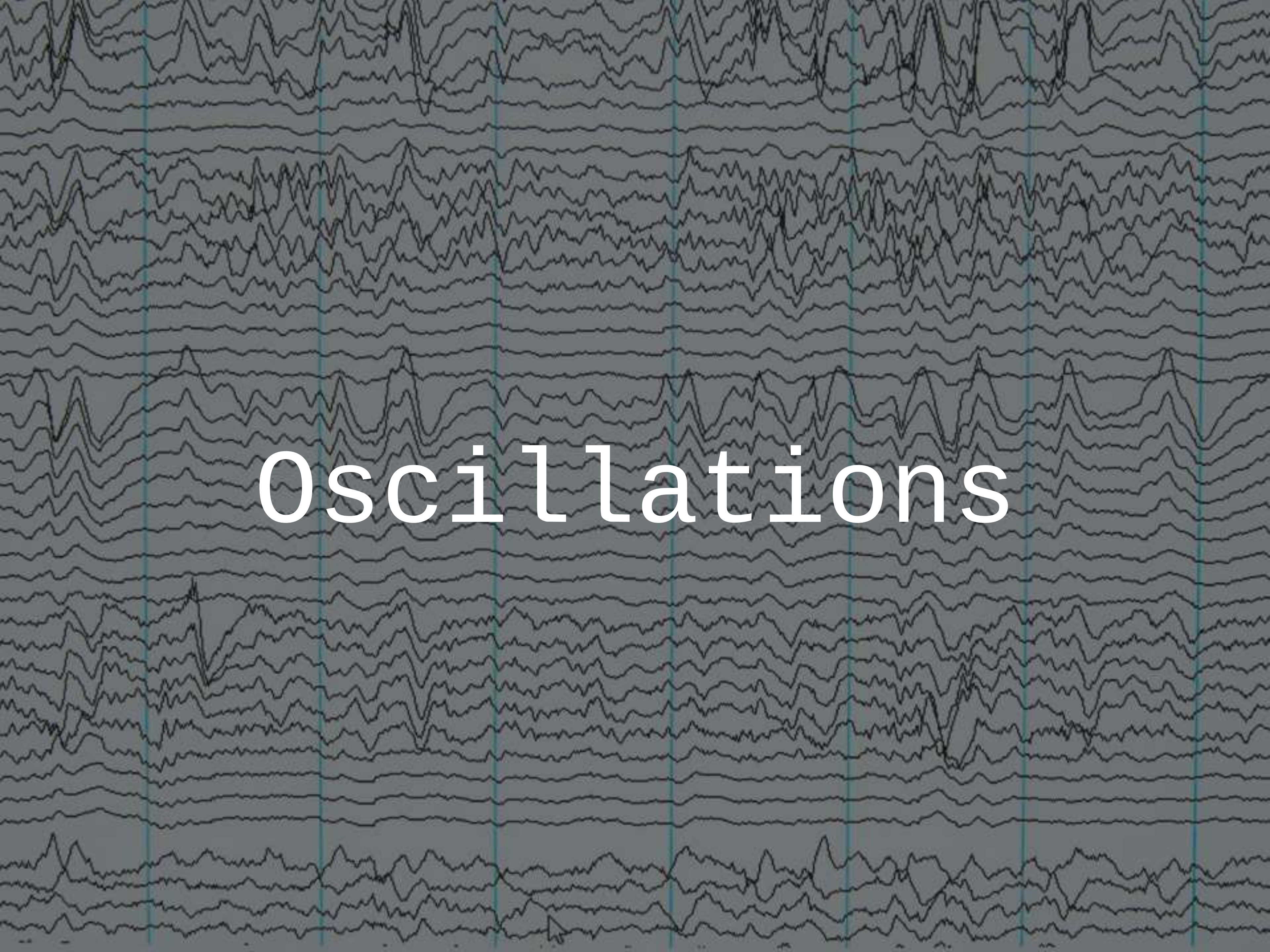


Oscillations

Clo 02

CLO:2 Use oscillations and analyze different types of waves graphically & mathematically.

Application of Oscillations in CS

Clock Signals: Used to synchronize digital circuits; the oscillation frequency determines CPU and system speed.

Communication Systems: Wi-Fi, Ethernet, and radio signals use oscillating electromagnetic waves to transmit data.

Signal Processing: Oscillations (sinusoids) are the basis for Fourier analysis, filtering, and data compression.

Control Systems: Feedback loops manage oscillations to keep automated systems stable and prevent overshoot.

Computer Graphics & Animation: Oscillatory motion creates realistic effects like waves, swaying trees, or blinking lights.

Applications in daily life

AC Electricity: Homes run on 50/60 Hz alternating voltage/current.

Timekeeping: Quartz crystals oscillate to keep watches/clocks accurate.

Sound & Music: Speech and instrument notes are pressure waves (air vibrations).

Wireless & Radio: Phones/Wi-Fi transmit data via oscillating electromagnetic waves.

Vehicles & Buildings: Suspensions and dampers control vibrations to improve comfort and safety.

Oscillations

Oscillations are repeated back-and-forth variations of a quantity around an equilibrium point, typically periodic in time, characterized by **amplitude** (size), **frequency** f in hertz, **period** $T = 1/f$, and **phase** (position within the cycle).

Simple Harmonic Oscillators

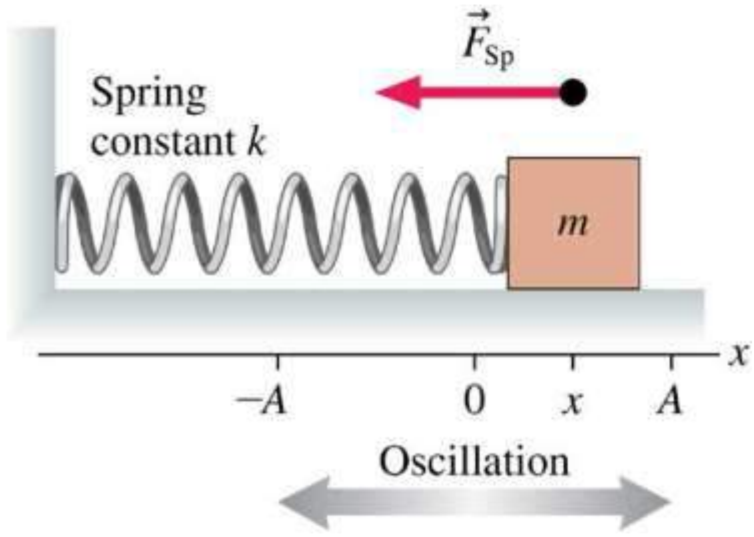
- **Simple Harmonic Oscillator (SHO):**
- A **Simple Harmonic Oscillator** is a system that moves back and forth (oscillates) about an equilibrium position such that the **restoring force is directly proportional to the displacement** and always acts **toward the mean position**.
-

- **Examples:**
- Mass attached to a spring
- Simple pendulum (for small angles)
- Vibrating tuning fork

Dynamics of Simple Harmonic Motion

- Consider a mass m oscillating on a horizontal spring with no friction.
- The spring force is

$$(F_{Sp})_x = -k\,\Delta x$$



- Since the spring force is the net force, Newton's second law gives

$$(F_{net})_x = (F_{Sp})_x = -kx = ma_x$$

$$a_x = -\frac{k}{m}x$$

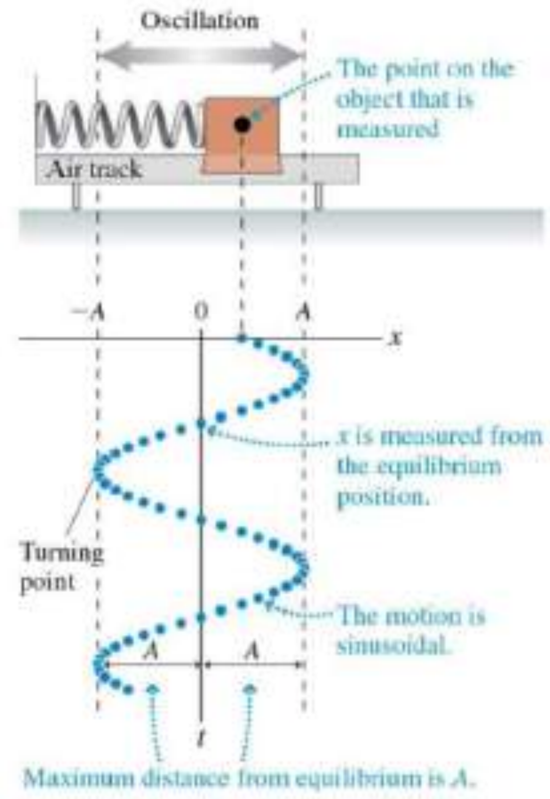
- Since $a_x = -\omega^2 x$, the angular frequency must be $\omega = \sqrt{\frac{k}{m}}$.

Restoring Force

- According to Hooke’s Law: $F = -k\,x$
- Here, F = restoring force, k = force constant, x = displacement.
- Negative sign indicates that the force acts opposite to the direction of displacement.

Equation of Motion

- By Newton's second law: $F = m\,a = m\,(d^2x/dt^2)$
- So, $m\,(d^2x/dt^2) = -k\,x$
- $\Longrightarrow (d^2x/dt^2) + (k/m)\,x = 0$
- Let $\omega^2 = k/m$, then the equation becomes: $(d^2x/dt^2) + \omega^2 x = 0$



Solution of the Differential Equation

- The general solution is: **$x(t) = A \sin(\omega t + \varphi)$**, or **$A \cos(\omega t + \varphi)$**,
- A = amplitude, $\omega = \sqrt{(k/m)}$ = angular frequency, φ = phase constant.
- This represents simple harmonic motion with periodic oscillations.

Velocity and Acceleration

- Velocity: $v = dx/dt = A\omega \cos(\omega t + \varphi)$
- Acceleration: $a = d^2x/dt^2 = -A\omega^2 \sin(\omega t + \varphi) = -\omega^2 x$
- Hence, acceleration is directly proportional to displacement and opposite in direction.

