_\alpha^\beta xp'(x)dx$$

$$= \frac{1}{\int_\alpha^\beta p(x)dx}\left[\int_\alpha^\beta xp(x)dx \int_\alpha^\beta p'(x)dx - \int_\alpha^\beta xp'(x)dx \int_\alpha^\beta p(x)dx\right]. \tag{17}$$

These integrals may now be interpreted as double integrals leading to the following equalities:

$$\sigma\left[\int_\alpha^\beta p(x)dx\right] = 2\int_\alpha^\beta\int_\alpha^\beta [xp(x)p'(y) - yp'(y)p(x)]dxdy$$

$$= 2\int_\alpha^\beta\int_\alpha^\beta (x-y)p(x)p'(y)dxdy$$

$$= 2\int_\alpha^\beta\int_\alpha^\beta (y-x)p(y)p'(x)dxdy$$

$$= \int_\alpha^\beta\int_\alpha^\beta (x-y)[p(x)p'(y) - p(y)p'(x)]dxdy. \tag{18}$$

In view of the fact that $\int_\alpha^\beta p(x)dx > 0$, it follows that a sufficient condition for $\sigma > 0$ is that

$$[p(x)p'(y) - p(y)p'(x)] > 0$$

whenever

$$(x-y) > 0. \tag{19}$$

Dividing the inequality by the positive quantity $p(x)p(y)$, the following is obtained:

$$\frac{p'(y)}{p(y)} > \frac{p'(x)}{p(x)} \quad \text{whenever} \quad x > y \tag{20}$$

This is equivalent to

$$\frac{d}{dx}[\ell n\, p(x)] = \text{decreasing} \tag{21}$$

and finally,

$$\frac{d^2}{dx^2}[\ell n\, p(x)] < 0 \qquad \text{Q.E.D.} \tag{15}$$

An equivalent sufficient condition, which follows immediately from (15), is:

$$[p'(x)]^2 > p(x)p''(x). \tag{22}$$

Suppose now that a given probability density obeys inequality (15) throughout its range. Then, the matrix (10) is positive definite for arbitrary sets of (x_k) and (e_k) which obey the necessary conditions (2) and (3).

Therefore, all stationary points of the noise power curve must be minima. It can then be shown that this implies a unique minimum. More precisely, the following is true:

Theorem: Let a given continuous probability density of finite second moment obey the equality $[\ell n\, p(x)]'' < 0$ over its total support. Then, the noise power of an N-level quantizer (N arbitrary) has a unique stationary point.

Details of the proof are given in the appendix and in a paper not yet published.

5. Examples

Simple computation shows that the Gaussian distribution has a unique quantizer:

$$\frac{d^2}{dx^2}\left[\ell n \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right] = -1 < 0. \tag{23}$$

The case of Laplacian probability density

$$p(x) = \left[\frac{1}{2\sqrt{2}} e^{-\frac{|x|}{\sqrt{2}}}\right] -\infty < x < \infty \tag{24}$$

requires more care. It is easy to show that

$$\frac{d^2}{dx^2}\left[\ell n \frac{1}{2\sqrt{2}} e^{-\frac{|x|}{2}}\right] = 0, \qquad x \neq 0, \tag{25}$$

i.e., the inequality degenerates to an equality. In order to show that a unique quantizer exists, consider the following family of functions:

$$p(x,\alpha) = e^{-|x|}\left[1 - \alpha e^{-\frac{1-\alpha}{\alpha}|x|}\right]. \tag{26}$$

It is clear that in the limit as $\alpha \to 0$, $p(x,\alpha)$ approaches the Laplacian distribution of (23) uniformly (disregarding the scale factor). This family of curves is shown in Fig. 5. Consider now the following:

$$\frac{d}{dx}\ell n\, p(x,\alpha) = \frac{d}{dx}\left[-|x| + \ell n\left(1-\alpha e^{-\frac{1-\alpha}{\alpha}|x|}\right)\right]$$

$$= (\operatorname{sgn} x)\left[-1 + \frac{(1-\alpha)e^{-\frac{1-\alpha}{\alpha}|x|}}{1-\alpha e^{-\frac{1-\alpha}{\alpha}|x|}}\right]. \tag{27}$$

Certainly, for all $\alpha < 1$, the above expression is strictly monotonically decreasing. Therefore, in accordance with (21), the probability of (26) possesses a unique optimal quantizer regardless of how small α is. Since the noise power is a continuous function of α, the same conclusion will be true in the limit.

