Choosing to Rank

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Abstract

Ranking data arises in a wide variety of application areas, generated both by complex algorithms and by human subjects, but remains difficult to model, learn from, and predict. Particularly when generated by humans, ranking datasets often feature multiple modes, intransitive aggregate preferences, or incomplete rankings, but popular probabilistic models such as the Plackett-Luce and Mallows models are too rigid to capture such complexities. In this work, we frame ranking as a sequence of discrete choices and then leverage recent advances in discrete choice modeling to build flexible and tractable models of ranking data. The basic building block of our connection between ranking and choice is the idea of repeated selection, first used to build the Plackett-Luce ranking model from the multinomial logit (MNL) choice model by repeatedly applying the choice model to a dwindling set of alternatives. We derive conditions under which repeated selection can be applied to other choice models to build new ranking models, addressing specific subtleties with modeling mixed-length top-k rankings as repeated selection. We translate several choice axioms through our framework, providing structure to our ranking models inherited from the underlying choice models. To train models from data, we transform ranking data into choice data and employ standard techniques for training choice models. We find that our ranking models provide higher out-of-sample likelihood when compared to Plackett-Luce and Mallows models on a broad collection of ranking tasks including food preferences, ranked-choice elections, car racing, and search engine relevance ranking data.

1 Introduction

Ranking data arise naturally in a wide variety of application areas including recommender systems, social choice, competitions, and information retrieval. Working with ranking data presents considerable computational challenges that derive from the complex structure of the space of permutations S_n, also known as the symmetric group, where models of ranking data are essentially parametric families of distributions in this space. As a result many such models are simplistic by necessity, and still inference may be very difficult.

There are extensive literatures on both choice and ranking with known connections. A ranking model should be thought of as a family of probability distributions over complete rankings of n items, whereas a choice model should be thought of as a a family of sets of probability distributions over alternatives of all subsets of n items. Although the space of subsets of a universe U of alternatives and the space of rankings of elements of U are different, models of choice and ranking face similar combinatorial challenges that makes balancing flexibility and tractability a shared concern. Discrete choice theory has wide applicability in domains that model consumer behavior, including demand forecasting [30, 4], assortment optimization [13], and revenue management [39]. In particular, there is a rich history of formulating discrete choice models as arising from underlying preference rankings [19, 7, 41], while we study probabilistic models for rankings that derive from discrete choice models.

A seminal connection between modeling rankings and choice is the construction of the Plackett-Luce (PL) ranking model from the Multinomial Logit (MNL) model by building rankings top-down with repeated applications of the MNL model [25, 33]. This work we develop more general principles under which this process can transform other choice models into choice-based ranking models that can be inferred efficiently from data. We call this process *repeated selection*, defined as a function that maps rankings to sets of choices, and show that assigning each permutation the product of the probabilities of its affiliated choices yields a nicely normalized distribution for any choice model. We further give a natural extension for this process to top-k partial rankings. As a result, our framework is well suited to handle the common practical complication of mixed-length top-k ranking data.

A separate approach to modeling ranking data, not usually viewed as choice-theoretic, involves characterizing probability distributions over the space of rankings in terms of distances away from a reference permutation on the symmetric group. The most popular such distance, Kendall's τ distance $\tau(\sigma, \sigma_0)$ counts the number of "inversions" between two rankings: the number of pairs $i < j$ where $\sigma(i) < \sigma(j)$ but $\sigma_0(i) > \sigma_0(j)$, underlies the popular Mallows model, which assigns each permutation a probability that decreases exponentially in some scaling of its Kendall's τ distance from reference permutation σ_0 [27]. While popular, learning a Mallows model from data poses many inferential challenges. Computing the most likely σ_0 from some input ranking data $\sigma_1, \ldots, \sigma_k$ is equivalent to Kemeny rank aggregation, which is known to be a NP-hard problem for $k \geq 4$ input rankings [16]. The Mallows model can, in fact, also be derived as repeated selection from a choice model by noting that the distance τ can be decomposed into the sum of inversions caused by successive items in a ranking [12], used by Qin et al. give a greedy approximation algorithm which produces a locally (under τ) optimal reference permutation for top-k lists [34].

Although we can study the Mallows and Plackett-Luce models through the shared lens of repeated selection, our central aim is to apply repeated selection to flexible and tractable choice models to obtain choice-based ranking models that inherit the flexibility and tractability of the choice models. We specifically focus on two recent models: the Pairwise Choice Markov Chain (PCMC) model, which frames choice probabilities as the stationary distribution a of continuous time Markov chain on the choice set [35], and is among a recent of wave in modeling progess in discrete choice theory using Markov chains [7, 31]. We also consider the Context Dependent Model (CDM), which frames choice probabilities as arising from adding pairwise utilities for alternatives within the choice set [32]. Both the PCMC and CDM models include the MNL choice model as a special case, so repeated selection with these models can be conceived as two different generalizations of the Plackett-Luce ranking model.

Axiomatic approaches are important in the development of discrete choice models. For example, the MNL model was derived from Luce's choice axiom [25], which is also commonly referred to as the Independence of Irrelevant Alternatives (IIA). The flexibility of the PCMC and CDM choice models arise largely by eschewing many common choice axioms, but they still exhibit important structure. As part of this work we develop two weaker versions of IIA, *local IIA* and *nested IIA*, which we show hold for some PCMC and CDM models. We translate these axioms through repeated selection to characterize structural properties of even our most flexible choice-based ranking models.

Another broad and important family of choice models known as Random Utility Models (RUMs) affiliate a "utility" distribution with each alternative and define the selection probability of an alternative as the probability that a draw from its utility distribution is the largest within the choice set. A natural ranking model arises from a RUM choice model by ordering the alternatives sorted by their utility, a process of *joint selection*. Joint selection transforms the MNL choice model into the Plackett-Luce models (the same correspondence as for repeated selection), but yields different ranking models for other RUMs [8]. Joint selection can only be applied to RUMs, and neither the CDM or PCMC choice model are RUMs. Some work has found success using the joint selection distribution of RUMs for modeling rankings [2], but our focus in this work is on choice-based models of ranking.

Lastly, we note the large body of work on "learning to rank" in the information retrieval literature [43, 23], which focuses on outputting a ranking to maximize metrics such as machine average precision (MAP), discounted cumulative gain (DCG), or mean reciprocal rank (MRR). Many learning to rank methods highlight the use of features of ranked alternatives or rankers or external quality scores [43, 37]. Our present work differs from the typical learning to rank framework both because we are learning a distribution over rankings (rather than outputting a single ranking) and because we do not focus on using relevance labels.

We bring our choice-based ranking models to data, focusing both on predicting out-of-sample full and top-k rankings as well as predicting subsequent entries of such rankings as the top entries are revealed. We find that the flexible choice-based ranking models we introduce in this work, repeated selection applied to the PCMC and CDM models, achieve significantly higher out-of-sample likelihood compared to the Plackett-Luce and Mallows models across a wide range of applications. Our datasets include ranked choice voting from elections, lists of ranked sushi preferences, Nascar race results, and search engine results. This collection of datasets demonstrates the broad efficacy of our approach across not only application areas but also differing dataset characteristics: these datasets range greatly in size, number of alternatives, how many rankings each alternative appears in, and uniformity of ranking length.

Our empirical results show that the best model on each dataset considered is repeated selection either with the PCMC model or with the CDM model. The Mallows model (more specifically, a common greedy approximation of it) performs poorly in all out-of-sample prediction tasks, typically much worse than a uniform distribution over rankings. For human preferences on sushi and in elections, we find that repeated selection with a CDM model increases log-likelihood significantly. For nascar races and search rankings, repeated selection with the PCMC model leads to large gains in log-likelihood over the Plackett-Luce model. By looking at the improvements in position-level predictions we find that the more complex CDM and PCMC models make specific improvements in predicting 2nd and 3rd place entires of ranked vote ballots much better than the simpler Plackett-Luce model. Separately, on the nascar dataset the PCMC model outperforms all other models in predicting the 25th through 35th place racers (once earlier finishing racers have been revealed).

2 Choosing to rank

In this section we formalize our translation for models of discrete choice data into models of ranking data. Throughout this work we will use $\sigma \in S_n$ to represent a permutation or ranking and σ^{-1} to represent the inverse mapping, where $\sigma^{-1}(k)$ is the item ranked at position k by σ. We begin with definitions formalizing the notion of a choice, choice model, and choice representation for this work.

Definition 1. *A* **choice** *is an ordered pair (i, S) where i is an element chosen from subset S of the universe of alternatives U.*

Definition 2. *A* **choice probability** *$p(i, S)$ is the probability of choosing i from S such that $\sum_{i \in S} p(i, S) = 1$. We write $p_\theta(i, S)$ when the choice model is parameterized by some parameter(s) θ.*

Definition 3. *A* **choice model** *p is a collection of probability distributions $p(\cdot, S)$ over all subsets S of some universe of alternatives U.*

Definition 4. *A* **ranking** *σ of a universe of alternatives U with $|U| = n$ is a bijection from U to $\{1, 2, \ldots, n\}$.*

Definition 5. *A* **choice representation** *is a mapping c from a ranking σ of alternatives in U to a set of choices (i, S).*

We can think of a ranking σ as resulting from a series of choices between alternatives in U, with the choice representation c giving us the translation from rankings into corresponding choices. Several choice representations that map to pairwise choices (only) arise in the literature on "rank breaking" [1], which considers e.g. the set of pairwise choices implied by σ as $c_{pair}(\sigma) = \{(i, \{i, j\}) : \sigma(i) < \sigma(j)\}$. For example, if σ ranks the set $\{A, B, C\}$ and $\sigma(A) = 2, \sigma(B) = 1, \sigma(C) = 3$, this choice representation gives $c_{pair}(\sigma) = \{(B, \{B, A\}), (B, \{B, C\}), (A, \{A, C\})\}$. This work on rank breaking has leveraged these representation of rankings as collections of pairwise choices to develop generalized method of moments estimators for model parameters [21, 1, 11]. One view of this paper's contribution is that it generalizes rank breaking to choice representations other than just pairwise choices, thereby opening up a broader range of connections between ranking and choice.

Our use of general choice sets allows us to naturally extend the rank breaking concept to apply it to models that violate the independence of irrelevant alternatives where choices from larger sets can not be reduced to pairwise comparisons. The heart of our *choosing to rank* approach is thus the following translation from a choice model and a choice representation to a ranking distribution.

Definition 6. *For a choice representation c and discrete choice model p over n alternatives, the c_p distribution over rankings is given by*

$$P_{c,p}(\sigma) = \frac{1}{Z(p,c)} \prod_{(i,S)\in c(\sigma)} p(i,S),$$

which defines a ranking distribution up to a normalization constant $Z(p,c)$:

$$Z(p,c) = \sum_{\sigma\in S_n} \prod_{(i,S)\in c(\sigma)} p(i,S). \tag{1}$$

When the choice model comes from a parametric family p_θ for $\theta \in \Theta$, we bypass p in the notation: $P_{c,\theta}(\sigma) \propto \prod_{(i,S)\in c(\sigma)} p_\theta(i,S)$.

