Question 1. Evaluate the following limit:

$$\lim_{x\to 0^+} \frac{\ln(cscx) - lnx}{cot\, x}$$

Solution:

As $x \to 0^+$

the numerator $\ln(\csc\, x) - \ln(x) \to +\infty - (-\infty) = +\infty$.

the denominator $\cot\, x \to +\infty$.

Thus, the limit is of the indeterminate form $\frac{\infty}{\infty}$, the limit becomes:

$$L = \lim_{x\to 0^+} \frac{\ln(\frac{1}{x\sin\, x})}{\cot x}$$

First application of L'Hôpital's Rule

Let $N(x) = \ln\,(\frac{1}{x\sin\, x})$ and $D(x) = \cot\, x$.

Now apply L'Hôpital's Rule:

$$L = \lim_{x\to 0^+} \frac{N'(x)}{D'(x)} = \lim_{x\to 0^+} \frac{-\frac{1}{x} - \cot\, x}{-\csc^2 x} = \lim_{x\to 0^+} \frac{\frac{1}{x} + \cot\, x}{\csc^2 x}$$

As $x \to 0^+$:

- Numerator: $\frac{1}{x} + \cot\, x \to \infty$.
- Denominator: $\csc^2 x \to \infty$.

  the limit is of the indeterminate form $\frac{\infty}{\infty}$

Second application of L'Hôpital's Rule and simplification

Now apply L'Hôpital:

$$L = \lim_{x\to 0^+} \frac{N_1'(x)}{D_1'(x)} = \lim_{x\to 0^+} \frac{-\frac{1}{x^2} - \csc^2 x}{-2\cot\, x\csc^2 x} = \lim_{x\to 0^+} \frac{\frac{1}{x^2} + \csc^2 x}{2\cot\, x\csc^2 x}$$

So the limit becomes:

$$L = \lim_{x\to 0^+} \frac{\csc^2 x(\frac{\sin^2 x}{x^2} + 1)}{2\cot\, x\csc^2 x} = \lim_{x\to 0^+} \frac{\frac{\sin^2 x}{x^2} + 1}{2\cot\, x}$$

As $x \to 0^+, \frac{\sin^2 x}{x^2} \to 1, 2\cot x \to \infty$

Therefore,

$L = \frac{2}{\infty} = 0$

Question 2.

The validation error of a predictive model, as a function of a regularization parameter x, is modeled as: $\mathrm{E(x)} = \mathrm{x}^3 - 3\mathrm{x}^2 - 9\mathrm{x} + 10$. Interpret and find all values of x at which the tangent to the curve $y = E(x)$ is horizontal.

Solution:

$$E(x) = x^3 - 3x^2 - 9x + 10$$

We need horizontal tangents ⇒ set the derivative $E'(x) = 0$.

$$E'(x) = 3x^2 - 6x - 9 = 0$$

$$(x - 3)(x + 1) = 0$$

So $x = 3$ or $x = -1$.

Interpret:

At $x = -1$ and $x = 3$, the derivative is zero, so the tangent line to $E(x)$ is horizontal.

These are critical points of $E(x)$, possibly local maxima, minima.

Q.3

$$P(t) = \begin{cases} 2t+1, & t \le 3 \\ t^2-2t+4, & t>3 \end{cases}$$

To determine if $P(t)$ is differentiable at $t=3$ first check continuity at $t=3$.

$$\lim_{t\to 3^-} P(t) = \lim_{t\to 3^-} (2t+1) = 7$$

$$\lim_{t\to 3^+} P(t) = \lim_{t\to 3^+} (t^2-2t+4) = 7$$

$$P(3) = 7$$

As, $P(3) = \lim_{x\to 3^-} P(t) = \lim_{x\to 3^+} P(t)$, $P(t)$ is continuos at $t=3$.

$$p'(t) = \begin{cases} 2, & t \le 3 \\ 2t-2, & t>3 \end{cases}$$

$$p'(3^-) = 2, \quad p'(3^+) = 4$$

$$p'(3^-) \neq p'(3^+)$$

So, $P(t)$ is not differentiable at $t=3$.

