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On the Liapounoff Limit of Error in the Theory of Probability.

By

CARL-GUSTAV ESSEEN.

With one figure in the text.

Communicated November 12th 1941 by T. CARLEMAN and ARNE BEURLING.

Introduction. Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables with the mean value zero and the finite dispersion σ_i, defined by

$$\sigma_i^2 = \int_{-\infty}^{\infty} x^2 \, d\, F_i(x), \qquad (i = 1, 2, \ldots, n),$$

where $F_i(x)$ is the distribution function, corresponding to X_i. The random variable $X_1 + X_2 + \cdots + X_n$ has the mean value zero and the dispersion s_n, where $s_n^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$. By $\overline{F_n}(x)$ we denote the distribution function which corresponds to the variable

$$\frac{X_1 + X_2 + \cdots + X_n}{s_n} \tag{1}$$

with the mean value zero and the dispersion 1. Then under certain general conditions[1] by the so-called central limit theorem of the theory of probability $\lim_{n \to \infty} \overline{F_n(x)} = \Phi(x)$, $\Phi(x)$ being the

[1] J. W. LINDEBERG, (1).

normal distribution function:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy.$$

It is of great interest, especially in regard to the statistical applications, to investigate the difference $\overline{F_n(x)} - \Phi(x)$. A. LIAPOUNOFF[1] has proved the following well-known theorem.

Let $X_1, X_2, \ldots, X_n$ be independent random variables with the same distribution function $F(x)$, the mean value zero, the dispersion σ and the finite absolute third moment

$$\beta_3 = \int_{-\infty}^{\infty} |x|^3 \, d\, F(x).$$

$\overline{F_n(x)}$ being defined as above, then

$$(2) \qquad |\overline{F_n(x)} - \Phi(x)| \leq C \cdot \frac{\beta_3}{\sigma^3} \frac{\log n}{\sqrt{n}},$$

C being an absolute constant which may be taken $= 3$.

It has been supposed by several authors that $\log n$ may be omitted in (2), and in some special cases this has been proved by CRAMÉR.[2] In this investigation I shall prove some general theorems concerning the behaviour of the difference $\overline{F_n(x)} - \Phi(x)$, among other things that $\log n$ is always superfluous in (2). The proof is mainly based on theorem 1. For the sake of brevity I shall only treat the case of equal distribution functions.

1. Definitions. General properties of distribution functions. In the following LEBESGUE or LEBESGUE-STIELTJES integrals are used.

The distribution function $F(x)$ is defined in the usual manner.

The moments α_ν and the absolute moments β_ν are determined by

$$\alpha_\nu = \int_{-\infty}^{\infty} x^\nu \, d\, F(x), \quad \beta_\nu = \int_{-\infty}^{\infty} |x|^\nu \, d\, F(x), \qquad (\nu = 1, 2, \ldots).$$

[1] A. LIAPOUNOFF, (1), (2).
[2] H. CRAMÉR, (2), (3).

When investigating the convergence of distribution functions it is often useful to introduce characteristic functions. The characteristic function $f(t)$, corresponding to $F(x)$, is defined by

$$f(t) = \int_{-\infty}^{\infty} e^{itx}\, d\, F(x)$$

for real values of t. $f(t)$ is a uniformly continuous function and $|f(t)| \leq f(0) = 1$.

For later purposes it is of importance to find out whether there exists a finite value $t = t_0 \neq 0$ such that $|f(t_0)| = 1$. It is easily seen that this can only occur if $F(x)$ is a step function and if any two discontinuities of $F(x)$ differ by an entire multiple of $\frac{2\pi}{t_0}$. In this and only this case $|f(t)|$ is periodic.

The characteristic function of $\Phi(x)$ is

$$e^{-\frac{t^2}{2}}. \tag{3}$$

As is well known the distribution function $\overline{F_n(x)}$, defined by (1), is obtained by the convolution of the functions

$$F_1(s_n x),\ F_2(s_n x),\ \ldots,\ F_n(s_n x),$$

or

$$\overline{F_n(x)} = F_1(s_n x) * F_2(s_n x) * \qquad * F_n(s_n x). \tag{4}$$

As to the corresponding characteristic functions we have

$$\overline{f_n(t)} = \prod_{i=1}^{n} f_i\left(\frac{t}{s_n}\right). \tag{5}$$

In the following $\overline{F_n(x)}$ and $\overline{f_n(t)}$ are always defined as in (4) and (5).