Motion Equations for Simple Harmonic Motion

$$x(t) = A\cos(\omega t + \phi)$$

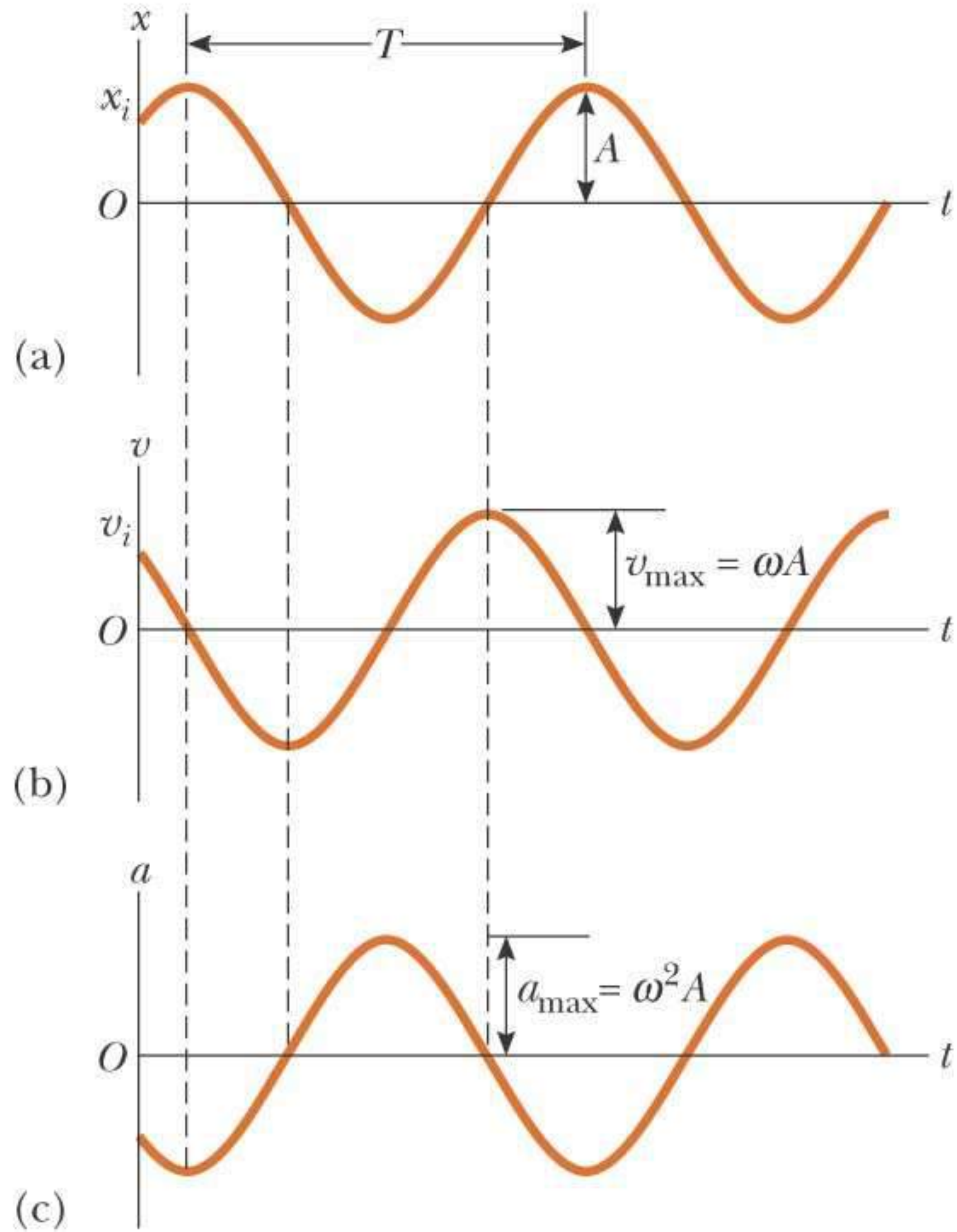
$$v = \frac{dx}{dt} = -\omega A \sin(\omega\, \mathrm{t} + \phi)$$

$$a = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega\, \mathrm{t} + \phi)$$

- Amplitude A= x_m
- Then $a = -\omega^2 x$
- Simple harmonic motion is one-dimensional and so directions can be denoted by + or - sign
- Remember, simple harmonic motion is **not** uniformly accelerated motion

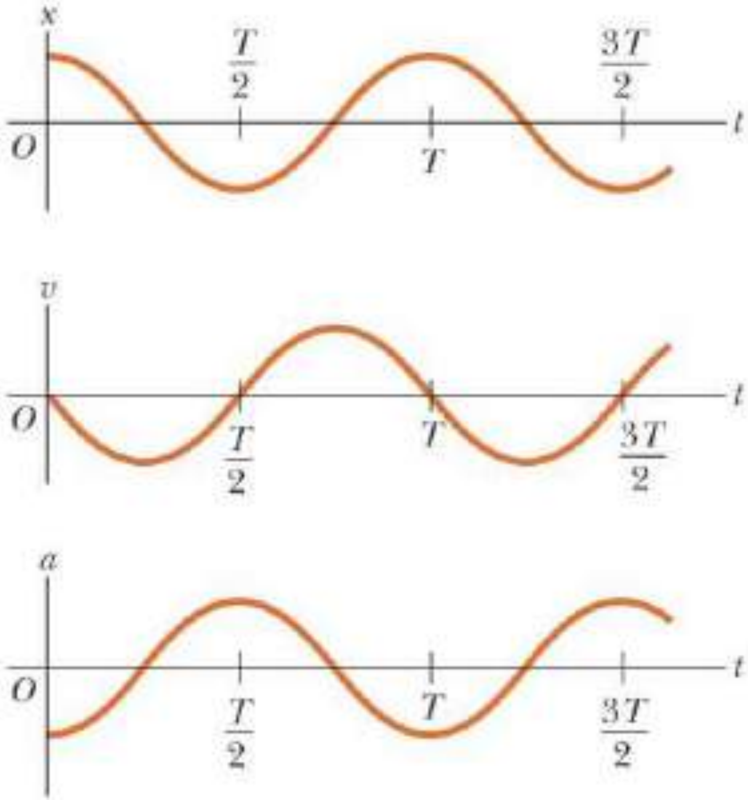
Graphs

- The graphs show:
  - (a) displacement as a function of time
  - (b) velocity as a function of time
  - (c) acceleration as a function of time
- The velocity is 90º out of phase with the displacement and the acceleration is 180º out of phase with the displacement



SHM Example 1

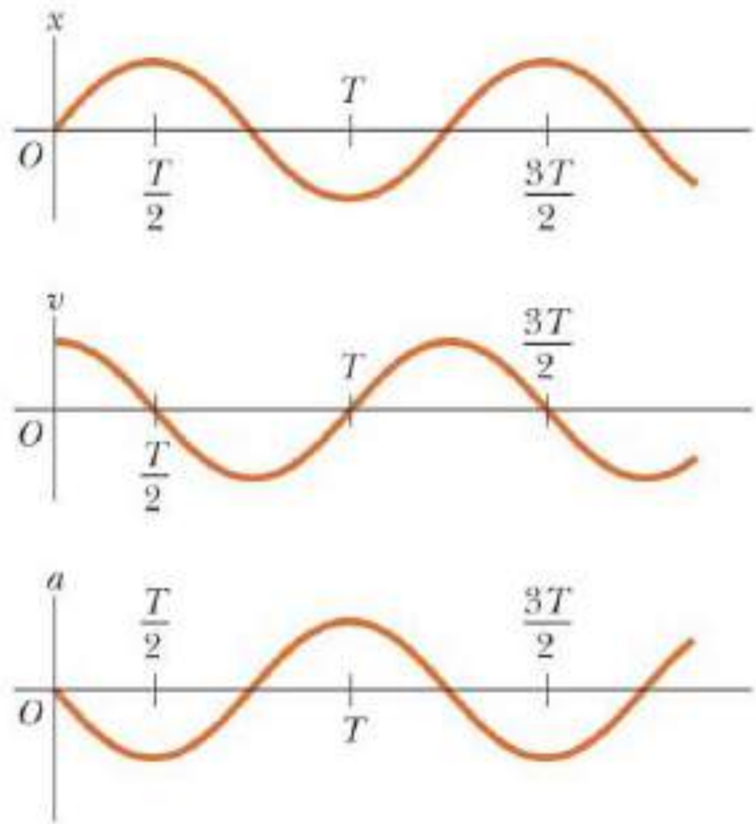
- Initial conditions at t = 0 are
  - x (0)= A
  - v (0) = 0
- This means ϕ = 0
- The acceleration reaches extremes of $\pm\,\omega^2 A$
- The velocity reaches extremes of $\pm\,\omega A$



SHM Example 2

- Initial conditions at

  $t = 0$ are
  - $x\ (0)=0$
  - $v\ (0) = v_i$
- This means $\phi = -\pi/2$
- The graph is shifted one-quarter cycle to the right compared to the graph of x



SHM Phase Constant: Sine vs Cosine

How to choose sin or cos, and find the phase constant φ with a worked numerical example.

General SHM Equations

- Displacement can be written in either form:
- $x = A \sin(\omega t + \varphi)$
- $x = A \cos(\omega t + \varphi)$
- A: amplitude, ω: angular frequency, φ: phase constant
- Both forms are equivalent; they differ by a phase shift.

Choosing sin or cos

- Use sin when starting at mean position: $x(0) \approx 0 \rightarrow$ often $\varphi = 0$ in sine form.
- Use cos when starting at an extreme: $|x(0)| \approx A \rightarrow$ often $\varphi = 0$ or π in cosine form.
- Always use the given initial conditions to solve for φ.

Finding φ (Sine Form)

- If $x = A \sin(\omega t + \varphi)$:
- At $t = 0$: $x_0 = A \sin \varphi$, $v_0 = A \omega \cos \varphi$
- $\Rightarrow \tan \varphi = (\omega x_0) / v_0$ (be careful with quadrants using v_0 sign).

Finding φ (Cosine Form)

- If $x = A \cos(\omega t + \varphi)$:
- At $t = 0$: $x_0 = A \cos \varphi$, $v_0 = -A \omega \sin \varphi$
- $\Rightarrow \tan \varphi = - v_0 / (\omega x_0)$ (use v_0 sign to pick correct quadrant).

