CHAPTER 4

Motion in two and three dimensions , Projectile motion, Uniform circular motion

Application in Computer Sciences

- Object position control in computers games and computer animation.

Object Position Control in Computer Games

Motion control over different objects is an important factor in computer games. It enhances the quality of game and gives an interactive feeling to user.

Improved mathematical models and algorithms can be used to control position of objects in computer games, which can increase game quality and gamer's satisfaction.

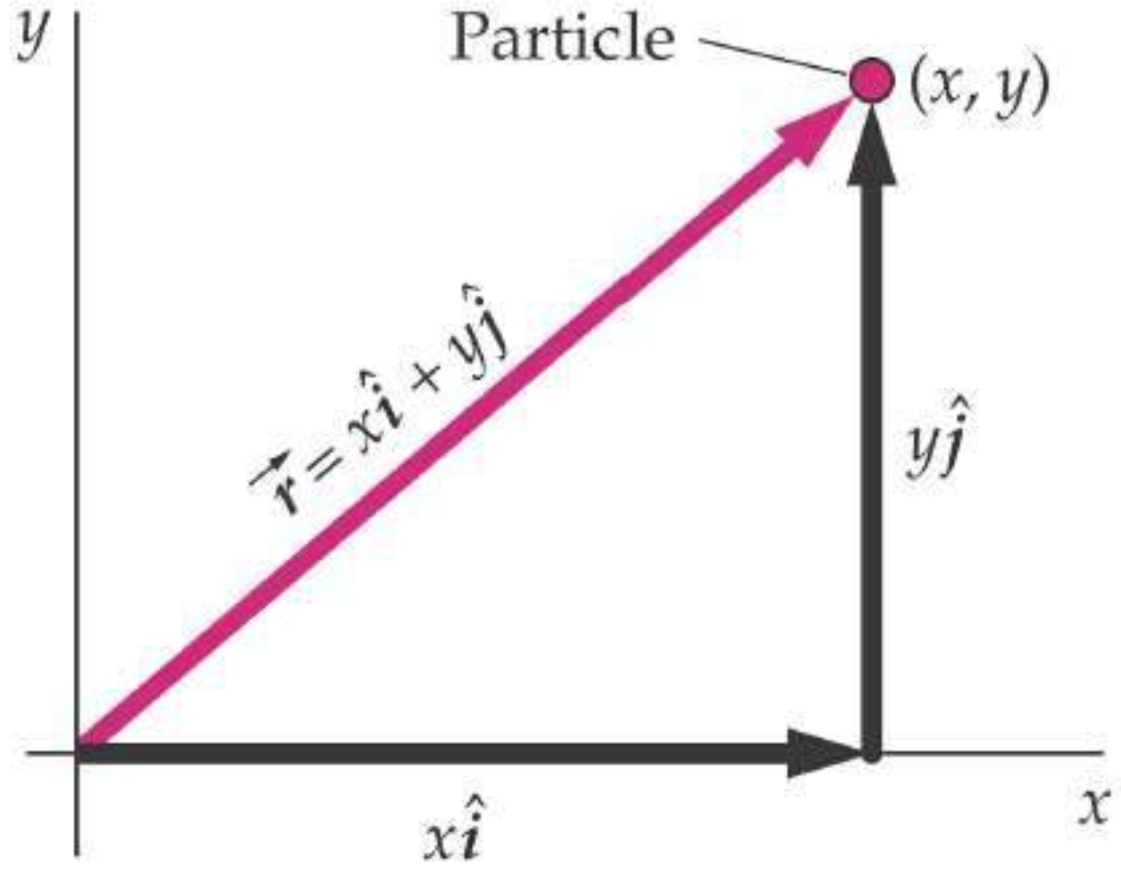
Real life examples of projectile motion

- A football kicked in a game.
- A cannonball fired from a cannon.
- A bullet fired from a gun.
- A disc thrown in the sport of discus throw.
- The flight of a golf ball.
- A jet of water escaping a hose.
- Motorcycles and cars jumping in extreme sports.

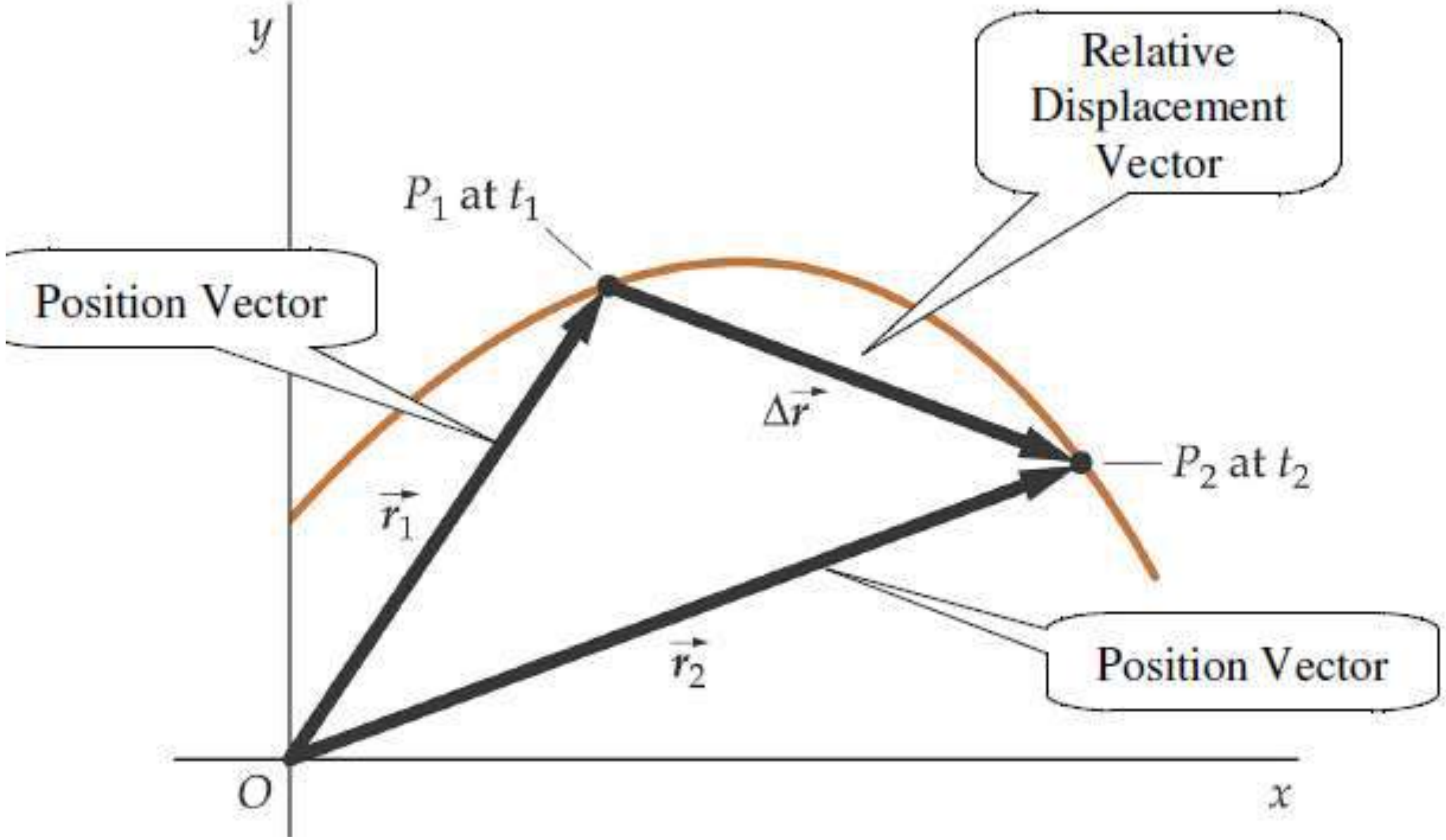
Motion in Two and Three Dimensions

- Displacement, Velocity and Speed
- Relative Motion
- Projectile Motion
- Circular Motion

