Chapter 1

Graph Theory

> The origins of graph theory are humble, even frivolous.
>
> Norman L. Biggs

1.1 Introduction

Graph theory is traditionally introduced using the Seven Bridges of Königsberg (see fig. 1.1). The scene is Königsberg, Prussia, 1700s. Leonhard Euler sees this area – two land masses separated by a river, with two islands in-between, all connected by 7 bridges – and thinks, *hmm, I wonder if I could walk through this city crossing each bridge only once.* Euler mathematically proved that no such walk exists due to the nature of this area. This was the birth of graph theory.

Euler did not brute-force the solution by examining all possible paths. Instead, Euler noticed that he could abstract out the large landmasses into single points, called vertices, and connect them by simple lines, called edges, which represent the bridges. This mathematical structure is a *graph*. The graph version of the Seven Bridges is presented in figure 1.2.

The beautiful thing about graphs is that we can move the vertices around in space, and so long as the edges remain connected to their respective vertices, without changing the fundamental structure of the graph!

Graphs are applicable in numerous fields, including

- Network (computer, water, electric) flow
- Facebook friendships
- Google Maps path-finding

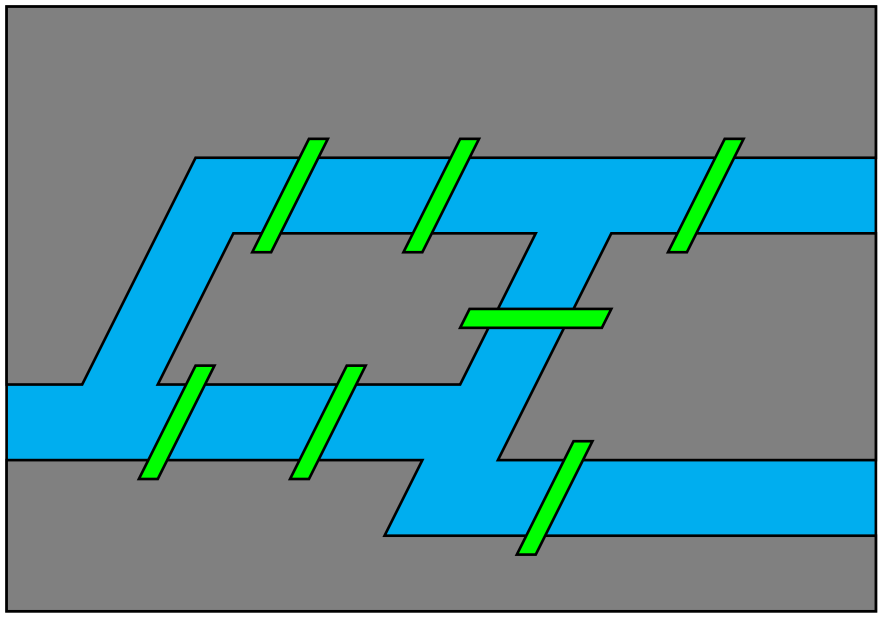

Figure 1.1: The Seven Bridges of Königsberg

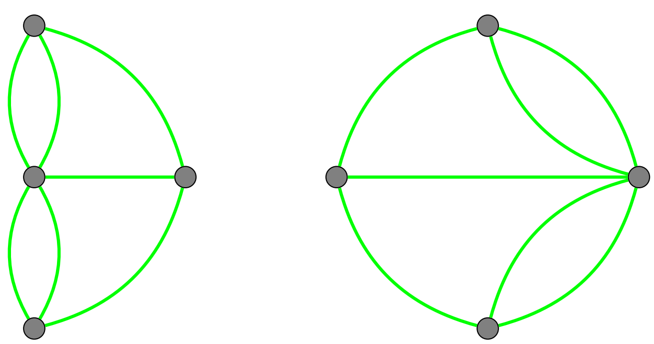

Figure 1.2: Two graphified versions of The Seven Bridges of Königsberg. The left graph is roughly a direct translation of the actual landmasses. The right graph is the same graph with the vertices moved around (take the middle-left vertex in the first graph, move it out to the left, then rotate the resulting graph 180 degrees)

- Game AI path-finding
- E-commerce similarity predictions
- Neural networks
- ...

1.2 Key Terms

Definition 1.2.1 Graph
A set of abstract points, called vertices V, and connections between those points, called edges E. We denote a graph as $G = \{V, E\}$ or $G = (V, E)$. The order that you list V and E does not matter, however typically V is listed before E

Definition 1.2.2 Vertex

Definition 1.2.3 Edge
Directed vs Un-directed

Definition 1.2.4 Di-graph
A graph where each edge is directed.

Definition 1.2.5 Cycle
A thing. Only defined for directed edges

Definition 1.2.6 DAG
A Directed Acyclic Graph – a di-graph with no cycles

1.3 Graph Representations

There exist multiple ways to represent a graph. Each way has benefits and drawbacks. Typically algorithmic analysis involving graphs uses the representation that provides the best run-time.

Definition 1.3.1 Adjacency Matrix
An $m \times m$ (square) symmetric matrix M of 0s and 1s. Each row/column represents a node, and the position i, j in M represents the connection. For an edge

Definition 1.3.2 Adjacency List

There are other graph representations, such as the *edge list* and the *incidence matrix*. We encourage you to take a data structures course to learn more about these.

1.4 Problems

1.4.1 Graph Coloring

Note that we can represent *maps* as graphs. A *map* is simply a plane that is separated into contiguous regions. To represent this as a graph, let the set of vertices be the contiguous regions and let an edge between two vertices exist if and only if the represented regions share a border.

Theorem 1.4.1 The Four Color Theorem
We can color the regions of a map in a maximum of four colors such that no two adjacent regions share the same color.

Remark 1.4.1
Deciding whether a map can be colored in three colors is NP-Complete.

1.4.2 Network Flow

1.5 Summary

1.6 Practice