Worked Example: Setup

- Given: $A = 10$ cm, $T = 2$ s $\Rightarrow$ $\omega = 2\pi/T = \pi$ rad/s.
- At $t = 0$: $x_0 = +5$ cm and the particle moves toward the mean position $\Rightarrow$ $v_0 < 0$.
- We will solve the same motion using both cosine and sine forms.

Worked Example (Cosine Form)

- Assume $x = A \cos(\omega t + \varphi)$.
- $x_0 = A \cos \varphi \Rightarrow 5 = 10 \cos \varphi \Rightarrow \cos \varphi = 0.5 \Rightarrow \varphi = \pi/3$ or $5\pi/3$.
- Check direction: $v_0 = -A \omega \sin \varphi$; need $v_0 < 0 \Rightarrow \sin \varphi > 0 \Rightarrow \varphi = \pi/3$.
- Result: $x(t) = 10 \cos(\pi t + \pi/3)$ cm.

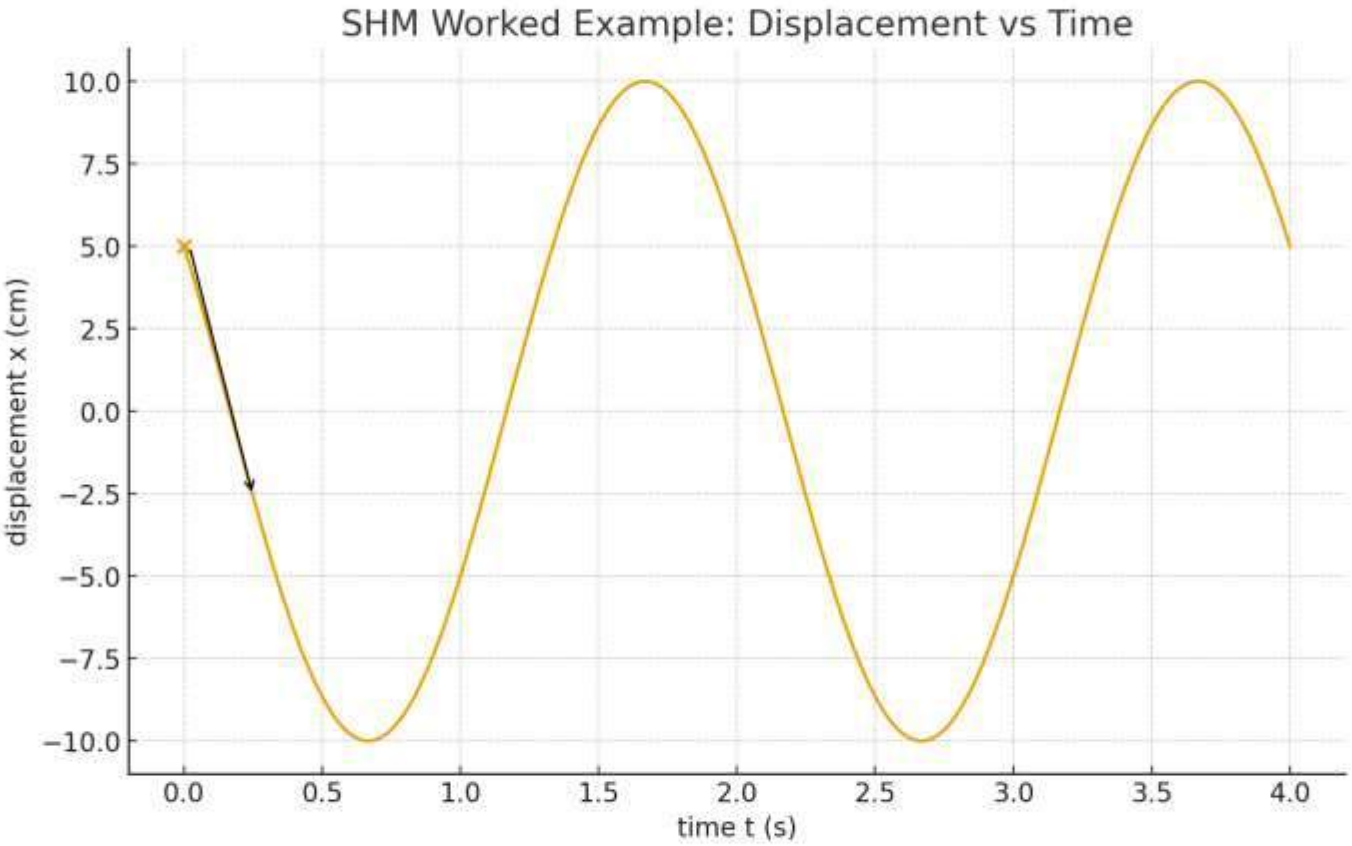
Worked Example (Sine Form)

- Assume $x = A \sin(\omega t + \varphi)$.
- $x_0 = A \sin \varphi \Rightarrow 5 = 10 \sin \varphi \Rightarrow \sin \varphi = \mathbf{0.5} \Rightarrow \varphi = \pi/6$ or $5\pi/6$.
- **Direction:** $v_0 = A \omega \cos \varphi$; need $v_0 < 0 \Rightarrow \cos \varphi < 0 \Rightarrow \varphi = 5\pi/6$.
- Equivalent result: $x(t) = 10 \sin(\pi t + 5\pi/6)$ cm.

Quick Numeric Check

- At $t = 0$ s: $10 \cos(\pi/3) = 5$ cm; $10 \sin(5\pi/6) = 5$ cm.
- At $t = 0.5$ s: $10 \cos(5\pi/6) = -(10\sqrt{3})/2 \approx -8.66$ cm;
- $10 \sin(4\pi/3) = -(10\sqrt{3})/2 \approx -8.66$ cm.
- Conclusion: Displacement matches at the same times; velocity and acceleration also match.
- Difference is only the phase constant ($\varphi\_sin = \varphi\_cos + \pi/2$).

Displacement–Time Plot (Worked Example)



Key Takeaways

- Sine vs cosine is a choice of reference; the physics is identical.
- Pick the form that simplifies the initial condition, then solve for φ.
- Use x_0 and v_0 (including their signs) to select the correct quadrant for φ.
- Your final answer should match reality: check direction at $t = 0$.

Sine vs Cosine: Same Motion (Our Example)

- Cosine form: $x(t) = 10 \cos(\pi t + \pi/3)$
- Sine form: $x(t) = 10 \sin(\pi t + 5\pi/6)$
- Identity: $\sin(\theta) = \cos(\theta - \pi/2) \Rightarrow$
- $10 \sin(\pi t + 5\pi/6) = 10 \cos(\pi t + 5\pi/6 - \pi/2) = 10 \cos(\pi t + \pi/3)$.
- Therefore, both equations describe the exact same displacement for all t.

Example 15.1 A System in Simple Harmonic Motion

EXAMPLE 15.1 | **A system in simple harmonic motion**

An air-track glider is attached to a spring, pulled 20.0 cm to the right, and released at $t = 0$ s. It makes 15 oscillations in 10.0 s.

a. What is the period of oscillation?
b. What is the object's maximum speed?
c. What are the position and velocity at $t = 0.800$ s?

MODEL An object oscillating on a spring is in SHM.

Example 15.1 A System in Simple Harmonic Motion

EXAMPLE 15.1 A system in simple harmonic motion

SOLVE a. The oscillation frequency is

$$f = \frac{15 \text{ oscillations}}{10.0 \text{ s}} = 1.50 \text{ oscillations/s} = 1.50 \text{ Hz}$$

Thus the period is $T = 1/f = 0.667$ s.

b. The oscillation amplitude is $A = 0.200$ m. Thus

$$v_{\max} = \frac{2\pi A}{T} = \frac{2\pi(0.200 \text{ m})}{0.667 \text{ s}} = 1.88 \text{ m/s}$$

Example 15.1 A System in Simple Harmonic Motion

EXAMPLE 15.1 A system in simple harmonic motion

SOLVE c. The object starts at $x = +A$ at $t = 0$ s. This is exactly the oscillation described by Equations 15.2 and 15.6. The position at $t = 0.800$ s is

$$x = A\cos\left(\frac{2\pi t}{T}\right) = (0.200\text{ m})\cos\left(\frac{2\pi(0.800\text{ s})}{0.667\text{ s}}\right)$$

$$= (0.200\text{ m})\cos(7.54\text{ rad}) = 0.0625\text{ m} = 6.25\text{ cm}$$

The velocity at this instant of time is

$$v_x = -v_{\max}\sin\left(\frac{2\pi t}{T}\right) = -(1.88\text{ m/s})\sin\left(\frac{2\pi(0.800\text{ s})}{0.667\text{ s}}\right)$$

$$= -(1.88\text{ m/s})\sin(7.54\text{ rad}) = -1.79\text{ m/s} = -179\text{ cm/s}$$

At $t = 0.800$ s, which is slightly more than one period, the object is 6.25 cm to the right of equilibrium and moving to the *left* at 179 cm/s. Notice the use of radians in the calculations.

Example 15.2 Finding the Time

EXAMPLE 15.2 Finding the time

A mass oscillating in simple harmonic motion starts at $x = A$ and has period T. At what time, as a fraction of T, does the object first pass through $x = \frac{1}{2}A$?

