

## Assignment-3 Solution

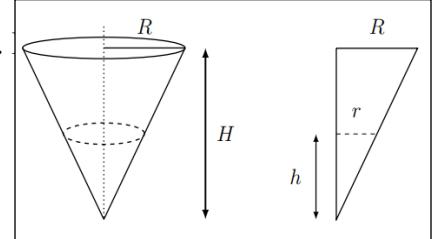
### Question 1

Let  $h(t)$  be the depth of water at time  $t$  and  $r(t)$  the radius of the water surface.

$$\frac{r}{h} = \frac{R}{H} \Rightarrow r = \frac{R}{H} h.$$

Volume of water (a cone) is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{R}{H} h\right)^2 h = \frac{1}{3}\pi \frac{R^2}{H^2} h^3.$$



Differentiate with respect to  $t$ :

$$\frac{dV}{dt} = \frac{1}{3}\pi \frac{R^2}{H^2} \cdot 3h^2 \frac{dh}{dt} = \pi \frac{R^2}{H^2} h^2 \frac{dh}{dt}.$$

At the instant the cup is full  $h = H$ , so

$$\frac{dV}{dt} \Big|_{h=H} = \pi \frac{R^2}{H^2} H^2 \frac{dh}{dt} = \pi R^2 \frac{dh}{dt}.$$

If the volume is decreasing at constant rate  $k$  (i.e.  $\frac{dV}{dt} = -k$  with  $k > 0$ ), then

$$\pi R^2 \frac{dh}{dt} = -k \Rightarrow \frac{dh}{dt} = -\frac{k}{\pi R^2}.$$

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### Question 2

- a) Find the rate of change in  $z$  at the instant when  $y = 50$

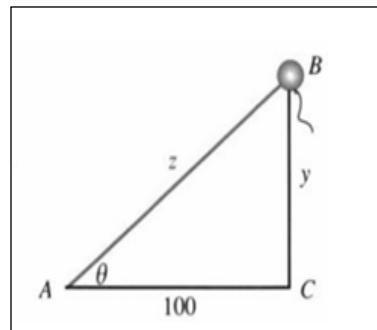
We know:

$$z^2 = y^2 + 100^2$$

Differentiate both sides with respect to  $t$ :

$$2z \frac{dz}{dt} = 2y \frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{y dy}{z dt}$$



Substitute  $y = 50$  and  $\frac{dy}{dt} = 3$ :

$$z = \sqrt{50^2 + 100^2} = 50\sqrt{5}$$

Now:

$$\frac{dz}{dt} = \frac{50}{50\sqrt{5}}(3) = \frac{3\sqrt{5}}{5}$$

b) Find the rate of change in the area of right triangle BCA when  $y = 50$ .

The area of the right triangle is:

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(100)(y) = 50y$$

Differentiate with respect to  $t$ :

$$\frac{dA}{dt} = 50 \frac{dy}{dt}$$

Substitute  $\frac{dy}{dt} = 3$ :

$$\frac{dA}{dt} = 50(3) = 150$$

c) Find the rate of change in  $\theta$  when  $y = 50$ .

We know:

$$\tan \theta = \frac{y}{100}$$

Differentiate both sides:

$$\begin{aligned} \sec^2 \theta \frac{d\theta}{dt} &= \frac{1}{100} \frac{dy}{dt} \\ \frac{d\theta}{dt} &= \frac{1}{100} \frac{dy}{dt} \cos^2 \theta \end{aligned}$$

When  $y = 50$ :

$$\cos \theta = \frac{100}{z} = \frac{100}{50\sqrt{5}} = \frac{2}{\sqrt{5}}$$

Now substitute:

$$\frac{d\theta}{dt} = \frac{1}{100} \left(3\right) \left(\frac{2}{\sqrt{5}}\right)^2 = \frac{3}{100} \cdot \frac{4}{5} = \frac{3}{125}$$

### Question 3

Given

$$\frac{dW}{dt} = \frac{1}{25}(W - 300) \text{ and } W(0) = 1400$$

Find the slope at  $t = 0$

$$\frac{dW}{dt} = \frac{1}{25}(W - 300)$$

Substitute  $W(0) = 1400$ :

$$\begin{aligned}\frac{dW}{dt} \Big|_{t=0} &= \frac{1}{25}(1400 - 300) \\ &= \frac{1}{25}(1100) \\ &= 44\end{aligned}$$

So, the slope of the tangent line at  $t = 0$  is 44 tons per year.