2. We now proceed to the proof of

Theorem 1. *Let T, A and ε be arbitrary positive constants and $F(x)$ and $G(x)$ distribution functions, $f(t)$ and $g(t)$ the corresponding characteristic functions such that*

1) *$G'(x)$ exists everywhere and $G'(x) \leq A$.*

$$(6) \qquad 2) \quad \int_{-T}^{T} \left| \frac{f(t) - g(t)}{t} \right| d\,t = \varepsilon .$$

Then to every number $k > 1$ there corresponds a number $c(k)$, only depending on k, such that

$$(7) \qquad |F(x) - G(x)| \leq k \cdot \frac{\varepsilon}{2\pi} + c(k) \cdot \frac{A}{T} \cdot$$

Proof. By

$$f(t) - g(t) = \int_{-\infty}^{\infty} e^{itx}\, d\,(F(x) - G(x))$$

and integrating by parts we obtain:

$$(8) \qquad \frac{f(t) - g(t)}{-i\,t} = \int_{-\infty}^{\infty} e^{itx} (F(x) - G(x))\, d\,x .$$

Thus $\frac{f(t) - g(t)}{-i\,t}$ is the FOURIER transform of $F(x) - G(x)$. Suppose first that

$$\int_{-\infty}^{\infty} \left| \frac{f(t) - g(t)}{t} \right| d\,t = \varepsilon ,$$

i. e. $T = \infty$ in (6). By the FOURIER inversion formula we obtain by (8):

$$F(x) - G(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{f(t) - g(t)}{-i\,t}\, d\,t .$$

Hence

$$|F(x) - G(x)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{f(t) - g(t)}{t} \right| d\,t = \frac{\varepsilon}{2\pi} \cdot$$

Thus $c(k) \cdot \frac{A}{T}$ in (7) can be interpreted as a rest term, corresponding to a finite value T. Generally, however, $\frac{f(t)-f(t)}{-it}$ is not absolutely integrable $(-\infty, \infty)$ and another method must be used.

In the proof we may take $A = 1$, $T = 1$. We put

$$F_1(x) = \frac{T}{A} \cdot F\left(\frac{x}{T}\right), \quad G_1(x) = \frac{T}{A} \cdot G\left(\frac{x}{T}\right),$$

$f_1(t)$ and $g_1(t)$ being the corresponding characteristic functions. Then $G_1'(x) \leq 1$ and

$$\int_{-1}^{1} \left| \frac{f_1(t) - g_1(t)}{t} \right| dt = \frac{T\varepsilon}{A}.$$

The inequality to be proved is

$$| F_1(x) - G_1(x) | \leq k \cdot \frac{T}{A} \cdot \frac{\varepsilon}{2\pi} + c(k)$$

or

$$| F(x) - G(x) | \leq \frac{k\varepsilon}{2\pi} + \frac{c(k)A}{T}.$$

$F_1(x)$ and $G_1(x)$ are not distribution functions in the ordinary sense, as $F_1(+\infty) = G_1(+\infty) = \frac{T}{A}$. This, however, is of no account.

Thus in the following we suppose $A = T = 1$.

We first define an auxiliary function $h(t)$ such that

1) $h(t)$ is an even, real, continuous function for real values of t.

$$h(0) = 1. \quad h(t) = 0 \text{ for } |t| \geq 1.$$

2) The Fourier transform $H(x)$ of $h(t)$,

$$H(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} h(t)\, dt, \tag{9}$$

is non-negative. Further it follows that $H(x)$ is even.

3) $\int\limits_{-\infty}^{\infty} H(x)\,dx < \infty$. Hence

$$h(t) = \int\limits_{-\infty}^{\infty} e^{itx} H(x)\,dx, \tag{10}$$

and by (10), $H(x)$ being non-negative:

$$|h(t)| \leq h(0) = \int\limits_{-\infty}^{\infty} H(x)\,dx = 1\,. \tag{11}$$

4) $\int\limits_{-\infty}^{\infty} |x|\,H(x)\,dx < \infty$.

To obtain an example of such a function we may proceed as follows. Let

$$k(t) = \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{for } |t| \geq 1\,. \end{cases}$$

Hence the Fourier transform

$$K(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-itx} k(t)\,dt = \frac{1}{2\pi} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2.$$

Putting

$$h(t) = \int\limits_{-\infty}^{\infty} k(2t - s)\,k(s)\,ds,$$

$h(t)$ and $H(x)$ have the required properties. We get

$$H(x) = \frac{1}{2\pi} \left(\frac{\sin \frac{x}{4}}{\frac{x}{4}} \right)^4.$$

We now proceed to the proof of the theorem. Suppose the largest value of the difference $|F(x) - G(x)|$ to be Δ. Let $|F(x) - G(x)|$ take the value Δ for $x = 0$, since a translation x_0 of x is equivalent to a multiplication of the transform with e^{itx_0} of modulus 1. Further we may suppose $F(0) > G(0)$.