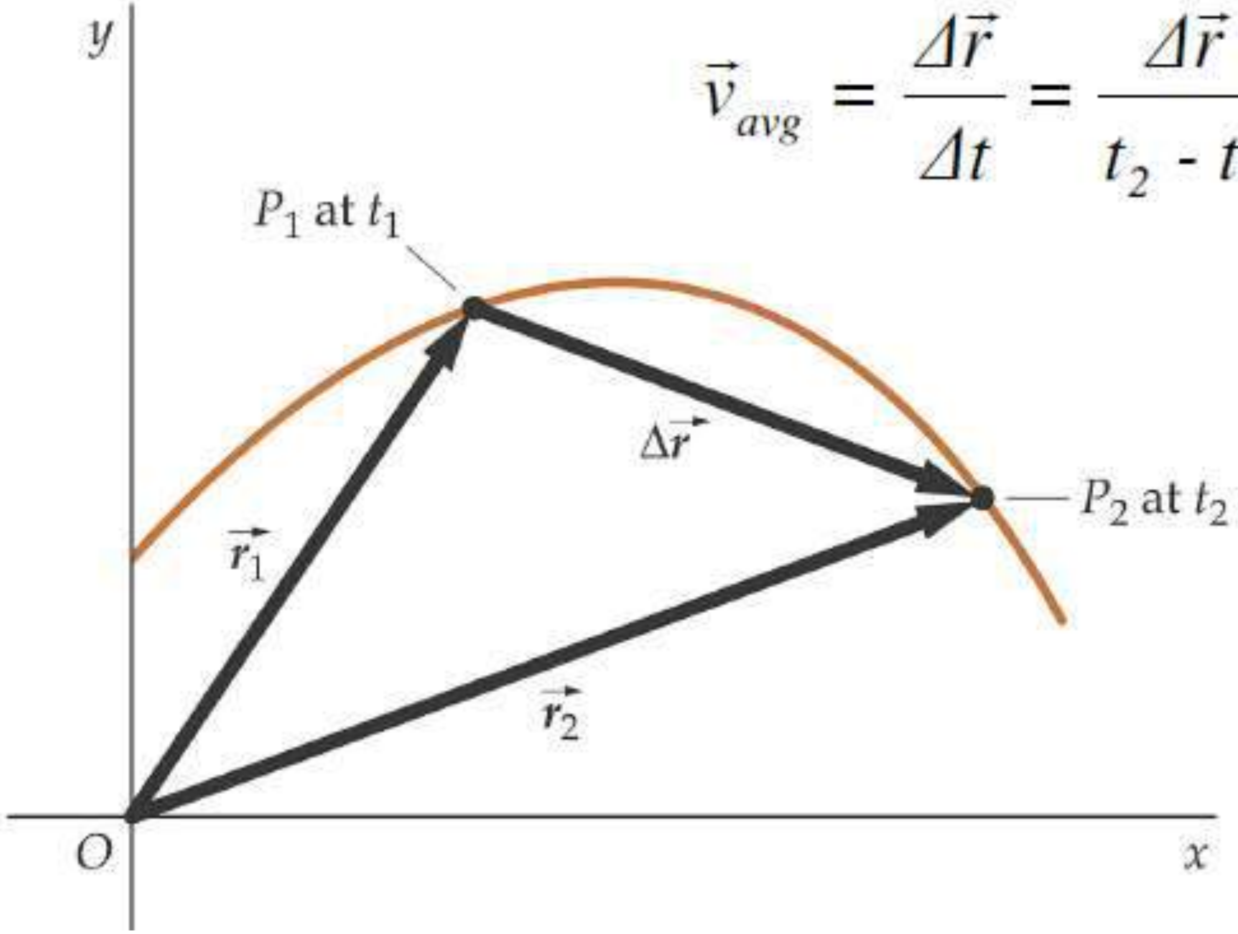
Particle Displacement



Relative Displacement



Average Velocity

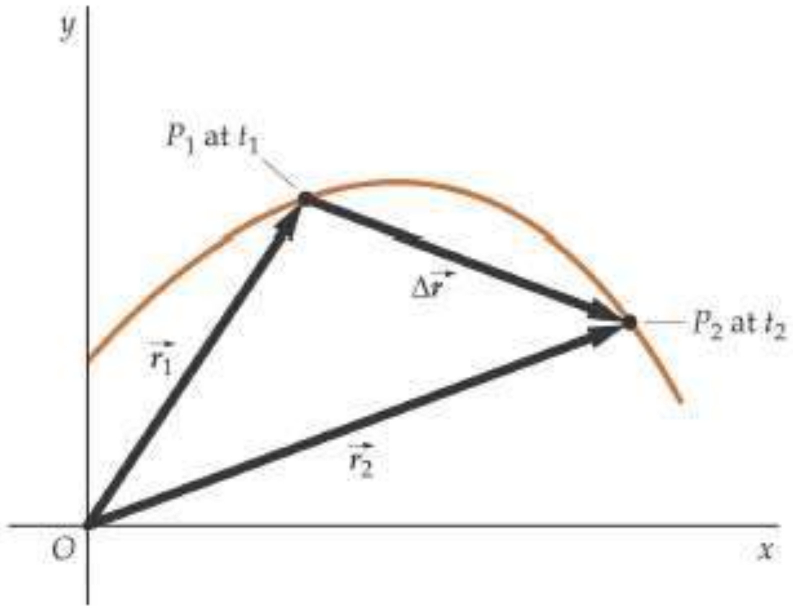


$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta\vec{r}}{t_2 - t_1}$$

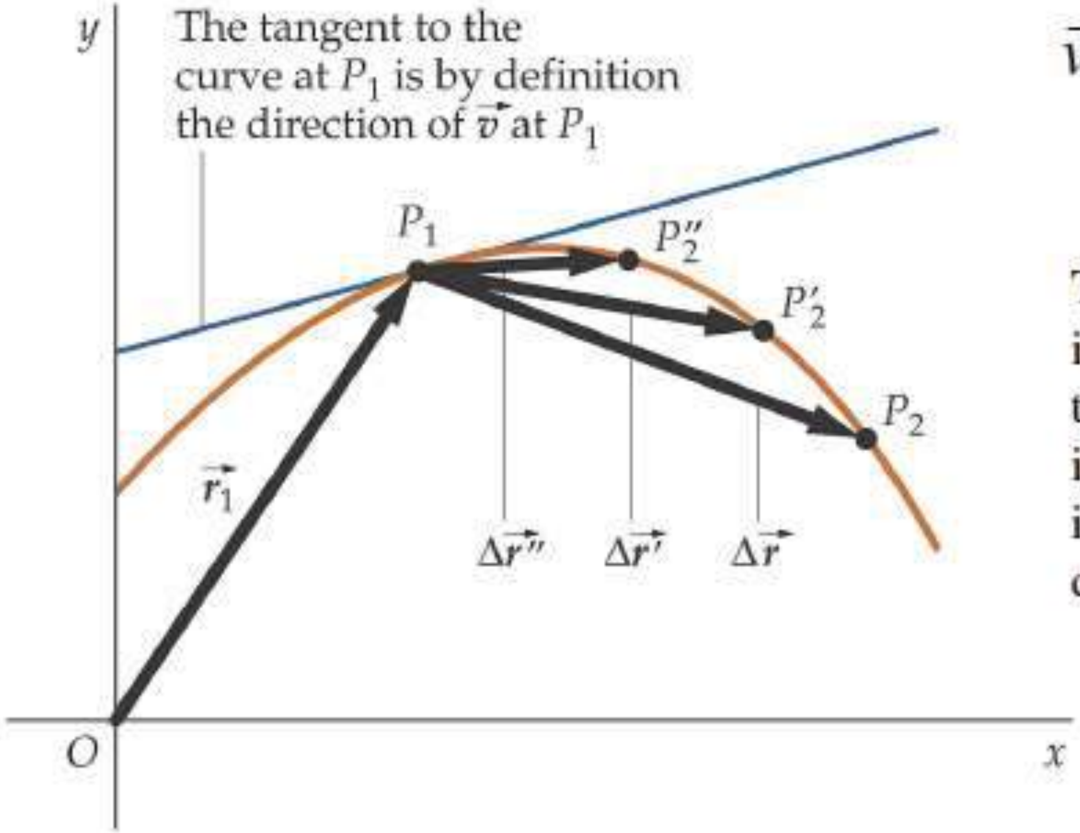
Average Velocity

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta\vec{r}}{t_2 - t_1} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{(x_2\hat{i} + y_2\hat{j}) - (x_1\hat{i} + y_1\hat{j})}{t_2 - t_1}$$

$$\vec{v}_{avg} = \frac{(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}}{t_2 - t_1}$$



Instantaneous Velocity Vector



$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$

There is not enough information presented here to actually calculate the instantaneous velocity. This is meant only to demonstrate the process.

Instantaneous Velocity Vector

Instantaneous velocity

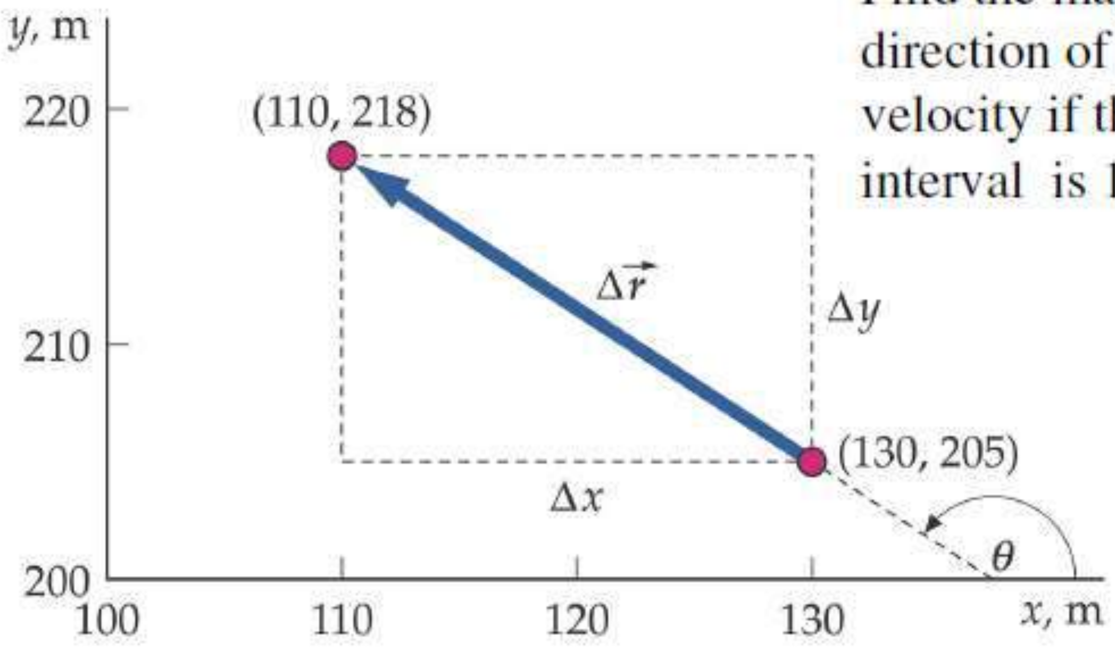
$$\vec{v}(t) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = v_x\hat{i} + v_y\hat{j}$$

Magnitude of the velocity

$$\left|\vec{v}(t)\right| = v(t) = \sqrt{v_x{}^2 + v_y{}^2}$$

Direction of the velocity

$$\Theta = tan^{-1}\left(\frac{v_y}{v_x}\right)$$



Ques:
Find the magnitude and direction of the average velocity if the time interval is 120s?

Problem

$$\vec{v}_{av} = v_{x\,av}\hat{i} + v_{y\,av}\hat{j}$$

where

$$v_{x\,av} = \frac{\Delta x}{\Delta t} = \frac{110\text{ m} - 130\text{ m}}{120\text{ s}} = -0.167\text{ m/s}$$

$$v_{y\,av} = \frac{\Delta y}{\Delta t} = \frac{218\text{ m} - 205\text{ m}}{120\text{ s}} = 0.108\text{ m/s}$$

so

$$\vec{v}_{av} = \boxed{-(0.167\text{ m/s})\hat{i} + (0.108\text{ m/s})\hat{j}}$$

$$v_{av} = \sqrt{(v_{x\,av})^2 + (v_{y\,av})^2} = \boxed{0.199\text{ m/s}}$$

$$\tan\theta = \frac{v_{y\,av}}{v_{x\,av}}$$

so

$$\theta = \tan^{-1}\frac{v_{y\,av}}{v_{x\,av}} = \tan^{-1}\frac{0.108\text{ m/s}}{-0.167\text{ m/s}} = -33.0^\circ + 180^\circ = \boxed{147^\circ}$$

Solution

Acceleration Vectors

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t}$$ Average acceleration

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$ Instantaneous acceleration

Where:

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

1. The position of an electron is given by $\mathbf{r} = 3.0t\mathbf{i} - 4.0t^2\mathbf{j} + 2.0\mathbf{k}$ (where t is in seconds and the coefficients have the proper units for $\mathbf{r}$ to be in meters). (a) What is $\mathbf{v}(t)$ for the electron? (b) In unit–vector notation, what is $\mathbf{v}$ at $t = 2.0\,\mathrm{s}$? (c) What are the magnitude and direction of $\mathbf{v}$ just then? [HRW5 4-9]

Problem

Solution

(a) The velocity vector $\mathbf{v}$ is the time–derivative of the position vector $\mathbf{r}$:

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(3.0t\mathbf{i} - 4.0t^2\mathbf{j} + 2.0\mathbf{k}) \\ &= 3.0\mathbf{i} - 8.0t\mathbf{j} \end{aligned}$$

where we mean that when t is in seconds, $\mathbf{v}$ is given in $\frac{\mathrm{m}}{\mathrm{s}}$.

(b) At $t = 2.00\,\mathrm{s}$, the value of $\mathbf{v}$ is

$$\mathbf{v}(t = 2.00\,\mathrm{s}) = 3.0\mathbf{i} - (8.0)(2.0)\mathbf{j} = 3.0\mathbf{i} - 16.\mathbf{j}$$

that is, the velocity is $(3.0\mathbf{i} - 16.\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}$.

