CLO 1

Use knowledge of scalars and vectors quantities along with operation of basic operators on it to help them in computer graphics

Chapter 3. Vector

1. Vectors and scalar by definition

2.Representation of vectors

3.Adding Vectors Geometrically

4. Components of Vectors

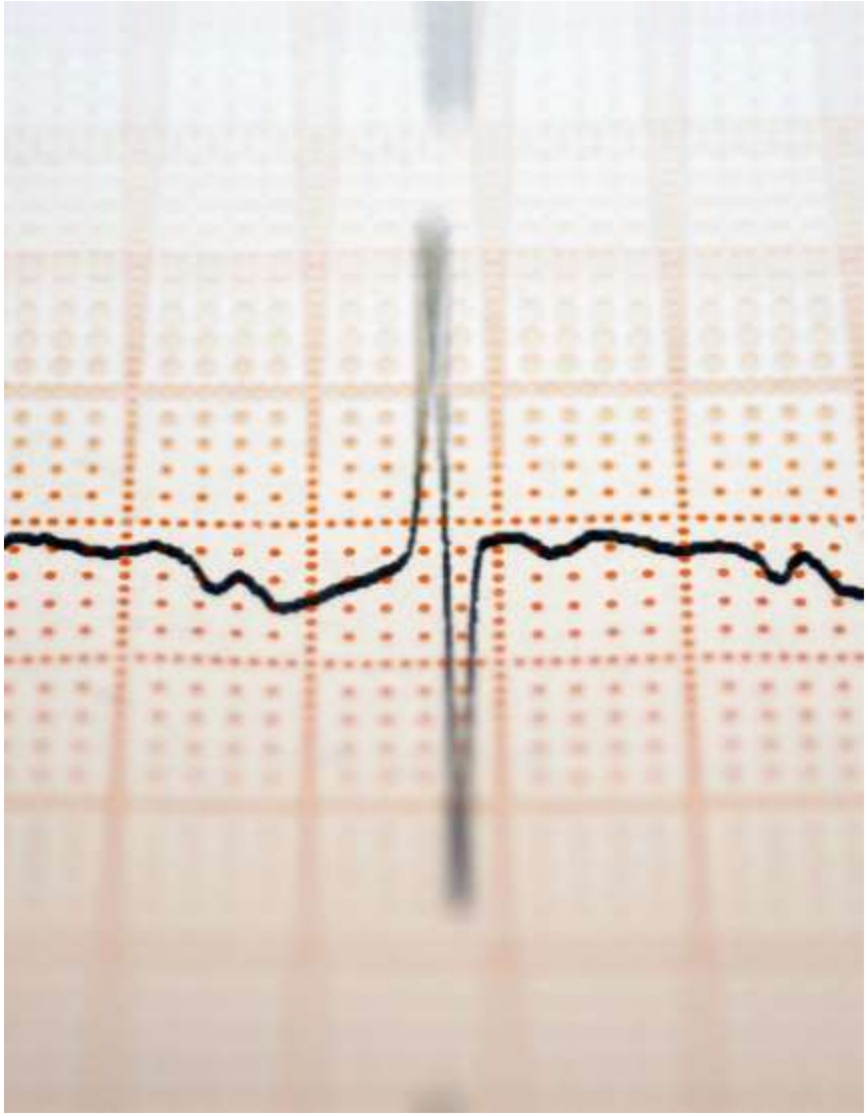
5. Unit Vectors

6. Adding Vectors by Components

7. Multiplying Vectors

Vectors and scalars

- A scalar quantity is defined as the physical quantity that has only magnitude, for example, mass and electric charge. On the other hand, a vector quantity is defined as **the physical quantity that has both magnitude as well as direction like force and weight**.



Difference between scalars and vectors

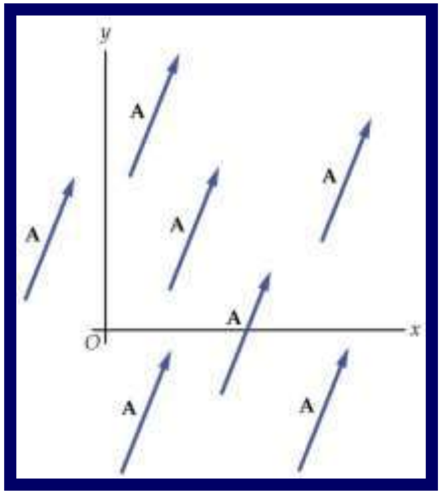
Aspect	Scalars	Vectors
Definition	Quantities with only magnitude	Quantities with both magnitude and direction
Example	Temperature (30°C), Mass (5 kg)	Velocity (30 km/h north), Force (10 N to the right)
Components	Single value	Multiple values (e.g., x, y, z components)
Mathematical Operations	Can be added, subtracted, multiplied, divided	Can be added, subtracted, and scaled; also have dot and cross products
Representation	Represented as a single number	Represented as an arrow or coordinate pair/triplet
Usage in Physics	Used to describe magnitude-only quantities like speed	Used to describe quantities with direction like force and velocity
Graphical Representation	Not typically visualized as an arrow	Typically visualized as an arrow with direction and length

Application in CS

- Scalars and vectors are fundamental mathematical concepts that play a crucial role in computer science. Understanding their properties and applications is essential for developing efficient and effective algorithms and systems in various domains, including graphics, game development, machine learning, and data analysis. By mastering these concepts, computer scientists can unlock new possibilities and create innovative solutions to complex problems.

Representation of vector

- The bold font: Vector A is **A**
- Or an **arrow** above the vector: $\vec{A}$
- In the pictures, we will always show vectors as arrows
- Arrows point the direction
- To describe the magnitude of a vector we will use absolute value sign: $\left|\vec{A}\right|$ or just A,
- Magnitude is always positive, the magnitude of a vector is equal to the length of a vector.



Vector representations

- Vectors can be represented in several ways depending on the context. Here are the main representations:

1. **Graphical Representation:**
    1. **Arrow:** A vector is often visualized as an arrow, where the length represents the magnitude and the direction represents the vector's direction. The arrow starts from an initial point and points to a terminal point.

2. Coordinate Form:

- **2D Vectors:** Represented as $\mathbf{v} = (x, y)$, where x and y are the components along the x and y axes, respectively.
- **3D Vectors:** Represented as $\mathbf{v} = (x, y, z)$, where x, y, and z are the components along the x, y, and z axes.

3. Component Form:

- **2D Vector:** $\mathbf{v} = x\hat{i} + y\hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors in the x and y directions.
- **3D Vector:** $\mathbf{v} = x\hat{i} + y\hat{j} + z\hat{k}$, where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors in the x, y, and z directions

4. Magnitude and Direction:

- **Polar Coordinates (2D):** Represented by magnitude r and angle θ, often written as $\mathbf{v} = (r, \theta)$.
- **Spherical Coordinates (3D):** Represented by magnitude r, polar angle θ, and azimuthal angle ϕ, often written as $\mathbf{v} = (r, \theta, \phi)$.