The tangent line approximation is:

$$\begin{aligned}L(t) &= W(0) + W'(0) \cdot (t - 0) \\ L(t) &= 1400 + 44t\end{aligned}$$

Estimate  $W$  at  $t = \frac{1}{4}$

$$\begin{aligned}L\left(\frac{1}{4}\right) &= 1400 + 44\left(\frac{1}{4}\right) \\ &= 1400 + 11 \\ &= 1411\end{aligned}$$

### Question 4

Let  $h(t)$  be the water height in inches. Radius is constant  $r = 10$  in, volume of water

$$V = \pi r^2 h = \pi(10)^2 h = 100\pi h.$$

Differentiate with respect to  $t$ :

$$\frac{dV}{dt} = 100\pi \frac{dh}{dt}.$$

Given the water is drained at  $3 \text{ in}^3/\text{min}$ , so  $\frac{dV}{dt} = -3 \text{ in}^3/\text{min}$ . Thus

$$\frac{dh}{dt} = \frac{1}{100\pi} \frac{dV}{dt} = \frac{-3}{100\pi} = -\frac{3}{100\pi} \text{ in/min.}$$

$$\frac{dh}{dt} \approx -0.00955 \text{ in/min}$$

So the water level is decreasing at  $-0.00955 \text{ in/min}$ .

## Question 5

a)  $g(x) = \left(\frac{x+2}{x}\right)$ , find  $(g^{-1})'(b)$

$$y = \frac{x+2}{x}$$

$$g^{-1}(y) = \frac{2}{y-1}$$

Find  $g'(x)$

$$g'(x) = -\frac{2}{x^2}$$

Apply the inverse derivative formula

$$(g^{-1})'(b) = \frac{1}{g'(g^{-1}(b))}$$

We already have  $g^{-1}(b) = \frac{2}{b-1}$ , so:

$$g'(g^{-1}(b)) = -\frac{2}{\frac{2}{(b-1)^2}} = -\frac{2(b-1)^2}{4} = -\frac{(b-1)^2}{2}$$

Now take the reciprocal:

$$(g^{-1})'(b) = \frac{1}{-\frac{(b-1)^2}{2}} = -\frac{2}{(b-1)^2}$$

b)  $f(x) = e^{2x} + 3$ , find  $(f^{-1})'(7)$

So  $f^{-1}(y) = \frac{1}{2} \ln(y-3)$ .

For  $y = 7$ :

$$a = f^{-1}(7) = \frac{1}{2} \ln 4 = \ln 2.$$

Derivative:  $f'(x) = 2e^{2x}$ .

Thus

$$(f^{-1})'(7) = \frac{1}{2e^{2\ln 2}} = \frac{1}{2 \cdot 4} = \frac{1}{8} = 0.125.$$

c)  $h(x) = \ln(x+4)$ , find  $(h^{-1})'(0)$

**Inverse:**  $y = \ln(x+4) \Rightarrow x+4 = e^y \Rightarrow h^{-1}(y) = e^y - 4$ .

For  $y = 0$ :  $a = f^{-1}(0) = e^0 - 4 = -3$ .

**Derivative:**  $f'(x) = \frac{1}{x+4}$ . So

$$(f^{-1})'(0) = \frac{1}{f'(-3)} = \frac{1}{-3+4} = 1$$

d)  $f(x) = \sin(x), \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ , find  $(f^{-1})'(\frac{\sqrt{3}}{2})$

**Inverse:**  $f^{-1}(y) = \arcsin y$

Here  $y = \frac{\sqrt{3}}{2} \Rightarrow a = \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ .

**Derivative:**  $f'(x) = \cos x$ . So

$$(f^{-1})' = \frac{1}{\cos(\pi/3)} = \frac{1}{1/2} = 2.$$

e)  $f(x) = \tan(x)$ ,  $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ , find  $(f^{-1})'(1)$

Inverse:  $f^{-1}(y) = \arctan y$ ,

$$f^{-1}(1) = \frac{\pi}{4}.$$

Derivative:  $f'(x) = \sec^2 x$ , so  $f'(\pi/4) = 2$ .

Thus

$$(f^{-1})'(1) = \frac{1}{\sec^2\left(\frac{\pi}{4}\right)} = \frac{1}{2}$$

### Question 6

The volume  $V$  of a sphere with radius  $r$  is given by:

$$V(r) = \frac{4\pi}{3}r^3$$

The linear approximation of  $V(r)$  near  $r = a$  is:

$$V(r) \approx L(r) = V(a) + V'(a)(r - a)$$

We will linearize around the initial radius  $a = 3$ .

First, find the volume at  $r = 3$ :

$$V(3) = \frac{4\pi}{3}(3)^3 = \frac{4\pi}{3} \cdot 27 = 36\pi$$

Next, find the derivative  $V'(r)$ , which gives the rate of change of volume:

$$V'(r) = 4\pi r^2$$

Evaluate the derivative at  $r = 3$ :

$$V'(3) = 4\pi(3)^2 = 36\pi$$

Substitute the values into the linearization formula:

$$V(r) \approx L(r) = 36\pi + 36\pi(r - 3)$$

**Apply the volume increase and solve for  $r$**

The new volume is the original volume plus an increase of  $\pi$  cubic inches:

$$V_{\text{new}} = 36\pi + \pi$$

Set the linear approximation  $L(r)$  equal to this new volume:

$$36\pi + 36\pi(r - 3) = 36\pi + \pi$$

$$r = 3 + \frac{1}{36}$$

Therefore, to increase the volume by  $\pi$  cubic inches, the radius must increase by approximately  $\frac{1}{36}$  inches.

### Question 7

We are given the model

$$L(t) = 0.01441t^3 - 0.4177t^2 + 2.703t + 1060.1$$

for  $t$  in months since January 1, 2012, over the year 2012, so  $t \in [0, 12]$ .

### Find critical points

We differentiate:

$$L'(t) = 0.04323t^2 - 0.8354t + 2.703$$

Set  $L'(t) = 0$ :

$$0.04323t^2 - 0.8354t + 2.703 = 0$$

### On Solving quadratic equations

$$t_1 = \frac{0.8354 - 0.4801}{0.08646} = \frac{0.3553}{0.08646} \approx 4.11$$
$$t_2 = \frac{0.8354 + 0.4801}{0.08646} = \frac{1.3155}{0.08646} \approx 15.21$$

### Check domain

Only  $t_1 \approx 4.11$  is in  $[0, 12]$ .  $t_2 \approx 15.21$  is outside 2012.

### Evaluate $L(t)$ at critical point and endpoints

$$L(0) = 1060.1$$

$$L(4.11) \approx 0.01441(69.38) - 0.4177(16.8921) + 2.703(4.11) + 1060.1$$

So  $L(4.11) \approx 1065.15$ .

$$L(12) = 0.01441(1728) - 0.4177(144) + 2.703(12) + 1060.1$$

So  $L(12) \approx 1057.29$ .

Maximum occurs at  $t \approx 4.11$  months, which is early May 2012.

### Question 8

Total length of wire: 4 ft.

Piece 1 → circle of radius  $r$ , circumference  $2\pi r$ .

Piece 2 → square of side  $x$ , perimeter  $4x$ .