(c) Using our answer from (b), at $t = 2.00\,\mathrm{s}$ the magnitude of $\mathbf{v}$ is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{(3.00\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (-16.\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (0)^2} = 16.\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

we note that the velocity vector lies in the xy plane (even though this is a three–dimensional problem!) so that we can express its direction with a single angle, the usual angle θ measured anti-clockwise in the xy plane from the x axis. For this angle we get:

$$\tan\theta = \frac{v_y}{v_x} = -5.33 \qquad \Longrightarrow \qquad \theta = \tan^{-1}(-5.33) = -79^\circ \ .$$

When we take the inverse tangent, we should always check and see if we have chosen the right quadrant for θ. In this case -79° *is* correct since v_y is negative and v_x is positive.

Solu.continued

2. A particle moves so that its position as a function of time in SI units is $\mathbf{r} = \mathbf{i} + 4t^2\mathbf{j} + t\mathbf{k}$. Write expressions for (a) its velocity and (b) its acceleration as functions of time. [HRW5 4-11]

Problem

(a) To clarify matters, what we mean here is that when we use the numerical value of t in *seconds*, we will get the values of $\mathbf{r}$ in *meters*. Since the velocity vector is the time–derivative of the position vector $\mathbf{r}$, we have:

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \frac{d}{dt}(\mathbf{i} + 4t^2\mathbf{j} + t\mathbf{k}) \\ &= 0\mathbf{i} + 8t\mathbf{j} + \mathbf{k} \end{aligned}$$

That is, $\mathbf{v} = 8t\mathbf{j} + \mathbf{k}$. Here, we mean that when we use the numerical value of t in seconds, we will get the value of $\mathbf{v}$ in $\frac{\mathrm{m}}{\mathrm{s}}$.

Solution

So $\mathbf{a} = 8\mathbf{j}$, where we mean that the value of $\mathbf{a}$ is in units of $\frac{\mathrm{m}}{\mathrm{s}^2}$. In fact, we should really include the units *here* and write:

$$\mathbf{a} = \left(8\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)\mathbf{j}$$

(b) The acceleration $\mathbf{a}$ is the time–derivative of $\mathbf{v}$, so using our result from part (a) we have:

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= \frac{d}{dt}(8t\mathbf{j} + \mathbf{k}) \\ &= 8\mathbf{j} \end{aligned}$$

Sol.continued

Sample problem 4.1

A rabbit runs across a parking lot on which a set of coordinate axes has, strangely enough, been drawn. The coordinates (meters) of the rabbit's position as functions of time t (seconds) are given by

$$x = -0.31t^2 + 7.2t + 28 \tag{4-5}$$

and

$$y = 0.22t^2 - 9.1t + 30. \tag{4-6}$$

(a) At $t = 15$ s, what is the rabbit's position vector $\vec{r}$ in unit-vector notation and in magnitude-angle notation?

Solution

Calculations: We can write

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}. \qquad (4\text{-}7)$$

(We write $\vec{r}(t)$ rather than $\vec{r}$ because the components are functions of t, and thus $\vec{r}$ is also.)

At $t = 15$ s, the scalar components are

$$x = (-0.31)(15)^2 + (7.2)(15) + 28 = 66 \text{ m}$$

and $$y = (0.22)(15)^2 - (9.1)(15) + 30 = -57 \text{ m},$$

so $$\vec{r} = (66 \text{ m})\hat{i} - (57 \text{ m})\hat{j}, \qquad \text{(Answer)}$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(66 \text{ m})^2 + (-57 \text{ m})^2}$$
$$= 87 \text{ m}, \qquad \text{(Answer)}$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-57 \text{ m}}{66 \text{ m}}\right) = -41°. \qquad \text{(Answer)}$$

P15

••15 SSM ILW A particle leaves the origin with an initial velocity $\vec{v} = (3.00\hat{i})$ m/s and a constant acceleration $\vec{a} = (-1.00\hat{i} - 0.500\hat{j})$ m/s^2. When it reaches its maximum x coordinate, what are its (a) velocity and (b) position vector?

S

15. **THINK** Given the initial velocity and acceleration of a particle, we're interested in finding its velocity and position at a later time.

EXPRESS Since the acceleration, $\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} = (-1.0\ \mathrm{m/s^2})\hat{\mathrm{i}} + (-0.50\ \mathrm{m/s^2})\hat{\mathrm{j}}$, is constant in both x and y directions, we may use Table 2-1 for the motion along each direction. This can be handled individually (for x and y) or together with the unit-vector notation (for $\Delta\vec{r}$).

Since the particle started at the origin, the coordinates of the particle at any time t are given by $\vec{r} = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$. The velocity of the particle at any time t is given by $\vec{v} = \vec{v}_0 + \vec{a}t$, where $\vec{v}_0$ is the initial velocity and $\vec{a}$ is the (constant) acceleration. Along the x-direction, we have

$$x(t) = v_{0x}t + \frac{1}{2}a_x t^2, \qquad v_x(t) = v_{0x} + a_x t$$

Similarly, along the y-direction, we get

$$y(t) = v_{0y}t + \frac{1}{2}a_y t^2, \qquad v_y(t) = v_{0y} + a_y t\,.$$

Known: $v_{0x} = 3.0\ \mathrm{m/s}$, $v_{0y} = 0$, $a_x = -1.0\ \mathrm{m/s^2}$, $a_y = -0.5\ \mathrm{m/s^2}$.

ANALYZE (a) Substituting the values given, the components of the velocity are

$$v_x(t) = v_{0x} + a_x t = (3.0\ \mathrm{m/s}) - (1.0\ \mathrm{m/s^2})t$$
$$v_y(t) = v_{0y} + a_y t = -(0.50\ \mathrm{m/s^2})t$$

When the particle reaches its maximum x coordinate at $t = t_m$, we must have $v_x = 0$. Therefore, $3.0 - 1.0t_m = 0$ or $t_m = 3.0$ s. The y component of the velocity at this time is

$$v_y(t=3.0\,\text{s}) = -(0.50\,\text{m/s}^2)(3.0) = -1.5\,\text{m/s}$$

Thus, $\vec{v}_m = (-1.5\,\text{m/s})\hat{\text{j}}$.

(b) At $t = 3.0$ s , the components of the position are

$$x(t=3.0\,\text{s}) = v_{0x}t + \frac{1}{2}a_x t^2 = (3.0\,\text{m/s})(3.0\,\text{s}) + \frac{1}{2}(-1.0\,\text{m/s}^2)(3.0\,\text{s})^2 = 4.5\ \text{m}$$
$$y(t=3.0\,\text{s}) = v_{0y}t + \frac{1}{2}a_y t^2 = 0 + \frac{1}{2}(-0.5\,\text{m/s}^2)(3.0\,\text{s})^2 = -2.25\ \text{m}$$

Using unit-vector notation, the results can be written as $\vec{r}_m = (4.50\text{ m})\hat{\text{i}} - (2.25\text{ m})\hat{\text{j}}$.

LEARN The motion of the particle in this problem is two-dimensional, and the kinematics in the x- and y-directions can be analyzed separately.

••16 The velocity $\vec{v}$ of a particle moving in the xy plane is given by $\vec{v} = (6.0t - 4.0t^2)\hat{i} + 8.0\hat{j}$, with $\vec{v}$ in meters per second and t (> 0) in seconds. (a) What is the acceleration when $t = 3.0$ s? (b) When (if ever) is the acceleration zero? (c) When (if ever) is the velocity zero? (d) When (if ever) does the speed equal 10 m/s?

(a) The acceleration as a function of time is

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\left(6.0t - 4.0t^2\right)\hat{\mathrm{i}} + 8.0\,\hat{\mathrm{j}}\right) = (6.0 - 8.0t)\,\hat{\mathrm{i}}$$

in SI units. Specifically, we find the acceleration vector at $t = 3.0\,\mathrm{s}$ to be $(6.0 - 8.0(3.0))\,\hat{\mathrm{i}} = (-18\ \mathrm{m/s^2})\hat{\mathrm{i}}.$

(b) The equation is $\vec{a} = (6.0 - 8.0t)\,\hat{\mathrm{i}} = 0$; we find $t = 0.75$ s.

(c) Since the y component of the velocity, $v_y = 8.0$ m/s, is never zero, the velocity cannot vanish.

(d) Since speed is the magnitude of the velocity, we have

$$v = |\vec{v}| = \sqrt{\left(6.0t - 4.0t^2\right)^2 + (8.0)^2} = 10$$

in SI units (m/s). To solve for t, we first square both sides of the above equation, followed by some rearrangement:

$$\left(6.0t - 4.0t^2\right)^2 + 64 = 100 \;\Rightarrow\; \left(6.0t - 4.0t^2\right)^2 = 36$$

Taking the square root of the new expression and making further simplification lead to

$$6.0t - 4.0t^2 = \pm 6.0 \;\Rightarrow\; 4.0t^2 - 6.0t \pm 6.0 = 0$$

Finally, using the quadratic formula, we obtain

$$t = \frac{6.0 \pm \sqrt{36 - 4(4.0)(\pm 6.0)}}{2(8.0)}$$

where the requirement of a real positive result leads to the unique answer: $t = 2.2$ s.

••17 A cart is propelled over an xy plane with acceleration components $a_x = 4.0\ \mathrm{m/s^2}$ and $a_y = -2.0\ \mathrm{m/s^2}$. Its initial velocity has components $v_{0x} = 8.0\ \mathrm{m/s}$ and $v_{0y} = 12\ \mathrm{m/s}$. In unit-vector notation, what is the velocity of the cart when it reaches its greatest y coordinate?

17. We find t by applying Eq. 2-11 to motion along the y axis (with $v_y = 0$ characterizing $y = y_{max}$):

$$0 = (12\ \mathrm{m/s}) + (-2.0\ \mathrm{m/s^2})t \;\Rightarrow\; t = 6.0\ \mathrm{s}.$$

Then, Eq. 2-11 applies to motion along the x axis to determine the answer:

$$v_x = (8.0\ \mathrm{m/s}) + (4.0\ \mathrm{m/s^2})(6.0\ \mathrm{s}) = 32\ \mathrm{m/s}.$$

Therefore, the velocity of the cart, when it reaches $y = y_{max}$, is $(32\ \mathrm{m/s})\hat{\mathrm{i}}$.

•••19 The acceleration of a particle moving only on a horizontal xy plane is given by $\vec{a} = 3t\hat{i} + 4t\hat{j}$, where $\vec{a}$ is in meters per second-squared and t is in seconds. At $t = 0$, the position vector $\vec{r} = (20.0\text{ m})\hat{i} + (40.0\text{ m})\hat{j}$ locates the particle, which then has the velocity vector $\vec{v} = (5.00\text{ m/s})\hat{i} + (2.00\text{ m/s})\hat{j}$. At $t = 4.00$ s, what are (a) its position vector in unit-vector notation and (b) the angle between its direction of travel and the positive direction of the x axis?