2.1 Repeated selection and repeated elimination

We proceed to define repeated selection (RS) as perhaps the most intuitive choice representation. For a choice model p, RS envisions $\sigma^{-1}(1)$ as selected according to $p(\cdot, U)$, $\sigma^{-1}(2)$ according to $p(\cdot, U - \sigma^{-1}(1))$, and so forth. We also define the natural complement of RS, repeated elimination (RE).

Definition 7. **Repeated selection** *(RS) is the choice representation*

$$RS(\sigma) = \{(\sigma^{-1}(i), \{\sigma^{-1}(i), \sigma^{-1}(i+1), \ldots, \sigma^{-1}(n)\})\}_{i=1}^{n}.$$

Definition 8. **Repeated elimination** *(RE) is the choice representation*

$$RE(\sigma) = \{(\sigma^{-1}(i), \{\sigma^{-1}(i), \sigma^{-1}(i-1), \ldots, \sigma^{-1}(1)\})\}_{i=1}^{n}.$$

Repeated selection was first proposed by Luce as the "ranking postulate," accompanied by a theorem establishing that when Luce's choice axiom holds for the choice model, the probability that $x \in U$ is ranked above $y \in U$ under the ranking model is the same as the probability x is selected from $\{x, y\}$ [25].

Keeping in mind that RS is conceptually a mapping from a choice model to a ranking model, and that choice models studied typically belong to a parametric family of models $\{p_\theta\}_{\theta\in\Theta}$, we adopt for a family of choice models $\mathcal{M}$ the notation $RS_{\mathcal{M}}$ to denote the family of ranking models arising from repeated selection with choice models $p \in \mathcal{M}$. For example, for the multinomial logit model, MNL, $RS_{MNL} = \{P_{RS,p} : \ p \ \text{ is a MNL choice model}\}$. We use the phrase "repeated selection with MNL" to refer to RS_{MNL}.

The past work most in the spirit of repeated elimination is the seminal Elimination by Aspects (EBA) choice model [40], where it is notable that EBA is usually thought of primarily as a choice model and not a ranking model. In the EBA model, each alternative $i \in U$ has some set of "aspects," and choices are made by randomly choosing an aspect shared by some of the alternatives and eliminating all of the alternatives lacking that aspect.

In this work we focus on repeated selection (RS) and repeated elimination (RE) as the primary choice representations of interest. Not only are repeated selection and repeated elimination the most natural ways to translate rankings into choices, as evidenced by their connections to existing literature, but we will also show that they belong to a relatively small family of choice representations for which the normalization constant $Z(p, c) = 1$ for all choice models p, which greatly simplifies many computational issues. RS and RE are by far the two most natural representations in this small family.

2.2 Label-invariance

Label-invariance is the idea that if we relabeled the items in U and applied that relabeling to all of our rankings, the choices arising from the relabeled rankings should be the same as if we had simply applied the relabeling to each choice and choice set.

To make this notion rigorous we define label-invariance, a property of choice representations that are invariant under relabelings of U, and show that when a choice representation is label-invariant then the normalization constant in Equation (1) is 1 for all choice models. The label-invariance of the RS and RE choice representations will later be critical to our ability to maximize the likelihood of a choice-based ranking model, given ranking data, in Section 2.4.

Definition 9 (Label-invariance)**.** *A choice representation c is label-invariant if for all $\sigma, \pi \in S_{|U|}$:*

$$c(\sigma\pi) = \{(\pi^{-1}(i), \pi^{-1}(S)) : (i, S) \in c(\sigma)\}$$

where $\pi^{-1}(S) = \{\pi^{-1}(j)\}_{j \in S}$.

Label-invariance is a common assumption in studying distributions on rankings [12], and it is easy to see that RS is label-invariant.

Proposition 1. *The repeated selection (RS) and repeated elimination (RE) choice representations are label-invariant.*

Proof. We give the proof for repeated selection; the proof for repeated elimination is equivalent up to the substitution of definitions. Note that for permutations, where composition order matters, $(\sigma\pi)^{-1} = \pi^{-1}\sigma^{-1}$.

$$\begin{aligned} RS(\sigma\pi) &= \{(\sigma\pi)^{-1}(k), \{(\sigma\pi)^{-1}(k)\}_{j \geq k}\}_{k=1}^{n} \\ &= \{(\pi^{-1}(\sigma^{-1}(k)), \{\pi^{-1}(\sigma^{-1}(j))\}_{j \geq k})\}_{k=1}^{n} \\ &= \{(\pi^{-1}(x), \pi^{-1}(S)) : (x, S) \in c(\sigma)\}. \end{aligned}$$

□

Label-invariance for choice representations is inspired by the property of "right invariance," sometimes called "shift invariance," a property of distances d on S_n whereby $d(\sigma, \tau) = d(\sigma\pi, \tau\pi)$ for all $\sigma, \pi, \tau \in S_n$ [14]. The property has a succinct justification that aligns with our justification for label-invariance: a distance between two rankings should be the same if we relabel the items with a permutation π. While label-invariance seems like an obvious property of any non-pathological choice representation, it is a powerful enough property to provide the following theorem.

Theorem 1. *For label-invariant choice representation c, $Z(p,c) = 1$ for all choice models p if there exists some permutation π such that $c(\sigma) = RS(\sigma\pi)$. In particular, $Z(RS,p) = Z(RE,p) = 1$ for all choice models p.*

Proof. Suppose WLOG that we are ranking $U = [n]$. We begin by considering the RS choice representation and showing that for any choice model p, $\sum_{\sigma\in S_n}\prod_{(i,S)\in RS(\sigma)} p(i,S) = 1$.

We proceed by induction on n, where $n = 1$ and $n = 2$ are trivial. Assuming that the statement holds for rankings of $n-1$ items, we partition S_n into $A_j = \{\sigma : \sigma(j) = 1\}$ for $j \in [n]$. Then

$$\begin{aligned}
\sum_{\sigma\in S_n}\sum_{(i,S)\in RS(\sigma)} p(i,S) &= \sum_j \sum_{\sigma\in A_j}\prod_{(i,S)\in RS(\sigma)} p(i,S) \\
&= \sum_j p(j,U)\sum_{\sigma\in A_j}\prod_{(i,S)\in RS(\sigma)-(j,U)} p(i,S) \\
&= \sum_j p(j,U)\sum_{\sigma\in S_{U-j}}\prod_{(i,S)\in RS(\sigma)} p(i,S) \\
&= \sum_j p(j,U) = 1,
\end{aligned}$$

where the inductive hypothesis is applied in the second to last equality, noting that $|U-j| = n-1$, and the final inequality holds because $p(\cdot,U)$ is a probability distribution on U.

Now consider any permutation $\pi \in S_n$ and the choice representation

$$c_\pi(\sigma) = \{(\sigma^{-1}(\pi^{-1}(i)), \{\sigma^{-1}(\pi^{-1}(j))\}_{j\geq i})\}_{i=1}^n.$$

By label-invariance we have that $c_\pi(\sigma\pi) = RS(\sigma)$ and $c_\pi(\sigma) = RS(\sigma\pi^{-1})$. Because composition with π^{-1} is an automorphism of S_n, we have for any function $f : S_n \to \mathbb{R}$ that $\sum_\sigma f(\sigma) = \sum_\sigma f(\sigma\pi^{-1})$. Taking $f(\sigma) := \prod_{(i,S)\in RS(\sigma)} p(i,S)$, we thus have

$$\sum_\sigma \prod_{(i,S)\in c_\pi(\sigma)} p(i,S) = \sum_\sigma f(\sigma\pi^{-1}) = \sum_\sigma f(\sigma) = 1.$$

Letting π be the permutation that reverses σ gives the result for repeated elimination. □

When π is the identity permutation, we have that the normalization constant for RS is always 1, and when π reverses its inputs, we have that the normalization constant for RE is always 1. The remaining π describe choice models where we choose the $\pi^{-1}(1)$-th position from U first, then the $\pi^{-1}(2)$-th position from what remains, and so forth. As a result, we can conclude that RS and RE are the two "natural" choice representations with this property. We are not aware of any choice representations beyond the scope of this family that have a normalization constant of one.

2.3 Partial (top-k) rankings

Throughout this work we generally focus on complete rankings, unless otherwise noted. Because data consisting of incomplete "top-k" ratings are commonplace, we provide a natural extension of repeated selection to such partial rankings. We show in a theorem analogous to Theorem 1 that this extension has a normalization constant of one for top-k rankings for any $k \in [n]$ under repeated selection with any choice model p. The normalization constant for top-k rankings under repeated elimination or other permuted selection processes is not necessarily one.

Definition 10. *A* **top-k ranking** σ *of universe* U *with* $|U| = n$ *is a 1-1 mapping from some subset* S *of* U *with size* k *to* $[k]$*. We refer to the set of top-*k *rankings of* n *items as* $S_{k,n}$*.*

Top-k rankings are common in many empirical datasets we consider. Assuming these rankings came from some repeated selection ranking distribution, we can still obtain the first k choices in the choice representation $RS(\sigma)$ given $\sigma(1), \ldots, \sigma(k)$ and U.

Definition 11. Repeated selection for top-k rankings σ *on universe* U *is the choice representation*

$$RS(\sigma) = \{(\sigma^{-1}(i), U - \{\sigma^{-1}(j)\}_{j \geq i})\}_{i=1}^{k}. \tag{2}$$

A top-n ranking is simply a full ranking and the extension agrees with our previous definition of repeated selection in these instances. More abstractly, we could equivalently characterize this extension as the intersection of $RS(\tilde{\sigma})$ for complete lists $\tilde{\sigma}$ which match σ's first k entries. Given that we've extended RS to top-k rankings, we have a natural extension of RS_p from Definition 6 for top-k rankings.

Definition 12. *For discrete choice model* p *over* n *alternatives* U*, the* RS_p *distribution over top-*k *rankings* σ *is given by*

$$P_{RS,p,k}(\sigma) = \frac{1}{Z_k(p, RS)} \prod_{(i,S) \in RS(\sigma)} p(i, S), \tag{3}$$

which defines a top-k *ranking distribution up to a normalization constant* $Z_k(p, RS)$*:*

$$Z_k(p, RS) = \sum_{\sigma \in S_{k,n}} \prod_{(i,S) \in RS(\sigma)} p(i, S). \tag{4}$$

We also have a natural analog of Theorem 1 in the specific case of the choice representation RS_p. The proof of this theorem, which handles only RS_p and not RE_p or other permuted choice representations, is simpler than for Theorem 1 and so we include it here.

Theorem 2. *For any choice model* p *on* n *alternatives,* $Z_k(p, RS) = 1$ *for every* $k \in [n]$*.*

Proof. We induct on k. For $k = 1$, we note that a top-1 ranking has a single choice $(\sigma^{-1}(1), U)$. Because p is a choice model, $\sum_{x \in U} p(x, U) = 1$ gives $Z^1(p, RS) = 1$.