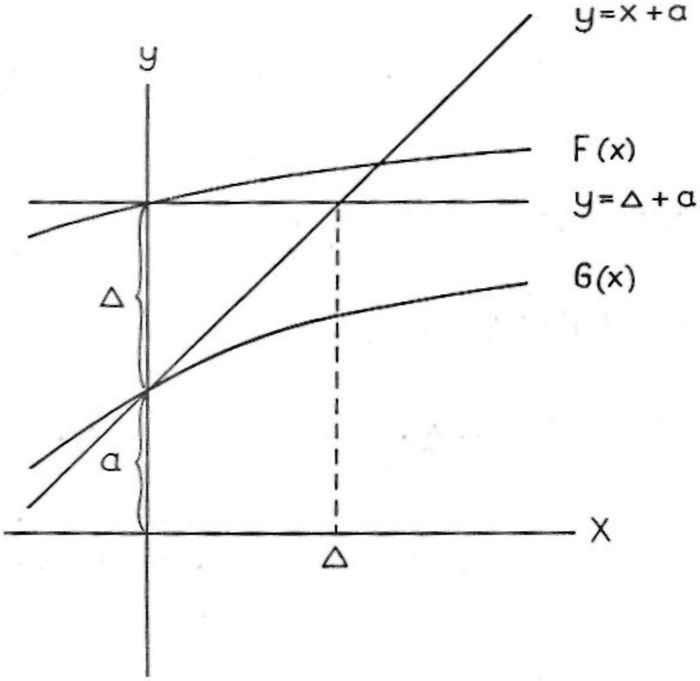

Fig. 1.

As $F(x)$ and $G(x)$ are non-decreasing and $G'(x) \leq 1$ it is easily seen in the fig. that

$$F(x) - G(x) \geq \Delta - x \quad \text{for} \quad 0 < x < \Delta. \tag{12}$$

Consider the integral

$$\int_{-\infty}^{\infty} H(x-y)\, (F(y) - G(y))\, dy,$$

where x is a quantity, later to be determined. As $H(x-y)$ belongs to $L(-\infty, \infty)$ and $(F(y) - G(y))$ is of bounded variation $(-\infty, \infty)$, and tends to 0 for $y \to \pm\infty$, we obtain by a theorem, regarding the validity of the PARSEVAL formula[1]:

[1] See e. g. E. C. TITCHMARSH, Intr. to the theory of FOURIER integrals, Theorem 38.

$$(13)\quad \int_{-\infty}^{\infty} H(x-y)\,(F(y)-G(y))\,dy = \frac{1}{2\pi}\int_{-1}^{1} e^{-itx}\cdot\frac{f(t)-g(t)}{-it}\,h(t)\,dt.$$

By (13) and (12) we have

$$\int_{0}^{\triangle} (\triangle - y)\,H(x-y)\,dy - \int_{\triangle}^{\infty} H(x-y)\,|F(y)-G(y)|\,dy -$$

$$-\int_{-\infty}^{0} H(x-y)\,|F(y)-G(y)|\,dy \leq \frac{1}{2\pi}\int_{-1}^{1}\left|\frac{f(t)-g(t)}{t}\,h(t)\right|dt \leq \frac{\varepsilon}{2\pi}.$$

But $|F(y)-G(y)| \leq \triangle$. Hence

$$\int_{0}^{\triangle} (\triangle - y)\,H(x-y)\,dy - \triangle\cdot\int_{\triangle}^{\infty} H(x-y)\,dy - \triangle\cdot$$

$$\cdot\int_{-\infty}^{0} H(x-y)\,dy \leq \frac{\varepsilon}{2\pi}$$

or

$$(14)\quad \int_{-x}^{\triangle - x} (\triangle - x - y)\,H(y)\,dy - \triangle\cdot\int_{\triangle - x}^{\infty} H(y)\,dy - \triangle\cdot\int_{-\infty}^{-x} H(y)\,dy \leq \frac{\varepsilon}{2\pi}.$$

Since by (11)

$$\int_{0}^{\infty} H(y)\,dy = \int_{-\infty}^{0} H(y)\,dy = 1/2$$

we obtain

$$\int_{\triangle - x}^{\infty} H(y)\,dy = 1/2 - \int_{0}^{\triangle - x} H(y)\,dy,\quad \int_{-\infty}^{-x} H(y)\,dy = 1/2 - \int_{-x}^{0} H(y)\,dy$$