5. Matrix Form:

- **Column Vector:** Represented as a column matrix. For example, a 3D vector $\mathbf{v}$ can be represented as:

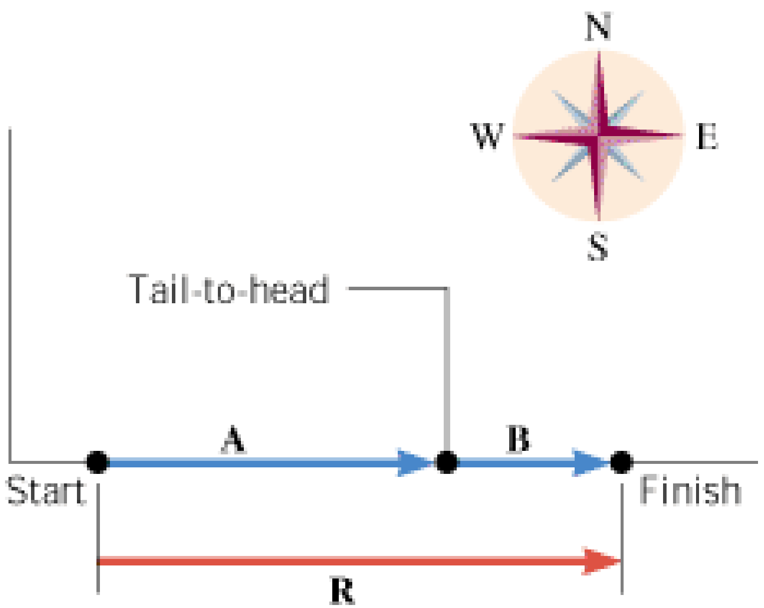
$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

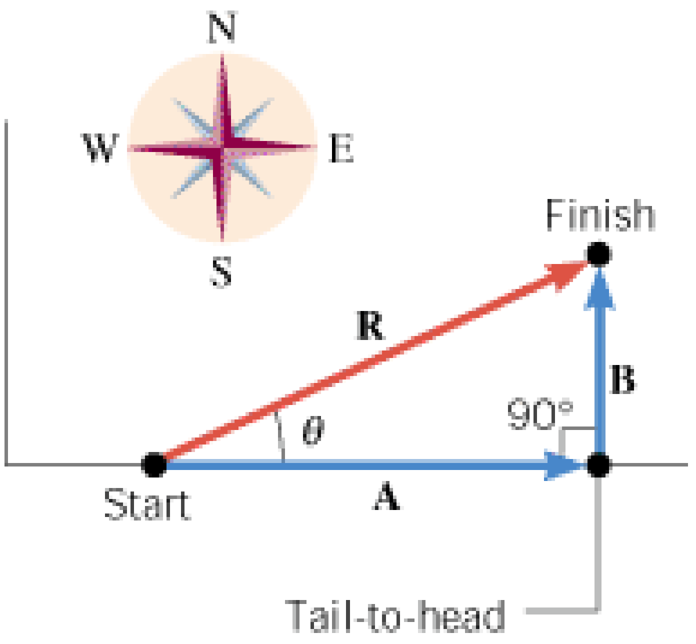
Addition of vectors

GRAPHICAL ADDITION OF VECTORS

ALGEBRAIC ADDITION OF VECTORS

Examples


N
W
E
S
Tail-to-head
A
B
Start
Finish
R


N
W
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Finish
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90°
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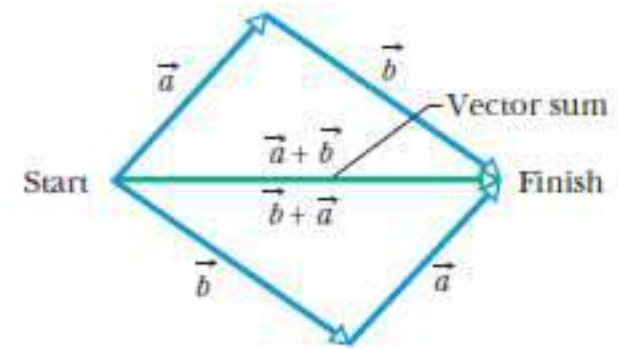
Two important properties of vector additions

(1) Commutative law:

$$\vec{a}+\vec{b}=\vec{b}+\vec{a}$$

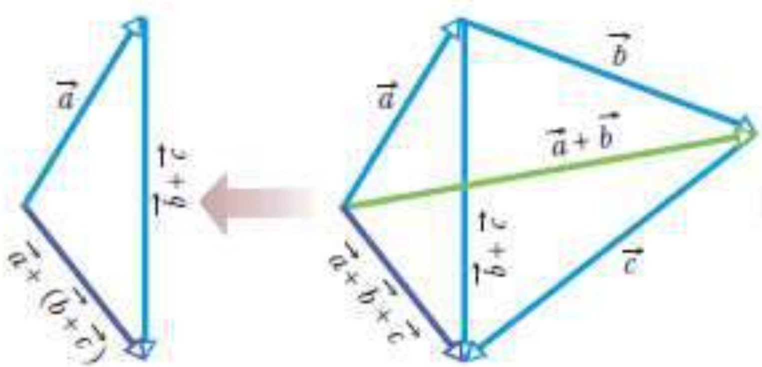


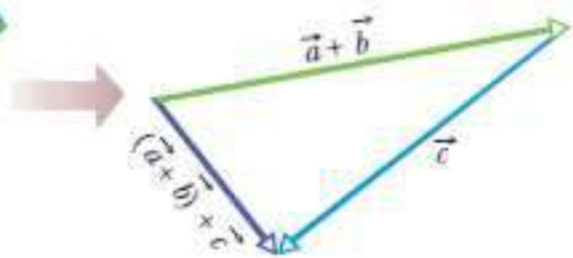
You get the same vect
result for either order o
adding vectors.

(2) Associative law:

$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$$

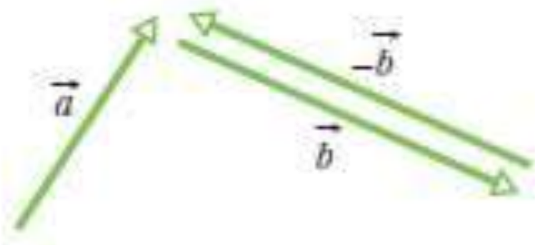
You get the same vector result for any order of adding the vectors.



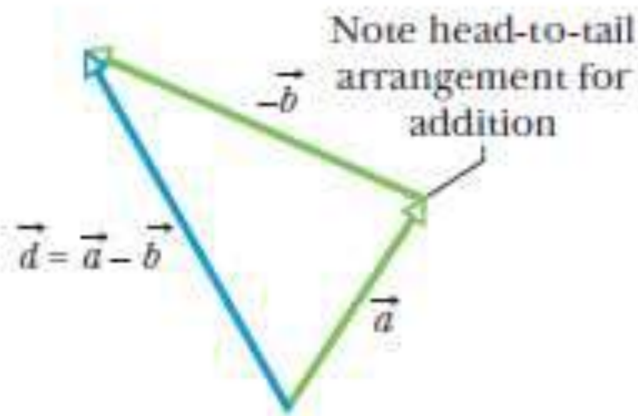


Subtraction of Vectors

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$



(*a*)