SOLVE Figure 15.3 showed that the object passes through the equilibrium position $x = 0$ at $t = \frac{1}{4}T$. This is one-quarter of the total distance in one-quarter of a period. You might expect it to take $\frac{1}{8}T$ to reach $\frac{1}{2}A$, but this is not the case because the SHM graph is not linear between $x = A$ and $x = 0$. We need to use $x(t) = A\cos(2\pi t/T)$. First, we write the equation with $x = \frac{1}{2}A$:

$$x = \frac{A}{2} = A\cos\left(\frac{2\pi t}{T}\right)$$

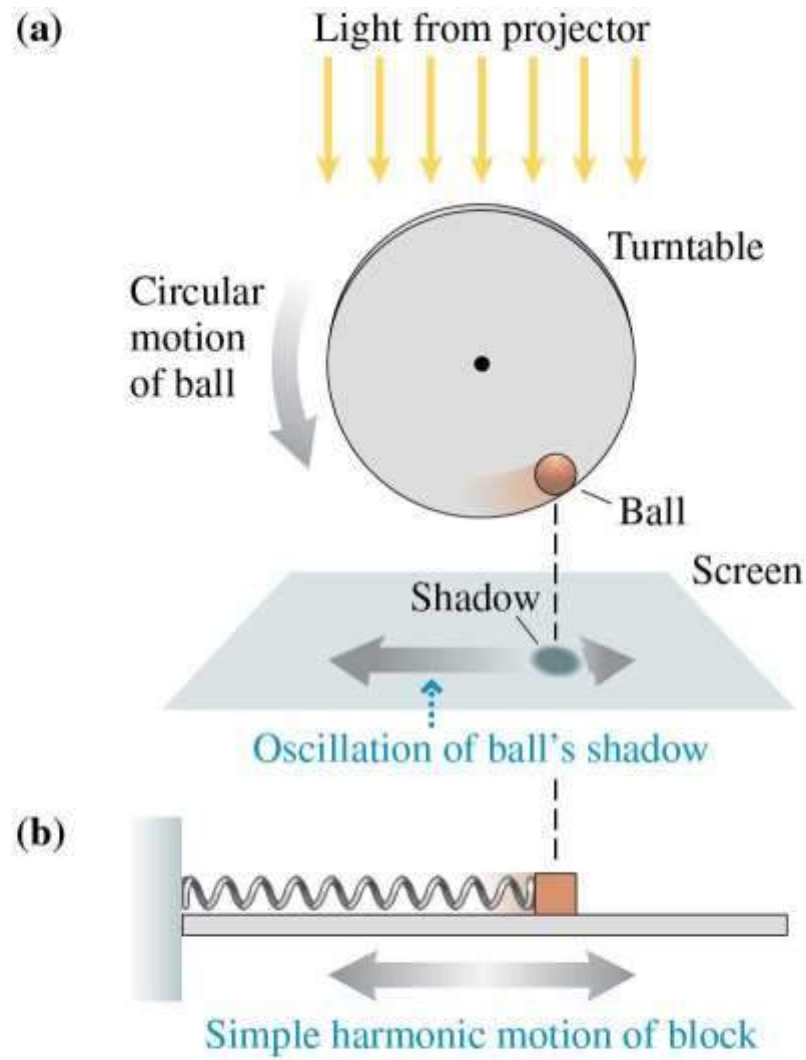
Then we solve for the time at which this position is reached:

$$t = \frac{T}{2\pi}\cos^{-1}\left(\frac{1}{2}\right) = \frac{T}{2\pi}\frac{\pi}{3} = \frac{1}{6}T$$

ASSESS The motion is slow at the beginning and then speeds up, so it takes longer to move from $x = A$ to $x = \frac{1}{2}A$ than it does to move from $x = \frac{1}{2}A$ to $x = 0$. Notice that the answer is independent of the amplitude A.

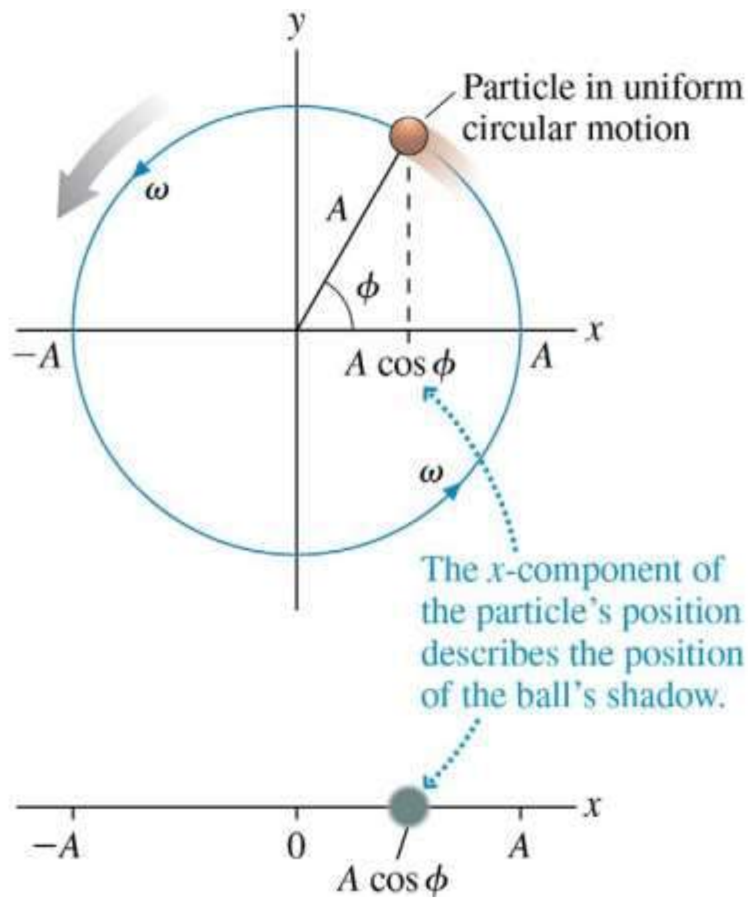
Simple Harmonic Motion and Circular Motion

- Figure (a) shows a “shadow movie” of a ball made by projecting a light past the ball and onto a screen.
- As the ball moves in uniform circular motion, the shadow moves with simple harmonic motion.
- The block on a spring in figure (b) moves with the same motion.



The Phase Constant

- What if an object in SHM is not initially at rest at $x = A$ when $t = 0$?
- Then we may still use the cosine function, but with a **phase constant** measured in radians.
- In this case, the two primary kinematic equations of SHM are:



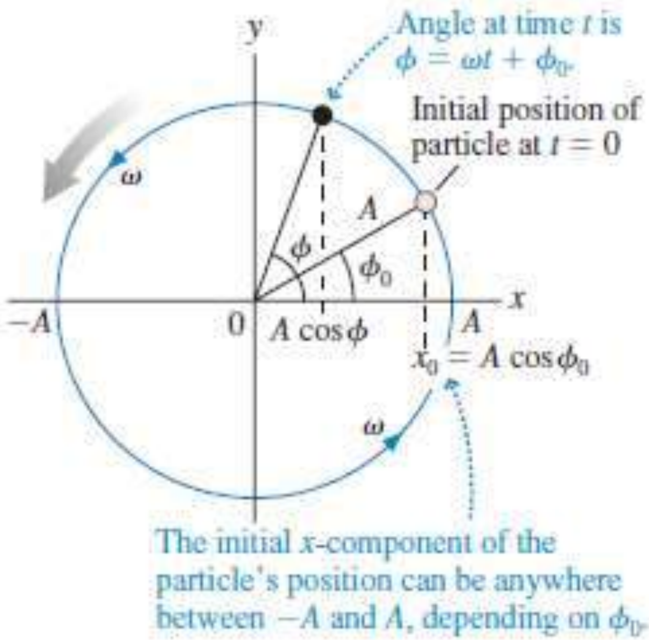
$$x(t) = A\cos(\omega t + \phi_0)$$

$$v_x(t) = -\omega A \sin(\omega t + \phi_0) = -v_{max}\sin(\omega t + \phi_0)$$

The Phase constant

The quantity $\phi = \omega t + \phi_0$, which steadily increases with time, is called the **phase** of the oscillation. The phase is simply the *angle* of the circular-motion particle whose shadow matches the oscillator. The constant ϕ_0 is called the **phase constant.** It specifies the *initial conditions* of the oscillator.

FIGURE 14.8 A particle in uniform circular motion with initial angle ϕ_0.