Assume the proposition holds for $j \leq k$. Recall that $S_{j,n}$ is the set of top-j rankings of $[n]$. For every top-$(k+1)$ ranking in σ $S_{k+1,n}$ there exists some unique top-k ranking σ' in $S_{k,n}$ such that σ' is the prefix of σ. Furthermore, $RS(\sigma) = RS(\sigma') \cup \{(\sigma^{-1}(k), U - \{\sigma^{-1}(j)\}_{j<k})\}$,

and each of the $n-k+1$ rankings σ which share σ' as a prefix is represented uniquely by $\sigma^{-1}(k)$ being chosen from $U - \{\sigma^{-1}(j)\}_{j<k})\}$, those items unranked by σ' . Thus we have

$$\sum_{\sigma \in S_{k+1,n}} \prod_{(i,S)\in RS(\sigma)} p(i,S) = \sum_{\sigma' \in S_{k,n}} \sum_{x \in U - \{\sigma'^{-1}(j)\}_{j<k}} \left(p(x, U - \{\sigma'^{-1}(j)\}_{j<k}\}) \prod_{(i,S)\in RS(\sigma')} p(i,S) \right).$$

Noting that we can pull the product out of the inner sum and that

$$\sum_{x \in U - \{\sigma'^{-1}(j)\}_{j<k}} p(x, U - \{\sigma^{-1}(j)\}_{j<k}\}) = 1,$$

because p is a choice model, we obtain

$$\sum_{\sigma \in S_{k+1,n}} \prod_{(i,S)\in RS(\sigma)} p(i,S) = \sum_{\sigma' \in S_{k,n}} \prod_{(i,S)\in RS(\sigma')} p(i,S),$$

and conclude that the sum is 1 for top-k rankings by the inductive hypothesis. □

A natural analogous extension of RE to top-k rankings σ is to again simply include all of the RE choices that would be in $RE(\sigma')$ where σ is a prefix of a complete ranking σ'. Unlike the RS case, this extension suffers from issues with normalization, as the sum over top-k lists of the product of choices in the extension is not 1 for $k < n$. Note that "bottom-k rankings," if such things were relevant objects of study, would have $Z_k(p, RE) = 1$.

The issues with extending RE to top-k lists are especially apparent for $k = 1$. For top-1 lists, $\sigma(x) = 1$, we can only determine that $(x, \{x\})$ lies in $RE(\sigma')$ for any complete list σ' that has σ as its prefix. Because $p(x, \{x\}) = 1$ for all choice models p, all rankings σ would be assigned equal probability under this extension, and the likelihood of the data would be the same for any choice model, making the extensions useless for $k = 1$.

2.4 Transforming ranking likelihoods to choice likelihoods

Consider a set $T = \{\sigma_j\}_{j=0}^m$ of rankings of items in a universe U. Then the likelihood of the choice model parameters θ for the ranking data T is:

$$\mathcal{L}_c(\theta; T) = \frac{1}{Z(p_\theta, c)} \prod_{\sigma \in T} \prod_{(i,S)\in c(\sigma)} p_\theta(i,S), \tag{5}$$

where $Z(p_\theta, c)$ is the normalization constant from Equation (1).

One of the most attractive features of our framework is that we can fit ranking models simply by fitting choice models when using repeated selection because the likelihood of ranking data decomposes into the likelihood of choice data. More specifically, when we have a set of choices $A = \{(i_k, S_k)\}_{k=0}^m$ (e.g. from a choice representation of ranking data, but not necessarily) and a parametric family of choice models $\{p_\theta\}_{\theta\in\Theta}$ we use the notation $\mathcal{L}(\theta; A) = \prod_{(i,S)\in A} p_\theta(i,S)$ to denote the likelihood of θ for that choice data. We justify the overloaded notation by noting that when the choice representation c is repeated selection or repeated elimination we have by Theorem 1 that the normalization constant in Equation (5) is 1, meaning that we can train a repeated selection model simply by training the underlying choice model on the union of the sets of choices given by applying the $RS(\cdot)$ choice representation to every ranking in the training set. Although this is straightforward from our existing formulations, we formalize it here for clarity.

Proposition 2. *For a parametric family of choice models $\{p_\theta\}_{\theta\in\Theta}$ and a choice representation c that is repeated selection or repeated elimination, let $\mathcal{L}_c(\theta; T)$ be the likelihood of a set of rankings T under when using choice representation c and choice model p_θ, and let $\mathcal{L}(\theta; A)$ be the likelihood for a set of choices A. Then $\mathcal{L}_c(\theta; T) = \mathcal{L}(\theta; \cup_{\sigma\in T} c(\sigma))$.*

Proof. We have from Theorem 1 that when c is repeated selection or repeated selection, $Z(p_\theta, c) = \sum_\sigma \prod_{(i,S)\in c(\sigma)} p_\theta(i, S) = 1$ for all $\theta \in \Theta$. The rest of the proof is simply reindexing with definitions: the definitions of $\mathcal{L}$ for sets of rankings and choices and from the definition $c(T) := \cup_{\sigma\in T} \cup_{(i,S)\in c(\sigma)} (i, S)$

$$\mathcal{L}_c(\theta; T) = \prod_{\sigma\in T} \prod_{(i,S)\in c(\sigma)} p_\theta(i, S) = \prod_{(i,S)\in \cup_{\sigma\in T} c(\sigma)} p_\theta(i, S) = \mathcal{L}(\theta; \cup_{\sigma\in T} c(\sigma)).$$

$\square$

When c is repeated selection, Proposition 2 can be extended to sets T of top-k rankings, appealing to Theorem 2 where $Z_k(p, RS) = 1$ for any $k \in [n]$ then as well. Because the likelihood of a repeated selection distribution is equivalent to the likelihood of the underlying choice model on a transformation of the rankings into choices, we can expect inference for a repeated selection *ranking model* to be effective and tractable when we can effectively and tractably infer the underlying *choice model*.

3 Ranking with repeated selection

We now take a detailed look into ranking distributions arising from repeated selection (RS) with choice models. We begin by examining the popular Plackett-Luce and Mallows ranking models and the choice models from which they arise with repeated selection. We then apply repeated selection to the PCMC and CDM choice models, yielding flexible models for rankings from the same conceptual vein as the Plackett-Luce model.

3.1 MNL and Plackett-Luce

The MNL model states that the probability of choosing alternative i from choice set S is proportional to some non-negative "quality scores" γ_i for all S. More precisely,

$$p_\gamma(i, S) = \frac{\gamma_i}{\sum_{j\in S} \gamma_j}. \tag{6}$$

Luce noted that Plackett's ranking model [33], since called the Plackett-Luce model, was a direct consequence of his choice axiom and his ranking postulate, which defined repeated selection for the MNL model. For the Plackett-Luce ranking model with its underlying MNL choice model parameterized by quality scores γ, we have:

$$P_{RS,\gamma}(\sigma) = \prod_{i=1}^{n} \frac{\gamma_{\sigma^{-1}(i)}}{\sum_{j \geq i} \gamma_{\sigma^{-1}(j)}}. \tag{7}$$

3.2 Mallows as repeated selection

The Mallows distribution assigns probabilities to rankings which decrease exponentially in the number of pairs of alternatives they "invert" relative to some reference permutation σ_0. More precisely, under a Mallows model with concentration parameter θ and reference permutation σ_0,

$$Pr(\sigma; \sigma_0, \theta) \propto e^{-\theta\tau(\sigma,\sigma_0)}, \tag{8}$$

where

$$\tau(\sigma, \sigma_0) = \sum_{i,j \in [n]: \sigma_0(i) < \sigma_0(j)} 1(\sigma(i) > \sigma(j)). \tag{9}$$

There is a known choice model for which repeated selection yields the Mallows ranking model that has been used in work on e.g. assortment optimization [13]. Let $\sigma_0 \in S_n$ be a reference permutation and let $\theta \in \mathbb{R}^+$ be a scalar concentration parameter. The probability that i is ranked first among a subset S for this choice model is then exponential in the number of elements in S ranked above i by the reference permutation σ_0.

[20] were among the first to note in that the Mallows model can be composed into what we can interpret as a product of choice probabilities: for choice model

$$p_{\theta,\sigma_0}(i, S) \propto \exp(-\theta \cdot |\{j \in S : \sigma_0(j) < \sigma_0(i)\}|), \tag{10}$$

then $\prod_{(i,S) \in RS(\sigma)} p_{\theta,\sigma_0}(i, S)$ is equivalent to the Mallows density $\frac{e^{-\theta\tau(\sigma,\sigma_0)}}{\sum_{\sigma' \in S_n} e^{-\theta\tau(\sigma',\sigma_0)}}$.

Although the Mallows model has a convenient form for repeated selection, finding the maximum likelihood reference permutation is NP-hard for as few as four input lists, as previously discussed. In our empirical comparisons between RS mappings of choice models later in this work we approximate the Mallows ranking distribution "greedily," following [34], by building the ranking from front to back by choices that minimize inversions.

3.3 Repeated selection with the PCMC choice model

The pairwise choice Markov chain (PCMC) model is a recent discrete choice model where the selection probability of an item i from a set S is the probability mass on i in the stationary distribution of a continuous time Markov Chain whose transition rates q_{ij} parameterize the model [35]. For all $S \subseteq U$, $i \in S$, the model says that $\sum_{j \in S-i} p(i, S)q_{ij} = \sum_{j \in S-i} p(j, S)q_{ji}$. The likelihood of choice data under this model can be maximized over its parameter space of non-negative rate matrices Q. The PCMC model is not a RUM.

Repeated selection from a PCMC model can be conceptualized as a "darting eye" construction where an individual takes a random walk on unranked alternatives. At some point the walk halts, returns the current state of the chain as an entry of their ranking, and then

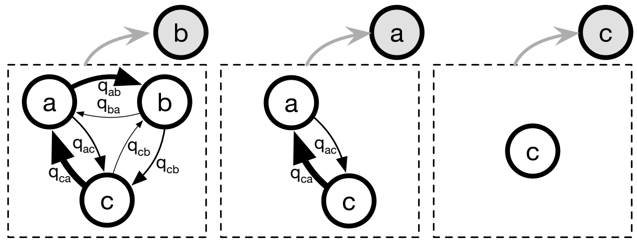

Figure 1: Repeated selection from a PCMC model to generate a distribution on rankings. First a choice is made from a CTMC on $\{a, b, c\}$, yielding $\sigma(1) = b$. The state space of the chain is subsetted to the unchosen elements $\{a, c\}$ and a new choice is made, yielding $\sigma(2) = a$. The last item is chosen last, $\sigma(3) = c$.

the walk continues over the unused alternatives, mixing fully before making the next halt. For a visualization of this ranking process, see Figure 1.