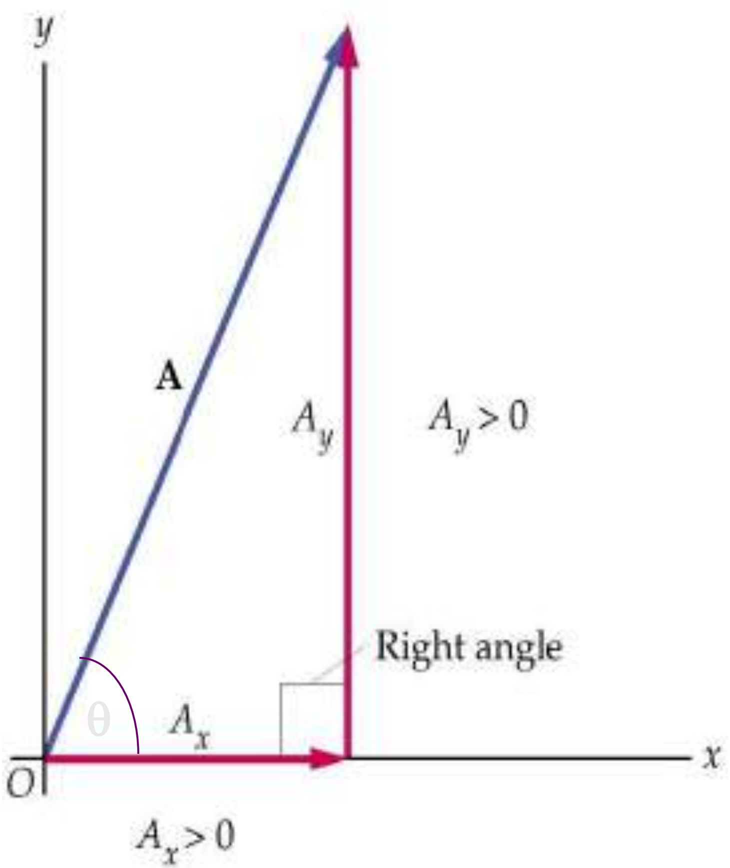


(*b*)

Resolution of vector

- The resolution of a vector involves breaking it down into its components along specified axes. This process helps in analyzing vectors in different directions and is crucial for solving problems in physics and engineering.
- **Applications:**
- **Physics:** For analyzing forces and motion in different directions.
- **Engineering:** To break down forces acting on structures.
- **Computer Graphics:** To determine movement and positioning in simulations.

Components of a Vector/Resolution



- The x-component of a vector is the projection along the x-axis

$$\cos\theta = \frac{A_x}{A} \qquad A_x = A\cos\theta$$

- The y-component of a vector is the projection along the y-axis

$$\sin\theta = \frac{A_y}{A} \qquad A_y = A\sin\theta$$

$$\vec{A} = \vec{A}_x + \vec{A}_y$$

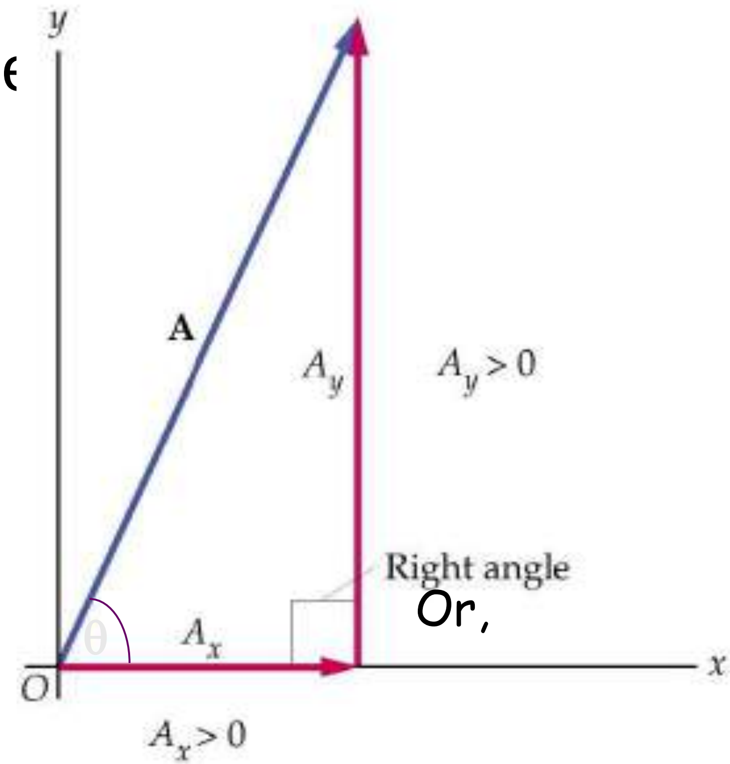
- $\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y$

More About Components

- The components are the legs of the right triangle whose hypotenuse is *A*

$$\begin{cases} A_x = A\cos(\theta) \\ A_y = A\sin(\theta) \end{cases}$$

$$\begin{cases} \left|\vec{A}\right| = \sqrt{(A_x)^2 + (A_y)^2} \\ \tan(\theta) = \dfrac{A_y}{A_x} \quad \text{or} \quad \theta = \tan^{-1}\left(\dfrac{A_y}{A_x}\right) \end{cases}$$

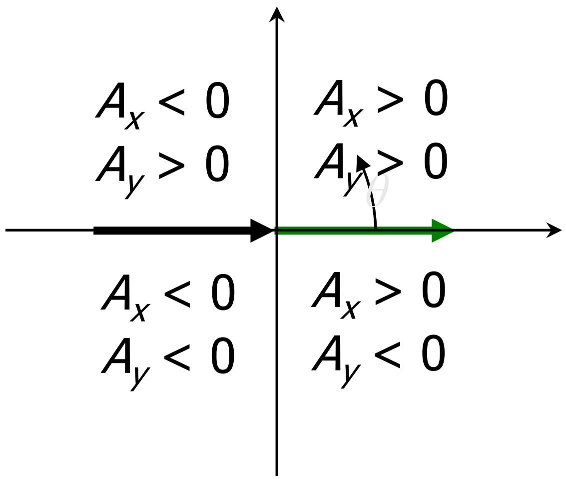


Or,

$$A = \sqrt{A_x^2 + A_y^2} \quad \textit{and} \quad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

Components of a Vector

- The previous equations are valid ***only if θ is measured with respect to the x-axis***
- The components can be positive or negative and will have the same units as the original vector



$$\theta = 0,\ A_x = A > 0,\ A_y = 0$$

$$\theta = 45^\circ,\ A_x = A\cos 45^\circ > 0,\ A_y = A\sin 45^\circ > 0$$

$$\theta = 90^\circ,\ A_x = 0,\ A_y = A > 0$$

$$\theta = 135^\circ,\ A_x = A\cos 135^\circ < 0,\ A_y = A\sin 135^\circ > 0$$

$$\theta = 180^\circ,\ A_x = -A < 0,\ A_y = 0$$

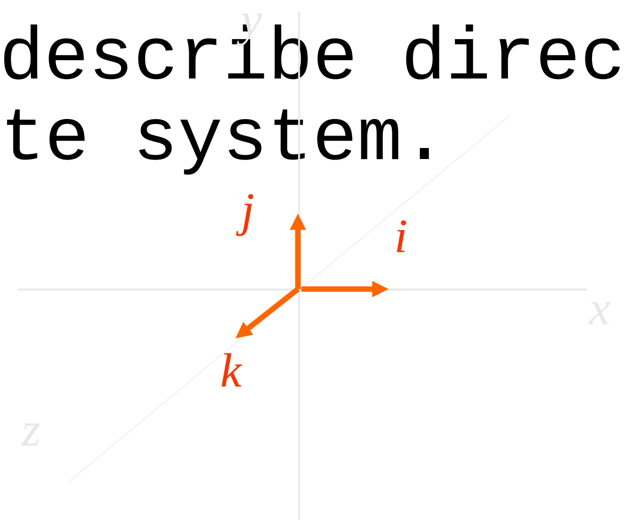
$$\theta = 225^\circ,\ A_x = A\cos 225^\circ < 0,\ A_y = A\sin 225^\circ < 0$$

$$\theta = 270^\circ,\ A_x = 0,\ A_y = -A < 0$$

$$\theta = 315^\circ,\ A_x = A\cos 315^\circ < 0,\ A_y = A\sin 315^\circ < 0$$

Unit vectors

- A unit vector is a vector with a magnitude of 1. It is used to specify a direction without changing the length or magnitude of the vector. Unit vectors are often used to describe directions in a coordinate system.



