Question 1

An animation director enters the position $f(t)$ of a character's head after t frames of the movie as given in the table.

t	200	220	240
$f(t)$	128	142	136

If the computer software uses interpolation to determine the intermediate positions, determine the position of the head at frame numbers (a) 208 and (b) 232.

Q1:- The closest 'x' value to $x = 208$ is $x = 200$ in the table. The linear approximation of $f(x)$ at $x = 200$ would look like

$$L(x) = f(200) + f'(200)(x - 200)$$

$$f'(200) \approx \frac{f(220) - f(200)}{220 - 200} = \frac{142 - 128}{20} = 0.7$$

The linear approximation is then

$$L(x) \approx 128 + 0.7(x - 200)$$

$$L(208) = 133.6$$

The closest x-value to $x = 232$ is $x = 220$ in the table. The linear approximation of $f(x)$ at $x = 220$ would be

$$L(x) = f(220) + f'(220)(x - 220)$$

$$f'(220) \approx \frac{f(240) - f(220)}{240 - 220} = \frac{136 - 142}{20} = -0.3$$

The linear approximation is then

$$L(x) \approx 142 - 0.3(x - 220)$$

$$L(232) = 138.4$$

Question 2

Volume of a Sphere

(a) Use differentials to approximate the volume of material needed to manufacture a hollow sphere if its inner radius is 2 m and its outer radius is 2.1 m.

(b) Is the approximation overestimating or underestimating the volume of material needed?

(c) Discuss the importance of knowing the answer to (b) if the manufacturer receives an order for 10,000 spheres.

Q2:- (a) Inner radius = 2m

outer radius = 2.1 m

Change in radius = dr = 0.1m

Vol of sphere = $V(r) = \frac{4}{3}\pi r^3$

Differential approximation is

$$dV \approx V'(r)dr$$

$$dV = 4\pi r^2 dr$$

$$dV \approx 4\pi (2)^2 (0.1)$$

$$dV \approx 1.6\pi m^3$$

(b) To check if the approximation is overestimation or underestimation find the exact value

$$\text{Exact value} = \frac{4}{3}\pi\left[(2.1)^3 - (2)^3\right]$$

$$\text{Exact value} \approx 5.28\,m^3$$

(b) The approximate change in volume underestimates the actual change by $0.26\,m^3$.

(c) If manufacturer receives an order of 10,000 spheres then the total extra material needed would be

$$10{,}000 \times 0.26 \approx 2600\,m^3$$

This known value of over or underestimation helps in costing & material planning.

Question 3

RECALL For motion that is circular, angular speed ω is defined as the rate of change of a central angle θ of the circle with respect to time. That is, $\omega = \frac{d\theta}{dt}$, where θ is measured in radians.

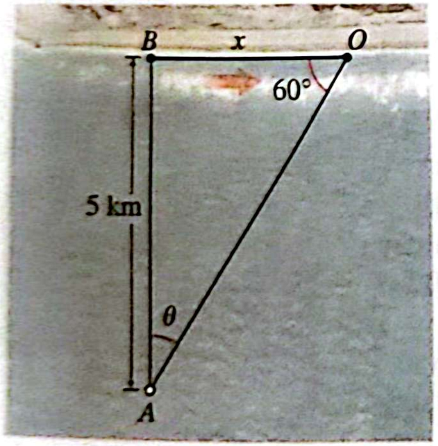

Figure 6

A revolving light, located 5 km from a straight shoreline, turns at a constant angular speed of 3 rad/min. With what speed is the spot of light moving along the shore when the beam makes an angle of 60° with the shoreline?

Solution Figure 6 illustrates the triangle that describes the problem.

$x =$ the distance (in kilometers) of the beam of light from the point B

$\theta =$ the angle (in radians) the beam of light makes with AB

Both variables x and θ change with time t (in minutes). The rates of change are

$$\frac{dx}{dt} = \text{the speed of the spot of light along the shore (in kilometers per minute)}$$

$$\frac{d\theta}{dt} = \text{the angular speed of the beam of light (in radians per minute)}$$

We are given $\frac{d\theta}{dt} = 3$ rad/min and we seek $\frac{dx}{dt}$ when the angle $AOB = 60°$.