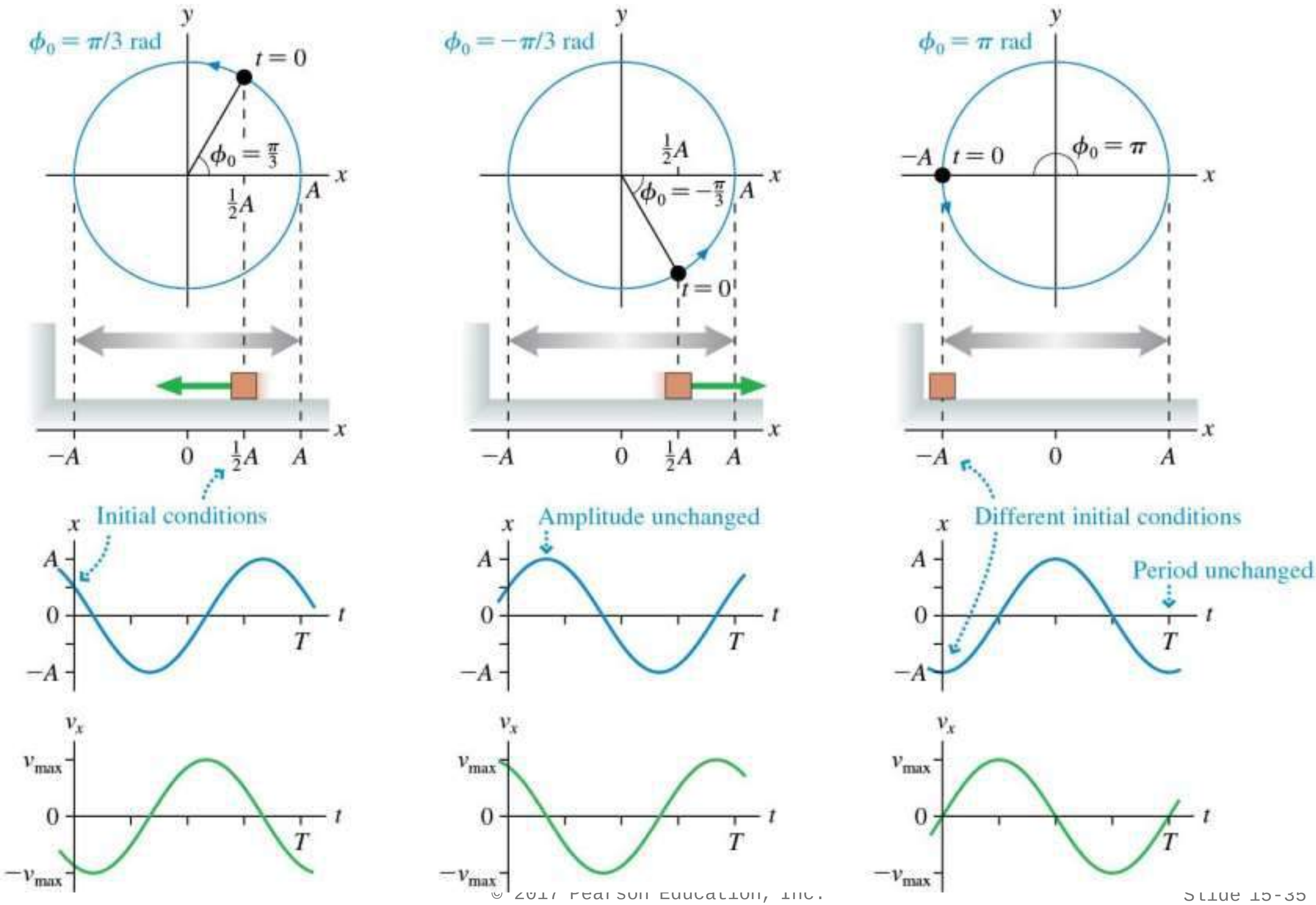


The angle at a later time t is then

$$\phi = \omega t + \phi_0$$

The Phase Constant

- Oscillations described by different values of the phase constant.



Phase constant

All values of the phase constant ϕ_0 between 0 and π rad correspond to a particle in the upper half of the circle and *moving to the left*. Thus v_{0x} is negative. All values of the phase constant ϕ_0 between π and 2π rad (or, as they are usually stated, between $-\pi$ and 0 rad) have the particle in the lower half of the circle and *moving to the right*. Thus v_{0x} is positive. If you're told that the oscillator is at $x = \frac{1}{2}A$ and moving to the right at $t = 0$, then the phase constant must be $\phi_0 = -\pi/3$ rad, not $+\pi/3$ rad.

Example

An object on a spring oscillates with a period of 0.80 s and an amplitude of 10 cm. At $t = 0$ s, it is 5.0 cm to the left of equilibrium and moving to the left. What are its position and direction of motion at $t = 2.0$ s?

SOLVE We can find the phase constant ϕ_0 from the initial condition $x_0 = -5.0 \text{ cm} = A\cos\phi_0$. This condition gives

$$\phi_0 = \cos^{-1}\left(\frac{x_0}{A}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \pm\frac{2}{3}\pi \text{ rad} = \pm 120^\circ$$

Because the oscillator is moving to the *left* at $t = 0$, it is in the upper half of the circular-motion diagram and must have a phase constant between 0 and π rad. Thus ϕ_0 is $\frac{2}{3}\pi$ rad. The angular frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.80 \text{ s}} = 7.85 \text{ rad/s}$$

Sol. con,,,

Thus the object's position at time $t = 2.0$ s is

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_0) \\ &= (10 \text{ cm})\cos\left((7.85 \text{ rad/s})(2.0 \text{ s}) + \frac{2}{3}\pi\right) \\ &= (10 \text{ cm})\cos(17.8 \text{ rad}) = 5.0 \text{ cm} \end{aligned}$$

The object is now 5.0 cm to the right of equilibrium. But which way is it moving? There are two ways to find out. The direct way is to calculate the velocity at $t = 2.0$ s:

$$v_x = -\omega A \sin(\omega t + \phi_0) = +68 \text{ cm/s}$$

The velocity is positive, so the motion is to the right. Alternatively, we could note that the phase at $t = 2.0$ s is $\phi = 17.8$ rad. Dividing by π, you can see that

$$\phi = 17.8 \text{ rad} = 5.67\pi \text{ rad} = (4\pi + 1.67\pi) \text{ rad}$$

••14 A simple harmonic oscillator consists of a block of mass 2.00 kg attached to a spring of spring constant 100 N/m. When $t = 1.00$ s, the position and velocity of the block are $x = 0.129$ m and $v = 3.415$ m/s. (a) What is the amplitude of the oscillations? What were the (b) position and (c) velocity of the block at $t = 0$ s?

14. Equation 15-12 gives the angular velocity:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100\ \text{N/m}}{2.00\ \text{kg}}} = 7.07\,\text{rad/s}.$$

Energy methods (discussed in Section 15-4) provide one method of solution. Here, we use trigonometric techniques based on Eq. 15-3 and Eq. 15-6.

(a) Dividing Eq. 15-6 by Eq. 15-3, we obtain

$$\frac{v}{x} = -\omega \tan(\omega t + \phi)$$

so that the phase ($\omega t + \phi$) is found from

$$\omega t + \phi = \tan^{-1}\left(\frac{-v}{\omega x}\right) = \tan^{-1}\left(\frac{-3.415\ \text{m/s}}{(7.07\ \text{rad/s})(0.129\ \text{m})}\right).$$

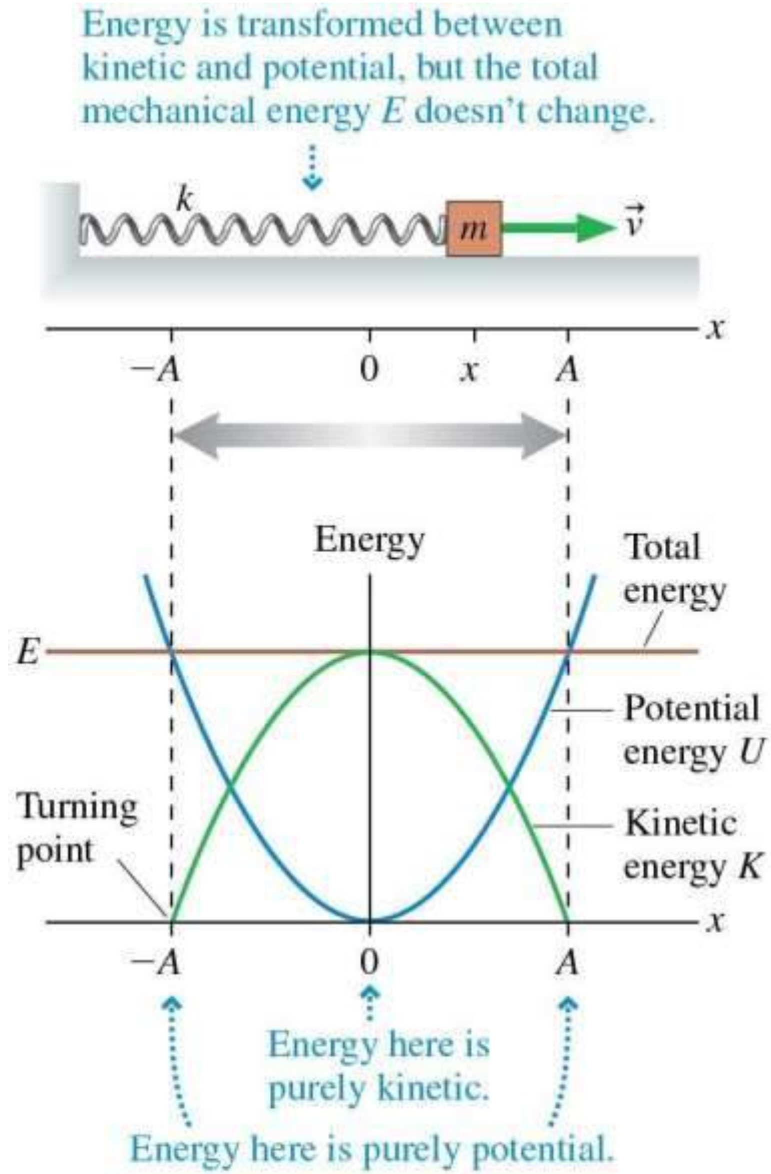
With the calculator in radians mode, this gives the phase equal to -1.31 rad. Plugging this back into Eq. 15-3 leads to $0.129\ \text{m} = x_m \cos(-1.31) \Rightarrow x_m = 0.500\ \text{m}$.

(b) Since $\omega t + \phi = -1.31$ rad at $t = 1.00$ s, we can use the above value of ω to solve for the phase constant ϕ. We obtain $\phi = -8.38$ rad (though this, as well as the previous result, can have 2π or 4π (and so on) added to it without changing the physics of the situation). With this value of ϕ, we find $x_o = x_m \cos\phi = -0.251$ m.

(c) And we obtain $v_o = -x_m\omega \sin\phi = 3.06$ m/s.

Energy in Simple Harmonic Motion

- An object of mass m on a frictionless horizontal surface is attached to one end of a spring of spring constant k.
- The other end of the spring is attached to a fixed wall.
- As the object oscillates, the energy is transformed between kinetic energy and potential energy, but the mechanical energy $E = K + U$ doesn't change.