Hence by (14)

$$\int_{-x}^{\triangle - x} (2\triangle - x - y)\,H(y)\,dy - \triangle \leq \frac{\varepsilon}{2\pi}$$

or

$$(15)\quad \int_{-x}^{\Delta - x} (2\Delta - x) H(y)\,dy - \Delta \leq \frac{\varepsilon}{2\pi} + \int_{-x}^{\Delta - x} y\, H(y)\,dy \leq \frac{\varepsilon}{2\pi} + b,$$

where

$$b = \int_{-\infty}^{\infty} |y|\, H(y)\,dy.$$

Let us put $x = m \cdot \Delta$, $(0 < m < 1)$. By (15)

$$(16)\qquad \Delta \left[(2 - m) \int_{-m\Delta}^{(1-m)\Delta} H(y)\,dy - 1 \right] \leq \frac{\varepsilon}{2\pi} + b.$$

Given an arbitrary number $k > 1$, we may choose the number $m(k)$ sufficiently small and the number $\alpha(k)$ sufficiently large so that

$$(17)\qquad (2 - m(k)) \int_{-m(k)\alpha(k)}^{(1-m(k))\alpha(k)} H(y)\,dy - 1 = 1/k.$$

Now two cases may occur:

1) $\Delta \leq \alpha(k)$
2) $\Delta > \alpha(k)$. Hence by (16) and (17)

$$\Delta \cdot 1/k \leq \frac{\varepsilon}{2\pi} + b \quad \text{or} \quad \Delta \leq \frac{k \cdot \varepsilon}{2\pi} + kb.$$

Thus

$$\Delta \leq \text{Max}\left(\frac{k\varepsilon}{2\pi} + kb,\ \alpha(k) \right) \leq \frac{k\varepsilon}{2\pi} + kb + \alpha(k) = \frac{k\varepsilon}{2\pi} + c(k),$$

$c(k)$ being a number only depending on k. Thus the proof is completed. We also observe that when $k \to 1$, $c(k) \to \infty$.

Remark. It is easily seen that we do not have to assume $G(x)$ to be non-decreasing. We only need $|G'(x)| \leq A$.

3. The convergence of distribution functions. In this section we shall consider the theorem of LIAPOUNOFF. We begin by proving this general theorem.

Theorem 2. *Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables with the same distribution function $F(x)$, the characteristic function $f(t)$, the mean value zero and the dispersion σ. Further let $f(t)$ have the expansion:*

$$f(t) = 1 - {}^1/_2 \cdot \sigma^2 t^2 + r(t), \tag{18}$$

where

$$|r(t)| \leq K \cdot |t|^{2+2\alpha},$$

K being a constant and $0 < \alpha \leq {}^1/_2$. Then

$$|\overline{F_n(x)} - \Phi(x)| \leq \frac{c(\sigma, K)}{n^\alpha}, \tag{19}$$

$c(\sigma, K)$ being a finite quantity, only depending on σ and K.

Remark. Let the absolute moment β_k, $(k \geq 3)$, be finite. By

$$f(t) = \int\limits_{-\infty}^{\infty} e^{itx}\, dF(x)$$

it immediately follows by the expansion of e^{itx} in series and the integration term by term that

$$f(t) = 1 - {}^1/_2\, \sigma^2 t^2 + \sum_{\nu=3}^{k} \frac{\alpha_\nu}{\nu!} (i\,t)^\nu + o(t^k) \tag{20}$$

for small values of t.

For the proof of theorem 2 we need two inequalities, analogous to two inequalities proved by CRAMÉR[1] and LIAPOUNOFF[2]. As my methods of proof do not differ much from those of CRAMÉR and LIAPOUNOFF I omit them here.

In the following $c_1, c_2, \ldots$ are bounded quantities, only depending on σ and K.

[1] For lemma 1, see H. CRAMÉR, (2), (3).
[2] For lemma 2, see A. LIAPOUNOFF, (2) or H. CRAMÉR, (2), (3).