Applications:

- **Direction Specification:** Used to indicate direction in calculations without affecting the magnitude.
- **Normal Vectors:** In computer graphics and physics, unit vectors are used as normals to surfaces.
- **Directional Cosines:** In vector analysis, unit vectors are used to compute the angles between vectors.

Properties of Unit Vectors

1. **Magnitude:**
   - A unit vector has a magnitude of 1.
   - Mathematically: $|\mathbf{u}| = 1$, where $\mathbf{u}$ is the unit vector.
2. **Direction:**
   - Unit vectors indicate direction but do not have a specific magnitude other
3. **Representation:**
   - **In 2D:** Unit vectors are often represented as $\hat{i}$ and $\hat{j}$, where:

$$\hat{i} = (1, 0) \quad \text{(along the x-axis)}$$

$$\hat{j} = (0, 1) \quad \text{(along the y-axis)}$$

- **In 3D**: Unit vectors are represented as $\hat{i}$, $\hat{j}$, and $\hat{k}$, where:

$$\hat{i} = (1, 0, 0) \quad \text{(along the x-axis)}$$

$$\hat{j} = (0, 1, 0) \quad \text{(along the y-axis)}$$

$$\hat{k} = (0, 0, 1) \quad \text{(along the z-axis)}$$

Example:

For a vector $\mathbf{v} = (3, 4)$:

1. **Calculate the Magnitude:**

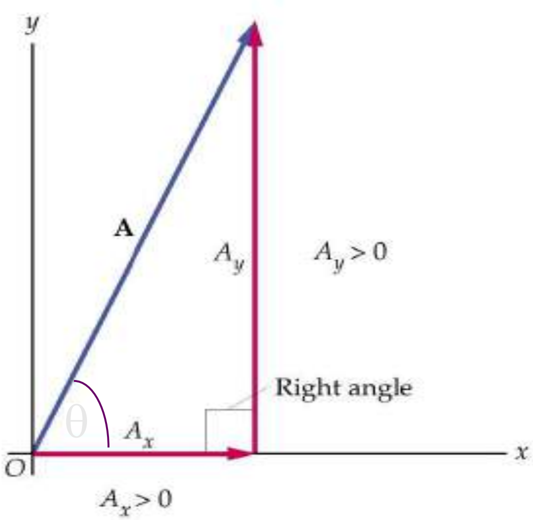
$$|\mathbf{v}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$

2. **Find the Unit Vector:**

$$\hat{v} = \left(\frac{3}{5}, \frac{4}{5}\right) = (0.6, 0.8)$$

The unit vector $\hat{v}$ points in the same direction as $\mathbf{v}$ but has a magnitude of 1.

Unit Vectors



- Components of a vector are vectors $\vec{A} = \vec{A}_x + \vec{A}_y$
- Unit vectors *i*-hat, *j*-hat, *k*-hat $\hat{i} \to x \quad \hat{j} \to y \quad \hat{k} \to z$
- Unit vectors used to specify direction
- Unit vectors have a magnitude of 1
- Then $\vec{A} = A_x \hat{i} + A_y \hat{j}$

Magnitude + Sign

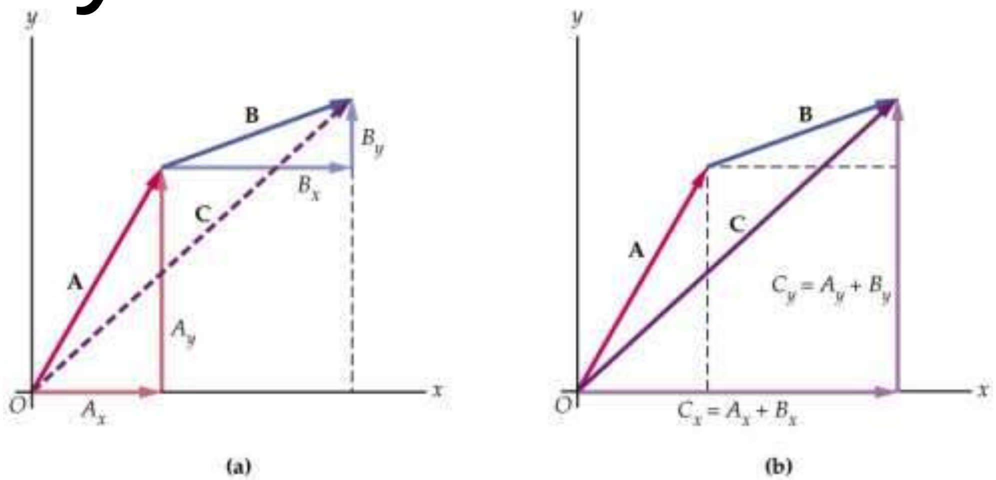
Unit vector

Adding Vectors Algebraically

- Consider two vectors

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j}$$



- Then

$$\vec{A} + \vec{B} = (A_x \hat{i} + A_y \hat{j}) + (B_x \hat{i} + B_y \hat{j})$$

$$= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

- If

$$\vec{C} = \vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

- so

$$C_x = A_x + B_x \qquad C_y = A_y + B_y$$

EXAMPLE 3.3 | Finding the components of an acceleration vector

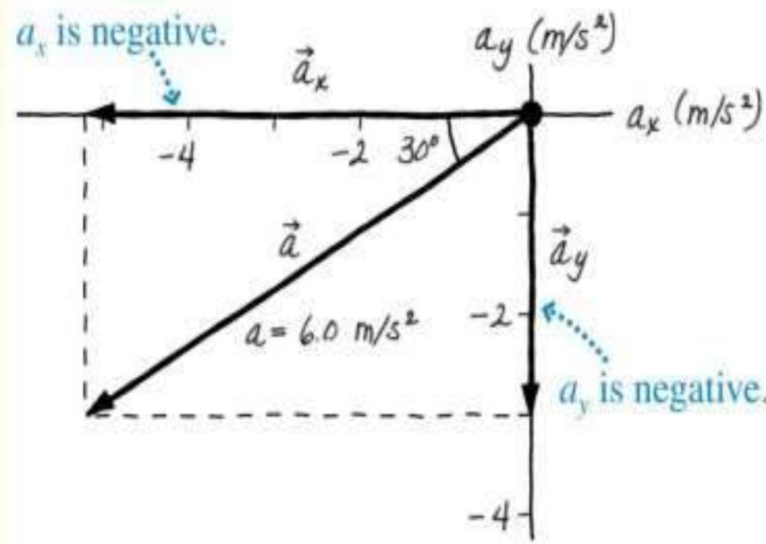
Seen from above, a hummingbird's acceleration is ($6.0\ \mathrm{m/s^2}$, 30° south of west). Find the x- and y-components of the acceleration vector $\vec{a}$.

VISUALIZE It's important to *draw* vectors. **FIGURE 3.13** establishes a map-like coordinate system with the x-axis pointing east and the y-axis north. Vector $\vec{a}$ is then decomposed into components parallel to the axes. Notice that the axes are "acceleration axes" with units of acceleration, not xy-axes, because we're measuring an acceleration vector.