From Figure 6,

$$\tan\theta = \frac{x}{5} \qquad \text{so} \qquad x = 5\tan\theta$$

Then

$$\frac{dx}{dt} = 5\sec^2\theta\frac{d\theta}{dt}$$

When $AOB = 60°$, angle $\theta = 30° = \frac{\pi}{6}$ radian. Since $\frac{d\theta}{dt} = 3$ rad/min,

$$\frac{dx}{dt} = \frac{5}{\cos^2\theta}\frac{d\theta}{dt} = \frac{5}{\left(\cos\frac{\pi}{6}\right)^2}\cdot 3 = \frac{15}{\frac{3}{4}} = 20$$

When $\theta = 30°$, the light is moving along the shore at a speed of 20 km/min. ■

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Question 4

An elevator in a building is located on the fifth floor, which is 25 m above the ground. A delivery truck is positioned directly beneath the elevator at street level. If, simultaneously, the elevator goes down at a speed of 5 m/s and the truck pulls away at a speed of 8 m/s, how fast will the elevator and the truck be separating 1 s later? Assume the speeds remain constant at all times.

Q4:- $y(t)$ = Vertical height of elevator

$x(t)$ = Horizontal distance the truck has driven from pt beneath the elevator

$s(t)$ = Distance b/w truck & elevator.

$$\frac{dx}{dt} = 8 \text{ m/s}$$

$$\frac{dy}{dt} = -5 \text{ m/s}$$

$$s^2 = x^2 + y^2$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left[x\frac{dx}{dt} + y\frac{dy}{dt}\right]$$

At $t = 1$ $\quad x = 8m \quad , y = 25 - (5)(1) = 20m$

$$s = \sqrt{(8)^2 + (20)^2} = \sqrt{464}$$

$$\frac{ds}{dt} = \frac{8(8) + (20)(-5)}{\sqrt{464}} = -1.67 \text{ m/s}$$

After 1sec the elevator & the truck are separating at the rate of 1.67m/s

Question 5

A town hangs strings of holiday lights across the road between utility poles. Each set of poles is 12 m apart. The strings hang in catenaries modeled by $y = 15\cosh\frac{x}{15} - 10$ with the poles at $(\pm 6, 0)$. What is the height of the string of lights at its lowest point?

Q5:- $y = 15\cosh\frac{x}{15} - 10 \qquad [-6, 6]$

If a ftn is continuous on closed & bounded interval ftn has its abs max/min value either at CP's or at endpts

$$y'(x) = 15\sinh\left(\frac{x}{15}\right)\cdot\frac{1}{15}$$

$$y'(x) = \sinh\left(\frac{x}{15}\right)$$

$$y'(x) = 0$$

$$\sinh\left(\frac{x}{15}\right) = 0$$

$$\frac{x}{15} = 0$$

$$\boxed{x = 0}$$

$$y(-6) = 15\cosh\left(-\frac{6}{15}\right) - 10$$

$$y(-6) \approx 15\cosh(-0.4) - 10$$

$$y(-6) \approx 6.216$$

$$y(6) \approx 6.216$$

$$y(0) = 15\cosh(0) - 10 = 15 - 10 = 5$$

So the min occurs at $x=0$ & the height is 5m.

Question 6

A variation of the von Liebig model states that the yield $f(x)$ of a plant, measured in bushels, responds to the amount x of potassium in a fertilizer according to the following square root model:

$$f(x) = -0.057 - 0.417x + 0.852\sqrt{x}$$

For what amounts of potassium will the yield increase? For what amounts of potassium will the yield decrease?

Solution The yield is increasing when $f'(x) > 0$.

$$f'(x) = -0.417 + \frac{0.426}{\sqrt{x}} = \frac{-0.417\sqrt{x} + 0.426}{\sqrt{x}}$$

Now $f'(x) > 0$ when

$$\begin{aligned} -0.417\sqrt{x} + 0.426 &> 0 \\ 0.417\sqrt{x} &< 0.426 \\ \sqrt{x} &< 1.022 \\ x &< 1.044 \end{aligned}$$

The crop yield is increasing when the amount of potassium in the fertilizer is less than 1.044 and is decreasing when the amount of potassium in the fertilizer is greater than 1.044. ■

Question 7

Unit monthly sales R of a new product over a period of time are expected to follow the logistic function

$$R = R(t) = \frac{20{,}000}{1+50e^{-t}} - \frac{20{,}000}{51} \qquad t \geq 0$$

where t is measured in months.