Energy of the SHM Oscillator

- Assume a spring-mass system is moving on a frictionless surface
- This tells us the total energy is constant
- The kinetic energy can be found by

  $K = \frac{1}{2} mv^2 = \frac{1}{2} m\omega^2 x_m^2 \sin^2(\omega t + \phi)$
- The elastic potential energy can be found by

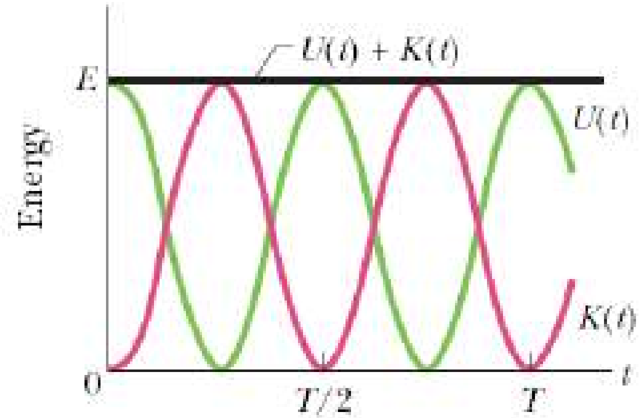
  $U = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \cos^2(\omega t + \phi)$
- The total energy is

$$E = K + U = \frac{1}{2} m\omega^2 x_m^2 \sin^2(\omega t + \phi) + \frac{1}{2} kx_m^2 \cos^2(\omega t + \phi)$$

As $m\omega^2 = k$,

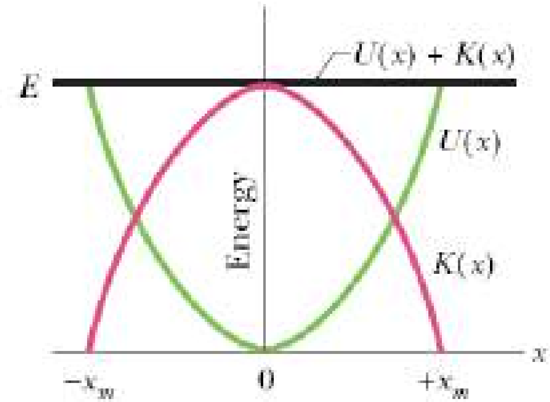
$$E = \frac{1}{2} kx_m^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)] = \frac{1}{2} kx_m^2$$

- Energy in SHM



- The total mechanical energy of the system:

$$E = U + K = \frac{1}{2}kx_m^2$$



Frequency of Simple Harmonic Motion

- In SHM, when K is maximum, $U = 0$, and when U is maximum, $K = 0$.
- $K + U$ is constant, so $K_{\text{max}} = U_{\text{max}}$:

$$\frac{1}{2}m(v_{\text{max}})^2 = \frac{1}{2}kA^2$$

- So $$v_{\text{max}} = \sqrt{\frac{k}{m}}A$$

- Earlier, using kinematics, we found that

$$v_{\text{max}} = \frac{2\pi A}{T} = 2\pi f A = \omega A$$

- So $$\omega = \sqrt{\frac{k}{m}} \qquad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

15. A block of unknown mass is attached to a spring with a spring constant of 6.50 N/m and undergoes simple harmonic motion with an amplitude of 10.0 cm. When the block is halfway between its equilibrium position and the end point, its speed is measured to be 30.0 cm/s. Calculate (a) the mass of the block, (b) the period of the motion, and (c) the maximum acceleration of the block.

P15.15 (a) Energy is conserved for the block-spring system between the maximum-displacement and the half-maximum points:

$$(K+U)_i = (K+U)_f \qquad 0 + \frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$\frac{1}{2}(6.50\ \text{N/m})(0.100\ \text{m})^2 = \frac{1}{2}m(0.300\ \text{m/s})^2 + \frac{1}{2}(6.50\ \text{N/m})(5.00\times10^{-2}\ \text{m})^2$$

$$32.5\ \text{mJ} = \frac{1}{2}m(0.300\ \text{m/s})^2 + 8.12\ \text{mJ} \qquad m = \frac{2(24.4\ \text{mJ})}{9.0\times10^{-2}\ \text{m}^2/\text{s}^2} = \boxed{0.542\ \text{kg}}$$

(b) $$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{6.50\ \text{N/m}}{0.542\ \text{kg}}} = 3.46\ \text{rad/s} \qquad \therefore T = \frac{2\pi}{\omega} = \frac{2\pi\ \text{rad}}{3.46\ \text{rad/s}} = \boxed{1.81\ \text{s}}$$

(c) $$a_{\max} = A\omega^2 = 0.100\ \text{m}(3.46\ \text{rad/s})^2 = \boxed{1.20\ \text{m/s}^2}$$

Example 15.4 Using Conservation of Energy

EXAMPLE 15.4 | **Using conservation of energy**

A 500 g block on a spring is pulled a distance of 20 cm and released. The subsequent oscillations are measured to have a period of 0.80 s.

a. At what position or positions is the block's speed 1.0 m/s?

b. What is the spring constant?

MODEL The motion is SHM. Energy is conserved.

Example 15.4 Using Conservation of Energy

EXAMPLE 15.4 **Using conservation of energy**

SOLVE a. The block starts from the point of maximum displacement, where $E = U = \frac{1}{2}kA^2$. At a later time, when the position is x and the speed is v, energy conservation requires

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2$$

Solving for x, we find

$$x = \sqrt{A^2 - \frac{mv^2}{k}} = \sqrt{A^2 - \left(\frac{v}{\omega}\right)^2}$$

where we used $k/m = \omega^2$ from Equation 15.24. The angular frequency is easily found from the period: $\omega = 2\pi/T = 7.85$ rad/s. Thus

$$x = \sqrt{(0.20\text{ m})^2 - \left(\frac{1.0\text{ m/s}}{7.85\text{ rad/s}}\right)^2} = \pm 0.15\text{ m} = \pm 15\text{ cm}$$

There are two positions because the block has this speed on either side of equilibrium.

Example 15.4 Using Conservation of Energy

EXAMPLE 15.4 | **Using conservation of energy**

SOLVE b. Although part a did not require that we know the spring constant, it is straightforward to find from Equation 15.24:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

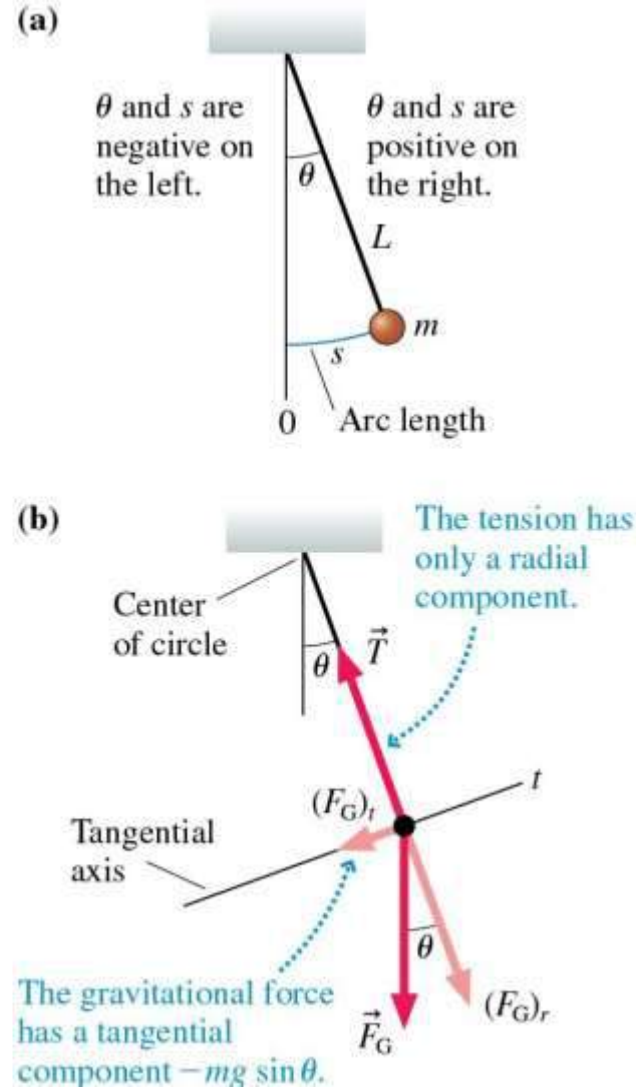
$$k = \frac{4\pi^2 m}{T^2} = \frac{4\pi^2(0.50\ \text{kg})}{(0.80\ \text{s})^2} = 31\ \text{N/m}$$

The Simple Pendulum

- Consider a mass m attached to a string of length L which is free to swing back and forth.
- If it is displaced from its lowest position by an angle θ, Newton's second law for the tangential component of gravity, parallel to the motion, is

$$(F_{\text{net}})_t = \sum F_t = (F_G)_t = -mg\sin\theta = ma_t$$

$$\frac{d^2s}{dt^2} = -g\sin\theta$$



The Simple Pendulum

- If we restrict the pendulum's oscillations to small angles ($< 10^{\circ}$), then we may use the **small angle approximation** $\sin\theta \approx \theta$, where θ is measured in radians.

$$(F_{\text{net}})_t = -mg\sin\theta \approx -mg\theta = -\frac{mg}{L}s$$

and the angular frequency of the motion is found to be

$$\omega = 2\pi f = \sqrt{\frac{g}{L}}$$

Example 15.7 The Maximum Angle of a Pendulum

EXAMPLE 15.7 | **The maximum angle of a pendulum**

A 300 g mass on a 30-cm-long string oscillates as a pendulum. It has a speed of 0.25 m/s as it passes through the lowest point. What maximum angle does the pendulum reach?

MODEL Assume that the angle remains small, in which case the motion is simple harmonic motion.

Example 15.7 The Maximum Angle of a Pendulum

EXAMPLE 15.7 **The maximum angle of a pendulum**

SOLVE The angular frequency of the pendulum is

$$\omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8\ \text{m/s}^2}{0.30\ \text{m}}} = 5.72\ \text{rad/s}$$

The speed at the lowest point is $v_{max} = \omega A$, so the amplitude is

$$A = s_{max} = \frac{v_{max}}{\omega} = \frac{0.25\ \text{m/s}}{5.72\ \text{rad/s}} = 0.0437\ \text{m}$$

The maximum angle, at the maximum arc length s_{max}, is

$$\theta_{max} = \frac{s_{max}}{L} = \frac{0.0437\ \text{m}}{0.30\ \text{m}} = 0.146\ \text{rad} = 8.3°$$

ASSESS Because the maximum angle is less than 10°, our analysis based on the small-angle approximation is reasonable.