SOLVE The acceleration vector points to the left (negative x-direction) and down (negative y-direction), so the components a_x and a_y are both negative:

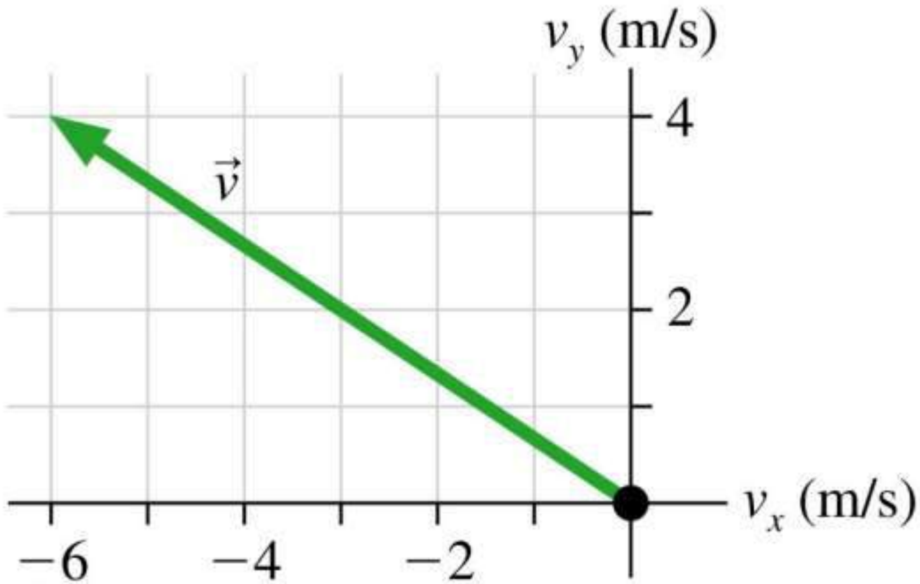
$$a_x = -a\cos 30^\circ = -(6.0\ \mathrm{m/s^2})\cos 30^\circ = -5.2\ \mathrm{m/s^2}$$
$$a_y = -a\sin 30^\circ = -(6.0\ \mathrm{m/s^2})\sin 30^\circ = -3.0\ \mathrm{m/s^2}$$

ASSESS The units of a_x and a_y are the same as the units of vector $\vec{a}$. Notice that we had to insert the minus signs manually by observing that the vector points left and down.



EXAMPLE 3.4 | **Finding the direction of motion**

FIGURE 3.14 shows a car's velocity vector $\vec{v}$. Determine the car's speed and direction of motion.



EXAMPLE 3.4 Finding the direction of motion

VISUALIZE FIGURE 3.15 shows the components v_x and v_y and defines an angle θ with which we can specify the direction of motion.

SOLVE We can read the components of $\vec{v}$ directly from the axes: $v_x = -6.0$ m/s and $v_y = 4.0$ m/s. Notice that v_x is negative. This is enough information to find the car's speed v, which is the magnitude of $\vec{v}$:

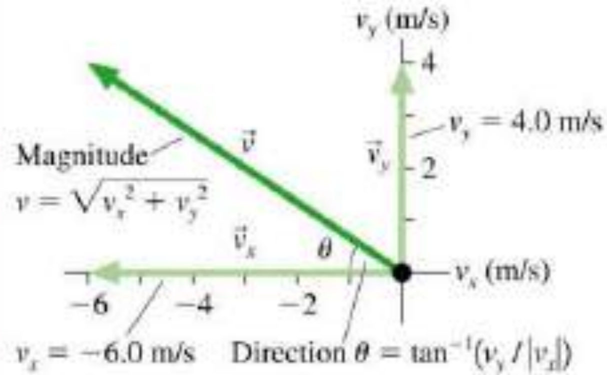
$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(-6.0\ \text{m/s})^2 + (4.0\ \text{m/s})^2} = 7.2\ \text{m/s}$$

From trigonometry, angle θ is

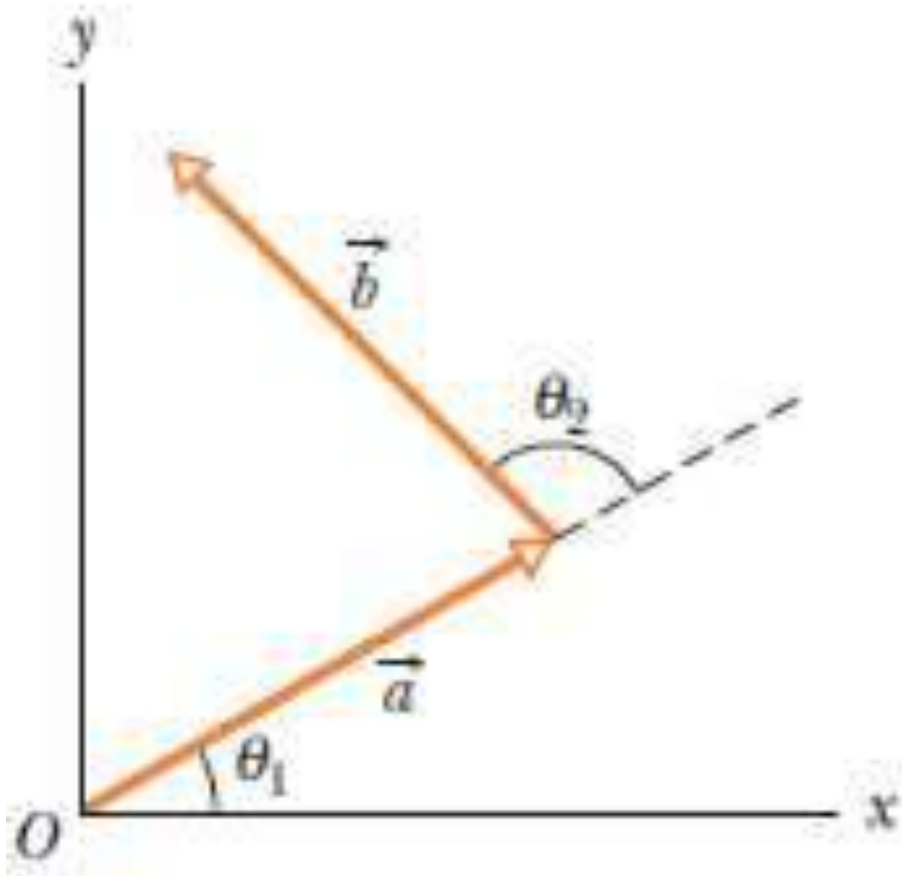
$$\theta = \tan^{-1}\left(\frac{v_y}{|v_x|}\right) = \tan^{-1}\left(\frac{4.0\ \text{m/s}}{6.0\ \text{m/s}}\right) = 34°$$

The absolute value signs are necessary because v_x is a negative number. The velocity vector $\vec{v}$ can be written in terms of the speed and the direction of motion as

$$\vec{v} = (7.2\ \text{m/s},\ 34° \text{ above the negative } x\text{-axis})$$



•15 SSM ILW WWW The two vectors $\vec{a}$ and $\vec{b}$ in Fig. 3-28 have equal magnitudes of 10.0 m and the angles are $\theta_1 = 30°$ and $\theta_2 = 105°$. Find the (a) x and (b) y components of their vector sum $\vec{r}$, (c) the magnitude of $\vec{r}$, and (d) the angle $\vec{r}$ makes with the positive direction of the x axis.



y
$\vec{b}$
θ_2
$\vec{a}$
θ_1
O
x

$\vec{r} = \vec{a} + \vec{b}$

$\vec{a} = a_x\hat{i} + a_y\hat{j}$

$a_x = |\vec{a}| \cdot \cos(\theta_1)$

$a_y = |\vec{a}| \cdot \sin(\theta_1)$

$\vec{a} = |\vec{a}| \cos(\theta_1)\hat{i} + |\vec{a}| \cdot \sin(\theta_1)\hat{j}$

$\vec{b} = |\vec{b}| \cos(\theta_1 + \theta_2)\hat{i} + |\vec{b}| \cdot \sin(\theta_1 + \theta_2)\hat{j}$

$\vec{r} = [|\vec{a}| \cos(\theta_1) + |\vec{b}| \cos(\theta_1 + \theta_2)]\hat{i} + [|\vec{a}| \cdot \sin\theta_1 + |\vec{b}| \cdot \sin(\theta_1 + \theta_2)]\hat{j} =$