Q4:- $A(\theta) = 16\sin\theta(\cos\theta + 1) \quad 0 \le \theta \le \frac{\pi}{2}$

$$A'(\theta) = 16\sin\theta(-\sin\theta) + 16\cos\theta(\cos\theta + 1)$$

$$= 16[-\sin^2\theta + \cos^2\theta + \cos\theta]$$

$$= 16[(\cos^2\theta - 1) + \cos^2\theta + \cos\theta]$$

$$= 16[2\cos^2\theta + \cos\theta - 1]$$

$$= 16(2\cos\theta - 1)(\cos\theta + 1)$$

For Critical pts put $A'(\theta) = 0$

$$16(2\cos\theta - 1)(\cos\theta + 1) = 0$$

$2\cos\theta - 1 = 0$ or $\cos\theta + 1 = 0$

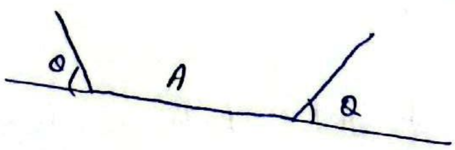
$\cos\theta = \frac{1}{2}$ $\qquad$ $\cos\theta = -1$

$\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ $\qquad$ $\theta = \pi$

only $\theta = \frac{\pi}{3}$ lies in the interval so it is the CP

θ	0	$\frac{\pi}{3}$	$\pi/2$
A	0	$12\sqrt{3} \approx 20.785$	16

If the aluminium is bent at an angle of $\frac{\pi}{3}$ which is $60°$, the area of opening is max. The max area is approximately $20.785\,in^2$.



Q5:- Initial radius $= R = 3cm$

Change in radius $= 2.971cm$

Vol of sphere $= V = \frac{4}{3}\pi R^3$

Approximate vol lost using differential

$dV = 4\pi R^2 dR$

dR or $\Delta R = 2.971 - 3 = -0.029\text{cm}$

$\Delta V \approx dV = (4\pi)(3)^2(-0.029) \approx -3.280\text{cm}^3$

The approximate loss in vol of bearing is 3.28cm^3

The actual loss in vol ΔV is

$$\Delta V = V(R+\Delta R) - V(R)$$

$$\Delta V = \frac{4}{3}\pi(2.971)^3 - \frac{4}{3}\pi\cdot(3)^3$$

$$\Delta V = \frac{4}{3}\pi(-0.7755)$$

$$\Delta V \approx -3.248\text{cm}^3$$

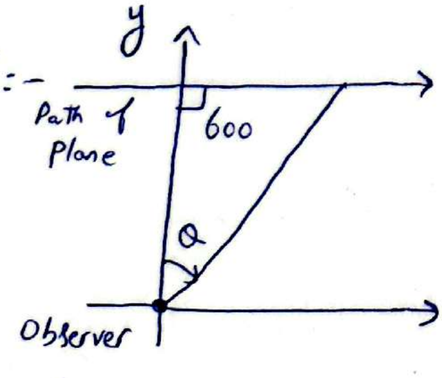
Actual loss is vol $= 3.248\text{cm}^3$

$$\text{\%age error} = \frac{|\text{Approx} - \text{Actual}|}{\text{Actual}} \times 100\%$$

$$= \frac{|3.28 - 3.25|}{3.25} \times 100\%$$

$$\approx 0.92\%$$

Q6:-



The spectator is at origin (0,0) & jet's path is from left to right on the line $y=600$ & 'θ' is the angle b/w +ive y-axis & line of sight. We will measure distance in 'ft' & time in 's' so we will convert the jet's speed to ft/s.

$$540\,\text{mi/h} = (540\,\text{mi/h})(5280\,\text{ft/mi})\left(\frac{1}{3600}\,\text{h/s}\right) = 792\,\text{ft/s}$$

$$\tan\theta(t) = \frac{x(t)}{y(t)} \quad \text{(All quantities are changing with time)}$$

$$[\sec^2 \theta(t)]\,\theta'(t) = \frac{x'(t)\,y(t) - x(t)\,y'(t)}{[y(t)]^2}$$

As the jet is moving from left to right along the line $y = 600$, we have $x'(t) = 792$, $y(t) = 600$ & $y'(t) = 0$. Substituting these quantities, we have

$$[\sec^2 \theta(t)]\,\theta'(t) = \frac{792(600)}{(600)^2} = 1.32$$

$$\theta'(t) = \frac{1.32}{\sec^2 \theta(t)}$$

$$\theta'(t) = 1.32 \cos^2 \theta(t)$$

The rate of change is a max when $\cos^2 \theta(t)$ is max. Since the max of Cosine ftn is 1, the max value of $\cos^2 \theta(t)$ is 1, occuring when $\theta = 0$. So we conclude that max rate of angle change is 1.32 rad/sec. This occurs when $\theta = 0$, that is, when the jet reaches its closest pt to the observer.