It is also easy to show that both the Rayleigh distribution:

$$p_3(x) = 0 \qquad x \leq 0$$

$$p_3(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \qquad x > 0 \qquad (28)$$

and the χ^2 distribution:

$$p_4(x) = 0 \qquad x \leq 0$$

$$p_4(x) = k_1 x^{(n/2)-1} e^{-x/2} \qquad x > 0 \qquad (29)$$

satisfy (15) and thus have unique quantizers.

A very general class of unimodal probability densities is the Pearson system.(9)* It includes the normal and the χ^2 distributions, as special cases. The class is defined by the following differential equation:

$$\frac{p'(x)}{p(x)} = [\ln p(x)]' = \frac{x+a}{b_2x^2+b_1x+b_o} = \frac{x+a}{D(x)} \qquad (30)$$

Applying the criterion of uniqueness yields the following:

$$\frac{d^2}{dx^2} \ln p(x) = \frac{d}{dx} \frac{x+a}{b_2x^2+b_1x+b_o}$$

$$= \frac{(b_2x^2+b_1x+b_o) - (x+a)(2b_2x+b_1)}{D^2(x)} < 0 \qquad (31)$$

This is equivalent to

$$b_2(x+a)^2 - D(-a) > 0 \qquad (32)$$

for all x on the support of the probability density generated. If $b_2 < 0$, the condition can be satisfied only in a neighborhood of $x = -a$. If, however, $b_2 \geq 0$, it is sufficient for uniqueness to have $D(-a) < 0$.

6. Summary

In the first part of this paper, a quick numerical method for checking the optimality of a given quantizer has been given.

By combining the necessary conditions for stationarity with a strengthened form of the sufficient conditions for a minimum, a condition under which a quantizer possesses a unique stationary point has been found. This condition is overly restrictive. Thus, a matrix may be positive definite even if not all its row-sums are positive. Furthermore, for lack of information on the locations of the $\{x_n\}$, which satisfy the necessary conditions, the row-sum condition had to be proved for every possible set of $\{x_n\}$. The sufficient condition that emerges is seen to define a class of function having the weak convexity property:

$$\frac{d^2p}{dx^2} < \frac{1}{p(x)}\left(\frac{dp}{dx}\right)^2 \qquad (30)$$

Unfortunately, there is no indication of the extent to which this class can be extended.

*This was pointed out to the author by M. R. Aaron.

References

1. S. P. Lloyd, "Least Squares Quantization in PCM", Unpublished paper, 1957.
2. J. Max, "Quantizing for Minimum Distortion," Transactions of P.G.I.T., March 1960, pp. 7-12.
3. A. J. Goldstein, Unpublished work.
4. A. J. Goldstein and E. S. Reading, "Computer Program for Optimal Quantizer Structure," Unpublished memorandum.
5. J. D. Bruce, "Optimum Quantization for a General Error Criterion," M.I.T.-R.L.E. Quarterly Progress Report, No. 69, 1963, pp. 135-141.
6. R. Bellman, "Introduction to Matrix Analysis," McGraw-Hill Book Company, Inc., 1960, p. 60.
7. Ibid., p. 74.
8. Ibid., pp. 294-295.
9. H. Cramer, "Mathematical Methods of Statistics," Princeton University Press, 1951, pp. 248-249.