Wire length constraint:

$$\begin{aligned} 2\pi r + 4x &= 4 \\ 4x &= 4 - 2\pi r \\ x &= 1 - \frac{\pi r}{2} \end{aligned}$$

Area of circle:  $A_c = \pi r^2$ .

Area of square:  $A_s = x^2 = (1 - \frac{\pi r}{2})^2$

Total area:  $A(r) = A_c + A_s$

$$A(r) = \pi r^2 + (1 - \frac{\pi r}{2})^2$$

#### Domain of $r$

From  $x \geq 0$ :

$$1 - \frac{\pi r}{2} \geq 0 \Rightarrow r \leq \frac{2}{\pi}$$

Also  $r \geq 0$ .

So  $r \in [0, \frac{2}{\pi}]$ .

#### Minimize $A(r)$

$$A(r) = \pi r^2 + 1 - \pi r + \frac{\pi^2 r^2}{4}$$

$$A(r) = 1 - \pi r + (\pi + \frac{\pi^2}{4})r^2$$

$$A'(r) = -\pi + 2\left(\pi + \frac{\pi^2}{4}\right)r = 0$$

$$(2\pi + \frac{\pi^2}{2})r = \pi$$

$$r = \frac{\pi}{2\pi + \frac{\pi^2}{2}}$$

$$r = \frac{2\pi}{4\pi + \pi^2} = \frac{2\pi}{\pi(4 + \pi)} = \frac{2}{4 + \pi}$$

### Check endpoints for minimization

At  $r = 0$ : all wire for square,  $x = 1, A = 1$ .

At  $r = \frac{2}{\pi}$ : all wire for circle,  $x = 0, A = \pi \cdot \frac{4}{\pi^2} = \frac{4}{\pi} \approx 1.273$ .

At  $r = \frac{2}{4+\pi}$ :

Let's compute  $A$ :

$$\text{First } x = 1 - \frac{\pi}{2} \cdot \frac{2}{4+\pi} = 1 - \frac{\pi}{4+\pi} = \frac{4+\pi-\pi}{4+\pi}.$$

$$x = \frac{4}{4 + \pi}$$

$$\text{So } A = \pi r^2 + x^2 = \pi \cdot \frac{4}{(4+\pi)^2} + \frac{16}{(4+\pi)^2}.$$

$$A = \frac{4\pi + 16}{(4 + \pi)^2} = \frac{4(\pi + 4)}{(4 + \pi)^2} = \frac{4}{4 + \pi}$$

$$\text{Numerically: } \frac{4}{4+\pi} \approx \frac{4}{7.14159} \approx 0.560.$$

This is less than 1 and less than  $4/\pi$ , so it's the minimum.

### Maximize $A(r)$

Since  $A(r)$  is a quadratic with positive coefficient of  $r^2$ , the maximum occurs at an endpoint.

We computed:

$$r = 0, A = 1$$

$$r = \frac{2}{\pi}, A \approx 1.273$$

So maximum at  $r = \frac{2}{\pi}$  (all wire to circle).

**Minimize total area:**  $r = \frac{2}{4+\pi}$

**Maximize total area:**  $r = \frac{2}{\pi}$

### Question 9

$$f'(x) = 3x^2 - 12x + 9 = (x - 1)(x - 3) = 0$$

Critical points:  $x = 1$  and  $x = 3$ .

Make a sign chart for  $f'(x)$

Test the sign of  $f'(x) = 3(x - 1)(x - 3)$  on the intervals determined by the critical points:

- Interval  $(-\infty, 1)$ . Pick  $x = 0$ :

$$f'(0) = 3(-1)(-3) = +9 > 0 \Rightarrow f \text{ is increasing on } (-\infty, 1).$$

- Interval  $(1, 3)$ . Pick  $x = 2$ :

$$f'(2) = 3(1)(-1) = -3 < 0 \Rightarrow f \text{ is decreasing on } (1, 3).$$

- Interval  $(3, \infty)$ . Pick  $x = 4$ :