PROBLEM

••17 A cart is propelled over an xy plane with acceleration components $a_x = 4.0\ \text{m/s}^2$ and $a_y = -2.0\ \text{m/s}^2$. Its initial velocity has components $v_{0x} = 8.0$ m/s and $v_{0y} = 12$ m/s. In unit-vector notation, what is the velocity of the cart when it reaches its greatest y coordinate?

Using $\vec{a} = 3t\hat{\mathrm{i}} + 4t\hat{\mathrm{j}}$, we have (in m/s)

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}\ dt = (5.00\hat{\mathrm{i}} + 2.00\hat{\mathrm{j}}) + \int_0^t (3t\hat{\mathrm{i}} + 4t\hat{\mathrm{j}})\ dt = \left(5.00 + 3t^2/2\right)\hat{\mathrm{i}} + \left(2.00 + 2t^2\right)\hat{\mathrm{j}}$$

Integrating using Eq. 4-10 then yields (in meters)

SOLUTION

$$\vec{r}(t) = \vec{r}_0 + \int_0^t \vec{v}\,dt = (20.0\hat{\mathrm{i}} + 40.0\hat{\mathrm{j}}) + \int_0^t [(5.00 + 3t^2/2)\hat{\mathrm{i}} + (2.00 + 2t^2)\hat{\mathrm{j}}]dt$$
$$= (20.0\hat{\mathrm{i}} + 40.0\hat{\mathrm{j}}) + (5.00t + t^3/2)\hat{\mathrm{i}} + (2.00t + 2t^3/3)\hat{\mathrm{j}}$$
$$= (20.0 + 5.00t + t^3/2)\hat{\mathrm{i}} + (40.0 + 2.00t + 2t^3/3)\hat{\mathrm{j}}$$

(a) At $t = 4.00$ s, we have $\vec{r}(t = 4.00\,\mathrm{s}) = (72.0\ \mathrm{m})\hat{\mathrm{i}} + (90.7\ \mathrm{m})\hat{\mathrm{j}}$.

(b) $\vec{v}(t = 4.00\,\mathrm{s}) = (29.0\ \mathrm{m/s})\hat{\mathrm{i}} + (34.0\ \mathrm{m/s})\hat{\mathrm{j}}$. Thus, the angle between the direction of travel and $+x$, measured counterclockwise, is $\theta = \tan^{-1}[(34.0\,\mathrm{m/s})/(29.0\,\mathrm{m/s})] = 49.5°$.

SOLUTION CONTINUE

THANK YOU

Kinematics: Driving Realistic Motion in Computer Science

Kinematics bridges physics and computation. It creates dynamic virtual worlds. It is essential for animation, robotics, and simulation. This presentation explores how kinematics unlocks natural movement in digital spaces.

A by Aisha Ijaz

CHAPTER 4

Motion in two and three dimensions , Projectile motion,

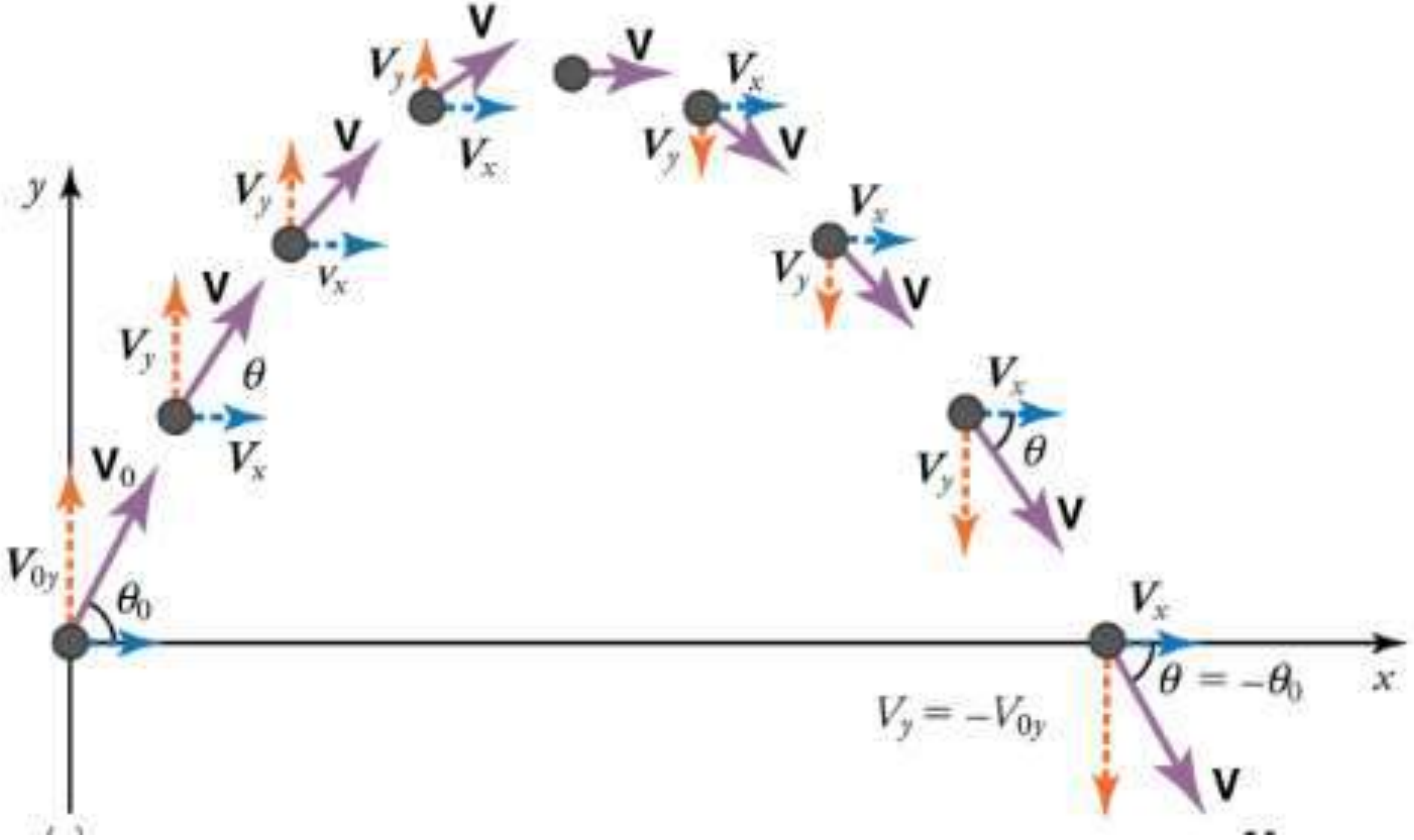
Application in Computer Sciences

- Object position control in computers games and computer animation.

Projectile Motion

A projectile is a particle moving near the Earth's surface under the influence of its weight only (directed downward).

Projectile motion is an extension of the free-fall motion. We will continue to neglect the influence of air resistance, leading to results that are a good approximation of reality for relatively heavy objects moving relatively slowly over relatively short distances.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y
x
$\theta = -\theta_0$
$V_y = -V_{0y}$
V

Trajectory of Projectile

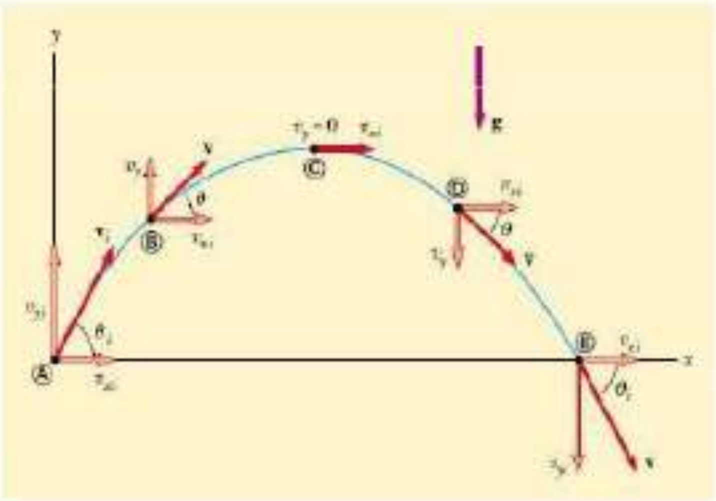
let us assume that at t = 0, the projectile leaves the origin (0, 0) with speed v_0 , as shown in Figure and vector v_0 makes an angle θ with the horizontal.

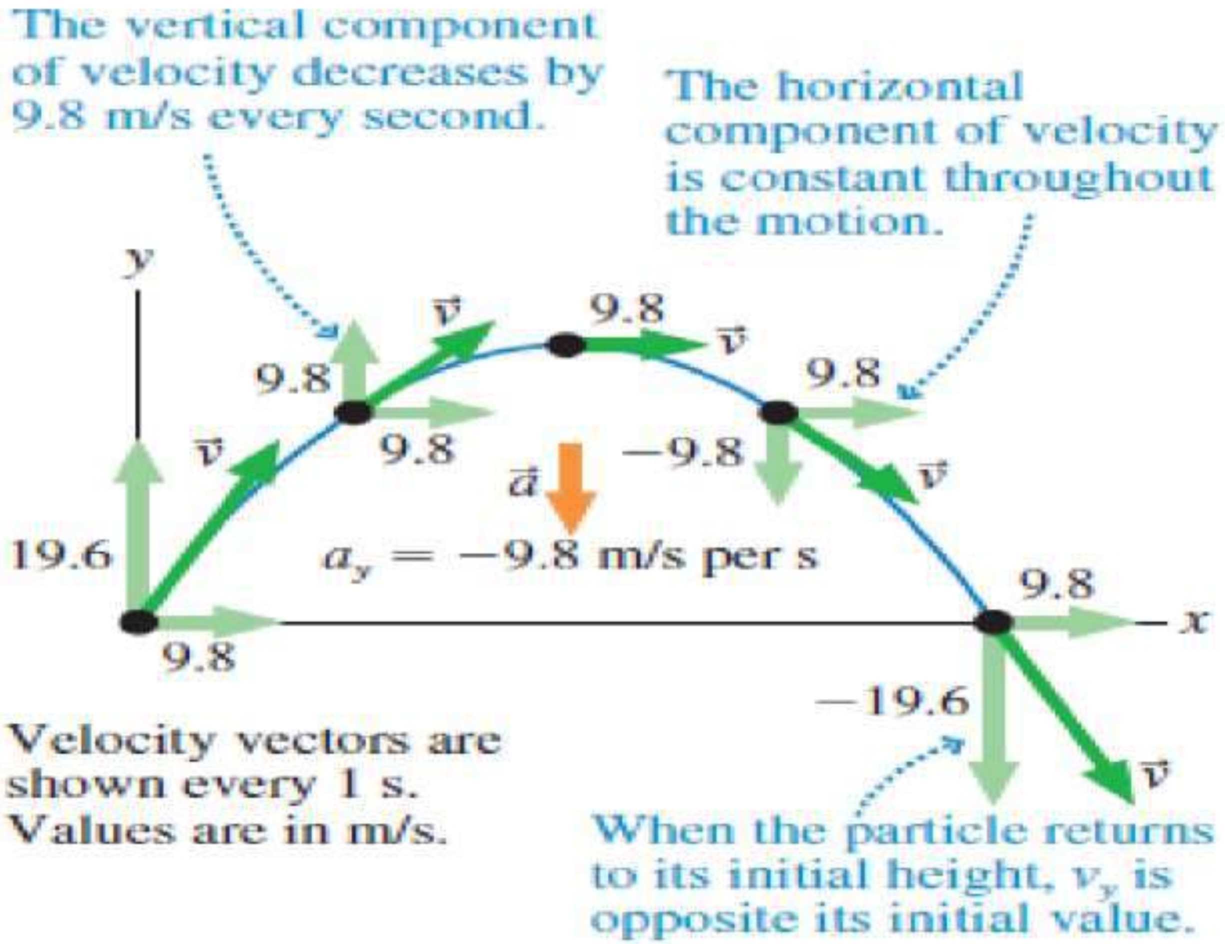
$$\vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j}$$

$$v_{0x} = v_0 \cos\theta$$

$$v_{0y} = v_0 \sin\theta$$

$$a_y = -g \quad a_x = 0$$





$$x = v_{0x}t + \frac{1}{2}a_x t^2 = (v_0 \cos\theta)t \qquad (1)$$

$$y = v_{0y}t + \frac{1}{2}a_y t^2 = (v_0 \sin\theta)t - \frac{1}{2}gt^2 \qquad (2)$$