(a) When are the monthly sales increasing? When are they decreasing?
(b) Find the rate of change of sales.
(c) When is the rate of change of sales R' increasing? When is it decreasing?
(d) When is the rate of change of sales a maximum?
(e) Find any inflection points of R.
(f) Interpret the result found in (e) in the context of the problem.

Solution (a) We find $R'(t)$ and use the Increasing/Decreasing Function Test.

$$R'(t) = \frac{d}{dt}\left(\frac{20{,}000}{1+50e^{-t}} - \frac{20{,}000}{51}\right) = 20{,}000 \cdot \frac{50e^{-t}}{(1+50e^{-t})^2} = \frac{1{,}000{,}000e^{-t}}{(1+50e^{-t})^2}$$

Since $e^{-t} > 0$ for all $t \geq 0$, then $R'(t) > 0$ for $t \geq 0$. The sales function R is an increasing function. So, monthly sales are always increasing.

(b) The rate of change of sales is given by the derivative $R'(t) = \dfrac{1{,}000{,}000e^{-t}}{(1+50e^{-t})^2}$, $t \geq 0$.

(c) Using the Increasing/Decreasing Function Test with R', the rate of change of sales R' is increasing when its derivative $R''(t) > 0$; $R'(t)$ is decreasing when $R''(t) < 0$.

$$R''(t) = \frac{d}{dt}R'(t) = 1{,}000{,}000\left[\frac{-e^{-t}(1+50e^{-t})^2 + 100e^{-2t}(1+50e^{-t})}{(1+50e^{-t})^4}\right]$$

$$= 1{,}000{,}000e^{-t}\left[\frac{-1-50e^{-t}+100e^{-t}}{(1+50e^{-t})^3}\right] = \frac{1{,}000{,}000e^{-t}}{(1+50e^{-t})^3}(50e^{-t}-1)$$

Since $e^{-t} > 0$ for all t, the sign of R'' depends on the sign of $50e^{-t} - 1$.

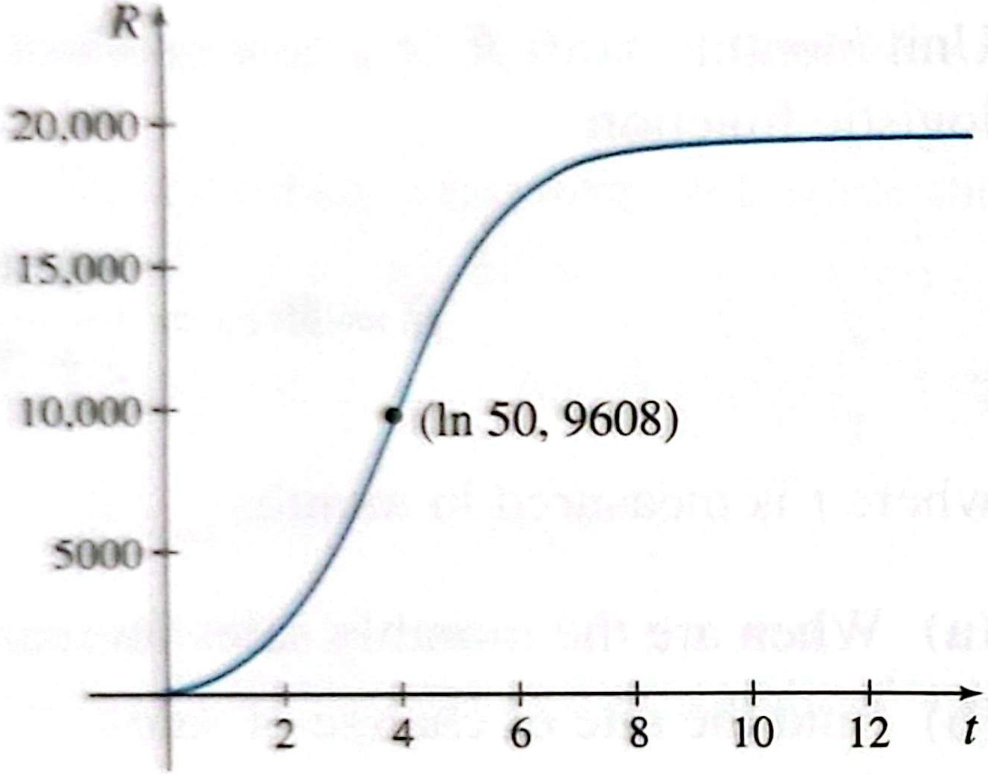
$$\begin{array}{cc} 50e^{-t} - 1 > 0 & 50e^{-t} - 1 < 0 \\ 50e^{-t} > 1 & 50e^{-t} < 1 \\ 50 > e^{t} & 50 < e^{t} \\ t < \ln 50 & t > \ln 50 \end{array}$$