Damped Oscillations

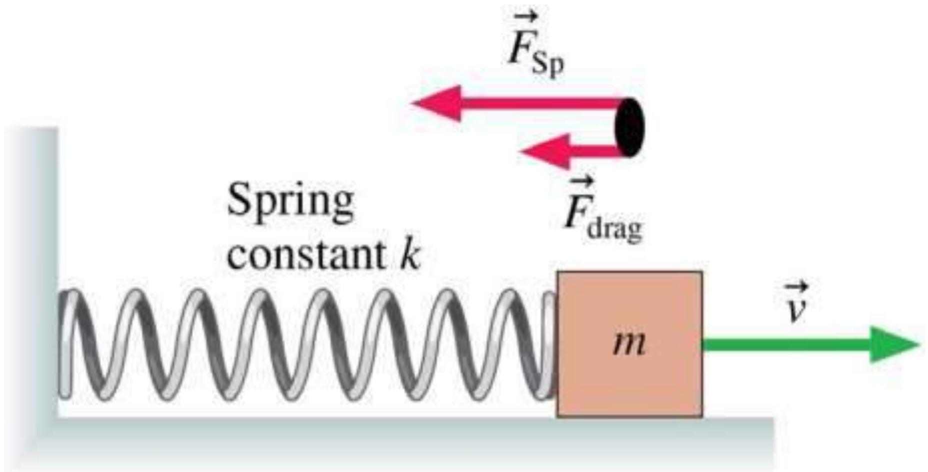
- An oscillation that runs down and stops is called a damped oscillation.
- The shock absorbers in cars and trucks are heavily damped springs.
- The vehicle's vertical motion, after hitting a rock or a pothole, is a damped oscillation.

- One possible reason for dissipation of energy is the drag force due to air resistance.
- The forces involved in dissipation are complex, but a simple *linear drag* model is

$$\vec{F}_{\text{drag}} = -b\vec{v} \quad \text{(model of the drag force)}$$

Damped Oscillations

- When a mass on a spring experiences the force of the spring as given by Hooke's Law, as well as a linear drag force of magnitude $|F_{\text{drag}}| = bv$, the solution is



$$x(t) = Ae^{-bt/2m}\cos(\omega t + \phi_0) \qquad \text{(damped oscillator)}$$

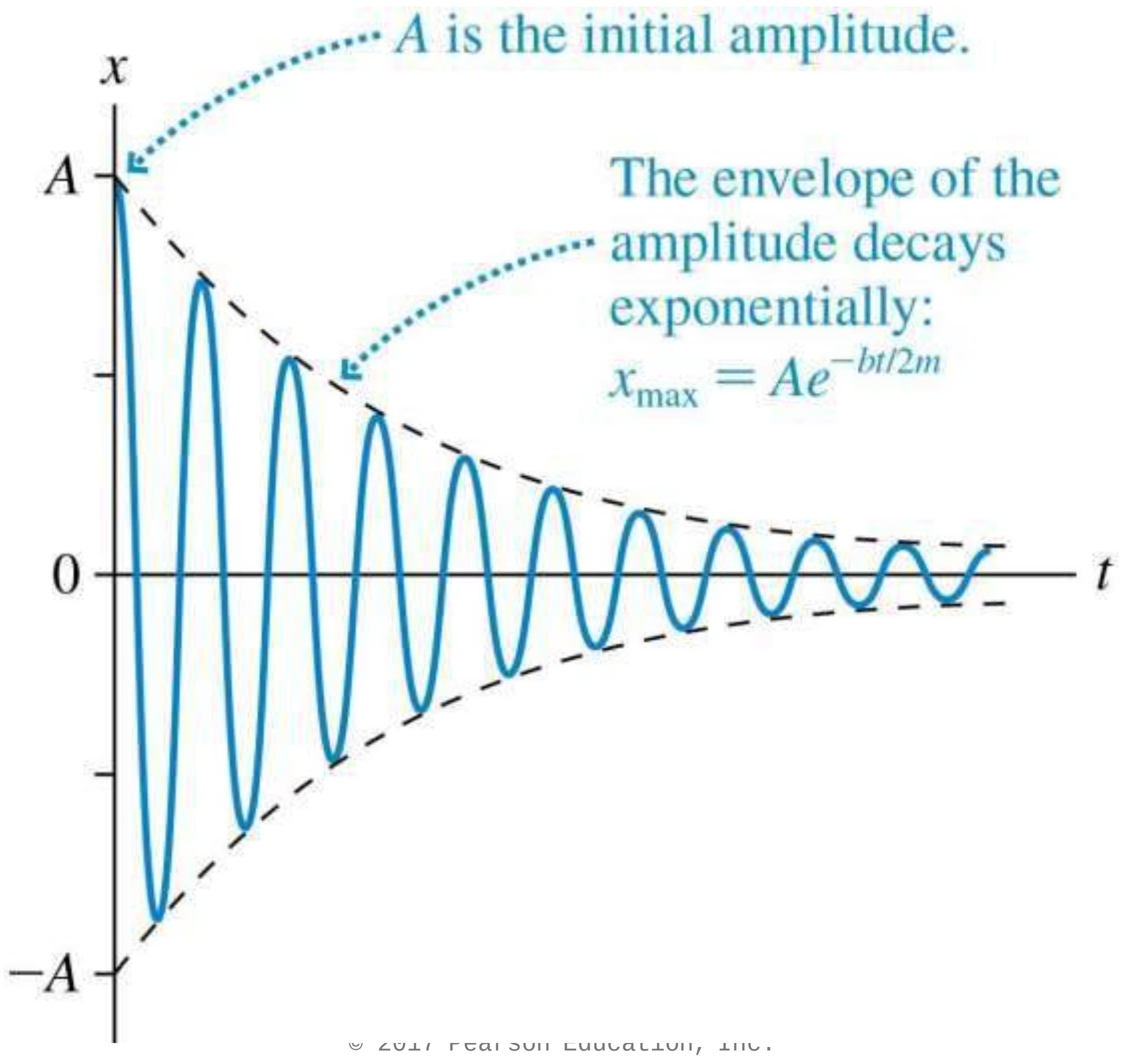
where the angular frequency is given by

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$$

- Here $\omega_0 = \sqrt{k/m}$ is the angular frequency of the undamped oscillator ($b = 0$).

Damped Oscillations

- Position-versus-time graph for a damped oscillator.

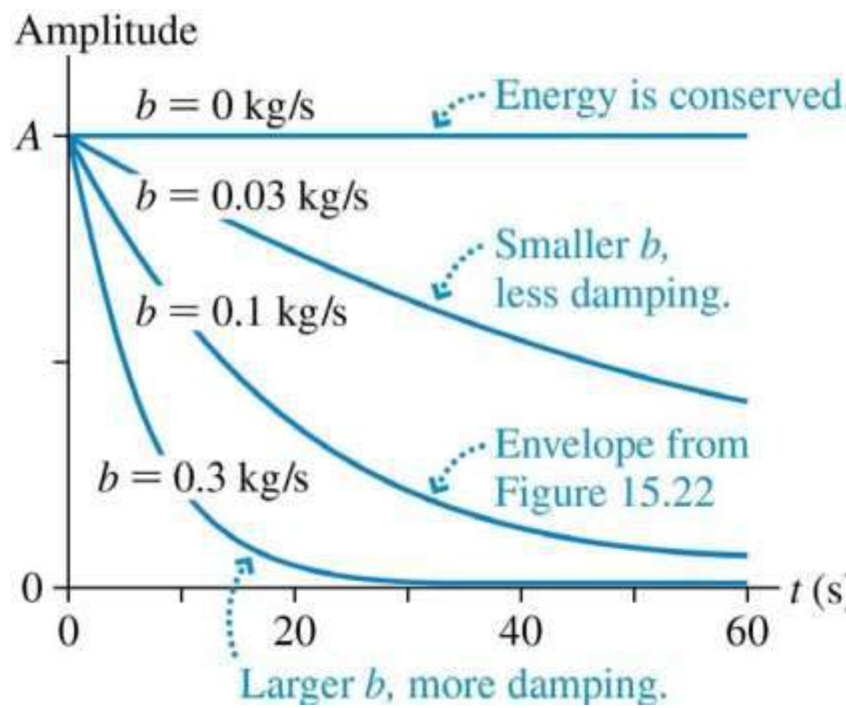


Damped Oscillations

- A damped oscillator has position $x = x_{max}\cos(\omega t + \phi_0)$, where

$$x_{max}(t) = Ae^{-bt/2m}$$

- This slowly changing function x_{max} provides a border to the rapid oscillations, and is called the **envelope**.
- The figure shows several oscillation envelopes, corresponding to different values of the damping constant b.

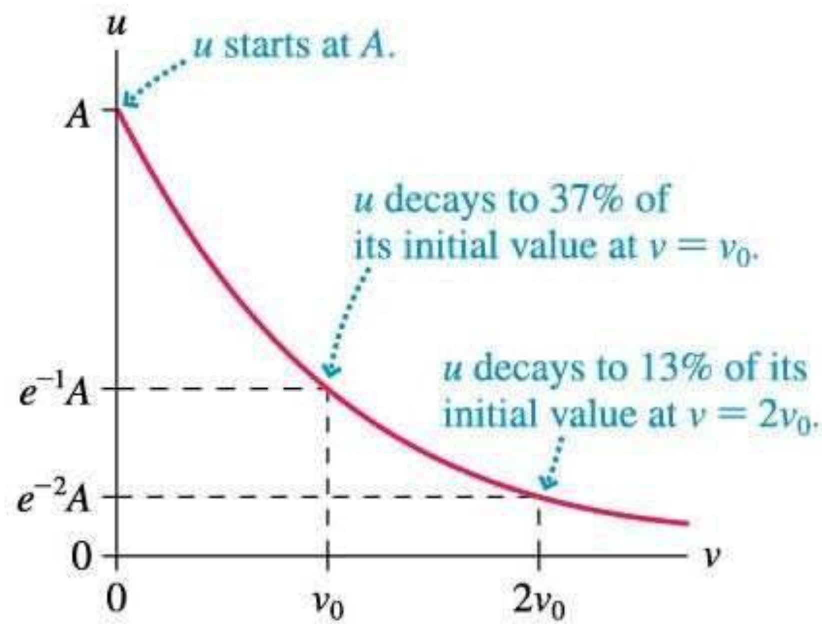


Mathematical Aside: Exponential Decay

- **Exponential decay** occurs in a vast number of physical systems of importance in science and engineering.
- Mechanical vibrations, electric circuits, and nuclear radioactivity all exhibit exponential decay.
- The graph shows the function:

$$u = Ae^{-v/v_0} = A\exp(-v/v_0)$$

where
  - $e = 2.71828\ldots$ is Euler's number.
  - exp is the *exponential function.*
  - v_0 is called the *decay constant.*