Lemma 1.

$$\left|\overline{f_n(t)} - e^{-\frac{t^2}{2}}\right| \leq c_1 \, |t|^{2+2\alpha} \cdot \frac{e^{-\frac{t^2}{2}}}{n^\alpha} \quad \text{for} \quad |t| \leq c_2 \cdot n^{\frac{\alpha}{2+2\alpha}}. \tag{21}$$

Lemma 2.

$$\left|\overline{f_n(t)}\right| \leq e^{-\frac{t^2}{3}} \quad \text{for} \quad |t| \leq c_3 \cdot n^\alpha. \tag{22}$$

Proof of theorem 2. Suppose first

$$c_3 \, n^\alpha > c_2 \, n^{\frac{\alpha}{2+2\alpha}}.$$

Consider the integral

$$I = \int_{-c_3 \cdot n^\alpha}^{c_3 \cdot n^\alpha} \left| \frac{\overline{f_n(t)} - e^{-\frac{t^2}{2}}}{t} \right| dt. \tag{23}$$

Let us put

$$I = \int_{-c_3 \cdot n^\alpha}^{-c_2 \cdot n^{\frac{\alpha}{2+2\alpha}}} + \int_{-c_2 \cdot n^{\frac{\alpha}{2+2\alpha}}}^{+c_2 \cdot n^{\frac{\alpha}{2+2\alpha}}} + \int_{c_2 \cdot n^{\frac{\alpha}{2+2\alpha}}}^{c_3 \cdot n^\alpha} = I_1 + I_2 + I_3.$$

By lemma 1,

$$I_2 \leq \frac{c_1}{n^\alpha} \int_{-c_2 \cdot n^{\frac{\alpha}{2+2\alpha}}}^{c_2 \cdot n^{\frac{\alpha}{2+2\alpha}}} |t|^{1+2\alpha} \cdot e^{-\frac{t^2}{2}} \, dt \leq \frac{c_1}{n^\alpha} \int_{-\infty}^{\infty} |t|^{1+2\alpha} \cdot e^{-\frac{t^2}{2}} \, dt = \frac{c_4}{n^\alpha}. \tag{24}$$

By lemma 2,

$$I_1 + I_3 \leq 2 \cdot \int_{c_2 \cdot n^{\frac{\alpha}{2+2\alpha}}}^{c_3 \cdot n^\alpha} \frac{e^{-\frac{t^2}{2}} + e^{-\frac{t^2}{3}}}{t} \, dt \leq \frac{c_5}{n^\alpha}. \tag{25}$$

Hence

$$I \leq \frac{c_4}{n^\alpha} + \frac{c_5}{n^\alpha} = \frac{c_6}{n^\alpha}. \tag{26}$$

We apply theorem 1 to (23) and (26) with

$$F(x) = \overline{F_n(x)}, \quad G(x) = \Phi(x)$$

$$f(t) = \overline{f_n(t)}, \qquad g(t) = e^{-\frac{t^2}{2}}$$

$$T = c_3 \cdot n^\alpha, \quad \varepsilon \leq \frac{c_6}{n^\alpha}, \quad A = \frac{1}{\sqrt{2\pi}}.$$

Hence

$$|\overline{F_n(x)} - \Phi(x)| \leq \frac{k}{2\pi} \cdot \frac{c_6}{n^\alpha} + \frac{\frac{1}{\sqrt{2\pi}}}{c_3 n^\alpha} c(k) = \frac{c_7}{n^\alpha}.$$

Thus the proof is completed.

If

$$c_2 \cdot n^{\frac{\alpha}{2+2\alpha}} > c_3 \cdot n^\alpha$$

only a trivial change is needed.

Remarks.

a. Let us apply theorem 2 to the theorem of LIAPOUNOFF (p. 2). By (20) we obtain the expansion $f(t) = 1 - {}^1/_2\, \sigma^2 t^2 + r(t)$, where $|r(t)| \leq \frac{\beta_3 |t|^3}{6}$. Hence $\alpha = {}^1/_2$ in (18), and by theorem 2

$$|\overline{F_n(x)} - \Phi(x)| \leq \frac{c(\sigma, \beta_3)}{\sqrt{n}}.$$

It is easily shown that $c(\sigma, \beta_3)$ is of the form $C \cdot \frac{\beta_3}{\sigma^3}$, C being an absolute constant.[1] Thus

$$|\overline{F_n(x)} - \Phi(x)| \leq C \cdot \frac{\beta_3}{\sigma^3 \sqrt{n}}. \tag{27}$$

[1] By simple, though rather extensive calculations I have found that C may be taken $= 12$. This is of course not the best value.

Thus $\log n$ may be omitted in (2) without any new conditions being added.

b. Let X_i, $(i = 1, 2, \ldots n)$ have different distribution functions but finite third absolute moments β_{3i}. Then (27) is easily extended to this general case. Let us with CRAMÉR[1] introduce the quantities

$$(28) \quad B_{3n} = \frac{1}{n}(\beta_{31} + \beta_{32} + \cdots + \beta_{3n}), \; \varrho_{3n} = \frac{B_{3n}}{s_n^3}, \; T_{3n} = \frac{\sqrt{n}}{4\,\varrho_{3n}}.$$

Then, by applying theorem 1 and the lemmas of CRAMÉR and LIAPOUNOFF, already referred to, it is possible to show shat

$$(29) \qquad |\overline{F_n(x)} - \Phi(x)| \leq \frac{C \cdot \varrho_{3n}}{\sqrt{n}},$$

C being an absolute constant.[2]

It may be shown that by the same methods it is also possible to prove some asymptotic expansions of $\overline{F_n(x)}$, given by CRAMÉR.[1]

4. **An asymptotic expansion.** In this section we shall obtain a further improvement of the theorem of LIAPOUNOFF, holding except in a special case.