Since $R''(t) > 0$ for $t < \ln 50 \approx 3.912$ and $R''(t) < 0$ for $t > \ln 50 \approx 3.912$, the rate of change of sales is increasing for the first 3.9 months and is decreasing from 3.9 months on.

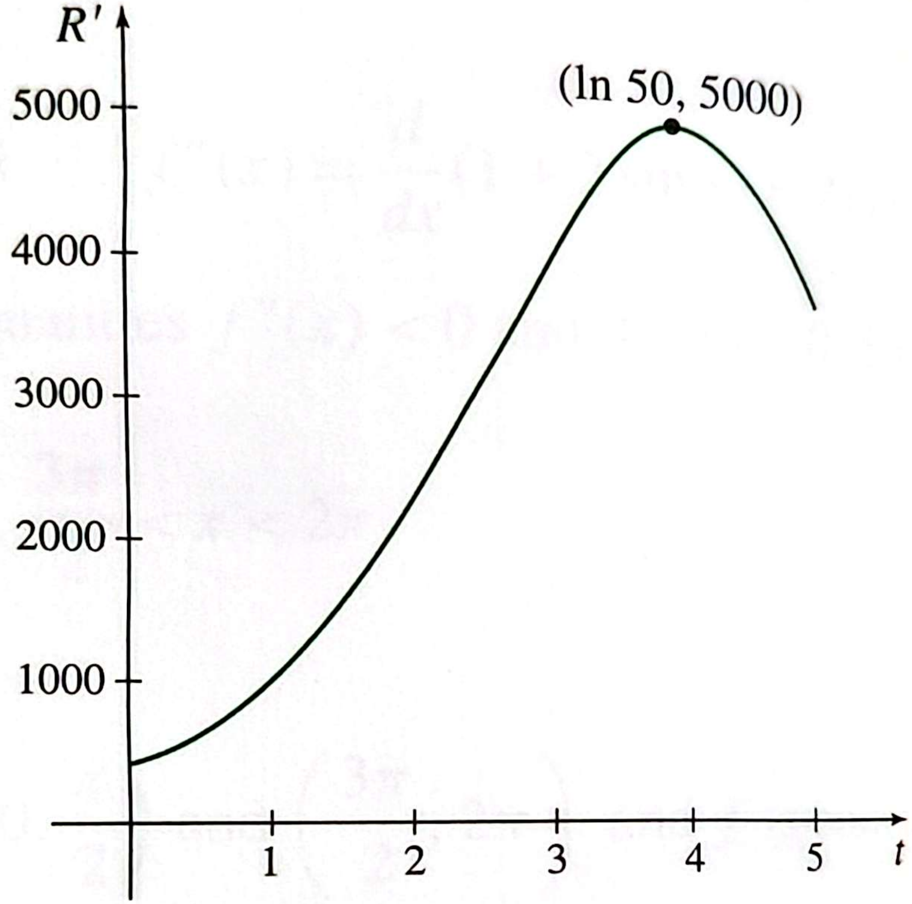
(d) The critical number of R' is $\ln 50 \approx 3.912$. Using the First Derivative Test, the rate of change of sales is a maximum about 3.9 months after the product is introduced.

(e) Since $R''(t) > 0$ for $t < \ln 50$ and $R''(t) < 0$ for $t > \ln 50$, the point $(\ln 50, 9608)$ is the inflection point of R.

(f) The sales function R is an increasing function, but at the inflection point $(\ln 50, 9608)$ the rate of change in sales begins to decrease. ■



(a) $R(t) = \dfrac{20{,}000}{1 + 50e^{-t}} - \dfrac{20{,}000}{51}$



(b) $R'(t) = \dfrac{1{,}000{,}000\, e^{-t}}{(1 + 50e^{-t})^2}$

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Question 8a

Graph $f(x) = 4x^{1/3} - x^{4/3}$.