From (1) $t = x / v_0 \cos\theta$

So (2) $$y = (v_0 \sin\theta)\frac{x}{v_0 \cos\theta} - \frac{1}{2}g\frac{x^2}{v_0^2 \cos^2\theta}$$

$$\boxed{y = (\tan\theta)x - \frac{g}{2v_0^2 \cos^2\theta}x^2}$$

Parabola

Time of Flight

- The time of flight **t** of a projectile is the time interval the projectile has traversed when it returns to it initial (launch) height.
- Substitute y=0 in equation (2)

$$0 = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

$$t = 0$$ **Launched time**

$$\boxed{t = \frac{2v_0 \sin\theta}{g}}$$ **Returning time**

Horizontal Range

- The horizontal range R of a projectile is the horizontal distance the projectile has travelled when it returns to it initial (launch) height.
- Substitute x=R in equation (1)

$$R = (v_0 \cos\theta)t$$

$$R = (v_0 \cos\theta)\frac{2v_0 \sin\theta}{g}$$

$$\boxed{R = \frac{v_0^2 \sin 2\theta}{g}}$$

Maximum Height

- Time taken by projectile to reach maximum height will be half of time of flight, that is,

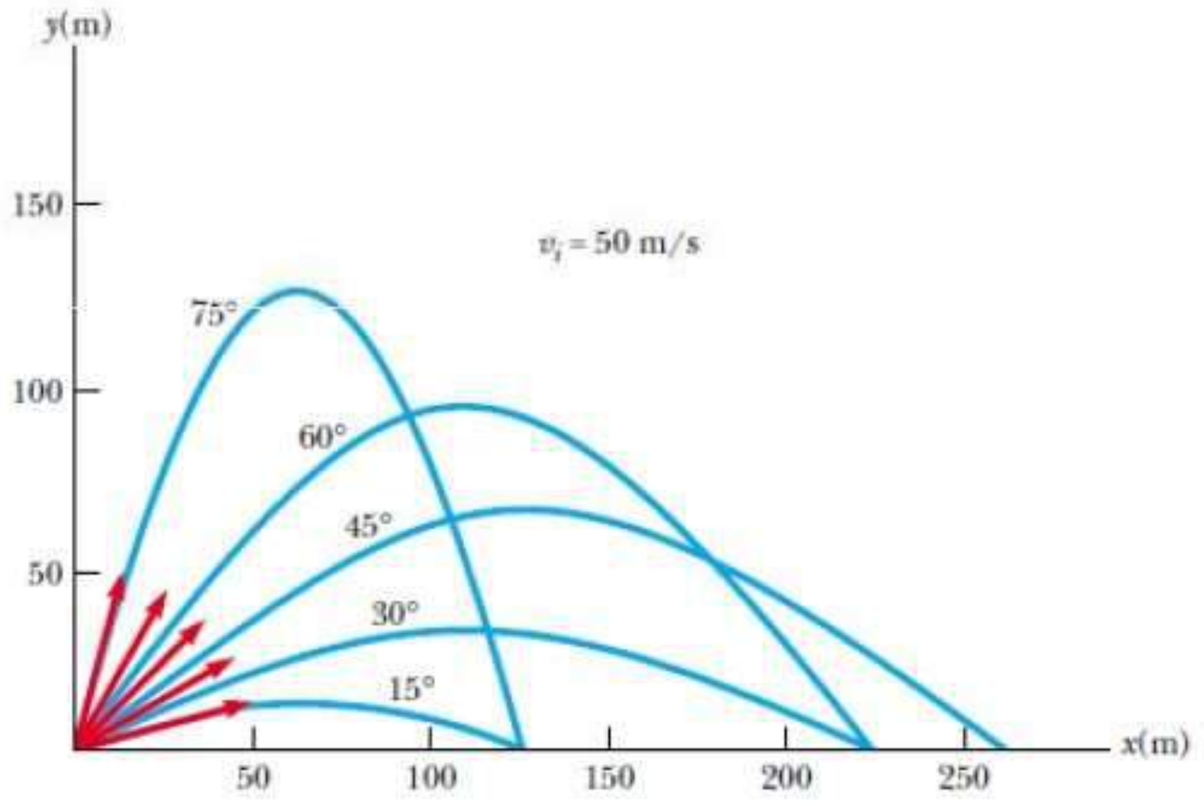
$$t = \frac{v_0 \sin\theta}{g}$$

- Substitute y=H in equation (2)

$$H = (v_0 \sin\theta)\frac{v_0 \sin\theta}{g} - \frac{1}{2} g \frac{v_0^2 \sin^2\theta}{g^2}$$

$$\boxed{H = \frac{v_0^2 \sin^2\theta}{2g}}$$

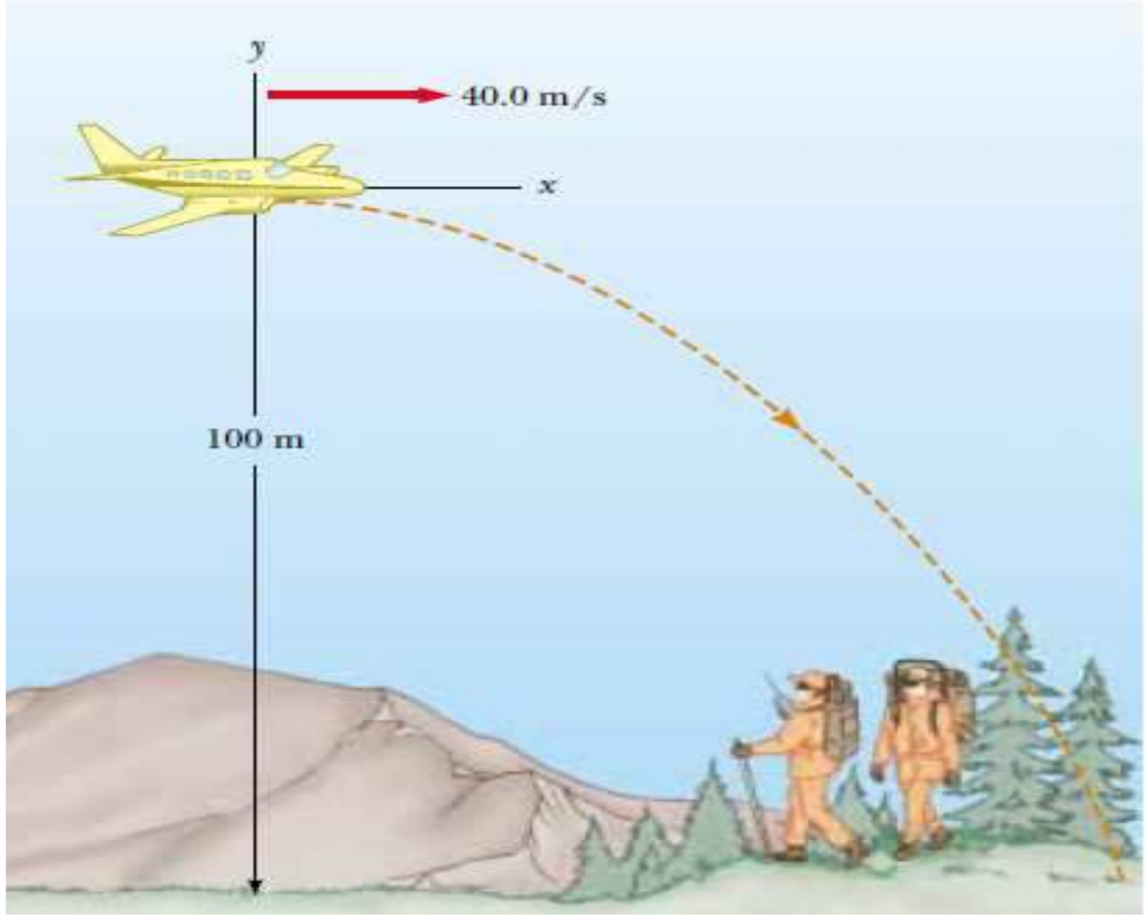
A projectile launched from the origin with an initial speed of 50 m/s at various angles of projection. Note that complementary values of θ result in the same value of R (range of the projectile).



PROBLEM SOLVING

Projectile motion

A plane drops a package of supplies to a party of explorers, as shown in Figure 4.15. If the plane is traveling horizontally at 40.0 m/s and is 100 m above the ground, where does the package strike the ground relative to the point at which it is released?