Observation 1. *All Plackett-Luce models are RS_{PCMC} models.*

This observation follows from noting that MNL choice model is a special case of the PCMC choice model with specific transition rates: given an MNL model with parameters $\gamma = (\gamma_1, ..., \gamma_n)$ and setting $q_{ij} = \gamma_i/(\gamma_i + \gamma_j)$, $\forall i, j$, the stationary distribution of the Markov chain is equivalent to the choice probabilities of the given MNL model by Proposition 2 of [35]. Because Plackett-Luce models are RS_{MNL} models, it follows in turn that Plackett-Luce models are RS_{PCMC} models.

3.4 Repeated selection with the CDM choice model

The context-dependent model (CDM) is another recently developed model for choice which focuses on contextual utility [32]. A d-dimensional CDM includes $2dn$ parameters, with each alternative $i \in U$ having a feature vector $a_i \in \mathbb{R}^d$ and a context vector $b_i \in \mathbb{R}^d$. The utility of alternative i in the context of S is simply the sum of the dot product of i's features with each context vector in the choice set. The choice probabilities are proportional to the exponential of these contextual utilities, i.e.

$$p(i, S) \propto e^{u_{iS}}, \ u_{iS} := \sum_{j \in S} a_i \cdot b_j. \tag{11}$$

An equivalent characterization of the CDM model considers pairwise contextual utilities $u_{ij} := a_i \cdot b_j$ as the full-rank parameterization of the model and notes that $u_{iS} = \sum_{j \in S} u_{ij}$. For d-dimensional feature and context vectors, it is equivalent to constraining the rank of the matrix of contextual pairwise utilities $\mathcal{U} = \{u_{ij}\}_{i \in U, j \in U}$ (recall that U here is the universe of alternatives). Because the contextual utilities depend on what is in the choice set S, the CDM is not a RUM.

Observation 2. *All Plackett-Luce models are RS_{CDM} models.*

As with Observation 1, this observation stems from MNL models being a subset of CDM models. One construction of a CDM model equivalent to an MNL model with parameters γ

is to set $u_{ij} = \log \gamma_j$ for all $i, j \in U$.

3.5 Repeated selection and choice axioms

One appeal of choice-based ranking is that existing knowledge about a choice model may provide structure to the resulting ranking distribution under that choice model. Here we translate several choice axioms through the repeated selection choice representation.

3.5.1 Luce's choice axiom (IIA)

Luce's choice axiom [25], also known as the independence of irrelevant alternatives (IIA), states that (i) the probability of choosing an element i from S conditioned on choosing an element from some subset S' of S is equal to the probability of choosing i from S', and (ii) if i is never chosen from S, i is never chosen from any set containing S. Luce showed that any model satisfying the choice axiom was equivalent to some MNL model, which defines the selection probability of i in S proportional to some latent quality γ_i for each choice set S (in Luce's words, every such model admits a *ratio scale representation* [26]). Thus the repeated selection ranking distribution of any choice model satisfying IIA will be equivalent to some Plackett-Luce distribution with parameters γ.

3.5.2 Local and Nested IIA

Although celebrated as a seminal choice axiom, IIA often fails to hold for empirical data [5, 9]. A popular generalization of MNL that escapes IIA is the Nested Multinomial Logit model (NMNL) that partitions the universe into subsets called nests for which IIA holds [29].

Recall that we let $p(i, T)$ represent the probability that i is chosen from T and $p(S, T) = \sum_j p(j, T)$ for $S \subseteq T$. If we unpack Luce's axiom with Bayes' rule we see that if x is the chosen element,

$$Pr(x = i | x \in S) = \frac{Pr(x \in S | x \in T) Pr(x = i | x \in T)}{Pr(x \in S | x \in T)} = \frac{p(i, T)}{p(S, T)},$$

and from the choice axiom for subset S of T we have $P(x = i | x \in S) = p(i, S)$. We thus have $p(i, T) = p(S, T)p(i, S)$. For the MNL model, this relationship holds for any $S \subseteq T \subseteq U$. For other models it may not hold for all pairs of S and T but may still hold for many pairs in a way that provides instructive structure. The benefit of this formulation is that it allows us to express the axiom entirely with our adopted notation for the choice model, p.

Definition 13. *For a choice model p on universe U and subsets S and T with $S \subseteq T \subseteq U$, (p, S, T) exhibits* **local IIA** *if for all $i \in S$:*

$$p(i, S) = p(S, T)p(i, T). \tag{12}$$

While the NMNL model is usually considered for data consisting of a single choice set, this work focuses on repeated selection from a choice model, which will typically use all subsets of U as choice sets. Because of this, we want to focus on choice models where the nesting structure is not destroyed by the removal of some of the alternatives.

Definition 14. *For a choice model p on universe U and subsets S and T with $S \subseteq T \subseteq U$, (p, S, T) exhibits* **nested IIA** *if for all $S' \subseteq S$:*

$$p(i, T - S') = p(S - S', T - S')p(i, S - S').$$

Both the PCMC model and the CDM model can exhibit nested IIA under reasonable conditions, giving structure to both their choice probabilities and their repeated selection distributions. Proofs of both propositions appear in the e-companion.

Proposition 3. *For a PCMC choice model p parameterized by rate matrix Q and a partition $S_1, \ldots, S_k$ of U where for all $i \in S_I, j \in S_J$ with $I \neq J$, $q_{ij} = \lambda_{IJ}$, then for each I, (p, S_I, U) exhibits nested IIA.*

Proposition 4. *For a CDM choice model p parameterized by pairwise contextual utilities $\mathcal{U} = \mathbf{f}^T\mathbf{c}$ with $\mathcal{U}_{ij}$ fixed for all $i \in S, j \notin S$, (p, S, U) exhibits nested IIA.*

Nested IIA identifies sets for which we can easily decompose the repeated selection probabilities into choosing a subset and choosing from within that subset for every choice in the choice representation $RS(\sigma)$. If (p, S, U) exhibits local IIA, then for any $i \in S$ we can decompose the choice probabilities as in Equation (12), so for the choice probabilities for the first item $\sigma^{-1}(1)$ in a ranking σ from repeated selection, the decomposition applies to $p(x, U)$ for $x \in S$.

Local IIA alone only applies to $p(\cdot, U)$, though, whereas $RS(\sigma)$ contains choices from other choice sets, so we cannot say anything about $P_{RS,p}(\sigma)$ with Local IIA on U alone. Nested IIA gives us that the structure in $p(\cdot, U)$ extends to the subsets of U which appear as choice sets in $RS(\sigma)$, so Nested IIA gives us enough structure to p to make statements about $P_{RS,p}$.

Consider the second choice in a ranking under in a repeated selection model, choosing $\sigma^{-1}(2)$ from $U - \sigma^{-1}(1)$, where $\sigma^{-1}(1)$ was in some subset S exhibiting local IIA for U. If S and U exhibited nested IIA for p, then for $i \in S - \sigma^{-1}(1)$, we have

$$p(i, U - \sigma^{-1}(1)) = \left(\sum_{j \in S - \sigma^{-1}(i)} p(j, U - \sigma^{-1}(1)) \right) p(i, S - \sigma^{-1}(1)),$$

but if the local IIA was not nested, we cannot say anything about the choice probabilities for $i \in S$ from $U - \sigma^{-1}(1)$ when $\sigma^{-1}(1) \in S$, as the local IIA may have been "destroyed" when $\sigma^{-1}(1)$ was removed.

When (p, S, U) further exhibit nested IIA, however, we can apply the decomposition in Equation (12) to $S \cap T$ for every choice set T in $RS(\sigma)$. As a result, for σ from $P_{RS,p}$, the relative ordering of the items in S is independent of the set of positions at which they land, $\{\sigma(i)\}_{i \in S}$, and as a result restricting σ to alternatives in S yields a repeated selection distribution with p restricted to subsets of S. We thus have the follow proposition, a proof of which appears in the e-companion.

Proposition 5 (RS within RS for nested IIA)**.** *For a choice model p on universe U, suppose (p, S, U) exhibits nested IIA. For σ drawn from $P_{RS,p}$, let σ_S be the restriction of σ to S and $\tilde{p}_S$ be the restriction of p to S. Then σ_S is distributed according to $P_{RS,\tilde{p}_S}$.*

3.5.3 Regularity

Regularity for a choice model stipulates that for all $S' \subseteq S$ and all $S \subseteq U$, $p(i, S') \geq p(i, S)$. As a result, $p(i, S) \geq p(i, U)$ for all $i \in S$ for any subset S of U, giving us a simple bound on the probability of any ranking that is most useful when $\min_{i \in U} p(i, U)$ is not too small.

Proposition 6. *For any σ drawn from repeated selection according to a choice model p exhibiting regularity,*

$$Pr(\sigma) \geq \frac{\prod_{i \in U} p(i, U)}{\max_{i \in U} p(i, U)}. \tag{13}$$

Proof. For every $k \in [n]$, $p(\sigma^{-1}(k), \{\sigma^{-1}(j)\}_{j>k}) \geq p(\sigma^{-1}(k), U)$ by regularity, as $\{\sigma^{-1}(j)\}_{j>k} \subseteq U$. Furthermore, for $k = n$, the choice probability is 1, as we are choosing from a singleton. Because σ^{-1} is a bijection from $[n]$ to U,

$$\prod_{k=1}^{n} p(\sigma^{-1}(k), \{\sigma^{-1}(j)\}_{j>k}) \geq \prod_{k=1}^{n-1} p(\sigma^{-1}(k), U) = \prod_{i \in U : \sigma^{-1}(i) < n} p(i, U) \geq \frac{\prod_{i \in U} p(i, U)}{\max_{i \in U} p(i, U)}.$$

□

Corollary 1. *If σ is drawn from repeated selection on a choice model p exhibiting regularity and with $p(i, U) \geq \epsilon$ for all $i \in U$ and some $\epsilon > 0$, then $P_{RS,p}(\sigma) \geq \epsilon^{n-1}$. Furthermore, any independent RUM model with random utilities having full support on $\mathbb{R}$ will admit a ranking model through repeated selection with full support on S_n.*

Proof. This corollary falls out of Proposition 6 simply by applying $p(i, U) \geq \epsilon$, but can also be shown directly. Let i_{min} be the element of U minimizing $p(i, U)$. Then for any subset S and $i \in S$, by regularity $p(i, S) \geq p(i, U) \geq p(i_{min}, U) \geq \epsilon$. Noting that the last choice in repeated selection is always a probability 1 choice from a singleton, the probability of σ is at least the probability of the first $n - 1$ choices, all bounded below by ϵ. □

Regularity holds for RUMs in particular, giving strong restrictions on the repeated selection ranking distribution based on any RUM. Independent random utilities are typically modeled as continuous distributions with full support, e.g. Gumbel as in the MNL model or Normal as in the Thurstone model, and thus have strictly positive choice probabilities for each alternative in U.

4 Empirical results

Here we compare the performance of repeated selection and repeated elimination models when training and making predictions on empirical datasets. The datasets span a wide variety of human decision domains including ranked elections and food preferences, while also including (search) rankings made by algorithms. We specifically highlight the performance of the RS_{PCMC} and RS_{CDM} ranking models for their ability to learn complex distributions to represent human ranking data.