Solution

Step 1 The domain of f is all real numbers. Since $f(0) = 0$, the y-intercept is 0. Now $f(x) = 0$ when $4x^{1/3} - x^{4/3} = x^{1/3}(4 - x) = 0$ or when $x = 0$ or $x = 4$. So, the x-intercepts are 0 and 4. Plot the intercepts (0, 0) and (4, 0).

Step 2 Since $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} [x^{1/3}(4 - x)] = -\infty$, there is no horizontal asymptote. Since the domain of f is all real numbers, there is no vertical asymptote.

Step 3
$$f'(x) = \frac{d}{dx}(4x^{1/3} - x^{4/3}) = \frac{4}{3}x^{-2/3} - \frac{4}{3}x^{1/3} = \frac{4}{3}\left(\frac{1}{x^{2/3}} - x^{1/3}\right)$$
$$= \frac{4}{3} \cdot \frac{1 - x}{x^{2/3}}$$
$$f''(x) = \frac{d}{dx}\left(\frac{4}{3}x^{-2/3} - \frac{4}{3}x^{1/3}\right) = -\frac{8}{9}x^{-5/3} - \frac{4}{9}x^{-2/3} = -\frac{4}{9}\left(\frac{2}{x^{5/3}} + \frac{1}{x^{2/3}}\right)$$
$$= -\frac{4}{9} \cdot \frac{2 + x}{x^{5/3}}$$

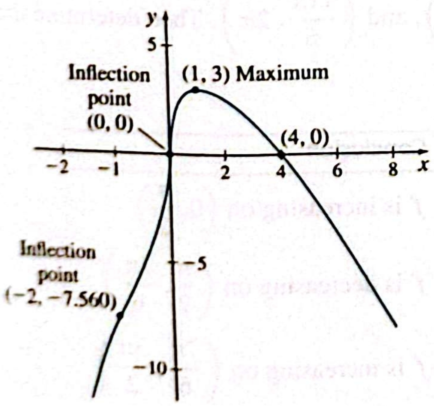
Since $f'(x) = \frac{4}{3} \cdot \frac{1 - x}{x^{2/3}} = 0$ when $x = 1$ and $f'(x)$ does not exist at $x = 0$, the critical numbers are 0 and 1. At the point (1, 3), the tangent line to the graph is horizontal; at the point (0, 0), the tangent line is vertical. Plot these points.

Step 4 Use the critical numbers 0 and 1 to form three intervals on the x-axis: $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$. Then determine the sign of $f'(x)$ on each interval.

Interval	Sign of f'	Conclusion
$(-\infty, 0)$	positive	f is increasing on $(-\infty, 0)$
$(0, 1)$	positive	f is increasing on $(0, 1)$
$(1, \infty)$	negative	f is decreasing on $(1, \infty)$

Step 5 By the First Derivative Test, $f(1) = 3$ is a local maximum value and $f(0) = 0$ is not a local extreme value.

Step 6 Now test for concavity by using the numbers -2 and 0 to form three intervals on the x-axis: $(-\infty, -2)$, $(-2, 0)$ and $(0, \infty)$. Then determine the sign of $f''(x)$ on each interval.



Interval	Sign of f''	Conclusion
$(-\infty, -2)$	negative	f is concave down on the interval $(-\infty, -2)$
$(-2, 0)$	positive	f is concave up on the interval $(-2, 0)$
$(0, \infty)$	negative	f is concave down on the interval $(0, \infty)$

The concavity changes at -2 and at 0. Since
$$f(-2) = 4(-2)^{1/3} - (-2)^{4/3} = 4\sqrt[3]{-2} - \sqrt[3]{16} \approx -7.560,$$
the inflection points are $(-2, -7.560)$ and (0, 0). Plot the inflection points.