Let $X_1, X_2, \ldots X_n$ be a sequence of independent random variables with the same distribution function $F(x)$, satisfying the conditions in the theorem of LIAPOUNOFF. Then by (28) $\varrho_{3n} = \frac{\beta_3}{\sigma^3}$ is independent of n, while T_{3n} is proportional to $\sqrt{n}$.

It is very easy to obtain the lemma of CRAMÉR, frequently mentioned, in a somewhat different form. Under the above conditions it runs as follows:

Lemma 3. For $|t| \leq \sqrt[3]{T_{3n}}$ the following expansion holds:

$$(30) \qquad e^{\frac{t^2}{2}} \cdot \overline{f_n(t)} = 1 - \frac{i\,\alpha_3}{6\,\sigma^3}\frac{t^3}{\sqrt{n}} + \delta_n(t) \cdot \frac{t^3}{T_{3n}};$$

$|\delta_n(t)|$ tends uniformly to zero as $n \to \infty$.

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[1] H. CRAMÉR, (2), (3).

[2] Here too C may be taken $= 12$.

We also need the lemma of LIAPOUNOFF:

Lemma 4. Under the given conditions

$$|f_n(t)| \leq e^{-\frac{t^2}{3}} \quad \text{for} \quad |t| \leq T_{3n}. \tag{31}$$

We denote by $\Phi(x)^{(\nu)}$ the ν:th derivative of $\Phi(x)$. By the relation

$$\int_{-\infty}^{\infty} e^{itx}\, d\,\Phi(x)^{(\nu)} = (-it)^{\nu}\, e^{-\frac{t^2}{2}}$$

we find that $\frac{-i\alpha_3}{6\sigma^3}\frac{t^3}{\sqrt{n}}e^{-\frac{t^2}{2}}$ is the FOURIER-STIELTJES transform of

$$P_1(x) = \frac{\alpha_3}{6\sigma^3\sqrt{2\pi}} \; \frac{(1-x^2)\,e^{-\frac{x^2}{2}}}{\sqrt{n}}. \tag{32}$$

Regarding the asymptotic behaviour of $\overline{F_n(x)} - \Phi(x) - P_1(x)$ we may prove the following statement:

Theorem 3. *Let $X_1, X_2 \ldots, X_n$ be a sequence of independent random variables with the same distribution function $F(x)$, the characteristic function $f(t)$, the mean value zero, the dispersion σ, the third moment α_3 and the finite third absolute moment β_3. Further we assume $|f(t)|$ NOT to be periodic. Then*

$$\overline{F_n(x)} = \Phi(x) + \frac{\alpha_3}{6\sigma^3\sqrt{2\pi}} \cdot \frac{(1-x^2)\,e^{-\frac{x^2}{2}}}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) \tag{33}$$

as $n \to \infty$.

Proof. By $c_1, c_2, \ldots$ we denote bounded quantities, independent of n and t. We denote by $\lambda_1(\tau)$ and $\lambda_2(\tau)$ functions ≥ 1, later to be determined. Let n be so large that

$$T_{3n} = \frac{\sqrt{n}\cdot\sigma^3}{4\beta_3} \geq 1.$$

Consider the integral

$$(34) \qquad I = \int\limits_{-T_{3n}\lambda_2(T_{3n})}^{T_{3n}\lambda_1(T_{3n})} \left| \frac{\overline{f_n(t)} - e^{-\frac{t^2}{2}} + \dfrac{i\alpha_3 t^3 e^{-\frac{t^2}{2}}}{6\sigma^3\sqrt{n}}}{t} \right| dt.$$