We evaluate predictions based on the negative log-likelihood for out-of-sample ranking data, handled as choice data in its choice representation. One upside of choice-based ranking is that it enables evaluation of the ranking distributions through the choices at each position. For repeated selection, we can use our inferred choice models $p_{\hat{\theta}}(\cdot, S)$ to predict the next entry of each $\sigma \in T$ at each position, given that S is the set of items we haven't yet chosen from σ. We can measure the error at the k-th position of a ranking σ given the set of already ranked items each by adding up some distance between the choice probabilities p for the corresponding choice sets and the empirical distribution of those choices in the data. We define the *position-level log-likelihood* at each position k as $\ell(k, \theta; \sigma) := \log p_\theta(\sigma^{-1}(k), \{\sigma^{-1}(j)\}_{j \geq k})$.

When averaging ℓ over a test set T we obtain the average position-level log-likelihood:

$$\ell(k; \theta, T) := \frac{1}{|T|} \sum_{\sigma \in T : len(\sigma) \geq k} \ell(k, \theta; \sigma), \tag{14}$$

where $len(\sigma)$ is n for a full ranking and k for a top-k ranking.

4.1 Datasets

We consider a wide variety of application domains to demonstrate that repeated selection and repeated elimination are appropriate and practical for general ranking problems. Many of our datasets can be found in the Preflib library[1], and we include all 96 Preflib datasets that contain partial or complete rankings of up to 20 items and at most 1000 rankings, a total of 35,967 rankings. These thresholds were decided arbitrarily for computational reasons. We call this collection of datasets `PREF-SOI` [28]. We separately study the subset of 10 datasets comprised of complete rankings, referred to as `PREF-SOC`, which contain a total 5,116 rankings. These complete rankings are suitable for both repeated selection and repeated elimination.

In our evaluation we place a particular emphasis on four widely studied datasets of human choices. First, the `sushi` dataset, consisting of 5,000 complete rankings of 10 types of sushi. Next, three election datasets, which consists of ranked choice votes given for three 2002 elections in Irish cities: the `dublin-north` (`dub-n` abbr.) election had 12 candidates and 43,942 votes for lists of varying length, `meath` had 14 candidates and 64,081 votes, and `dublin-west` (`dub-w` abbr.) had 9 candidates and 29,988 votes. Note that all of these datasets have more rankings than the thresholds set for the collections `PREF-SOI` and `PREF-SOC`, so while all election datasets and the `sushi` dataset appear in the Preflib library, they have been excluded from our other analyses.

We also explore the popular `LETOR` collection of datasets, which consists of ranking data arising from search engines. Although the `LETOR` data arises from algorithmic rather than human choices, it demonstrates the efficacy of our algorithms in large sparse data regimes.

[1]Preflib data is available at: http://www.preflib.org/

After removing datasets with fewer than 10 rankings and more than 100 alternatives, the `LETOR` collection includes 762 datasets with a total of 15,349 rankings of between 3 and 100 alternatives. We again decided these thresholds arbitrarily for computational reasons.

We round out our datasets with the `nascar` dataset representing competitions, which consists of the partial ordering given by finishing drivers in each race of the 2002 Winston Cup. The data includes 74 drivers (alternatives) and 35 races (rankings).

4.2 Training

We use the stochastic gradient-based optimization method Adam [24] implemented in Pytorch to train the MNL, PCMC, and CDM-based models in this paper. We run Adam with the default parameters ($lr = 0.001$, $\beta = (0.9, 0.999)$, $\epsilon = 1e-8$). We use 5 epochs of training unless otherwise specified. We cannot use Adam (or any simple gradient-based method), for the Mallows model as the reference permutation σ_0 parameter lives in a discrete space. Instead we select the reference permutation via the Mallows Greedy Approximation (MGA) as in [34]. Given a reference permutation, the concentration parameter θ has a tidy maximum likelihood estimate. Our results broadly show that the Mallows model performs poorly compared to all the other models, including even the uniform distribution (a naive baseline), so we exclude it from some of the more detailed evaluations.

For all datasets we use 5-fold cross validation for evaluating test metrics. Using the `sushi` dataset as an example, for each choice model we train on RS and RE choice representations for each of 5 folds of the 5,000 rankings in the dataset. Across 5 choice models, this amounts to fitting 10 models on 5 folds of 5,000 rankings, represents training on a total of 200,000 rankings that represent 2 million choices. The training process can be easily guided to exploit sparsity, parallelization, and batching. Repeated elimination (RE) models are trained analogously to repeated selection (RS) models, simply reversing the training rankings.

4.3 Log-likelihood for ranking data

In Table 1 we report out-of-sample log-likelihood for a variety of datasets and the `LETOR` and `PREFLIB-SOI` collections of datasets. On the `sushi` dataset we find that repeated selection with the one dimensional CDM and the PCMC model offer slight improvements over the MNL model, while repeated selection with higher dimensional CDM models offer significant additional improvements. We see a similar pattern for all three election datasets, `dublin-north`, `dublin-west`, and `meath`. For all datasets, the Mallows Greedy Approximation (MGA) is markedly worse than every other model.

For the `nascar` dataset, we find that the CDM models and MNL models are relatively equal, while repeated selection with the PCMC model preforms significantly better. There are more alternatives/racers in this dataset than the others, but few rankings/races, so the PCMC model appears to be benefitting from its larger number of parameters to represent this complex data. Another more remote possibility is that the default parameters for Adam somehow suit the RS_{PCMC} models better than the RS_{CDM} models. Brief explorations gave no indication that this is the case, as other Adam parameter values did not meaningfully change the relative performance of these methods. We did not extensively explore the myriad hyperparameters of Adam for fear of overfitting, instead relying on default values.

Dataset	RS_{MNL}	$RS_{CDM,d=1}$	$RS_{CDM,d=4}$	$RS_{CDM,d=8}$	RS_{PCMC}	MGA
`sushi`	14.24 ± 0.02	13.94 ± 0.02	13.57 ± 0.02	**13.47** ± 0.02	13.91 ± 0.02	34.94 ± 0.06
`dub-n`	8.36 ± 0.02	8.18 ± 0.02	7.61 ± 0.02	**7.59** ± 0.02	8.15 ± 0.02	28.70 ± 0.05
`dub-w`	6.36 ± 0.02	6.27 ± 0.02	5.87 ± 0.02	**5.86** ± 0.01	6.12 ± 0.02	16.88 ± 0.07
`meath`	8.46 ± 0.02	8.23 ± 0.02	7.59 ± 0.02	**7.56** ± 0.02	8.05 ± 0.02	34.31 ± 0.07
`nascar`	112.0 ± 1.23	112.4 ± 1.48	111.7 ± 1.69	110.5 ± 1.89	**101.6** ± 1.2	3484.6 ± 7.2
`LETOR`	17.14 ± 1.46	17.51 ± 1.59	16.61 ± 1.69	17.02 ± 1.95	**13.54** ± 1.2	175.7 ± 16.8
`PREF-SOI`	5.47 ± 0.24	5.45 ± 0.024	5.40 ± 0.28	5.43 ± 0.30	**5.23** ± 0.22	14.24 ± 0.70

Table 1: Mean out-of-sample negative log-likelihood for the MLE of repeated selection models across different datasets (lowercase names) or collections of datasets (uppercase names), ± standard errors (of the mean). The best result for each dataset appears in bold. Results are averaged across five folds.

Dataset	MNL	CDM,d=1	CDM,d=4	CDM,d=8	PCMC	MGA
`sushi` RS	14.24 ± 0.02	13.94 ± 0.02	13.57 ± 0.02	**13.47** ± 0.03	13.91 ± 0.03	34.9 ± 0.06
`sushi` RE	14.13 ± 0.02	13.97 ± 0.03	13.53 ± 0.02	**13.49** ± 0.03	13.94 ± 0.03	35.4 ± 0.12
`PREF-SOC` RS	5.52 ± 0.08	5.53 ± 0.07	5.55 ± 0.14	5.54 ± 0.15	**5.20** ± 0.10	16.7 ± 2.3
`PREF-SOC` RE	5.59 ± 0.06	5.56 ± 0.08	5.29 ± 0.10	**5.17** ± 0.11	6.13 ± 0.15	16.8 ± 2.3

Table 2: Mean out-of-sample negative log-likelihood for the MLE of repeated selection (RS) models and repeated elimination (RE) models for the `sushi` dataset and the `PREF-SOC` collection of datasets, ± standard errors (of the mean). The best result for each dataset appears in bold. Results are averaged across five folds.

Similarly, we find for the `LETOR` collection of datasets that the CDM models perform roughly as well as the MNL model across all dimensions, but we see that the PCMC model performs significantly better. The dip in out-of-sample performance for the eight-dimensional CDM relative to the four-dimensional CDM may suggest that RS_{CDM} begins to overfit the data when given more parameters while RS_{PCMC}, which has many more parameters, does not.

Table 2 gives a comparison of repeated selection and repeated elimination on datasets with complete rankings. On the `sushi` datasets we interestingly find that the RE_{MNL} model slightly outperforms the RS_{MNL} (Plackett-Luce) model, providing practical evidence for the importance of the non-equivalence between the family of ranking distributions that arise under MNL with the RS vs. RE choice representations (Proposition 7). Noting that the `sushi` data has previously been distributed and analyzed in the "wrong" order, it's interesting that it is in fact more predictable (under Plackett-Luce) in the wrong/reversed order than in the correct order. We do not find significant difference between repeated selection and repeated elimination for the other models, all of which outperform repeated selection and repeated elimination with MNL.

For the collection of datasets in `PREF-SOC`, we find that the repeated selection and repeated elimination models generally perform quite comparably, suggesting that the simple Plackett-Luce model is well-suited for most of the datasets in this collection.