Appendix 1

A Sufficient Condition for Positive Definiteness

The following theorem quoted in the main text will be proved here:

Theorem: A sufficient condition for the matrix G to be positive definite is $\sigma_i > 0$, $i = 1,2,\ldots,N$.

Proof: The symbols used above are defined in the text (10) - (13). To prove this theorem, consider the quadratic form corresponding to G:

$$Q(\chi) = \chi^* G \chi \qquad (A1)$$

This may be written as:

$$Q(\chi) = \sum_{i=1}^{N} \sigma_i x_i^2 + \sum_{i=1}^{N-1} b_i (x_{i+1} - x_i)^2. \qquad (A2)$$

Clearly, $Q(\chi) \geq 0$. Equality occurs only if all $\sigma_i = 0$. This certainly proves the theorem. It is interesting to note that if all $\sigma_i = 0$, the matrix is positive semidefinite. The multiplicity of the zero eigenvalue is 1, since the corresponding eigenvector must satisfy the N-1 equations:

$$x_1 = x_2 = x_3 = \ldots = x_N. \qquad (A3)$$

Appendix 2

On the Uniqueness of the Minimum

The following theorem, quoted in the text, is proved here:

Theorem: Let a given continuous probability density of finite second moment obey the inequality $[\ln p(x)]'' < 0$. Then the noise power of an N-level quantizer (N arbitrary) has a unique stationary point. This point is a relative and absolute minimum.

Proof: The following Lemma will be needed:

Let C_1 be a connected open region in N-dimensional Euclidian space and let C be a convex closed region in C_1. Let $P(x)$ be a function defined in C_1 which has the following properties:

1. Grad $P(x)$ exists and is continuous in C_1.
2. At every stationary point (a point where grad $P(x) = 0$), the function attains a strict local minimum.
3. At every point on the boundary of C there exists a vector pointing into C along which the directional derivative of $P(x)$ is negative.

Then, in the region C, $P(x)$ possesses a unique stationary point. The point is interior to C; it is a relative and absolute minimum of $P(x)$ in C.

A proof of this Lemma will be given in a forthcoming paper.

We will consider the support of the probability density to be arbitrary but finite. Since the second moment of the probability is finite, the possible effect of this on the noise power can be made arbitrarily small. In the text it was shown that for an N-level quantizer, the minimal (with respect to end points) quantizing noise power is given by:

$$P_o = \sum_{k=1}^{N} \int_{\varepsilon_{k-1}}^{\varepsilon_k} (x - x_k)^2 p(x)dx \qquad (A2\text{-}1)$$

where

$$\varepsilon_o = \text{left end-point of support}$$

$$\varepsilon_k = \frac{x_k + x_{k+1}}{2} \qquad (A2\text{-}2)$$

$$\varepsilon_N = \text{right end-point of support.}$$

P_o is a function of the N variables $x_1, x_2, \ldots, x_N$. Although on physical grounds it is clear that $x_1 < x_2 < \ldots < x_N$, the definition (A2-1) and the subsequent operations are perfectly valid without that assumption. The necessary conditions for minimizing P_o have been shown to be

$$\frac{\partial P_o}{\partial x_k} = -2 \int_{\varepsilon_{k-1}}^{\varepsilon_k} (x - x_k) p(x) dx = 0. \qquad (A2\text{-}3)$$

Note that (A2-2) and (A2-3) imply that

$$e_o \leq x_1 \leq x_2 \leq \cdots \leq x_N \leq e_N. \qquad (A2\text{-}4)$$

To summarize, P_o and grad P_o are defined at all points of an N-dimensional Euclidian space. The gradient is continuous, since $p(x)$ is continuous. The gradient does not vanish outside the convex region C, defined in (A2-4).* At critical points inside or on C, the matrix of second derivatives exists and is positive definite (since $[\ln p(x)]'' < 0$). Thus, no critical points lie outside C and all critical points inside or on C are strict local minima.