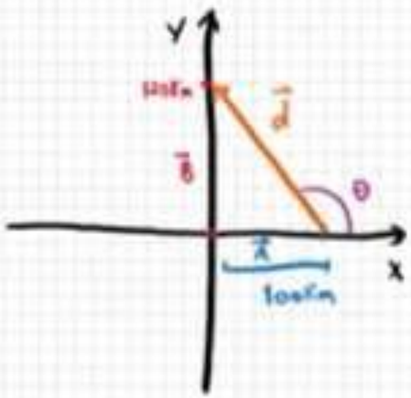
$= (1.59\hat{i} + 12.1\hat{j})\,m$

$r_x = 1.59m \;;\; r_y = 12.1m$

$|\vec{r}| = \sqrt{(1.59)^2 + (12.1)^2} = 12.2m$

$\theta = tg^{-1}\left(\frac{r_y}{r_x}\right) = tg^{-1}\left(\frac{12.1}{1.59}\right) = 82.5°$

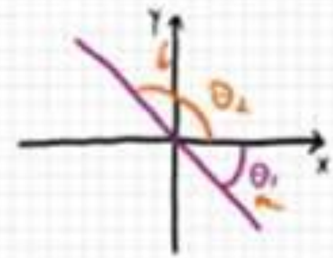
•5 A ship sets out to sail to a point 120 km due north. An unexpected storm blows the ship to a point 100 km due east of its starting point. (a) How far and (b) in what direction must it now sail to reach its original destination?



$\vec{A} + \vec{d} = \vec{B}$

$\vec{d} = \vec{B} - \vec{A} = (120\hat{j} - 100\hat{i})$ Km

$|\vec{d}| = \sqrt{(120^2) + (-100)^2} = 156$ Km



$\vec{A} = (100\hat{i})$ Km

$\vec{B} = (120\hat{j})$ Km

b) $tg(\theta) = \frac{d_y}{d_x}$

$\theta = tg^{-1}\left(\frac{d_y}{d_x}\right) = tg^{-1}\left(\frac{120}{-100}\right) = \begin{matrix} -50.2° \\ 129.8° \end{matrix}$

CHALLENGE EXAMPLE 3.8 **Finding the net force**

FIGURE 3.23 shows three forces acting at one point. What is the net force $\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$?

VISUALIZE Figure 3.23 show the forces and establishes a tilted coordinate system.

SOLVE The vector equation $\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$ is really two simultaneous equations:

$$(F_{net})_x = F_{1x} + F_{2x} + F_{3x}$$

$$(F_{net})_y = F_{1y} + F_{2y} + F_{3y}$$

The components of the forces are determined with respect to the axes. Thus

$$F_{1x} = F_1 \cos 45° = (50\text{ N})\cos 45° = 35\text{ N}$$

$$F_{1y} = F_1 \sin 45° = (50\text{ N})\sin 45° = 35\text{ N}$$

$\vec{F}_2$ is easier. It is pointing along the y-axis, so $F_{2x} = 0$ N and $F_{2y} = 20$ N. To find the components of $\vec{F}_3$, we need to recognize—because $\vec{F}_3$ points straight down—that the angle between $\vec{F}_3$ and the x-axis is 75°. Thus

$$F_{3x} = F_3 \cos 75° = (57\text{ N})\cos 75° = 15\text{ N}$$

$$F_{3y} = -F_3 \sin 75° = -(57\text{ N})\sin 75° = -55\text{ N}$$

The minus sign in F_{3y} is critical, and it appears not from some formula but because we recognized—from the figure—that the y-component of $\vec{F}_3$ points in the $-y$-direction. Combining the pieces, we have

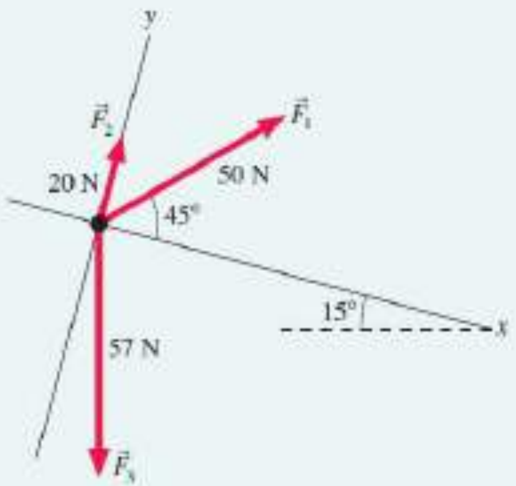
$$(F_{net})_x = 35\text{ N} + 0\text{ N} + 15\text{ N} = 50\text{ N}$$

$$(F_{net})_y = 35\text{ N} + 20\text{ N} + (-55\text{ N}) = 0\text{ N}$$

Thus the net force is $\vec{F}_{net} = 50\hat{\imath}$ N. It points along the x-axis of the tilted coordinate system.

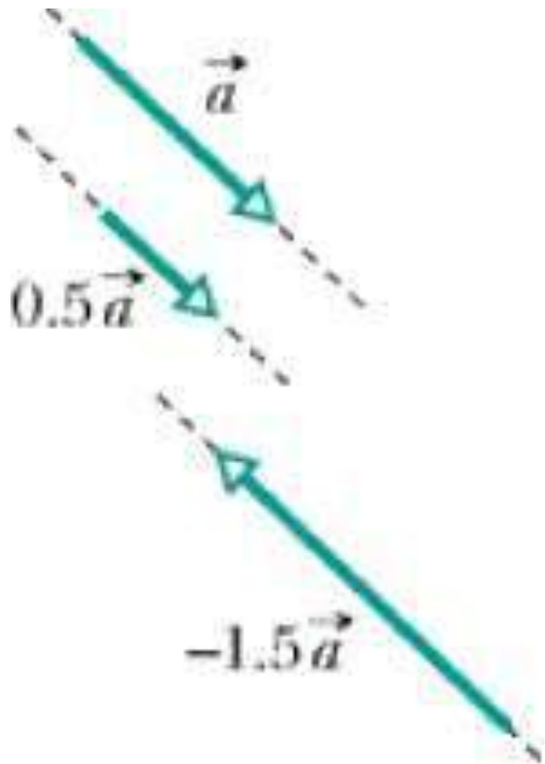
ASSESS Notice that all work was done with reference to the axes of the coordinate system, not with respect to vertical or horizontal.

FIGURE 3.23 Three forces.