Let us put

$$I = \int\limits_{-T_{3n}\lambda_2(T_{3n})}^{-T_{3n}} + \int\limits_{-T_{3n}}^{-\sqrt[3]{T_{3n}}} + \int\limits_{-\sqrt[3]{T_{3n}}}^{\sqrt[3]{T_{3n}}} + \int\limits_{\sqrt[3]{T_{3n}}}^{T_{3n}} + \int\limits_{T_{3n}}^{T_{3n}\lambda_1(T_{3n})} =$$

$$= I_1 + I_2 + I_3 + I_4 + I_5.$$

By lemma 3 we obtain:

$$(35) \qquad I_3 \leq c_1 \cdot \frac{\operatorname{Max} |\delta_n(t)|}{T_{3n}},$$

where $\operatorname{Max} |\delta_n(t)|$ is the largest value of $|\delta_n(t)|$ for $|t| \leq \sqrt[3]{T_{3n}}$. By lemma 3 $\lim\limits_{n\to\infty} \operatorname{Max} |\delta_n(t)| = 0$. By lemma 4:

$$(36) \qquad I_2 + I_4 \leq \frac{c_2}{T_{3n}^2}.$$

Further

$$I_5 \leq \int\limits_{T_{3n}}^{T_{3n}\lambda_1(T_{3n})} \frac{|\overline{f_n(t)}|}{t} dt + \frac{c_3}{T_{3n}^2} = \int\limits_{T_{3n}}^{T_{3n}\lambda_1(T_{3n})} \frac{\left| f\left(\dfrac{t}{\sigma\sqrt{n}}\right)\right|^n}{t} dt + \frac{c_3}{T_{3n}^2} =$$

$$= \int\limits_{\frac{1}{4\sigma\varrho_{3n}}}^{\frac{1}{4\sigma\varrho_{3n}}\lambda_1\left(\frac{\sqrt{n}}{4\varrho_{3n}}\right)} \frac{|f(t)|^n}{t} dt + \frac{c_3}{T_{3n}^2}$$

Hence

$$(37) \qquad I_5 \leq \int\limits_{\frac{1}{4\sigma\varrho_{3n}}}^{\varkappa(n)} \frac{|f(t)|^n}{t}\,dt + \frac{c_3}{T_{3n}^2},$$

where $\varkappa(n)$ is defined by:

$$(38) \qquad \varkappa(n) = \frac{1}{4\sigma\varrho_{3n}}\,\lambda_1\left(\frac{\sqrt{n}}{4\varrho_{3n}}\right).$$

Suppose first that $\overline{\lim\limits_{t\to+\infty}}\,|f(t)| < 1$. Hence $|f(t)| < e^{-c}$, $(c > 0)$, for $t \geq \frac{1}{4\sigma\varrho_{3n}}\cdot$ By (37) we obtain

$$(39) \qquad I_5 \leq \int\limits_{\frac{1}{4\sigma\varrho_{3n}}}^{\varkappa(n)} \frac{e^{-cn}}{t}\,dt + \frac{c_3}{T_{3n}^2} \leq \frac{c_4}{T_{3n}^2}\cdot$$

Here we may put $\varkappa(n) = n$.

Now suppose $\overline{\lim\limits_{t\to+\infty}}\,|f(t)| = 1$, $|f(t)|$ not being periodic. Thus, (p. 3), $|f(t)| \neq 1$ for all finite positive values of t. We define the function $\eta(t)$ by

$$(40) \qquad 1 - \frac{1}{\eta(t)} = \underset{\frac{1}{4\sigma\varrho_{3n}} \leq \tau \leq t}{\mathrm{Max}} |f(\tau)| \quad \text{for} \quad t \geq \frac{1}{4\sigma\varrho_{3n}}\cdot$$

From this we see that $\eta(t)$ is a continuous, non-decreasing function, finite for finite values of t. As

$$\overline{\lim_{t\to+\infty}}\,|f(t)| = 1, \ \lim_{t\to+\infty} \eta(t) = \infty.$$

In (37) we put

$$(41) \qquad I_6 = \int\limits_{\frac{1}{4\sigma\varrho_{3n}}}^{\varkappa(n)} \frac{|f(t)|^n}{t}\,dt \leq \int\limits_{\frac{1}{4\sigma\varrho_{3n}}}^{\varkappa(n)} \frac{\left|1 - \frac{1}{\eta(t)}\right|^n}{t}\,dt \;.$$

For a given value of n we distinguish two cases:

1) $\eta(n) \leq \sqrt{n}$. Putting $\varkappa(n) = n$, we obtain by (41)

$$I_6 \leq \int\limits_{\frac{1}{4\sigma\varrho_{3n}}}^{n} \frac{\left(1 - \frac{1}{\sqrt{n}}\right)^n}{t}\, d\,t \leq e^{-\frac{\sqrt{n}}{2}} \cdot \log\left(4\,\sigma\,\varrho_{3n} \cdot n\right) \leq \frac{c_6}{T_{3n}^2}. \tag{42}$$