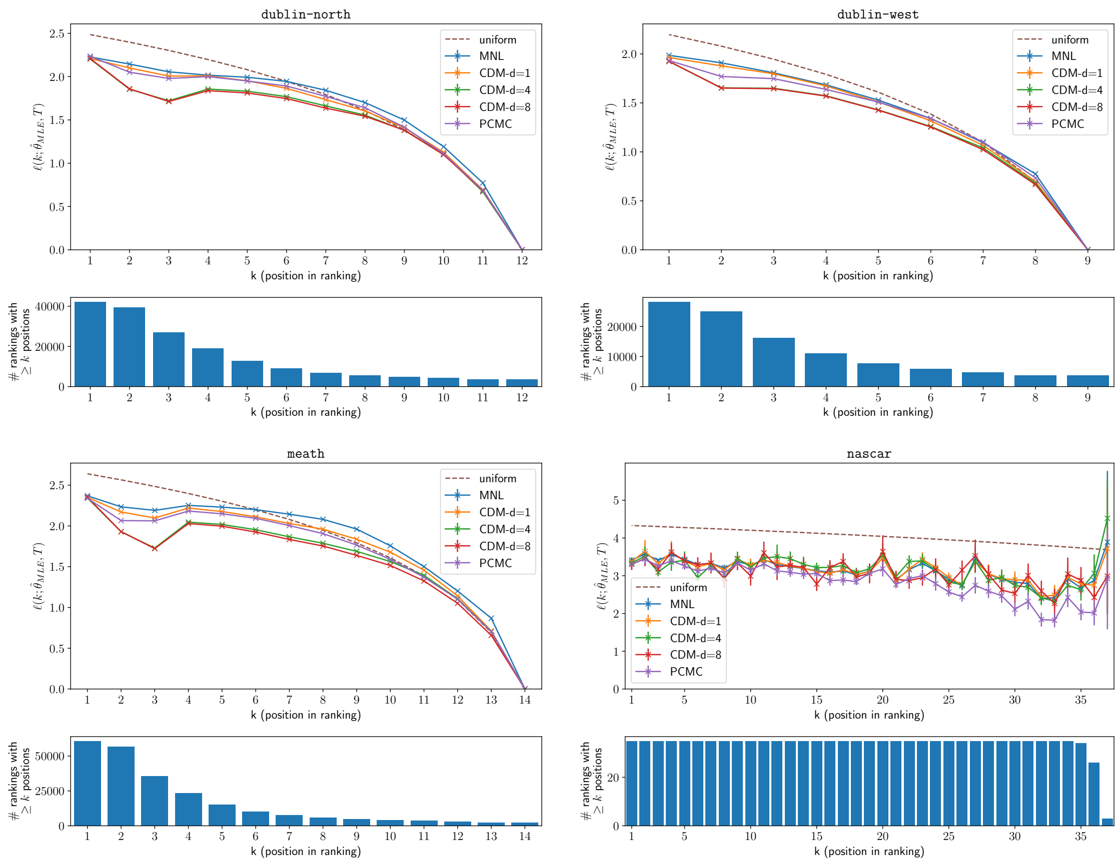

Figure 2: The average position-level log likelihood of choice probabilities and accompanying histograms of ranking lengths for the `dublin-north` (top left), `dublin-west` (top right), `meath` (bottom right), and `nascar` (bottom right) datasets.

4.4 Position-level evaluation results

In order to analyze the differences between choice-based ranking models, we examine the log-likelihood of the choices at individual positions, as defined in Equation (14), for the election, `nascar`, and `sushi` datasets. For repeated selection we look at the log probability of the choices at each position, reading from top to bottom as we reveal more of the prefix. For repeated elimination we look at the log probability of the entries reading from bottom to top as we reveal more of the suffix. Our plots include error bars denoting standard errors, though the error bars are typically very small because there are many test lists when averaging over the data.

In Figure 2 we analyze the election datasets, where we find that more nuanced choice models make significant gains relative to simpler models when predicting candidates near—but not at—the top of the list. We further notice that for both the CDM and PCMC-based models the performance is not monotonically decreasing in the number of remaining choices. Specifically, it is easier to guess the third-ranked candidate than the fourth, despite having fewer options in the latter scenario. A plausible explanation is that many voters rank candidates from a single political party and then stop ranking others, and the more nuanced

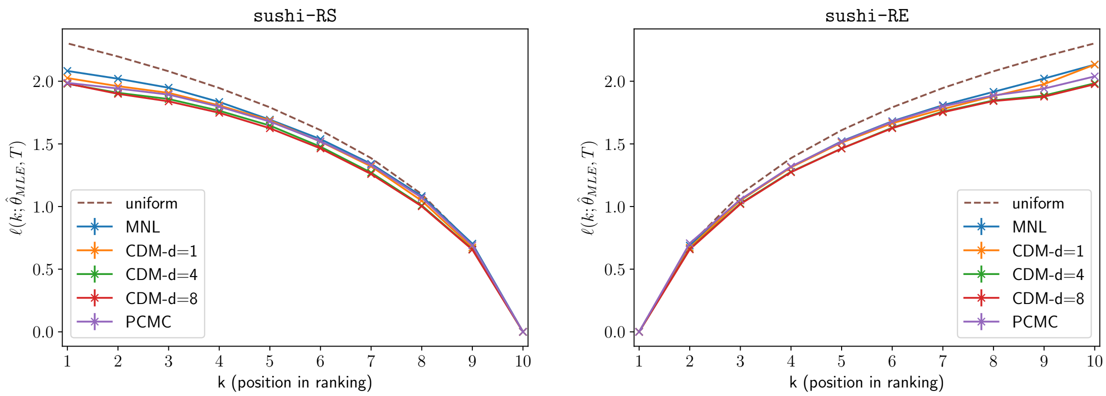

Figure 3: The average position-level log likelihood of choice probabilities under repeated selection (left) and repeated elimination (right) on the `sushi` dataset.

choice models are assigning high probability to candidates when other candidates in their political party are removed. For each dataset we show a histogram of the number of rankings of length $\geq k$. We see that there are many lists that rank the top few candidates and very few that rank all of the candidates, so the MLE objective is more concerned with performing well near the top of the ranking than near the bottom.

For the `nascar` data, also in Figure 2, we see that repeated selection with different choice models preforms roughly the same at the top positions, but PCMC outpaces the other models at predicting the lower finishers, particularly shining in the bottom 10 positions. The log probabilities exhibit much more variance across folds. There are far fewer races in a Winston Cup than ranked votes in the election datasets. The histogram shows us that nearly all of the races had between 34 and 37 racers finishing, of 76 total racers that finished at least one race in the cup.

In Figure 3 we use the `sushi` dataset, which contains complete rankings, to study the difference between RS and RE as a function of rank positions. For the RE model, $\sigma^{-1}(k)$ is chosen from $\{\sigma^{-1}(j)\}_{j\leq k}$, which has size k rather than $n-k$ as with RS. Thus for RS uniform choice model assigns probability $\frac{1}{n-k+1}$ to position k, as there are $n-k+1$ unranked items when making the choice of position k, while for RE the rankings are build from back to front and the uniform choice model assigns probability $\frac{1}{k}$ to each of the k unranked items under consideration for position k. We can see that the last few positions in the RE model were slightly easier to predict than the first few positions under RS.

5 Conclusions

In this work we contribute a general framework for choice-based ranking by interpreting rankings as collections of choices, giving a method for translating any probabilistic discrete choice model into a probabilistic ranking model. We introduce the notion of a choice representation, and focus our analysis on choice representations based on repeated selection, with some attention to repeated elimination.

Because both the Mallows model and the Plackett-Luce model can be obtained through repeated selection, this choice-based framework serves as a simple conceptual tie between the extensive bodies of work that surround these two models. The framework further allows us to develop new ranking distributions by applying e.g. repeated selection to the recently developed PCMC and CDM choice models. These models are not RUMs and can exhibit choice set effects (violations of the independence of irrelevant alternatives, IIA).

We examine the performance of several choice-based ranking models on a wide array of data including food preferences, elections, search engine rankings, and racing results, showing that repeated selection with the PCMC and CDM models outperform the seminal Plackett-Luce and (approximated) Mallows models on a wide variety of datasets. We thus find the RS_{CDM} and RS_{PCMC} to be tractable, flexible generalizations of the Plackett-Luce model with attractive structure and promising performance on a wide variety of datasets.

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Appendices

This supplement begins by providing proofs that were omitted in the main paper and providing an omitted derivation of the maximum likelihood estimate of the Mallows concentration parameter for partial rankings. We then transition to additional expositions that may be useful to the interested reader. We provide an elaboration on the differences between repeated selection and repeated elimination, showing that these two choice representations yield strictly different sets of ranking models when applied to any family of choice models subsuming the MNL model family for as few as four alternatives. We follow this with a deeper treatment of the differences between joint selection and repeated selection for choice models which can be expressed as independent RUMs. We conclude with an extended discussion of the connections between the present work and other Markov Chain-based models for rankings.

A Additional Proofs

Here we provide restatements and proofs of any proposition whose proof was omitted from the main paper.

Proposition (Proposition 3)**.** *For a PCMC choice model p parameterized by rate matrix Q and a partition $S_1, \dots, S_k$ of U where for all $i \in S_I, j \in S_J$ with $I \neq J$, $q_{ij} = \lambda_{IJ}$, then for each I, (p, S_I, U) exhibits nested IIA.*

Proof. The conditions given on Q make $S_1, \dots, S_k$ meet the criteria for a contractible partition (Definition 3, [35]), so by Proposition 3 of [35], for any subset T of U, $p(i, T) = p(i, S_I \cap T)p(S_I \cap T, T)$ where $p(i, S_I \cap T)$ are the choice probabilities given by the restriction of Q to $S_I \cap T$ and $p(S_I \cap T, T)$ follows a separate PCMC choice model on the $S_1, \dots, S_k$ whose rate matrix is a function of the sizes of $S_I \cap T$ and the λ_{IJ} giving the block structure. □ □

Proposition (Proposition 4)**.** *For a CDM choice model p parameterized by pairwise contextual utilities $\mathcal{U} = \mathbf{f}^T\mathbf{c}$ with $\mathcal{U}_{ij}$ fixed for all $i \in S, j \notin S$, (p, S, U) exhibits nested IIA.*

Proof. Let T be any subset of U and let $C = (|T| - |S|)c$. For any $i \in S$ we have that $\sum_{k \in T} u_{ik} = C + \sum_{k \in S} u_{ik}$, and thus that $\sum_{j \in S} e^{\sum_{k \in T} u_{jk}} = \sum_{j \in S} e^{\sum_{k \in S} u_{jk}} e^C$.

From these we have that

$$\begin{aligned} p(i, S) &= \frac{e^{\sum_{k \in T} u_{ik}}}{\sum_{j \in T} e^{\sum_{k \in T} u_{ik}}} = \frac{e^{\sum_{k \in S} u_i k} e^C}{\sum_{j \in T} e^{\sum_{k \in T} u_{ik}}} \cdot \frac{\sum_{j \in S} e^{\sum_{k \in S} u_{jk}}}{\sum_{j \in S} e^{\sum_{k \in S} u_{jk}}} \\ &= \frac{\sum_{j \in S} e^{\sum_{k \in T} u_{jk}}}{\sum_{j \in T} e^{\sum_{k \in T} u_{ik}}} \cdot \frac{e^{\sum_{k \in S} u_i k}}{\sum_{j \in S} e^{\sum_{k \in S} u_{jk}}} = p(S, T)p(i, S) \end{aligned}$$

where we move the e^C term from the sum of i's utility from $T - S$ into the sum for each element in S, which all have the same utility from $T - S$ as i by the assumption. □ □

Proposition (Proposition 5)**.** *For a choice model p on universe U, suppose (p, S, U) exhibits nested IIA. For σ drawn from $P_{RS,p}$, let σ_S be the restriction of σ to S and $\tilde{p}_S$ be the restriction of p to S. Then σ_S is distributed according to $P_{RS,\tilde{p}_S}$.*

Proof. Let $|S| = m$. We partition S_n into $\binom{n}{m}$ subsets X_v indexed by binary vectors v of length n with $\sigma \in X_v \Leftrightarrow v(i) = 1(\sigma^{-1}(i) \in S)$, i.e. the m entries of v with a one are the positions at which rankings in X_v assign items of S.