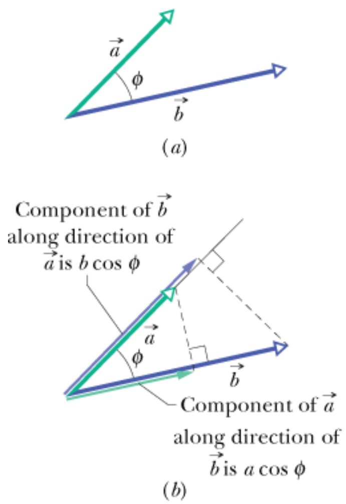


Multiplying and Dividing a Vector by a Scalar

$$e\vec{V} = e(\vec{V}_x + \vec{V}_y) = e(V_x\hat{i} + V_y\hat{j}) = (eV_x)\hat{i} + (eV_y)\hat{j}$$



The Scalar Product of Vectors
(**dot product**)



$$\vec{a} \cdot \vec{b} = ab\cos\phi,$$

$$\vec{a} \cdot \vec{b} = (a\cos\phi)(b) = (a)(b\cos\phi).$$

- The dot product is a scalar.
- If the angle   between two vectors is 0°, dot product is maximum
- If the angle   between two vectors is 90°, dot product is zero

The commutative law

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} .$$

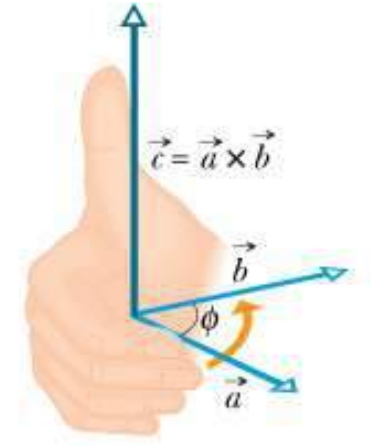
$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z .$$

The Vector Product (**cross product**)

$$\vec{c} = \vec{a} \times \vec{b}$$

(1) Cross production is a vector

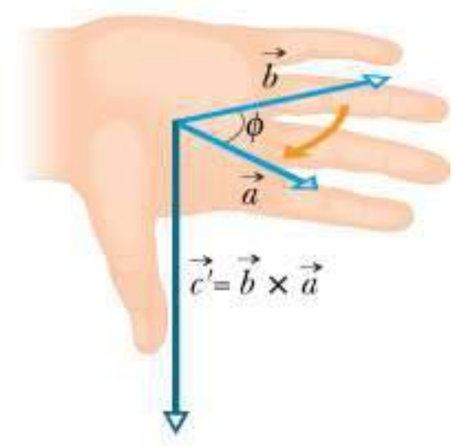
(3) Direction is determined by **right-hand rule**



(a)

(2) Magnitude is

$$c = ab\sin\phi$$



(b)

Property of vector cross product

- The order of the vector multiplication is important.

$$\vec{b} \times \vec{a} = -\left(\vec{a} \times \vec{b}\right).$$

$$\vec{a} \times \vec{b} = (a_y b_z - b_y a_z)\hat{\mathrm{i}} + (a_z b_x - b_z a_x)\hat{\mathrm{j}} + (a_x b_y - b_x a_y)\hat{\mathrm{k}}.$$

If two vectors are parallel or anti-parallel, $\vec{a} \times \vec{b} = 0$

If two vectors are perpendicular to each other , the magnitude of their cross product is maximum.

••44 In the product $\vec{F} = q\vec{v} \times \vec{B}$, take $q = 2$,

$$\vec{v} = 2.0\hat{i} + 4.0\hat{j} + 6.0\hat{k} \quad \text{and} \quad \vec{F} = 4.0\hat{i} - 20\hat{j} + 12\hat{k}.$$

What then is $\vec{B}$ in unit-vector notation if $B_x = B_y$? B_0

$\vec{F} = q\vec{v} \times \vec{B}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$q \cdot \vec{v} = 4\hat{i} + 8\hat{j} + 12\hat{k}$

$$q\vec{v} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 8 & 12 \\ B_x & B_y & B_z \end{vmatrix} = 8B_z\hat{i} + 12B_x\hat{j} + 4B_y\hat{k} -$$

$$- 8B_x\hat{k} - 12B_y\hat{i} - 4B_z\hat{j} =$$

$$= (8B_z - 12B_y)\hat{i} + (12B_x - 4B_z)\hat{j} + (4B_y - 8B_x)\hat{k}$$

$4\hat{i} - 20\hat{j} + 12\hat{k}$

$$\begin{cases} 4 = 8B_z - 12B_0 \\ -20 = 12B_0 - 4B_z \\ 12 = 4B_0 - 8B_0 \end{cases}$$

$12 = -4B_0$

$B_0 = -3$

$4 = 8B_z - 12 \cdot (-3)$

$4 = 8B_z + 36$

$-32 = 8B_z$

$B_z = -4$

$\vec{B} = -3\hat{i} - 3\hat{j} - 4\hat{k}$

74 Vector $\vec{a}$ lies in the yz plane 63.0° from the positive direction of the y axis, has a positive z component, and has magnitude 3.20 units. Vector $\vec{b}$ lies in the xz plane 48.0° from the positive direction of the x axis, has a positive z component, and has magnitude 1.40 units. Find (a) $\vec{a} \cdot \vec{b}$, (b) $\vec{a} \times \vec{b}$, and (c) the angle between $\vec{a}$ and $\vec{b}$.

74. The two vectors $\vec{a}$ and $\vec{b}$ are given by

$$\vec{a} = 3.20(\cos 63^\circ\,\hat{\mathrm{j}} + \sin 63^\circ\,\hat{\mathrm{k}}) = 1.45\,\hat{\mathrm{j}} + 2.85\,\hat{\mathrm{k}}$$

$$\vec{b} = 1.40(\cos 48^\circ\,\hat{\mathrm{i}} + \sin 48^\circ\,\hat{\mathrm{k}}) = 0.937\hat{\mathrm{i}} + 1.04\,\hat{\mathrm{k}}$$

The components of $\vec{a}$ are $a_x = 0$, $a_y = 3.20 \cos 63^\circ = 1.45$, and $a_z = 3.20 \sin 63^\circ = 2.85$. The components of $\vec{b}$ are $b_x = 1.40 \cos 48^\circ = 0.937$, $b_y = 0$, and $b_z = 1.40 \sin 48^\circ = 1.04$.

(a) The scalar (dot) product is therefore

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z = (0)(0.937) + (1.45)(0) + (2.85)(1.04) = 2.97.$$

(b) The vector (cross) product is

$$\begin{aligned}\vec{a}\times\vec{b} &= \left(a_y b_z - a_z b_y\right)\hat{\mathrm{i}} + \left(\mathrm{a}_z b_x - a_x b_z\right)\hat{\mathrm{j}} + \left(a_x b_y - a_y b_x\right)\hat{\mathrm{k}} \\ &= \left((1.45)(1.04) - 0\right)\hat{\mathrm{i}} + \left((2.85)(0.937) - 0\right)\hat{\mathrm{j}} + \left(0 - (1.45)(0.937)\right)\hat{\mathrm{k}} \\ &= 1.51\hat{\mathrm{i}} + 2.67\,\hat{\mathrm{j}} - 1.36\hat{\mathrm{k}}.\end{aligned}$$

(c) The angle θ between $\vec{a}$ and $\vec{b}$ is given by

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{ab}\right) = \cos^{-1}\left(\frac{2.97}{(3.20)(1.40)}\right) = 48.5^\circ.$$

Mathematical Operators

Differential Operators:

d/dx (Derivative): Represents the rate of change of a function with respect to a variable.

∇ (Nabla or Gradient): Indicates the vector of partial derivatives (gradient) of a scalar field.

∂/∂x (Partial Derivative): Represents the rate of change of a function with respect to one variable, holding others constant.

∇^2 (Laplacian): A scalar operator that describes the divergence of the gradient of a field.

Del(Nabla)Operator

The **nabla (∇) operator** is crucial in computer science for computing gradients in optimization (like in machine learning), detecting edges in images, simulating physical phenomena (e.g., fluid dynamics), and calculating surface normals in 3D graphics. It’s a fundamental tool for analyzing and manipulating scalar and vector fields in various computational applications.

Scalar and vector field

Scalar Field: A function that assigns a single value (magnitude) to every point in space.

- **Example**: Temperature distribution in a room.

Vector Field: A function that assigns a vector (magnitude and direction) to every point in space.

- **Example**: Wind velocity in the atmosphere.