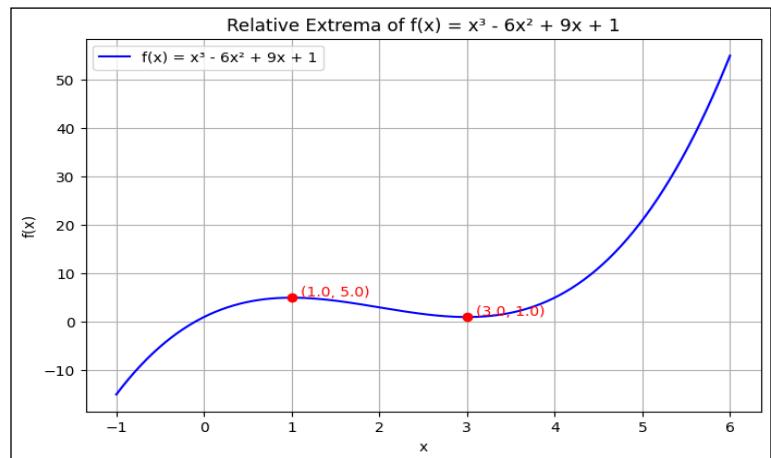
$$f'(4) = 3(3)(1) = +9 > 0 \Rightarrow f \text{ is increasing on } (3, \infty).$$

### Apply the First Derivative Test

- At  $x = 1$ :  $f'$  changes **from + to -** (increasing  $\rightarrow$  decreasing)  $\Rightarrow$  **local maximum** at  $x = 1$ .
- At  $x = 3$ :  $f'$  changes **from - to +** (decreasing  $\rightarrow$  increasing)  $\Rightarrow$  **local minimum** at  $x = 3$ .

Value  $f(1) = 1 - 6 + 9 + 1 = 5$ .

Value  $f(3) = 27 - 54 + 27 + 1 = 1$ .



## Question 10

### Python Solution

```
import numpy as np
import matplotlib.pyplot as plt
import sympy as sp

# Define the function and find its linearization at x=0
def f(x):
    return 1 / (1 + np.exp(-x))

# Calculate f(0)
x0 = 0
f0 = f(x0)
print(f"f(0) = {f0}")

# Calculate derivative using symbolic differentiation
x_sym = sp.Symbol('x')
f_sym = 1 / (1 + sp.exp(-x_sym))
f_prime_sym = sp.diff(f_sym, x_sym)

# Evaluate derivative at x=0
f_prime_0 = float(f_prime_sym.subs(x_sym, 0))
print(f"f'(0) = {f_prime_0}")

# Linearization: L(x) = f(0) + f'(0)(x - 0)
def L(x):
    return f0 + f_prime_0 * (x - x0)

print(f"\nLinearization at x=0:")
print(f"L(x) = {f0:.4f} + {f_prime_0:.4f}(x - {x0})")
print(f"L(x) = 0.5000 + 0.2500x")
```

### Python Output:

```
→ f(0) = 0.5
    f'(0) = 0.25

    Linearization at x=0:
L(x) = 0.5000 + 0.2500(x - 0)
L(x) = 0.5000 + 0.2500x
```

## AI Tool Solution (ChatGPT)

I used ChatGPT to solve the same problem:

**Prompt:** "Find the linearization of  $f(x) = 1/(1+e^{-x})$  at  $x=0$ "

### ChatGPT Response:

To find the linearization of  $f(x) = 1/(1+e^{-x})$  at  $x=0$ , we use the formula:

$L(x) = f(a) + f'(a)(x - a)$ , where  $a = 0$ .

