Q1: Alaina wants to get to the bus stop as quickly as possible. The bus stop is across a grassy park, 2000 feet west and 600 feet north of her starting position. Alaina can walk west along the edge of the park on the sidewalk at a speed of 6 ft/sec. She can also travel through the grass in the park, but only at a rate of 4 ft/sec. Construct a function of single variable that **describes** the total time of the trip and state the realistic domain. **[10 marks]**

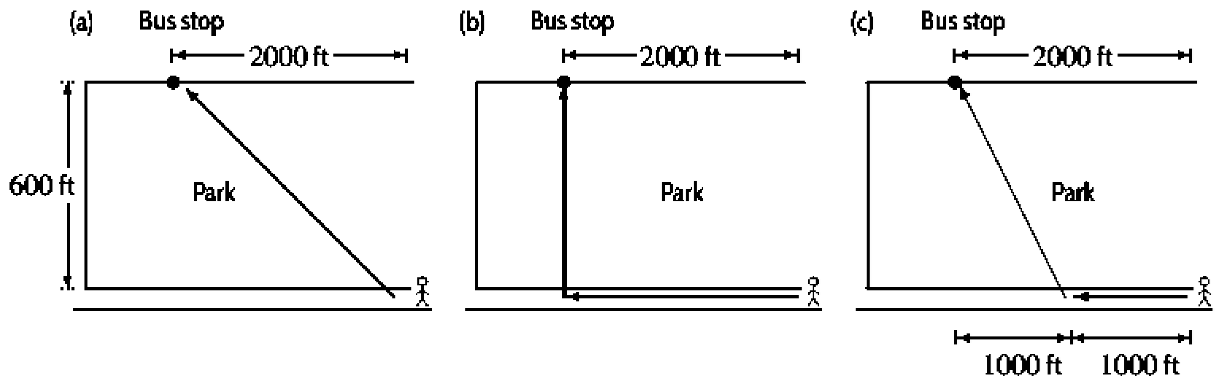


Figure Three possible paths to the bus stop

We might first think that she should take a path that is the shortest distance. Unfortunately, the path that follows the shortest distance to the bus stop is entirely in the park, where her speed is slow. (See Figure 4.34(a).) That distance is $\sqrt{2000^2 + 600^2} = 2088$ feet, which takes her about 522 seconds to traverse. She could instead walk quickly the entire 2000 feet along the sidewalk, which leaves her just the 600-foot northward journey through the park. (See Figure) This method would take $2000/6 + 600/4 \approx 483$ seconds total walking time.

But can she do even better? Perhaps another combination of sidewalk and park gives a shorter travel time. For example, what is the travel time if she walks 1000 feet west along the sidewalk and the rest of the way through the park? (See Figure) The answer is about 458 seconds.

We make a model for this problem. We label the distance that Alaina walks west along the sidewalk x and the distance she walks through the park y, as in Figure 4.35. Then the total time, t, is

$$t = t_{\text{sidewalk}} + t_{\text{park}}.$$

Since

$$\text{Time} = \text{Distance/Speed},$$

and she can walk 6 ft/sec on the sidewalk and 4 ft/sec in the park, we have

$$t = \frac{x}{6} + \frac{y}{4}.$$

Now, by the Pythagorean Theorem, $y = \sqrt{(2000 - x)^2 + 600^2}$. Therefore

$$t = \frac{x}{6} + \frac{\sqrt{(2000 - x)^2 + 600^2}}{4} \qquad \text{for } 0 \le x \le 2000.$$

Q2: National Highways Authority (NHA) charges tolls (in dollars) for vehicles based on the distance x (in kilometres) traveled on the tolled network:

- For the first 10km the toll is a flat \$5.
- For the portion above 10km and up to 50km, the toll is \$0.20 per km.
- For the portion above 50km and up to 200km, the toll is \$0.15 per km.
- For any distance beyond 200km, the toll is \$0.10 per km.