PROBLEM

SOLUTION

$$x_f = (40.0\ \mathrm{m/s})(4.52\ \mathrm{s}) = \boxed{181\ \mathrm{m}}$$

$$y_f = -\tfrac{1}{2}gt^2$$

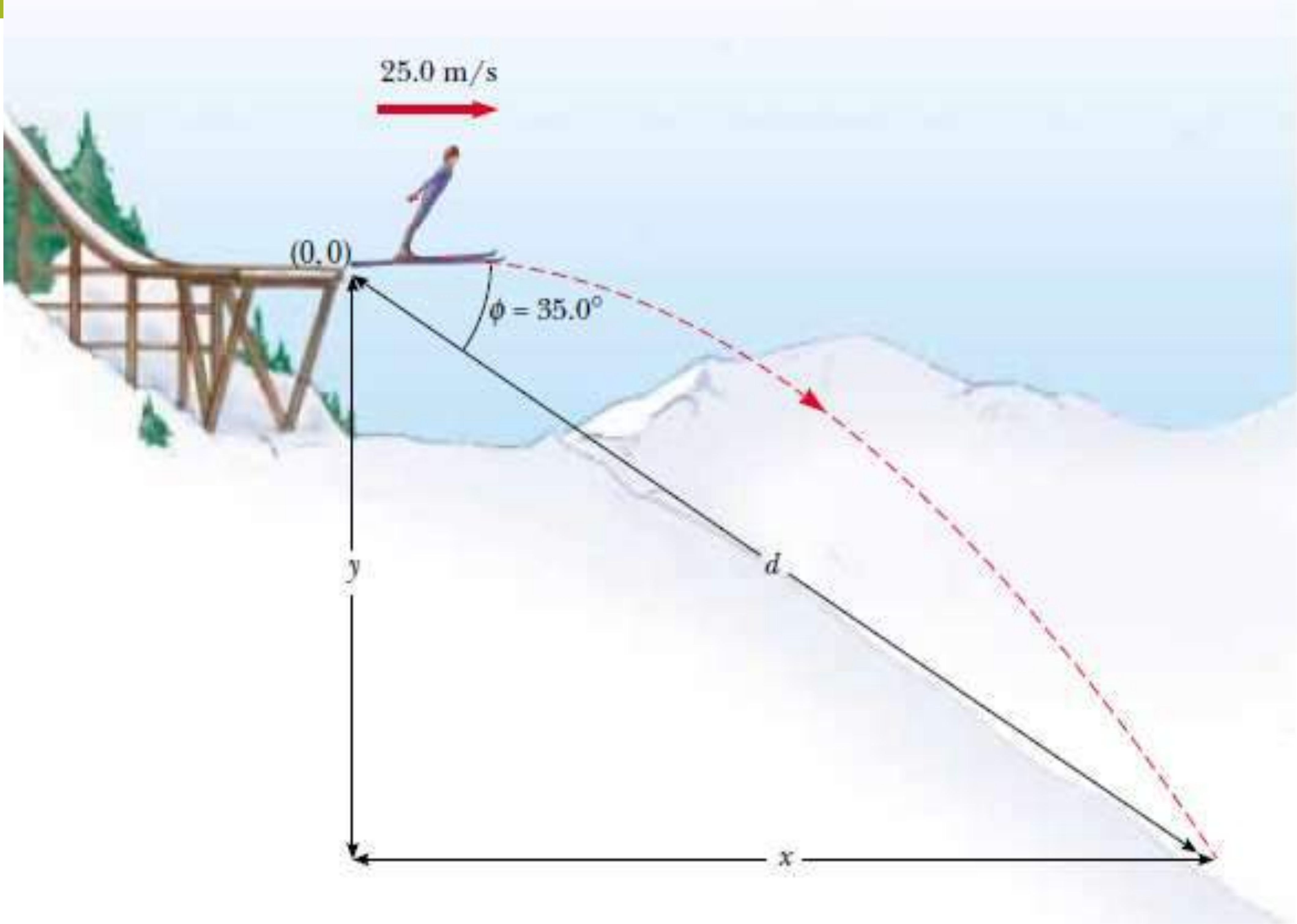
$$-100\ \mathrm{m} = -\tfrac{1}{2}(9.80\ \mathrm{m/s^2})t^2$$

$$t = 4.52\,\mathrm{s}$$

$$x_f = (40.0\ \mathrm{m/s})\,t$$

A ski-jumper leaves the ski track moving in the horizontal direction with a speed of 25.0 m/s, as shown in Figure 4.16. The landing incline below him falls off with a slope of 35.0°. Where does he land on the incline?

PROBLEM


25.0 m/s
(0, 0)
$\phi = 35.0°$
y
d
x

$$(1) \qquad x_f = v_{xi}t = (25.0\ \text{m/s})t$$

$$(2) \qquad y_f = v_{yi}t + \tfrac{1}{2}a_y t^2 = -\tfrac{1}{2}(9.80\ \text{m/s}^2)t^2$$

From the right triangle in Figure 4.16, we see that the jumper's x and y coordinates at the landing point are $x_f = d\cos 35.0°$ and $y_f = -d\sin 35.0°$. Substituting these relationships into (1) and (2), we obtain

$$(3) \qquad d\cos 35.0° = (25.0\ \text{m/s})t$$

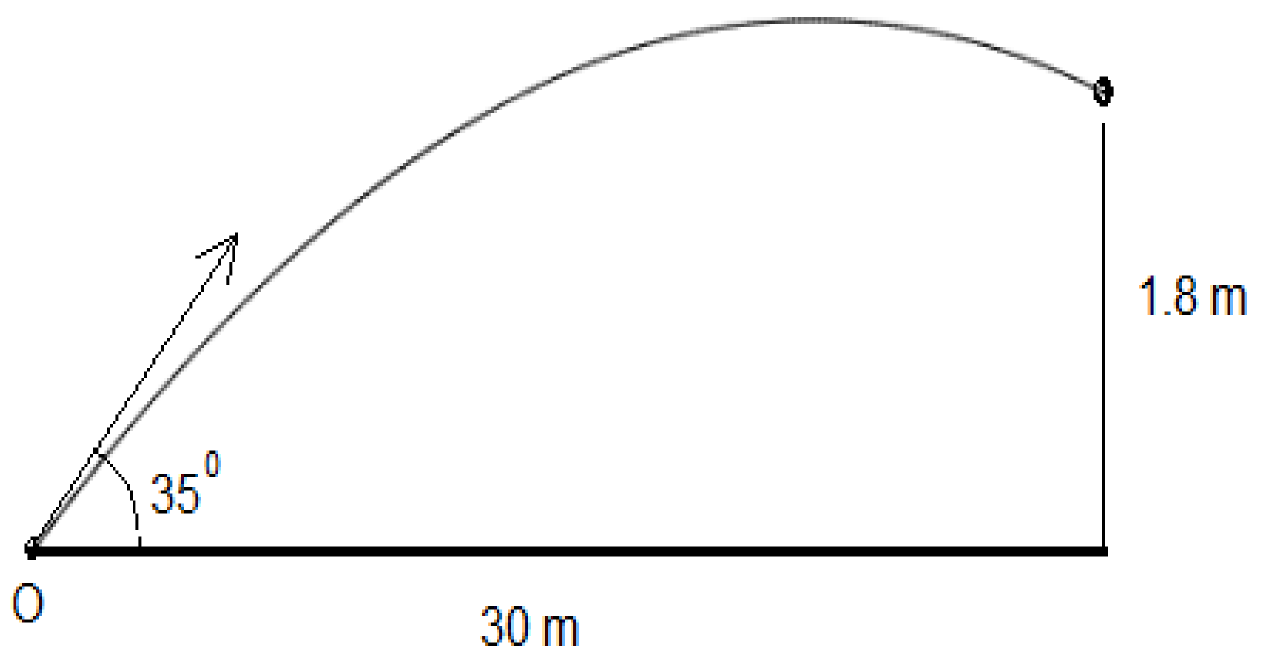
$$(4) \qquad -d\sin 35.0° = -\tfrac{1}{2}(9.80\ \text{m/s}^2)t^2$$

Solving (3) for t and substituting the result into (4), we find that $d = 109$ m. Hence, the x and y coordinates of the point at which the skier lands are

$$x_f = d\cos 35.0° = (109\ \text{m})\cos 35.0° = \boxed{89.3\ \text{m}}$$

$$y_f = -d\sin 35.0° = -(109\ \text{m})\sin 35.0° = \boxed{-62.5\ \text{m}}$$

A ball is kicked at an angle of 35° with the ground.
a) What should be the initial velocity of the ball so that it hits a target that is 30 meters away at a height of 1.8 meters?
b) What is the time for the ball to reach the target?



PROBLEM

Solution

a)

$x = V_0 \cos(35°)\, t$

$30 = V_0 \cos(35°)\, t$

$t = 30 / V_0 \cos(35°)$

$1.8 = -(1/2)\, 9.8\, (30 / V_0 \cos(35°))^2 + V_0 \sin(35°)(30 / V_0 \cos(35°))$

$V_0 \cos(35°) = 30 \sqrt{[\, 9.8 / 2(30 \tan(35°)-1.8)\,]}$

$V_0 = 18.3$ m/s

b)

$t = x / V_0 \cos(35°) = 2.0$ s

••28 GO In Fig. 4-34, a stone is projected at a cliff of height h with an initial speed of 42.0 m/s directed at angle $\theta_0 = 60.0°$ above the horizontal. The stone strikes at A, 5.50 s after launching. Find (a) the height h of the cliff, (b) the speed of the stone just before impact at A, and (c) the maximum height H reached above the ground.

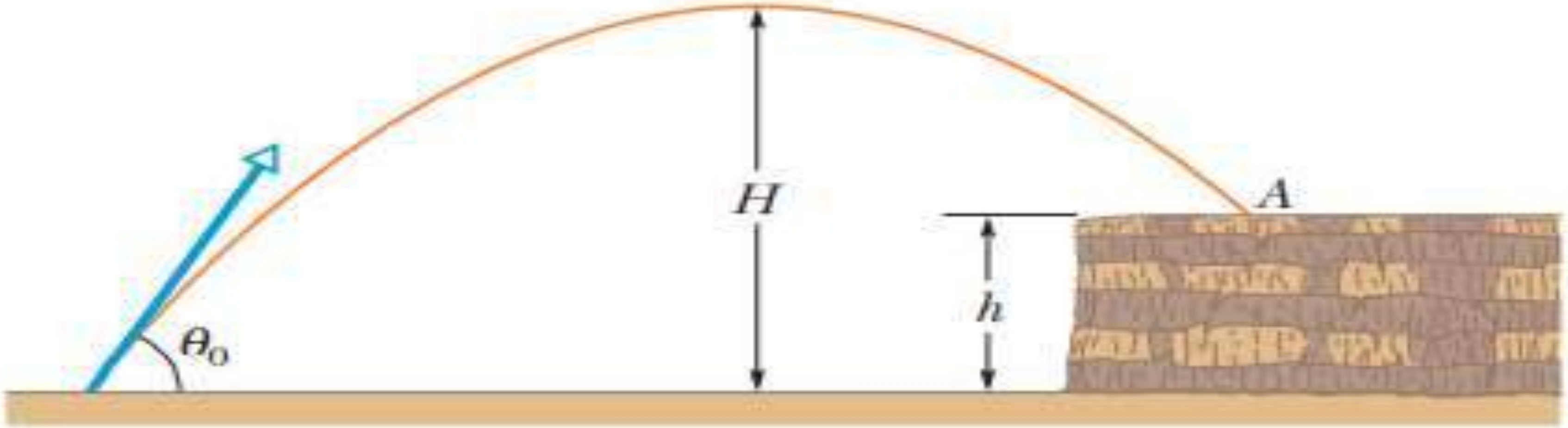


Figure 4-34 Problem 28.

PROBLEM

28. (a) Using the same coordinate system assumed in Eq. 4-22, we solve for $y = h$:

$$h = y_0 + v_0 \sin\theta_0 t - \frac{1}{2} g t^2$$

which yields $h = 51.8$ m for $y_0 = 0$, $v_0 = 42.0$ m/s, $\theta_0 = 60.0°$, and $t = 5.50$ s.