Step 7 The graph of f is given ■

Question 8b

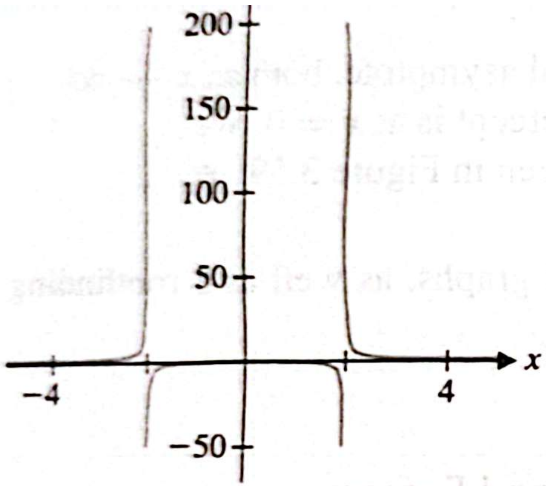


FIGURE 3.58a

$$y = \frac{x^2}{x^2 - 4}$$

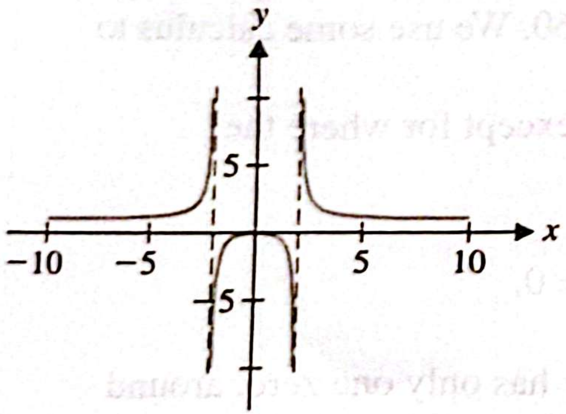


FIGURE 3.58b

$$y = \frac{x^2}{x^2 - 4}$$

Draw a graph of $f(x) = \dfrac{x^2}{x^2 - 4}$ showing all significant features.

Solution The default graph produced by our computer algebra system is seen in Figure 3.58a, while the default graph drawn by most graphing calculators looks like th graph seen in Figure 3.58b. Notice that the domain of f includes all x except $x = \pm 2$ (since the denominator is zero at $x = \pm 2$). Figure 3.58b suggests that there are vertical asymptotes at $x = \pm 2$, but let's establish this carefully. We have

$$\lim_{x \to 2^+} \frac{x^2}{x^2 - 4} = \lim_{x \to 2^+} \frac{\overset{+}{x^2}}{\underset{+}{(x-2)}\underset{+}{(x+2)}} = \infty. \qquad (5.7$$

Similarly, we get

$$\lim_{x \to 2^-} \frac{x^2}{x^2 - 4} = -\infty, \qquad \lim_{x \to -2^+} \frac{x^2}{x^2 - 4} = -\infty \qquad (5.8)$$

and

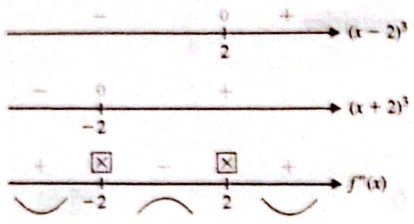
$$\lim_{x \to -2^-} \frac{x^2}{x^2 - 4} = \infty. \qquad (5.9)$$

Thus, there are vertical asymptotes at $x = \pm 2$. Next, we have

$$f'(x) = \frac{2x(x^2 - 4) - x^2(2x)}{(x^2 - 4)^2} = \frac{-8x}{(x^2 - 4)^2}.$$

Since the denominator is positive for $x \neq \pm 2$, it is a simple matter to see that

$$f'(x) \begin{cases} > 0, \text{ on } (-\infty, -2) \text{ and } (-2, 0) & f \text{ increasing.} \\ < 0, \text{ on } (0, 2) \text{ and } (2, \infty). & f \text{ decreasing.} \end{cases} \qquad (5.10)$$



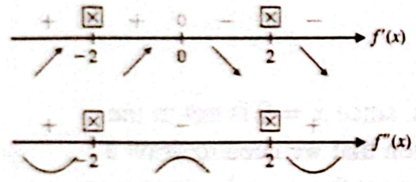
In particular, notice that the only critical number is $x = 0$ (since $x = -2, 2$ are not in the domain of f). Thus, the only local extremum is the local maximum located at $x = 0$. Next, we have

$$f''(x) = \frac{-8(x^2-4)^2 + (8x)2(x^2-4)^1(2x)}{(x^2-4)^4} \qquad \text{Quotient rule.}$$

$$= \frac{8(x^2-4)[-(x^2-4)+4x^2]}{(x^2-4)^4} \qquad \text{Factor out } 8(x^2-4).$$

$$= \frac{8(3x^2+4)}{(x^2-4)^3} \qquad \text{Combine terms.}$$

$$= \frac{8(3x^2+4)}{(x-2)^3(x+2)^3}. \qquad \text{Factor difference of two squares.}$$