Energy in Damped Systems

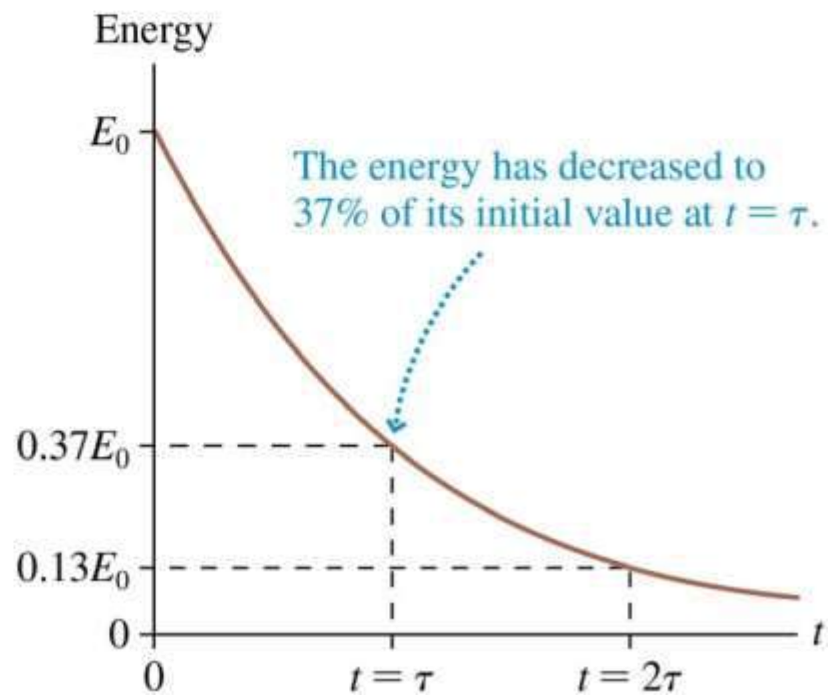
- Because of the drag force, the mechanical energy of a damped system is no longer conserved.
- At any particular time we can compute the mechanical energy from

$$E(t) = E_0 e^{-t/\tau}$$

- Where the decay constant of this function is called the **time constant** τ, defined as

$$\tau = \frac{m}{b}$$

- The oscillator's mechanical energy decays exponentially with time constant τ.



A 500 g mass swings on a 60-cm-string as a pendulum. The amplitude is observed to decay to half its initial value after 35.0 s.

a. What is the time constant for this oscillator?

b. At what time will the *energy* have decayed to half its initial value?

MODEL The motion is a damped oscillation.

SOLVE a. The initial amplitude at $t = 0$ is $x_{max} = A$. At $t = 35.0$ s the amplitude is $x_{max} = \frac{1}{2}A$. The amplitude of oscillation at time t is given by Equation 14.57:

$$x_{max}(t) = Ae^{-bt/2m} = Ae^{-t/2\tau}$$

In this case,

$$\frac{1}{2}A = Ae^{-(35.0\text{ s})/2\tau}$$

Notice that we do not need to know A itself because it cancels out. To solve for τ, we take the natural logarithm of both sides of the equation:

$$\ln\left(\frac{1}{2}\right) = -\ln 2 = \ln e^{-(35.0\text{ s})/2\tau} = -\frac{35.0\text{ s}}{2\tau}$$

This is easily rearranged to give

$$\tau = \frac{35.0\text{ s}}{2\ln 2} = 25.2\text{ s}$$

If desired, we could now determine the damping constant to be $b = m/\tau = 0.020$ kg/s.

$$E(\text{at } t = t_{1/2}) = \frac{1}{2}E_0 = E_0 e^{-t_{1/2}/\tau}$$

The E_0 cancels, giving

$$\frac{1}{2} = e^{-t_{1/2}/\tau}$$

Again, we take the natural logarithm of both sides:

$$\ln\left(\frac{1}{2}\right) = -\ln 2 = \ln e^{-t_{1/2}/\tau} = -t_{1/2}/\tau$$

Finally, we solve for $t_{1/2}$:

$$t_{1/2} = \tau \ln 2 = 0.693\tau$$

This result that $t_{1/2}$ is 69% of τ is valid for any exponential decay. In this particular problem, half the energy is gone at

$$t_{1/2} = (0.693)(25.2\ \text{s}) = 17.5\ \text{s}$$

The energy at time t is given by

$$E(t) = E_0 e^{-t/\tau}$$

Problem

•••24 In Fig. 15-35, two springs are joined and connected to a block of mass 0.245 kg that is set oscillating over a frictionless floor. The springs each have spring constant k = 6430 N/m. What is the frequency of the oscillations?

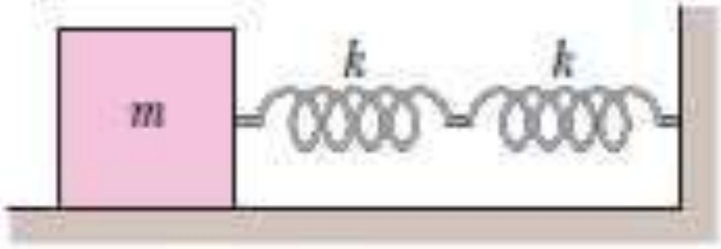


Figure 15-35 Problem 24.

24. We wish to find the effective spring constant for the combination of springs shown in the figure. We do this by finding the magnitude F of the force exerted on the mass when the total elongation of the springs is Δx. Then $k_{\text{eff}} = F/\Delta x$. Suppose the left-hand spring is elongated by Δx_ℓ and the right-hand spring is elongated by Δx_r. The left-hand spring exerts a force of magnitude $k\Delta x_\ell$ on the right-hand spring and the right-hand spring exerts a force of magnitude $k\Delta x_r$ on the left-hand spring. By Newton's third law these must be equal, so $\Delta x_\ell = \Delta x_r$. The two elongations must be the same, and the total elongation is twice the elongation of either spring: $\Delta x = 2\Delta x_\ell$. The left-hand spring exerts a force on the block and its magnitude is $F = k\Delta x_\ell$. Thus,

$$k_{\text{eff}} = k\Delta x_\ell / 2\Delta x_r = k/2.$$

The block behaves as if it were subject to the force of a single spring, with spring constant $k/2$. To find the frequency of its motion, replace k_{eff} in $f = (1/2\pi)\sqrt{k_{\text{eff}}/m}$ with $k/2$ to obtain

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{2m}}.$$

With $m = 0.245$ kg and $k = 6430$ N/m, the frequency is $f = 18.2$ Hz.

Problem

•••25 GO In Fig. 15-36, a block weighing 14.0 N, which can slide without friction on an incline at angle $\theta = 40.0°$, is connected to the top of the incline by a massless spring of unstretched length 0.450 m and spring constant 120 N/m. (a) How far from the top of the incline is the block's equilibrium point? (b)

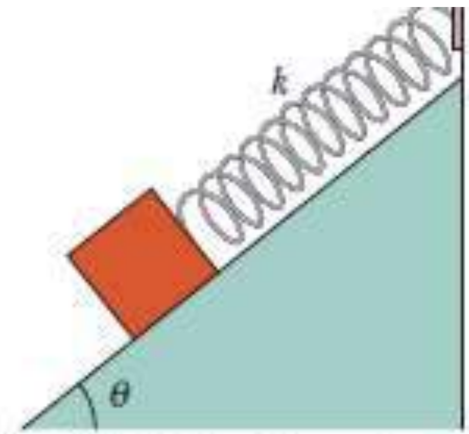


Figure 15-36 Problem 25.

If the block is pulled slightly down the incline and released, what is the period of the resulting oscillations?

Solution

25. (a) We interpret the problem as asking for the equilibrium position; that is, the block is gently lowered until forces balance (as opposed to being suddenly released and allowed to oscillate). If the amount the spring is stretched is x, then we examine force-components along the incline surface and find

$$kx = mg\sin\theta \;\Rightarrow\; x = \frac{mg\sin\theta}{k} = \frac{(14.0\text{ N})\sin 40.0°}{120\text{ N/m}} = 0.0750\text{ m}$$

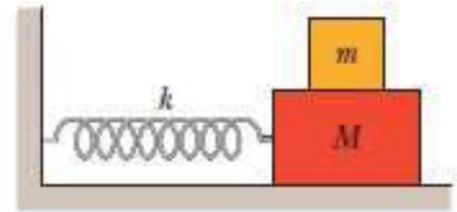
at equilibrium. The calculator is in degrees mode in the above calculation. The distance from the top of the incline is therefore (0.450 + 0.75) m = 0.525 m.

(b) Just as with a vertical spring, the effect of gravity (or one of its components) is simply to shift the equilibrium position; it does not change the characteristics (such as the period) of simple harmonic motion. Thus, Eq. 15-13 applies, and we obtain

$$T = 2\pi\sqrt{\frac{14.0\text{ N}/9.80\text{ m/s}^2}{120\text{ N/m}}} = 0.686\text{ s.}$$

Problem

••26 In Fig. 15-37, two blocks ($m = 1.8$ kg and $M = 10$ kg) and a spring ($k = 200$ N/m) are arranged on a horizontal, frictionless surface. The coefficient of static friction between the two blocks is 0.40. What amplitude of simple harmonic motion of the spring–blocks system puts the smaller block on the verge of slipping over the larger block?



Solution

26. To be on the verge of slipping means that the force exerted on the smaller block (at the point of maximum acceleration) is $f_{\max} = \mu_s mg$. The textbook notes (in the discussion immediately after Eq. 15-7) that the acceleration amplitude is $a_m = \omega^2 x_m$, where $\omega = \sqrt{k/(m+M)}$ is the angular frequency (from Eq. 15-12). Therefore, using Newton's second law, we have

$$ma_m = \mu_s mg \Rightarrow \frac{k}{m+M} x_m = \mu_s g$$

which leads to

$$x_m = \frac{\mu_s g(m+M)}{k} = \frac{(0.40)(9.8\ \text{m/s}^2)(1.8\ \text{kg} + 10\ \text{kg})}{200\ \text{N/m}} = 0.23\ \text{m} = 23\ \text{cm}.$$