(b) The horizontal motion is steady, so $v_x = v_{0x} = v_0 \cos\theta_0$, but the vertical component of velocity varies according to Eq. 4-23. Thus, the speed at impact is

$$v = \sqrt{(v_0 \cos\theta_0)^2 + (v_0 \sin\theta_0 - gt)^2} = 27.4 \text{ m/s}.$$

(c) We use Eq. 4-24 with $v_y = 0$ and $y = H$:

$$H = \frac{(v_0 \sin\theta_0)^2}{2g} = 67.5 \text{ m}.$$

SOLUTION

THANK YOU

CHAPTER 4

Motion in two and three dimensions , Projectile motion, Uniform circular motion

Application in Computer Sciences

- Object position control in computers games and computer animation.

Object Position Control in Computer Games

Motion control over different objects is an important factor in computer games. It enhances the quality of game and gives an interactive feeling to user.

Improved mathematical models and algorithms can be used to control position of objects in computer games, which can increase game quality and gamer's satisfaction.

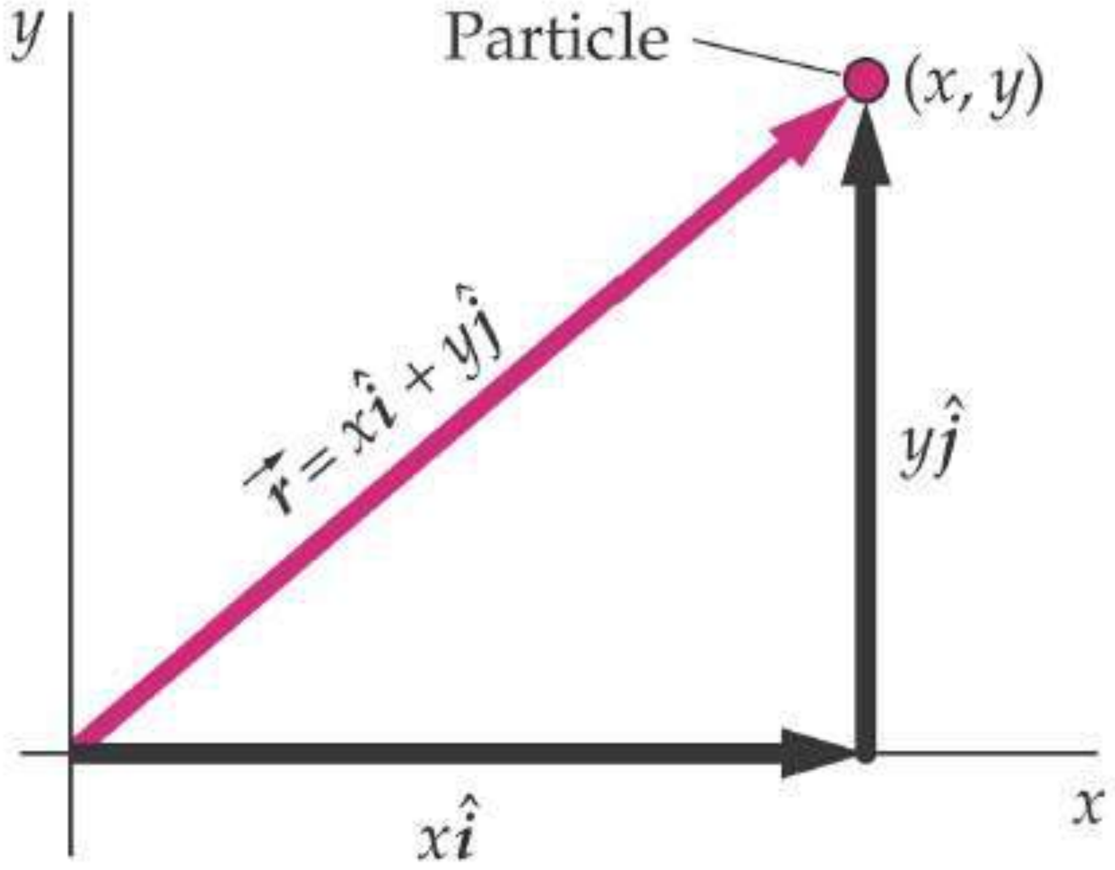
Real life examples of projectile motion

- A football kicked in a game.
- A cannonball fired from a cannon.
- A bullet fired from a gun.
- A disc thrown in the sport of discus throw.
- The flight of a golf ball.
- A jet of water escaping a hose.
- Motorcycles and cars jumping in extreme sports.

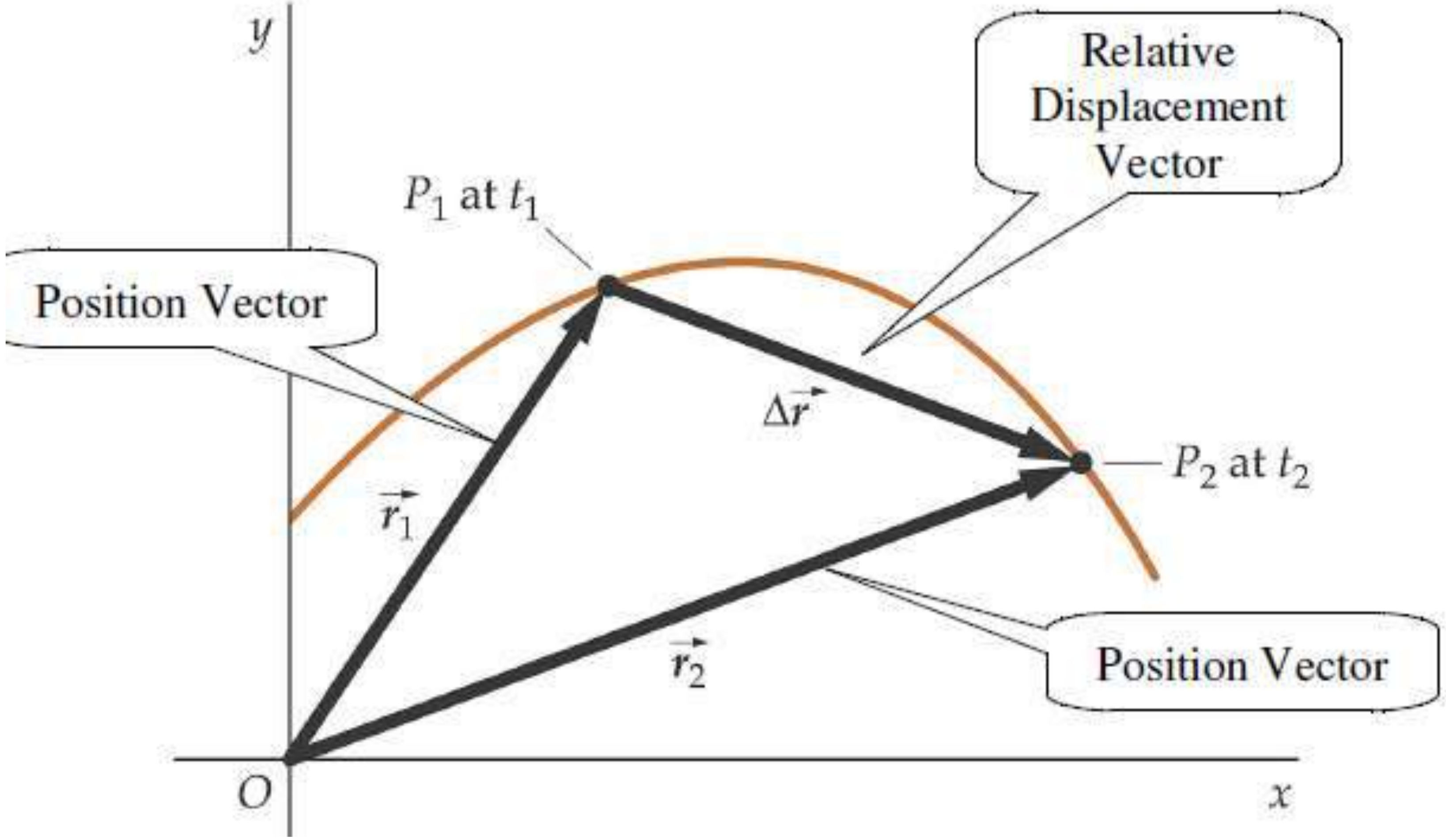
Motion in Two and Three Dimensions

- Displacement, Velocity and Speed
- Relative Motion
- Projectile Motion
- Circular Motion