Since the numerator is positive for all x, we need only consider the factors in the denominator, as seen in the margin. We then have

$$f''(x) \begin{cases} > 0, \text{ on } (-\infty, -2) \text{ and } (2, \infty) & \text{Concave up.} \\ < 0, \text{ on } (-2, 2). & \text{Concave down.} \end{cases} \tag{5.11}$$

However, since $x = 2, -2$ are not in the domain of f, there are no inflection points. It is an easy exercise to verify that

$$\lim_{x\to\infty} \frac{x^2}{x^2-4} = 1 \tag{5.12}$$

and

$$\lim_{x\to-\infty} \frac{x^2}{x^2-4} = 1. \tag{5.13}$$

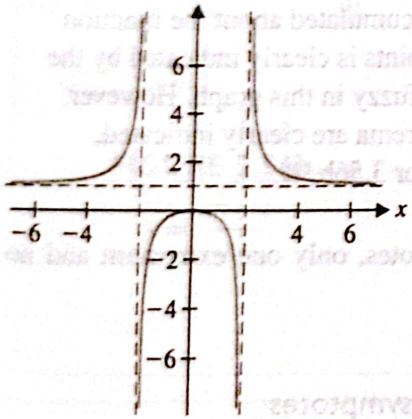


FIGURE 3.59

$$y = \frac{x^2}{x^2-4}$$

From (5.12) and (5.13), we have that $y = 1$ is a horizontal asymptote, both as $x \to \infty$ and as $x \to -\infty$. Finally, we observe that the only x-intercept is at $x = 0$. We summarize the information in (5.7)–(5.13) in the graph seen in Figure 3.59. ■

In example 5.4, we need to use computer-generated graphs, as well as a rootfinding method to determine the behavior of the function.

Question 9

Q:- Find the limits of the following ftns

(a) $\lim_{x \to +\infty} \dfrac{x^{-4/3}}{\sin(1/x)}$

Sol

Form $\left(\dfrac{0}{0}\right)$

$$\lim_{x\to+\infty} \frac{-\frac{4}{3}x^{-7/3}}{\left(\frac{-1}{x^2}\right)\cos\left(\frac{1}{x}\right)}$$

$$\lim_{x\to+\infty} \frac{\frac{4}{3}x^{-1/3}}{\cos\left(\frac{1}{x}\right)}$$

$$\frac{0}{1} = 0$$

(b) $$\lim_{x\to 0^+} \frac{\ln x}{\operatorname{Cosec} x}$$

Form $\left(\frac{\infty}{\infty}\right)$

$$\lim_{x\to 0^+} \frac{1/x}{-\operatorname{cosec} x \cot x} \quad \left(\frac{\infty}{\infty}\right)$$

$$\lim_{x\to 0^+} \left(-\frac{\sin x}{x} \cdot \tan x\right)$$

$$-\lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \tan x$$

$$-(1)\cdot(0)$$

$$= 0$$

(c) $$\lim_{x\to \frac{\pi}{4}} (1-\tan x)\sec 2x$$

<u>sol</u> Form $(0\cdot\infty)$

$$\lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{1/\sec 2x}$$

$$\lim_{x \to \frac{\pi}{4}} \frac{1 - \tan x}{\cos 2x}$$

$$\lim_{x \to \frac{\pi}{4}} \frac{-\sec^2 x}{-2\sin 2x}$$

$$\frac{-2}{-2} \Rightarrow 1$$

Question 10

Question:

A student writes a Python script to predict CPU temperature T(t) over time (in seconds):

$$T(t) = 50 + 20\sin(0.1t)$$

During a test run, they need a quick estimate of the CPU temperature at t = 32 seconds but the monitoring system only shows readings every 10 seconds.

Task:

1. Find the **linear approximation L(t)** of T(t) near t=30.
2. Use L(t) to estimate T(32).
3. Use **Python**:
    - To plot both T(t) and L(t) on the same graph between t = 20 and t = 40.
    - Compare visually how accurate the linear approximation is near t = 30.