Physical Significance of operators

The **gradient**, **divergence**, and **curl** are vector calculus operators with distinct physical significances:

- **Gradient**: Represents the rate and direction of the steepest increase of a scalar field, like temperature, indicating how and where the field changes most rapidly.
- **Divergence**: Measures the magnitude of a source or sink at a given point in a vector field, such as fluid flow, indicating how much the field is expanding (positive divergence) or contracting (negative divergence).
- **Curl**: Represents the rotation or swirling strength of a vector field, like the rotational motion of a fluid, indicating the tendency to rotate around a point.

These operators are crucial in analyzing physical phenomena such as fluid dynamics, electromagnetism, and more.

Application in CS

- **Gradient**: Used in optimization (e.g., machine learning), edge detection in images, and calculating surface normals in 3D graphics.
- **Divergence**: Applied in fluid simulation to measure flow expansion and in analyzing electromagnetic fields.
- **Curl**: Used in fluid dynamics for rotational flow and in electromagnetic simulations to compute field rotations.

These operators are key tools in optimization, simulations, and image processing in computer science.

Del Operator

- Operator ∇ is called vector differential operator defined as

$$\nabla = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right)$$

Gradient of scalar function

- If $\phi(x, y, z)$ is a scalar function of three variables and ϕ is differentiable, the gradient of ϕ is defined as

$$grad\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Where ,

ϕ is a scalar function

$\nabla\phi$ is a vector function

If $\phi = x^2yz^3 + xy^2z^2$,determine $grad\phi$ at point P=(1,3,2).

solution

$$\phi = x^2yz^3 + xy^2z^2$$

$$\frac{\partial\phi}{\partial x} = 2xyz^3 + y^2z^2, \frac{\partial\phi}{\partial y} = x^2z^3 + 2xyz^2, \frac{\partial\phi}{\partial z} = 3x^2yz^2 + 2xy^2z$$

Therefore,

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \left(2xyz^3 + y^2z^2\right)\hat{i} + \left(x^2z^3 + 2xyz^2\right)\hat{j} + \left(3x^2yz^2 + 2xy^2z\right)\hat{k}$$

at

$$P = (1,3,2)$$

$$\nabla\phi = 84\hat{i} + 32\hat{j} + 72\hat{k}$$

Grad Properties

If A and B are two scalars ,then

1) $\nabla(A \pm B) = \nabla A \pm \nabla B$

2) $\nabla(AB) = A(\nabla B) + B(\nabla A)$

Compute the directional derivative of $\phi = x^2z + 2xy^2 + yz^2$ at the point (1,2,-1) in the direction of the vector A=2i+3j-4k.

Solution

Directional derivative of ϕ in the direction of a

$$\frac{d\phi}{ds} = \hat{a}.grad\phi$$

Where,

$$grad\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

And, $\hat{a} = \frac{A}{|A|}$

$\phi = x^2z + 2xy^2 + yz^2$ Hence,

$$\nabla\phi = (2xz + 2y^2)\hat{i} + (4xy + z^2)\hat{j} + (x^2 + 2yz)\hat{k}$$

$At(1,2,-1),$

$$\nabla\phi = 6\hat{i} + 9\hat{j} - 3\hat{k}$$

Also given $A = 2\hat{i} + 3\hat{j} - 4\hat{k}$,then

$$|A| = \sqrt{2^2 + 3^2 + (-4)^2} = \sqrt{29}$$

Therefore,

$$\hat{a} = \frac{A}{|A|} = \frac{1}{\sqrt{29}}\left(2\hat{i} + 3\hat{j} - 4\hat{k}\right)$$

then,

$$\frac{d\phi}{ds} = \hat{a}.grad\,\phi$$

$$= \frac{51}{\sqrt{29}}$$

Divergence

Divergence in a physical context measures how much a vector field spreads out from or converges into a point. It represents the net rate of flow out of or into a point in space.

Physical Interpretation:

- **Positive Divergence**: Indicates a source, where more field lines are emanating from a point (e.g., water flowing out from a spring).
- **Negative Divergence**: Indicates a sink, where field lines are converging into a point (e.g., water flowing into a drain).
- **Zero Divergence**: Indicates no net flow in or out at a point, such as in a steady, incompressible fluid.The vector field whose divergence is zero called solenoidal.

If $A = x^2 y\hat{i} - xyz\hat{j} + yz^2\hat{k}$, determine $divA$ at point (1,2,3) .

solution

$$divA = \nabla . A$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) . (a_x\hat{i} + a_y\hat{j} + a_z\hat{k})$$

$$\Rightarrow \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}.$$

$$divA = 2xy - xz + 2yz$$

$$at(1,2,3)$$

$$divA = 2(1)(2) - (1)(3) + 2(2)(3)$$

$$divA = 13$$

Curl

Curl measures the rotation or swirling of a vector field around a point.

- **Physical Significance**: Indicates the presence and strength of rotational motion in a field.
- **Example**: In a whirlpool, the curl of the water's velocity field shows how fast and in which direction the water is rotating.
- If the curl of a vector field is zero, it is called **irrotational**. This means that there is no local rotation or swirling at any point in the field.

Curl of a vector field

- The curl of a vector field

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$$

is defined as

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Practice problems, Compute divergence and curl

1.

$$. \vec{F} = x^2 y \vec{i} - (z^3 - 3x)\vec{j} + 4y^2 \vec{k}$$

$$\text{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(3x - z^3) + \frac{\partial}{\partial z}(4y^2) = \boxed{2xy}$$

$$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & 3x - z^3 & 4y^2 \end{vmatrix}$$

$$= \frac{\partial}{\partial y}(4y^2)\vec{i} + \frac{\partial}{\partial z}(x^2 y)\vec{j} + \frac{\partial}{\partial x}(3x - z^3)\vec{k} - \frac{\partial}{\partial y}(x^2 y)\vec{k} - \frac{\partial}{\partial x}(4y^2)\vec{j} - \frac{\partial}{\partial z}(3x - z^3)\vec{i}$$

$$= 8y\vec{i} + 3\vec{k} - x^2\vec{k} + 3z^2\vec{i}$$

$$= \boxed{(8y + 3z^2)\vec{i} + (3 - x^2)\vec{k}}$$

Problem 2. compute div and curl

2. Compute $\mathbf{div}\vec{F}$ and $\mathbf{curl}\vec{F}$ for $\vec{F} = (3x + 2z^2)\ \vec{i} + \dfrac{x^3y^2}{z}\vec{j} - (z - 7x)\,\vec{k}$.

$$\mathrm{div}\vec{F} = \nabla\,.\,\vec{F} = \frac{\partial}{\partial x}(3x + 2z^2) + \frac{\partial}{\partial y}\left(\frac{x^3y^2}{z}\right) + \frac{\partial}{\partial z}(7x - z) = \boxed{2 + \frac{2x^3y}{z}}$$

$$\mathrm{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x + 2z^2 & \dfrac{x^3y^2}{z} & 7x - z \end{vmatrix}$$

$$= \frac{\partial}{\partial y}(7x - z)\,\vec{i} + \frac{\partial}{\partial z}(3x + 2z^2)\,\vec{j} + \frac{\partial}{\partial x}\left(\frac{x^3y^2}{z}\right)\vec{k}$$

$$- \frac{\partial}{\partial y}(3x + 2z^2)\,\vec{k} - \frac{\partial}{\partial x}(7x - z)\,\vec{j} - \frac{\partial}{\partial z}\left(\frac{x^3y^2}{z}\right)\vec{i}$$

$$= 4z\vec{j} + \frac{3x^2y^2}{z}\vec{k} - 7\vec{j} + \frac{x^3y^2}{z^2}\vec{i}$$

$$= \boxed{\frac{x^3y^2}{z^2}\vec{i} + (4z - 7)\,\vec{j} + \frac{3x^2y^2}{z}\vec{k}}$$

Thank you