Consider next the gradient at points on the boundary of C. Such points correspond to one or more equal signs occurring in (A2-4). Suppose, for example, that

$$x_{i-1} < x_i = x_{i+1} = \ldots = x_k < x_{k+1} \qquad (A2\text{-}5)$$

where

$$i \geq 2, \qquad k \leq N\text{-}1, \qquad k > i.$$

*This region is a simplex. Its outside consists of points which fail to satisfy all the relations in (A2-4). Thus, all the components of the gradient, (A2-3), cannot vanish.

Then, it follows from (A2-3) that

$$\frac{\partial P_o}{\partial x_i} > 0$$

$$\frac{\partial P_o}{\partial x_k} < 0 \qquad \text{(A2-6)}$$

and, if k - i > 1,

$$\frac{\partial P_o}{\partial x_j} = 0 \qquad i < j < k \qquad \text{(A2-7)}$$

Clearly, by the continuity of the partial derivatives, the directional derivative in the direction: $\Delta x_i < 0$ and

$$\frac{\Delta x_j}{\Delta x_k} = \frac{j-i}{k-i} \qquad i \leq j < k \qquad \text{(A2-8)}$$

is negative. Furthermore, this direction "points into the region."

A similar construction is possible for each "group" of equalities. The extension to the case when some of the representative points lie at one of the end-points (e_0 or e_N) is also immediate.

This shows that all the hypotheses of the lemma are satisfied. Since the gradient does not vanish outside C, the uniqueness of the minimum has been shown. This completes the proof.

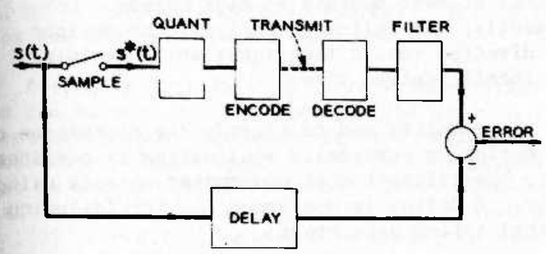

FIGURE 1.

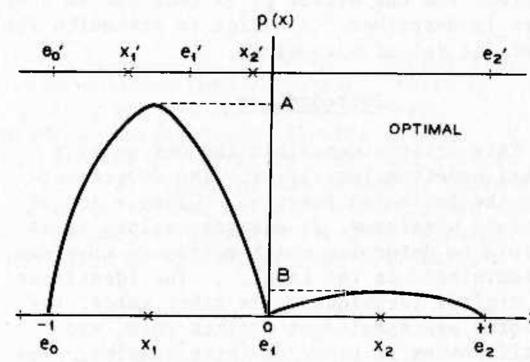

FIGURE 3.

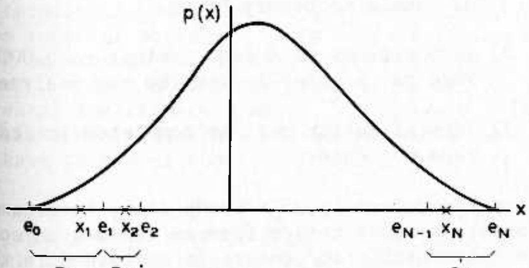

FIGURE 2.

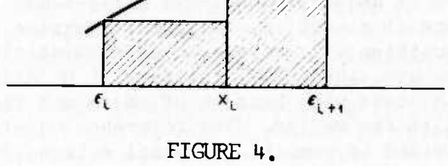

FIGURE 4.

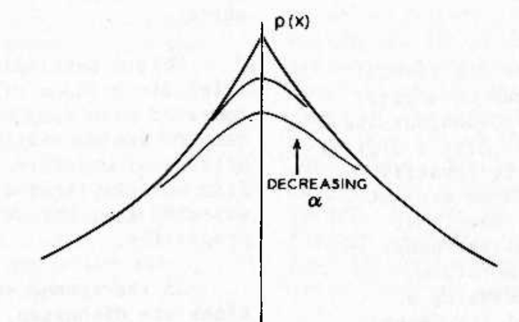

FIGURE 5.