4.Use **ChatGPT or any AI tool** to generate **one new but similar real-world scenario** where *linearization could be applied in computing or technology* (for example: predicting battery percentage, network latency, or cooling fan speed).

- Write the new function or situation that the AI suggested.

- Using your own understanding, **apply linearization** to estimate a value in your new scenario.

Answers:

Part 1 and 2)

1) Derivation — linearization at $t_0 = 30$

Linearization formula:

$$L(t) = T(t_0) + T'(t_0)\,(t - t_0).$$

Compute derivative:

$$T'(t) = 20 \cdot 0.1 \cos(0.1t) = 2\cos(0.1t).$$

At $t_0 = 30$:

$$T(30) = 50 + 20\sin(0.1 \cdot 30) = 50 + 20\sin(3) \approx 52.822400\ ^\circ\mathrm{C},$$

$$T'(30) = 2\cos(3) \approx -1.979985\ ^\circ\mathrm{C/s}.$$

So

$$L(t) = 52.822400 + (-1.979985)(t - 30).$$

Estimate at $t = 32$:

$$L(32) = 52.822400 + (-1.979985) \cdot 2 \approx 48.862430\ ^\circ\mathrm{C}.$$

Actual value for comparison:

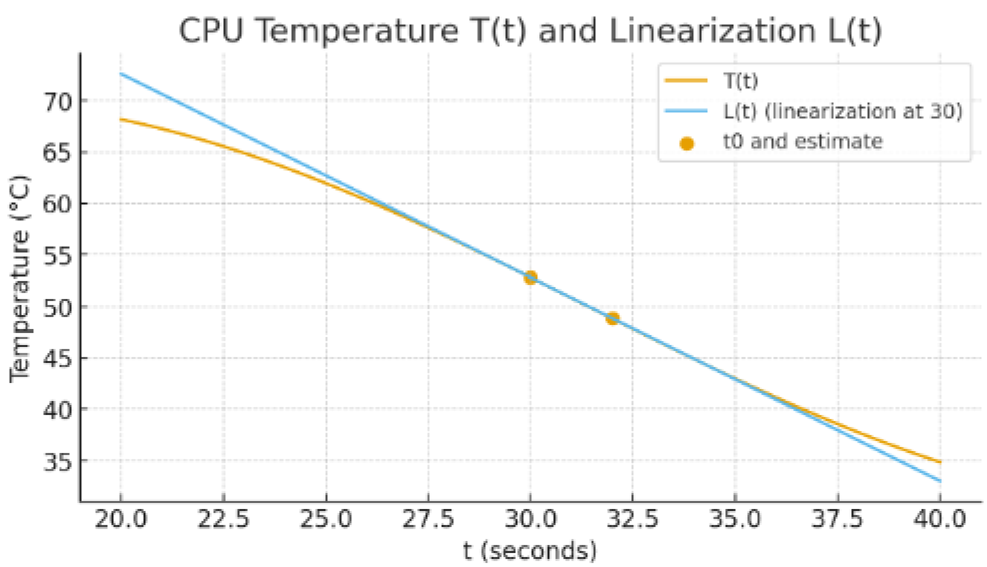
$$T(32) = 50 + 20\sin(3.2) \approx 48.832517\ ^\circ\mathrm{C}.$$

Absolute error:

$$|L(32) - T(32)| \approx 0.029913\ ^\circ\mathrm{C}.$$

Conclusion: The linear approximation gives 48.8624°C while the actual is 48.8325°C — error ≈ 0.03°C (very small), so linearization near $t = 30$ works well here for a two-second step.

Part 3)



I ran Python (NumPy + Matplotlib) to:

- compute $T(30)$, $T'(30)$, linear estimate $L(32)$, actual $T(32)$,
- plot $T(t)$ and $L(t)$ over $20 \le t \le 40$.

Outputs (numeric):

- $T(30) = 52.822400$ °C
- $T'(30) = -1.979985$ °C/s
- Linear estimate $L(32) = 48.862430$ °C
- Actual $T(32) = 48.832517$ °C
- Absolute error ≈ 0.029913 °C

(Plots were created — the CPU temperature curve and the linear tangent line are shown on the same graph. The linear line closely follows $T(t)$ near $t = 30$.)

Part 4 depends on the question the user generated so it varies.