Then for $i \in S$ and any subset T of U, we have by nested IIA that

$$p(i, T) = p(i, S \cap T)p(S \cap T, T),$$

and if we condition on our choice from T lying in S, the second term becomes 1 while the first is unchanged.

This thus gives for any v, the distribution of σ_S only depends on the choices made at the entries where alternatives in S are ranked according to v, and the nested IIA allows us to turn these choices probabilities into those from the subsets of S cleanly. Thus we have

$$\begin{aligned} Pr(\sigma_S|\sigma \in X_v) &= \prod_{\ell=1}^{n} v(\ell)p(\sigma^{-1}(\ell), \{\sigma^{-1}(j)\}_{j\geq \ell}) = \prod_{\ell=1}^{n} v(\ell)p(\sigma^{-1}(\ell), S \cap \{\sigma^{-1}(j)\}_{j\geq \ell}) \\ &= \prod_{k=1}^{m} p(\sigma_S^{-1}(k), \{\sigma_S(j)\}_{j\geq k}) = P_{RS,p}(\sigma_S) \end{aligned}$$

Note that this did not depend on v, so the proposition follows. □ □

The proposition gives a natural analogue of nested IIA for rankings in that just as choices within a nest exhibit independence from irrelevant alternatives outside the nest, repeated selection with a choice model exhibiting nested IIA yields independence between the relative orderings of each of the nests.

B Mallows concentration parameter estimates

Here we provide closed form estimates of the Mallow's concentration parameter for datasets with either full rankings or partial rankings. These estimates are derived given an estimate $\hat{\sigma}^*$ of the reference permutation, which may not be the maximum likelihood reference permutation (recall that finding the maximum likelihood permutation is NP-hard). Given an estimate $\hat{\sigma}^*$ we can find the maximum likelihood estimate of the concentration parameter *conditional* on $\hat{\sigma}^*$.

Recall that the Mallows model with concentration parameter θ and reference permutation σ^* assigns each permutation σ probability proportional to $\exp(-\theta\tau(\sigma, \sigma^*))$ where τ counts the number of inversions between σ and σ^*, i.e.

$$\tau(\sigma, \sigma^*) = \sum_{i<j} \mathbf{1}\big[\mathbf{1}[\sigma(i) < \sigma(j)] \neq \mathbf{1}[\sigma^*(i) < \sigma^*(j)]\big],$$

where $\mathbf{1}[\cdot]$ is the indicator function.

We first consider full rankings. Given the reference permutation, the number of inversions in a sample from the Mallows model is thus binomially distributed $\text{Bin}(\binom{n}{2}, p)$, one trial for each pair and success probability $p = e^{-\theta}$ for each pair. Further, the total number of inversions over k samples $\sigma_1, \ldots, \sigma_k$ is binomially distributed as $\text{Bin}(k\binom{n}{2}, e^{-\theta})$. It follows that the maximum likelihood estimate for θ can be derived from the corresponding Binomial distribution:

$$\hat{\theta}^{MLE} = \log\left(\frac{\sum_{j=1}^{k} \tau(\sigma_k, \hat{\sigma}^*)}{k\binom{n}{2}}\right).$$

When our data includes partial rankings, we can still enumerate the number of pairs for which a partial ranking makes assertions, and compute the probability under which a partial ranking orients a pair differently than $\hat{\sigma}^*$. Assume σ_j is of length ℓ_j. Then σ_j includes comparisons for $\binom{\ell_j}{2} + (n-\ell_j)\ell_j$ pairs of items and then

$$\hat{\theta}^{MLE}_{\text{partial}} = \log\left(\frac{\sum_{j=1}^{k} \tau(\sigma_k, \hat{\sigma}^*)}{\sum_{j=1}^{k} \binom{\ell_j}{2} + (n-\ell_j)\ell_j}\right).$$

For partial rankings we restrict the inversion count $\tau(\cdot,\cdot)$ to sum over those pairs for which the comparisons can be made.

C Repeated elimination and reversibility

The natural counterpart to repeated selection (RS) is repeated elimination (RE) (as introduced in Section 2) and rankings models built up using RS and RE applied to the same choice model will intuitively differ quite dramatically for almost any choice model p. Less obviously, it's not clear how similar we can make the ranking distribution induced under RS with choice model p to the ranking distribution induced under RE with some choice model p' from the same choice model family. In this section we show that even for simple families of choice models, most RS ranking models will have no RE counterpart that provides the same ranking distribution.

Recall that repeated elimination models the construction of a ranking σ as the choice of the last item, $\sigma^{-1}(n)$, from U, then the second to last item, $\sigma^{-1}(n-1)$, as chosen from $U - \sigma^{-1}(n)$, and so forth. The intuition behind this representation is that the agent building the ranking eliminates the worst alternative, placing it at the back of σ, and then proceeds by eliminating the worst remaining alternative from what remains repeatedly until only $\sigma^{-1}(1)$ is left to place at the front of the ranking σ.

We will show that choice-based ranking with repeated elimination can produce different classes of distributions than repeated selection, even when the choice models come from a simple MNL family over $n = 3$ items. In concrete terms, the family of Plackett-Luce ranking distributions is different from the family of Plackett-Luce ranking distributions "backwards."

Critchlow et al. give a general definition of reversibility that is a property of a family of ranking distributions and fits well into our framework. We will use this definition to define reversibility as a property of a family of choice models.

Definition 15 (Reversibility of a family of ranking models, [12])**.** *Let π be the permutation* *that reverses a ranking in the sense that $\pi(i) = n - i + 1$. A family $\mathcal{P}$ of distributions on S_n is reversible if for every distribution $P \in \mathcal{P}$ there exists $P' \in \mathcal{P}$ such that for all σ, $P(\sigma) = P'(\pi\sigma)$.*

Critchlow et al. showed that L-decomposable distributions on rankings, discussed in Section 3, are reversible only if they are also R-decomposable, where L-decomposability is to repeated selection as R-decomposability is to repeated elimination. A distribution D on S_n is R-decomposable if and only if

$$Pr_D(\sigma(i) = x|\sigma^{-1}(k) = x_k, k > i) = Pr_D\left(\sigma(i) = x|\{\sigma^{-1}(k)\}_{k>i}\right), \forall \sigma \in S_n.$$

We use this characterization of reversibility to extend the property to choice models, so that we can describe a family of choice models by whether or not the set of ranking distributions given by repeated selection with choice models in the family is the same as the set of ranking distributions given by repeated elimination with choice models in the family.

Definition 16 (Reversibility for choice models)**.** *A family of choice models $\mathcal{P}$ is reversible if for every $p \in \mathcal{P}$ there exists some $p' \in \mathcal{P}$ such that $P_{RS,p}(\sigma) = P_{RE,p'}(\sigma)$ for all $\sigma \in S_n$.*

Critchlow et al. showed that the Mallows ranking model is reversible, and it follows that the corresponding Mallows choice model exhibits reversibility. Specifically, for a Mallows choice model $p_{\sigma_0,\theta}$ as defined in Section 3, the Mallows choice model p' for which repeated elimination with p' matches repeated selection with $p_{\sigma_0,\theta}$ has the same concentration parameter θ and the reversal of σ_0 as its reference permutation.

Meanwhile, we find that none of the other choice models we consider are reversible, with RS and RE giving different families of ranking models for these choice models. The absence of reversibility in CDM and PCMC stems from the lack of reversibility for the MNL model, a special case of each. To prove this rigorously we will use the following proposition due to Luce, which is simple but widely considered to be surprising [42, 8, 25].

Proposition 7 (MNL is not reversible for 3 or more items [25])**.** *Repeated selection with MNL and repeated elimination with MNL yield the same distribution for rankings of at least three items only when both distributions are uniform.*

The notion of reversibility has also been applied to joint selection from RUM distributions. Yellott defines a family of RUMs as reversible if the ranking distributions that arises from sorting the random utilities in increasing order being the same as the family that arises from sorting the random utilities in decreasing order [42]. Yellott notes in particular that Plackett-Luce ranking models are not reversible unless all of the random utilities have the same location parameter (the same mean). Recalling that joint selection is equivalent to repeated selection with an MNL choice model [8], this provides an alternative proof that RS_{MNL} and RE_{MNL} are different distributions (Proposition 7).

When ranking three items, the family of RS_{MNL} ranking models is parameterized by a 3-simplex of normalized MNL parameters $(\gamma_1, \gamma_2, \gamma_3)$. As a concrete demonstration, Figure C.1 illustrates the minimum total variation distance between a RS_{MNL} model and the nearest RE_{MNL} model over this simplex,

$$\underset{\gamma'}{\operatorname{argmin}}\, d_{TV}(P_{RS,\gamma}, P_{RE,\gamma'}),$$

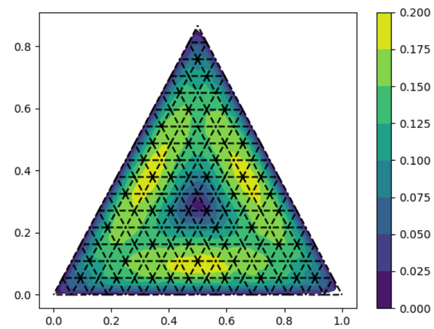

Figure C.1: The total variation distance from the RS_{MNL} distribution (Plackett-Luce) on three items with parameters $(\gamma_1, \gamma_2, \gamma_3)$ and the nearest RE_{MNL} distribution, plotted over the γ 3-simplex. We see that the two models are equivalent in the center (the uniform distribution) and in the corners of the simplex.

where $d_{TV}(p, q)$ is the total variation distance between distributions p and q on S_n. The minimum total variation distance between a given RS_{MNL} model and the nearest RE_{MNL} model can clearly be non-trivially greater than zero. Further exploration of this difference, including minimax bounds, would be an interesting direction for future research.

We can leverage the non-reversibility of MNL to show that models which include MNL as a special case are not reversible for $n \geq 4$ alternatives. We accomplish this by constructing a specific MNL model on four items whose repeated selection ranking distribution is not R-decomposable, and thus cannot be the repeated elimination distribution of any choice model.

Proposition 8. *Any family of choice models on $n \geq 4$ alternatives that include all MNL models on n alternatives is not reversible.*

Proof. We first give a simple counterexample to reversibility on four items. We then use show that this counterexample applies to all sets with $n \geq 4$ alternatives. The central idea is that R-decomposability constrains the ratio of probabilities of two similar lists in a way that cannot be captured by any choice model when those ranking probabilities come from certain Plackett-Luce models.