Step 1: Find  $f(0)$

$$f(0) = 1/(1+e^{-0}) = 1/(1+1) = 1/2 = 0.5$$

Step 2: Find  $f(x)$

Using the chain rule:

$$f(x) = (1+e^{-x})^{-1}$$

$$f(x) = -1 * (1+e^{-x})^{-2} * (-e^{-x})$$

$$= e^{-x} / (1+e^{-x})^2$$

Step 3: Find  $f(0)$

$$f(0) = e^0 / (1+e^0)^2 = 1 / (1+1)^2 = 1/4 = 0.25$$

Step 4: Write the linearization

$$L(x) = f(0) + f'(0)(x - 0)$$

$$= 0.5 + 0.25x$$

Therefore, the linearization is  $L(x) = 0.5 + 0.25x$

Both methods give identical results:  $L(x) = 0.5 + 0.25x$

## Question 11

Given:

$$s = \sqrt{4+t}$$

We are asked to find the approximate change in distance ( $ds$ ) when  $t$  increases from 5 to 5.1.  
Use the differential formula

$$ds \approx s'(t) dt$$

Here:

$$t = 5, dt = 5.1 - 5 = 0.1$$

Substitute values

$$s'(5) = \frac{1}{2\sqrt{4+5}} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

Then,

$$ds \approx \frac{1}{6}(0.1) = 0.0167$$

### AI Tool Verification (ChatGPT)

**Prompt:** "A robot uses a distance sensor with reading  $s = \sqrt{4+t}$  where  $t$  is time in seconds. Find the approximate change in distance  $ds$  when  $t$  increases from 5 to 5.1 seconds using differentials."

### ChatGPT Response:

To find the approximate change using differentials, we use:

$$ds \approx (ds/dt) * dt$$

$$\text{Given: } s(t) = \sqrt{4+t} = (4+t)^{1/2}$$

#### Step 1: Find $ds/dt$

$$ds/dt = (1/2)(4+t)^{-1/2} * 1 = 1/(2\sqrt{4+t})$$

#### Step 2: Evaluate at $t=5$

$$ds/dt \text{ at } t=5 = 1/(2\sqrt{4+5}) = 1/(2\sqrt{9}) = 1/(2*3) = 1/6 \approx 0.166667$$

#### Step 3: Calculate $dt$

$$dt = 5.1 - 5 = 0.1$$

#### Step 4: Compute $ds$

$$ds \approx (1/6) * 0.1 = 0.1/6 = 0.016667$$

The approximate change using differentials is 0.016667, which is very close to the exact change of 0.016621.

## Comparison and Analysis

Method	Calculation	Result
Differentials	$(1/6) \times 0.1$	0.016667
AI Solution	$(1/6) \times 0.1$	0.016667

## Question 12

```
import numpy as np
import matplotlib.pyplot as plt

# Define the function
def s(t):
    return np.sqrt(4 + t)

# Given values
t0 = 5
t1 = 5.1
dt = t1 - t0

# Calculate the derivative ds/dt
def ds_dt(t):
    return 1 / (2 * np.sqrt(4 + t))

# Using differentials: ds ≈ (ds/dt) * dt
ds_approx = ds_dt(t0) * dt

# Exact change
s0 = s(t0)
s1 = s(t1)
ds_exact = s1 - s0

print("Using Differentials:")
print(f"Function: s(t) = √(4 + t)")
print(f"Initial time: t = {t0} s")
print(f"Final time: t = {t1} s")
print(f"Time change: dt = {dt} s")
print(f"\nDerivative: ds/dt = 1/(2√(4 + t))")
print(f"ds/dt at t={t0}: {ds_dt(t0):.6f}")
print(f"\nApproximate change using differentials:")
print(f"ds ≈ (ds/dt) * dt = {ds_dt(t0):.6f} * {dt} = {ds_approx:.6f}")
```

```
→ Using Differentials:
Function: s(t) = √(4 + t)
Initial time: t = 5 s
Final time: t = 5.1 s
Time change: dt = 0.0999999999999964 s

Derivative: ds/dt = 1/(2√(4 + t))
ds/dt at t=5: 0.166667

Approximate change using differentials:
ds ≈ (ds/dt) * dt = 0.166667 * 0.0999999999999964 = 0.016667
```