Let $T(x)$ be the total toll for traveling x km.

a) **State** a piecewise function $T(x)$.
b) **State** the domain of T.
c) **State** the average toll per km, $A(x) = T(x)/x$
d) **State** $\lim_{x\to\infty} A(x)$ and interpret it. **[09 marks]**

(a) Build $T(x)$ from the description.

1. For $0 < x \le 10$: base flat toll

$$T(x) = 5.$$

2. For $10 < x \le 50$: base \$5 plus \$0.20 per km over 10:

$$T(x) = 5 + 0.20(x - 10).$$

3. For $50 < x \le 200$: include charge from 10→50 first. Extra from 10→50 is $0.20(40) = 8$, so cost up to 50 is $5 + 8 = 13$. For $x > 50$ add \$0.15 per km over 50:

$$T(x) = 13 + 0.15(x - 50).$$

4. For $x > 200$: cost up to 200 is $13 + 0.15(150) = 13 + 22.5 = 35.5$. For $x > 200$ add \$0.10 per km over 200:

$$T(x) = 35.5 + 0.10(x - 200).$$

$$T(x) = \begin{cases} 5, & 0 < x \le 10, \\ 0.20x + 3, & 10 < x \le 50, \\ 0.15x + 5.5, & 50 < x \le 200, \\ 0.10x + 15.5, & x > 200. \end{cases}$$

(b) Domain.

Distance must be positive (you could include $x = 0$ with toll 0 if you prefer). Using the given scheme:

$$\text{Domain } = (0, \infty).$$

(c) Average toll per km

Define $A(x) = T(x)/x$. Writing the piecewise form:

$$A(x) = \begin{cases} \dfrac{5}{x}, & 0 < x \le 10, \\ \dfrac{5 + 0.20(x - 10)}{x}, & 10 < x \le 50, \\ \dfrac{13 + 0.15(x - 50)}{x}, & 50 < x \le 200, \\ \dfrac{35.5 + 0.10(x - 200)}{x}, & x > 200. \end{cases}$$

You can simplify the middle expressions if you like:

- For $10 < x \le 50$: $5 + 0.20(x - 10) = 0.20x + 3$ so $A(x) = \dfrac{0.20x + 3}{x} = 0.20 + \dfrac{3}{x}$.
- For $50 < x \le 200$: $13 + 0.15(x - 50) = 0.15x + 5.5$ so $A(x) = 0.15 + \dfrac{5.5}{x}$.
- For $x > 200$: $35.5 + 0.10(x - 200) = 0.10x + 15.5$ so $A(x) = 0.10 + \dfrac{15.5}{x}$.

(d) Limit at infinity

$$\lim_{x \to \infty} A(x) = 0.10.$$

Interpretation: For very long trips, the average toll per kilometre approaches **$0.10/km**, i.e., the asymptotic per-km rate for long-distance travel.

Q3: Graph the function $y = f(x)$, **State** where do the graphs appear to have vertical tangents? Confirm your findings with limit calculations.

$$y = \begin{cases} -\sqrt{|x+2|}, & x \leq -2 \\ \sqrt{x+2}, & x > -2 \end{cases}$$

[07 marks]

At $x = -2$,

From left:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{-\sqrt{|-2+h+2|} - (-\sqrt{|-2+2|})}{h}$$

$$\lim_{h\to 0} \frac{-\sqrt{|h|}}{h}$$

$$\lim_{h\to 0} \frac{-1}{-\sqrt{h}} = +\infty$$

From right:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{-2+h+2} - (\sqrt{|-2+2|})}{h}$$