2) $\eta(n) > \sqrt{n}$. Hence by (41)

$$I_6 \leq \int\limits_{\frac{1}{4\sigma\varrho_{3n}}}^{\varkappa(n)} \frac{\left(1 - \frac{1}{\eta(\varkappa(n))}\right)^n}{t}\, d\,t.$$

Choose $\varkappa(n) = \eta^{-1}(\sqrt{n})$, $\eta^{-1}(t)$ being the inverse function of $\eta(t)$. Obviously $\lim\limits_{t \to +\infty} \eta^{-1}(t) = \infty$, and $\varkappa(n) \leq n$. Thus

$$I_6 \leq \int\limits_{\frac{1}{4\sigma\varrho_{3n}}}^{n} \frac{\left(1 - \frac{1}{\sqrt{n}}\right)^n}{t}\, d\,t \leq \frac{c_6}{T_{3n}^2}. \tag{43}$$

In either case $\lim\limits_{n \to \infty} \lambda_1(T_{3n}) = \lim\limits_{n \to \infty} 4\,\sigma\,\varrho_{3n}\,\varkappa(n) = \infty$ and by (39), (42) and (43)

$$I_5 \leq \frac{c_7}{T_{3n}^2}. \tag{44}$$

In the same way

$$I_1 \leq \frac{c_8}{T_{3n}^2}, \tag{45}$$

$\lim\limits_{n \to \infty} \lambda_2(T_{3n}) = \infty$.

By (35), (36), (44) and (45) we obtain at last

$$(46) \qquad \int\limits_{-T_{3n}\lambda_2(T_{3n})}^{T_{3n}\lambda_1(T_{3n})} \left| \frac{\overline{f_n(t)} - e^{-\frac{t^2}{2}} + \frac{i\,\alpha_3}{6\,\sigma^3\sqrt{n}}\, t^3 e^{-\frac{t^2}{2}}}{t} \right| d\,t \leq \frac{c_9}{T_{3n}^2} + $$

$$+ c_1 \cdot \frac{\operatorname{Max} |\delta_n(t)|}{T_{3n}}.$$

Apply theorem 1 to (46) with

$$F(x) = \overline{F_n(x)}, \quad G(x) = \Phi(x) + P_1(x).$$

$$f(t) = \overline{f_n(t)}, \qquad g(t) = e^{-\frac{t^2}{2}} - \frac{i\,\alpha_3}{6\,\sigma^3\sqrt{n}}\, t^3 e^{-\frac{t^2}{2}}.$$

$$T = T_{3n} \operatorname{Min} (\lambda_1(T_{3n}),\ \lambda_2(T_{3n})), \quad A = \operatorname*{Max}_{-\infty < x < \infty} \left| \frac{d}{d\,x} (\Phi(x) + P_1(x)) \right|.$$

$$\varepsilon \leq \frac{c_9}{T_{3n}^2} + \frac{c_1}{T_{3n}} \operatorname{Max} |\delta_n(t)|.$$

As $\lim\limits_{n\to\infty} \operatorname{Max} |\delta_n(t)| = 0$ and $\lim\limits_{n\to\infty} \operatorname{Min}(\lambda_1, \lambda_2) = \infty$ we obtain

$$\overline{F_n(x)} - \Phi(x) - \frac{\alpha_3}{6\,\sigma^3\sqrt{2\pi}}\, \frac{(1 - x^2)\, e^{-\frac{x^2}{2}}}{\sqrt{n}} = o\left(\frac{1}{\sqrt{n}}\right).$$

Hence the proof is completed.

Remark. If $|f(t)|$ is periodic, theorem 3 does not hold, as is shown by the following example:

$$\text{Let } F(x) = \begin{cases} 0 & \text{for } x < -1, \\ 1/2 & \text{for } -1 \leq x < 1, \\ 1 & \text{for } x \geq 1. \end{cases}$$

Then $f(t) = \cos t$, and $|f(t)|$ is periodic. The saltus of $\overline{F_n(x)}$ for $x = 0$ and n an even entire number is asymptotically equal to $\sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}$ i. e. $F_n(x)$ cannot be written $\Phi(x) + \frac{1}{\sqrt{n}}$ (continuous function) $+ o\left(\frac{1}{\sqrt{n}}\right)$.

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While the present paper — a summary of an earlier work, which was completed in the autumn of 1940 — has been at the press, I have found in a review in Mathematical Reviews, 2, (1941), p. 228, that the relation (29) has been proved by A. C. BERRY, Trans. Amer. Math. Soc. 49, (1941), p. 122—136. This work is not yeat accessible in Sweden.

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