Q.7 $P(t) = -t^3 + 9t^2 + 24t \quad , \quad t \geq 0$

(a) $p'(t) = -3t^2 + 18t + 24$

(b) $p(t)$ is increasing when $p'(t) > 0$ and decreasing where $p'(t) < 0$

Solve for $p'(t) = 0$

$-3t^2 + 18t + 24 = 0$

$\Rightarrow \quad t = 3 \pm \sqrt{17}$

$t = -1.12, 7.12$

for $0 < t < 3$ performance is increasing $\{ p'(t) > 0$

for $t > 3$ performance is decreasing $\{ p'(t) < 0$

(c) Net change from $t=0$ to $t=6$

Total improvement $= P(6) - P(0) = 252$

(d)

$$p''(t) = 18 - 6t$$

$$p''(t)\Big|_{t=1} = 12$$

(e) Speeding up $p''(t) > 0$
speeding down $p''(t) < 0$

$$p''(t) = 0 \Rightarrow \boxed{t=3}$$

for $0 < t < 3$, $p''(t) > 0$, learning speed increasing

for $t > 3$, $p''(t) < 0$, learning speed decreasing

combine with (b)

$0 < t < 3$: increasing and speeding up

$3 < t < 3 + \sqrt{17}$: increasing but slowing down

$t > 3 + \sqrt{17}$: decreasing and slowing down

Q.8

$$T(x) = \frac{x^4}{4} - 2x^2 + 4$$

Input size x is non-negative, so, the physically meaningful domain is $x \geq 0$ or $[0, \infty)$

$$T'(x) = \frac{4x^3}{4} - 4x + 0$$

$$T'(x) = x^3 - 4x = x(x^2 - 4)$$

$$T''(x) = 3x^2 - 4$$

Critical points:-

$$T'(x) = 0$$

$$x(x^2 - 4) = 0$$

$$x = 0, \quad x = \pm 2$$

$T'(x)$ 0 $- -$ 2 $+ + +$

local minimum at $x = 2$

local maximum at $x = 0$

Decreasing : $(0, 2)$

Increasing : $(2, \infty)$

Inflection points and concavity:-

$$T''(x) = 0$$

$$3x^2 - 4 = 0$$

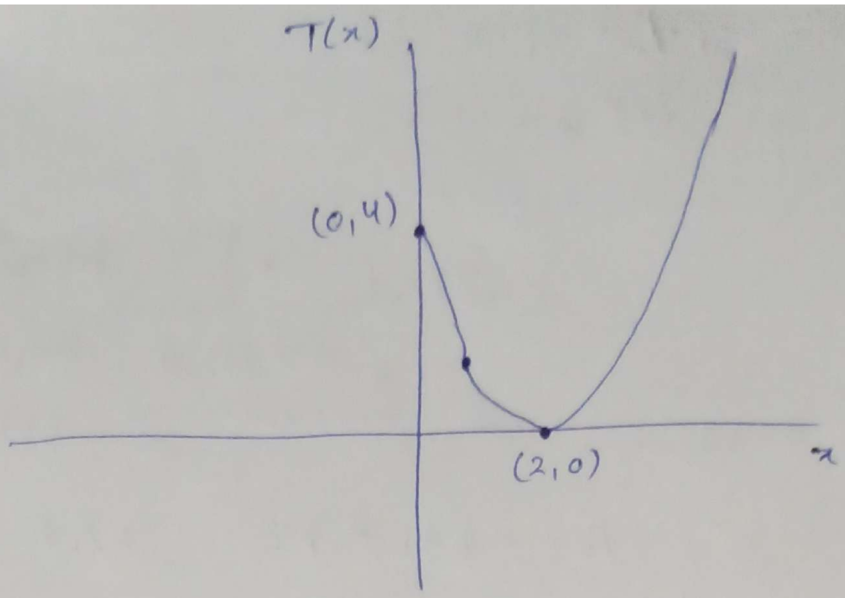
$$x = \pm \frac{2}{\sqrt{3}}$$

$T''(x)$ $- - -$ $\frac{2}{\sqrt{3}}$ $+ + +$

concave up : $\left(\frac{2}{\sqrt{3}}, \infty\right)$

concave down : $\left(0, \frac{2}{\sqrt{3}}\right)$

concavity changes at $x = \frac{2}{\sqrt{3}}$. So, it is an inflection point.



for $x>2$, $T(x)$ increases, eventualy growing like $\frac{x^4}{4}$.
for large x the quartic term dominates, this means very poor scalability for large inputs.