$$\lim_{h\to 0} \frac{\sqrt{h}}{h}$$

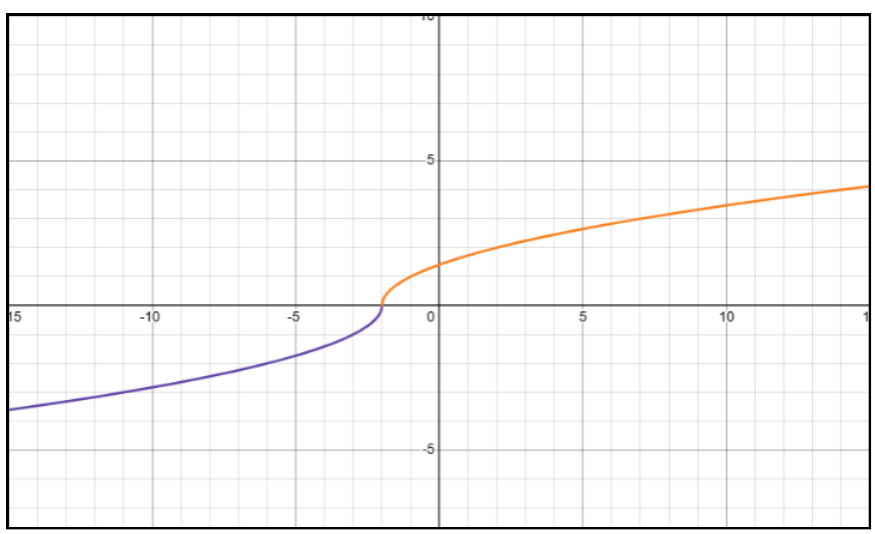
$$\lim_{h\to 0} \frac{1}{\sqrt{h}} = +\infty$$

Since,

Left hand derivative $\rightarrow +\infty$.

Right hand derivative $\rightarrow +\infty$.

Therefore, the graph has a vertical tangent at $x = -2$.



Q4: In a video game, the trajectory of a projectile is initially modelled by the function $y = -|x|$. To adjust the animation, the curve undergoes several transformations. First, it is shifted 2 units to the left. Next, the curve is stretched vertically by a factor of 2 to make the path steeper. After this modification, the entire trajectory is translated 5 units upward, and finally, the curve is reflected across the x-axis. **State** the equation of the final transformed graph and draw the graph at each step of the applied transformation. **[07 Marks]**

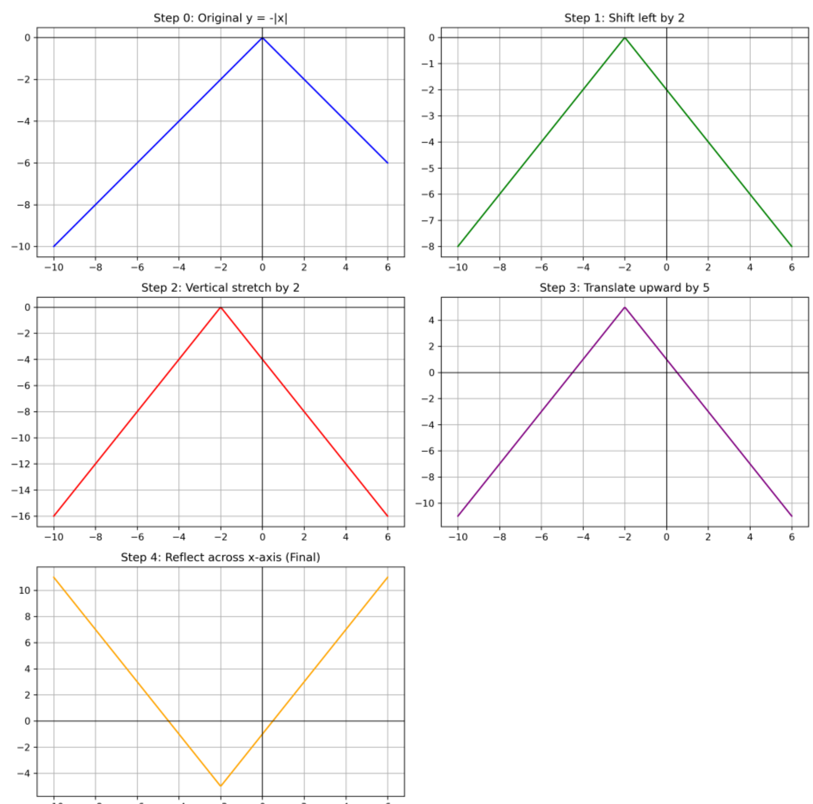
Transformations applied in order:

1. Shift 2 units to the left:
$$y(x) = -|x+2|.$$
2. Vertical stretch by factor 2:
$$y(x) = 2 \cdot y_1(x) = -2|x+2|.$$
3. Translate 5 units upward:
$$y(x) = -2|x+2| + 5.$$
4. Reflect across the x-axis:
$$y(x) = -\big(-2|x+2| + 5\big) = 2|x+2| - 5.$$

Therefore the final transformed graph has equation

$$\boxed{y = 2\,|x+2| - 5}$$

This is an upward-opening "V" with vertex at $(-2, -5)$.



Q5: When comparing algorithms, the decisive factor is how their running times behave as input grows without bound. Suppose two algorithms for searching have time complexity given as

$$A(n) = 3n\log n + 50n \quad ; \quad B(n) = n^2 + 5n$$

Use limits to **calculate** which algorithm is more efficient for very large input size. Show your work clearly. **[06 marks]**

Evaluate

$$\lim_{n\to\infty} \frac{A(n)}{B(n)} = \lim_{n\to\infty} \frac{3n\ln n + 50n}{n^2 + 5n}.$$

Divide numerator and denominator by n^2:

$$\frac{A(n)}{B(n)} = \frac{3\frac{\ln n}{n} + \frac{50}{n}}{1 + \frac{5}{n}}.$$

As $n \to \infty$, $\frac{\ln n}{n} \to 0$ and $\frac{50}{n} \to 0$, so the whole fraction tends to 0. Therefore

$$\boxed{\lim_{n\to\infty} \frac{A(n)}{B(n)} = 0.}$$

Interpretation: $A(n)$ grows strictly slower than $B(n)$; in notation, $A(n) = o(B(n))$.

Q6: During a rocket launch simulation, the stability ratio of the rocket's thrust to its structural load is modelled by

$$f(x) = \frac{7 - 3x}{\sqrt{16x^2 - 3}}$$

where x represents the scaled altitude factor.

As the altitude increases positively or negatively (due to simulation mirroring), engineers need to determine long-term stability. State the equation of all horizontal asymptotes of the stability ratio by **calculating** the limits of $f(x)$ as $x \to \infty$ and $x \to -\infty$. **[06 Marks]**

Limit as $x \to \infty$

For $x > 0$ we can write

$$\sqrt{16x^2 - 3} = x\sqrt{16 - \frac{3}{x^2}}.$$

Hence

$$f(x) = \frac{7 - 3x}{\sqrt{16x^2 - 3}} = \frac{7/x - 3}{\sqrt{16 - 3/x^2}}.$$

Taking the limit $x \to \infty$ (so $1/x \to 0$) gives

$$\lim_{x \to \infty} f(x) = \frac{-3}{\sqrt{16}} = -\frac{3}{4}.$$

Limit as $x \to -\infty$

For $x < 0$ we use $|x| = -x$, therefore

$$\sqrt{16x^2 - 3} = |x|\sqrt{16 - \frac{3}{x^2}} = -x\sqrt{16 - \frac{3}{x^2}}.$$

So

$$f(x) = \frac{7 - 3x}{-x\sqrt{16 - 3/x^2}} = \frac{-7 + 3x}{x\sqrt{16 - 3/x^2}} = \frac{-7/x + 3}{\sqrt{16 - 3/x^2}}.$$

As $x \to -\infty$ we again have $1/x \to 0$, hence

$$\lim_{x \to -\infty} f(x) = \frac{3}{\sqrt{16}} = \frac{3}{4}.$$

Therefore the horizontal asymptotes are

$$\boxed{y = -\tfrac{3}{4} \text{ as } x \to \infty, \qquad y = \tfrac{3}{4} \text{ as } x \to -\infty}.$$