Suppose there are exactly four alternatives, let $U = \{1, 2, 3, 4\}$ and let $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ be the parameters of an MNL model. We consider the repeated selection distribution on S_4 with γ, and examine the probability of a fixed $\pi = \pi^{-1}(1)\pi^{-1}(2)\pi^{-1}(3)\pi^{-1}(4) = 1243$ as well as the probability of the identity permutation $e = 1234$.

$$P_{RS,\gamma}(\pi) = \frac{\gamma_1}{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4} \cdot \frac{\gamma_2}{\gamma_2 + \gamma_3 + \gamma_4} \cdot \frac{\gamma_4}{\gamma_3 + \gamma_4}$$
$$P_{RS,\gamma}(e) = \frac{\gamma_1}{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4} \cdot \frac{\gamma_2}{\gamma_2 + \gamma_3 + \gamma_4} \cdot \frac{\gamma_3}{\gamma_3 + \gamma_4}$$

Noting that $P_{RS,\gamma}(\pi) = P_{RS,\gamma}(e) \Leftrightarrow \gamma_3 = \gamma_4$, we thus have for $\gamma_3 \neq \gamma_4$ that

$$P_{RS,\gamma}(\sigma(2) = 2 | \sigma(3) = 4, \sigma(4) = 3) = P_{RS,\gamma}(\pi) \neq P_{RS,\gamma}(e) = P_{RS,\gamma}(\sigma(2) = 2 | \sigma(3) = 3, \sigma(4) = 4).$$

It follows that $P_{RS,\gamma}$ is not R-decomposable, so it cannot be the repeated elimination distribution of any choice model p.

If n, the number of alternatives, is greater than 4, we can simply let $\tilde{\gamma} = (\gamma_1 - \epsilon, \gamma_2 - \epsilon, .\gamma_3 - \epsilon, \gamma_4 - \epsilon, \delta, \delta^2, \ldots, \delta^{n-4})$ where $\delta \sum_{i=1}^{n-4} \delta = 4\epsilon$ and as $\delta, \epsilon \to 0$ the probability of e and $\pi = 124356\ldots n$ will converge to the probabilities from the $n = 4$ case, giving the same violation of R-decomposability when $\gamma_3 \neq \gamma_4$. □ □

Corollary 2. *PCMC is not a reversible choice model for $n \geq 4$ items.*

Corollary 3. *CDM is not a reversible choice model for $n \geq 4$ items.*

The fact that RS and RE produce different ranking distributions for non-reversible choice models and that popular choice models such as the MNL model are non-reversible can have very important consequences in practical modeling. For example, the widely-studied sushi dataset that consists of preference rankings over types of sushi was originally reported as having low ranks for items with low priority, but was later corrected so that low ranks represent high priority. Rankings built from these scores prior to the correction were thus "backwards." (see the warning on [22].) And as we've seen in this section, the difference between learning a ranking distribution from forwards vs. backwards ranking data amounts to entirely different families of ranking distributions. Surprisingly, as part of our empirical results we find that the rankings provided in the sushi dataset (correctly ordered) are fit slightly better with RE distributions than with RS distributions.

D Joint selection from RUMs

An alternative and natural ranking distribution from a RUM is simply to rank alternatives by their random utilities when drawn from the joint distribution of the utility random variables X_i. We refer to the resulting distribution on rankings as the *joint selection* distribution of the RUM. Note that in general, the joint selection and repeated selection ranking distributions are not the same even for independent X_i variables, with Block and Marschak having proved that they are the same if and only if the independence of irrelevant alternatives (IIA) holds.

Theorem 3 (Joint Selection is Repeated Selection only under IIA [8])**.** *Let $\{X_i\}_{i \in U}$ be independent random utilities for a RUM model. Then for all rankings $\sigma \in S_n$,*

$$Pr(X_{\sigma^{-1}(1)} > X_{\sigma^{-1}(2)} > \cdots > X_{\sigma^{-1}(n)}) = \prod_{i=1}^{n} Pr(X_{\sigma^{-1}(i)} = \max_{j \geq i} X_{\sigma^{-1}(j)})$$

if and only if the RUM obeys IIA.

Unfortunately, the practical implications of the joint selection distributions of RUMs when IIA does not hold are unclear, as the generality of RUMs makes inference based on the joint selection rankings intractable. Of course, assumptions about the independence or distribution family of RUMs leaves a much more tractable picture for learning, and some work has found success using the JS distribution of RUMs for modeling rankings [10, 2].

[8] also constructed the so-called *Block-Marschak polynomials* that provide necessary conditions on the choice probabilities for a RUM given its joint selection distribution. Falmange

later proved that choice probabilities satisfying the Block-Marschak polynomials arising from some ranking distribution are also sufficient condition to construct a RUM whose JS distribution is that same ranking distribution [18], giving a full characterization theorem.

The notion of reversibility has also been applied to JS distributions. Yellott defines a family of RUMs as reversible if the ranking distributions that arises from sorting the random utilities in increasing order being the same as the family that arises from sorting the random utilities in decreasing order [42]. Yellott notes in particular that Plackett-Luce ranking models are not reversible unless all of the random utilities have the same location parameter (the same mean). Recalling that joint selection is repeated selection for MNL (Theorem 3), this provides an alternative proof that RS_{MNL} and RE_{MNL} are different distributions (Proposition 7).

E Markov chains on S_n

The study of self-organizing lists, originally motivated by research questions involving sequential access storage systems, introduced a number of Markov chains on S_n that provide an altogether different way of defining distributions over rankings (through the stationary distributions of these chains). A self-organizing list is a storage model where a list reindexes its elements based on which indices are more commonly accessed, moving popular items towards the front to facilitate faster access. A popular method for managing a self-organizing list is the *move to front* (MTF) method, which simply places the most recently accessed item at the front of the list, sliding any item which had been ranked ahead of it back one position [36].

If we assume that each item i is accessed with an independent probability γ_i, the MTF method induces a discrete time Markov chain on S_n (as opposed to a chain on the set U, where the PCMC chain and other discrete choice chains live) where the stationary distribution of that chain describes the probabilities that the state of the list follows a given ranking of the items. In the language of choice modeling, the distribution of this chain is equivalent to a Plackett-Luce distribution with quality parameters γ. We now give a more complete explanation of this connection.

We define the MTF Markov chain as follows. For ranking σ and alternative i let $mtf(i,\sigma)$ be a mapping that returns a ranking σ' with $\sigma'(i) = 1, \sigma'(j) = \sigma(j) + 1$ for j such that $\sigma(j) < \sigma(i)$ and $\sigma'(j) = \sigma(j)$ for j such that $\sigma(j) > \sigma(i)$. The transition probabilities of the Markov chain are thus

$$P(\sigma, \sigma') = \gamma_i \mathbf{1}(\sigma' = mtf(i,\sigma)). \tag{15}$$

We note that the Placett-Luce distribution, equivalent to repeated selection (RS) applied to the MNL model, can be written as a move to front chain.

Proposition 9 (MTF yields Plackett-Luce distributions [17])**.** *The stationary distribution of an MTF chain with parameters γ for items moving to the front is the same as the Plackett-Luce distribution with parameter vector γ:*

$$\Pr(\sigma; \gamma) = \prod_{i=1}^{n} \frac{\gamma_{\sigma^{-1}(i)}}{\sum_{k \geq i} \gamma_{\sigma^{-1}(k)}}. \tag{16}$$

A notable extension of the MTF chain, for our interests, is the Markov Move to Front (MMTF) model [15], which has the same walk behavior as the MTF chain except the rate at which items move to the front of the chain depends on the current first item of the list (i.e. the previous most recent item chosen). Thus instead of n parameters $\gamma_1, ..., \gamma_n$ there there are n^2 parameters $\{a_{ij}\}_{i,j\in U}$ where a_{ij} is the probability that i is moved to the front of a ranking with j currently in the front. The MMTF chain has the transition probabilities

$$P(\sigma, \sigma') = a_{i\sigma^{-1}(1)}\mathbf{1}(\sigma' = mtf(i, \sigma)). \tag{17}$$

Although this distribution seems intuitively similar to RS applied to PCMC, where choices are made according to a random walk that has jump probabilities dependent on the current state of the PCMC chain, and for which Plackett-Luce is also a special case, the stationary distribution of an MMTF chain is generally *not* that of a repeatedly selected PCMC model.

Proposition 10 (MMTF is RS only when it is MTF)**.** *When there are at least four alternatives, the Markov Move-To-Front (MMTF) chain's stationary distribution is L-decomposable if and only if it is an MTF chain and has a Plackett-Luce distribution.*

Proof. Let $\{a_{ij}\}$ be the parameters of an MMTF model, so that the probability that i jumps to the front when j is currently in the front is a_{ij} as in Equation (17). We divide MMTF chains into two cases, one where a_{ij} is fixed for all j, and another where it differs.

If $a_{ij} = \gamma_i$ for every j then the transition probabilities in Equation (17) collapse to those of an MTF chain, as in Equation (15). Then by Proposition 9, it is also equal to a PL model and in turn a RS_{PCMC} model.

Now suppose that the a_{ij} are not fixed for all i for every j. The second item in an MMTF ranking is the item that was accessed before the first item. Let σ be drawn from the stationary distribution of the MMTF. Then $\sigma^{-1}(1)$ is given by the state most recently visited by the chain on U. $\sigma^{-1}(2)$ is the second most recently visited state given $\sigma^{-1}(1)$ was the most recently visited. Note, however, that the chain may have self loops, but $\sigma^{-1}(1) \neq \sigma^{-1}(2)$, so more precisely, $\sigma^{-1}(2)$ was the state most recently visited before $\sigma^{-1}(1)$, not including $\sigma^{-1}(1)$. It further follows that $\sigma^{-1}(k)$ was the k-th most recently visited *unique* state from a random walk on U.

The distribution of $\sigma^{-1}(k)$ have been studied as *taboo probabilities* for the time-reversal of the chain on U, where the taboo probability of a state i in a taboo set S from a starting state $y \notin S$ is the probability that a random walk hits i before any other state in S [38]. Concretely, $Pr(\sigma^{-1}(k) = x|\sigma^{-1}(1), \dots, \sigma^{-1}(k-1))$ is the taboo probability of x when the taboo set is $U - \{\sigma^{-1}(j)\}_{j\leq k}$ for a random walk starting in $\sigma^{-1}(k)$.

Suppose we have at least four items, w, x, y and z, and wlog $a_{wx} \neq a_{wy}$. Then when $\sigma(x) = 1, \sigma(y) = 2$, the taboo probability of w for taboo set $U - \{x, y\}$ when starting from x is different than when starting from y. □ □

Corollary 4 (MMTF is not RS applied to PCMC)**.** *Repeated selection with the PCMC distribution and the Markov Move-To-Front chain produce the same ranking distribution if and only if both are some Plackett-Luce distribution.*

The corollary follows from Observation 1 (in Section 3), which shows that RS applied to PCMC subsumes Plackett-Luce.

While the MMTF chain does not fall within the choosing to rank framework, large-scale inference of these list-based Markov chains on S_n is an interesting research direction. Further exploration of these processes, including the stationary distribution of MMTF chains, appear in [3]. An additional difficulty the prevents the practical learning of distributions for complex Markov chains on S_n from ranking data is that many of these Markov chains have poor or unknown mixing times [6].