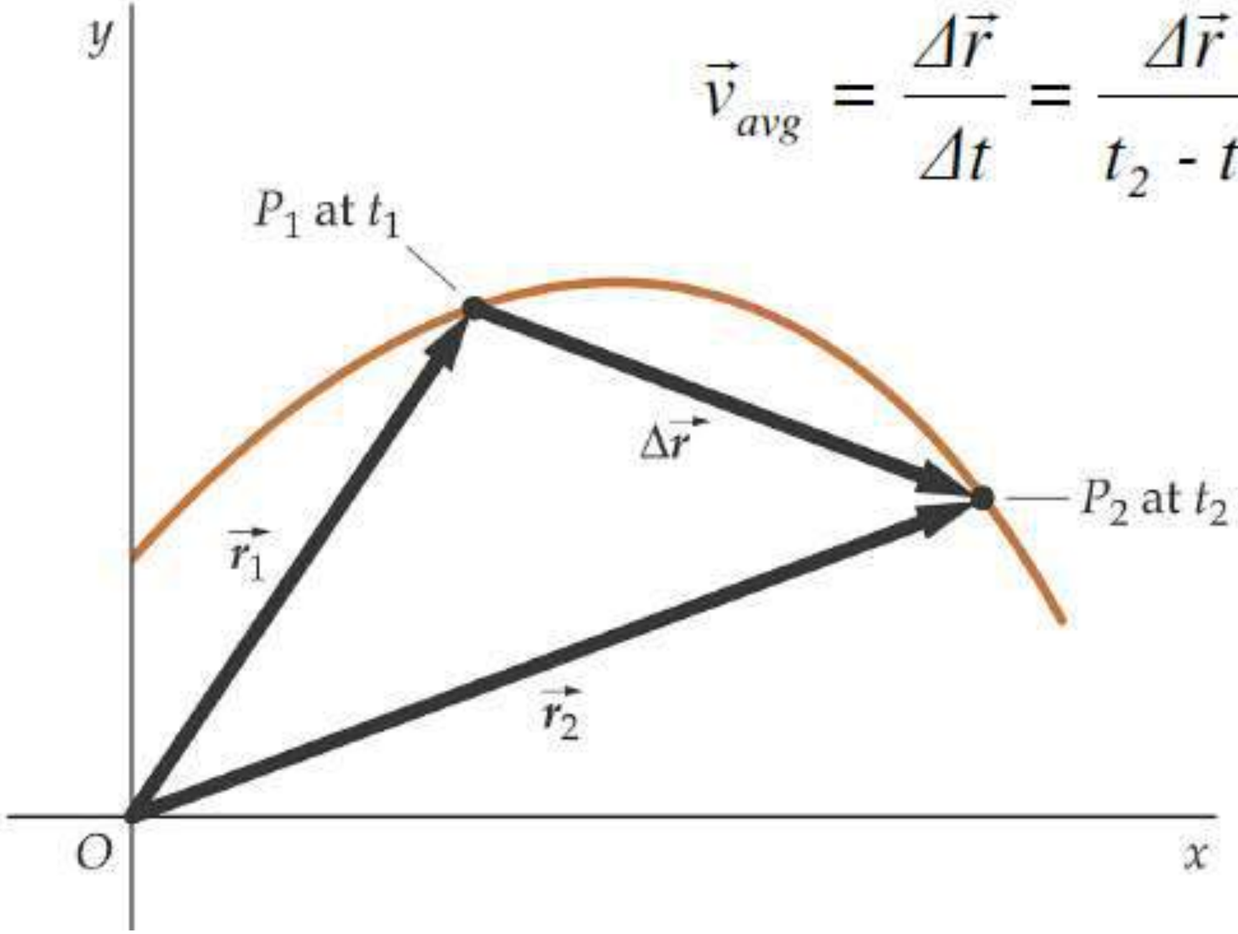
Particle Displacement



Relative Displacement



Average Velocity

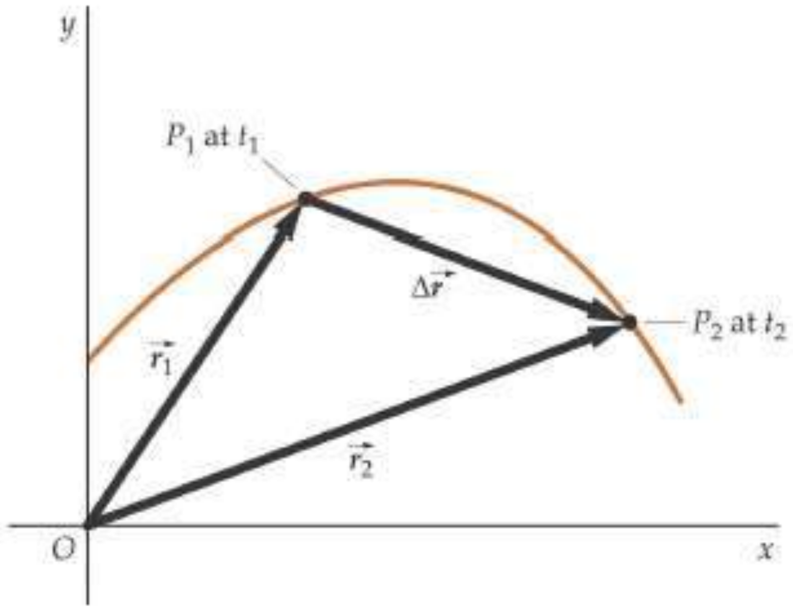


$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta\vec{r}}{t_2 - t_1}$$

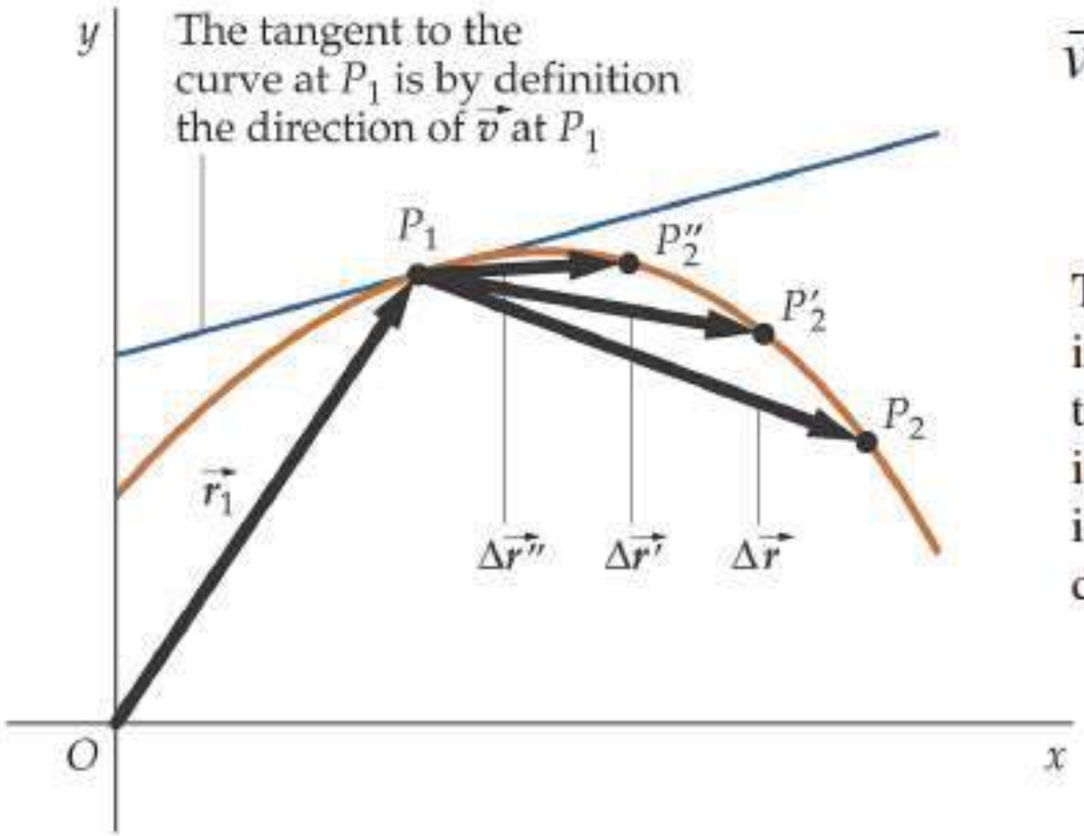
Average Velocity

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta\vec{r}}{t_2 - t_1} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{(x_2\hat{i} + y_2\hat{j}) - (x_1\hat{i} + y_1\hat{j})}{t_2 - t_1}$$

$$\vec{v}_{avg} = \frac{(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}}{t_2 - t_1}$$



Instantaneous Velocity Vector



$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$

There is not enough information presented here to actually calculate the instantaneous velocity. This is meant only to demonstrate the process.

Instantaneous Velocity Vector

Instantaneous velocity

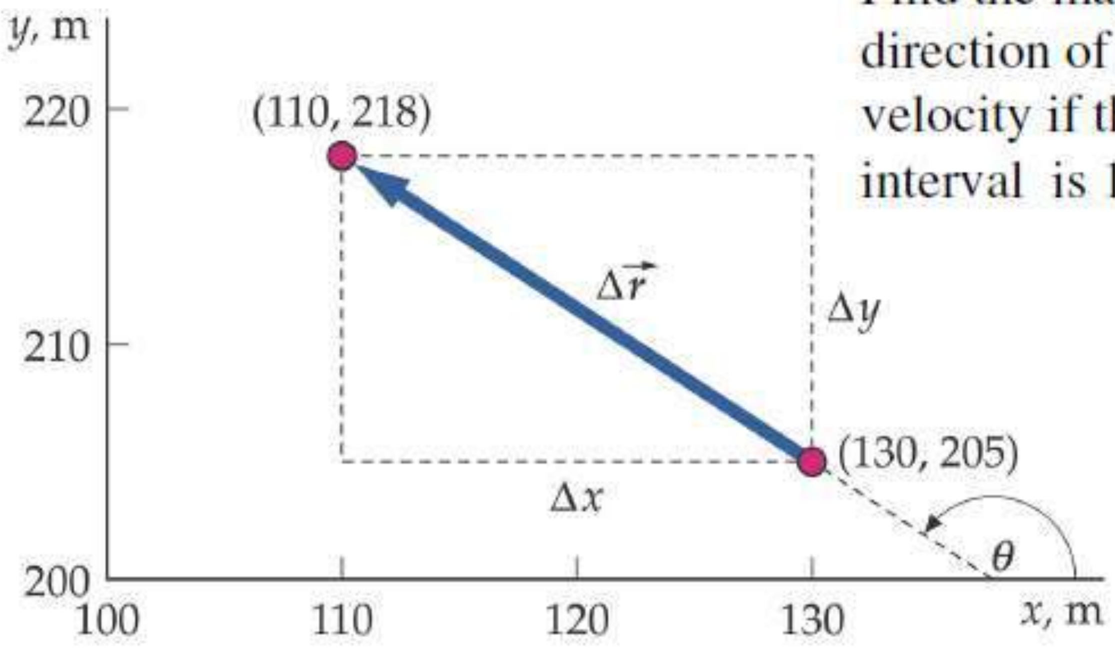
$$\vec{v}(t) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = v_x\hat{i} + v_y\hat{j}$$

Magnitude of the velocity

$$|\vec{v}(t)| = v(t) = \sqrt{v_x^{\,2} + v_y^{\,2}}$$

Direction of the velocity

$$\Theta = tan^{-1}(\frac{v_y}{v_x})$$



Ques:

Find the magnitude and direction of the average velocity if the time interval is 120s?

Problem

$$\vec{v}_{av} = v_{x\,av}\hat{i} + v_{y\,av}\hat{j}$$

where

$$v_{x\,av} = \frac{\Delta x}{\Delta t} = \frac{110\text{ m} - 130\text{ m}}{120\text{ s}} = -0.167\text{ m/s}$$

$$v_{y\,av} = \frac{\Delta y}{\Delta t} = \frac{218\text{ m} - 205\text{ m}}{120\text{ s}} = 0.108\text{ m/s}$$

so

$$\vec{v}_{av} = \boxed{-(0.167\text{ m/s})\hat{i} + (0.108\text{ m/s})\hat{j}}$$

$$v_{av} = \sqrt{(v_{x\,av})^2 + (v_{y\,av})^2} = \boxed{0.199\text{ m/s}}$$

$$\tan\theta = \frac{v_{y\,av}}{v_{x\,av}}$$

so

$$\theta = \tan^{-1}\frac{v_{y\,av}}{v_{x\,av}} = \tan^{-1}\frac{0.108\text{ m/s}}{-0.167\text{ m/s}} = -33.0^\circ + 180^\circ = \boxed{147^\circ}$$

Solution

Acceleration Vectors

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t}$$ Average acceleration

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$ Instantaneous acceleration

Where:

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

1. The position of an electron is given by $\mathrm{r} = 3.0t\mathrm{i} - 4.0t^2\mathrm{j} + 2.0\mathrm{k}$ (where t is in seconds and the coefficients have the proper units for r to be in meters). (a) What is $\mathrm{v}(t)$ for the electron? (b) In unit–vector notation, what is v at $t = 2.0\,\mathrm{s}$? (c) What are the magnitude and direction of v just then? [HRW5 4-9]

Problem

Solution

(a) The velocity vector $\mathbf{v}$ is the time–derivative of the position vector $\mathbf{r}$:

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(3.0t\mathbf{i} - 4.0t^2\mathbf{j} + 2.0\mathbf{k}) \\ &= 3.0\mathbf{i} - 8.0t\mathbf{j}\end{aligned}$$

where we mean that when t is in seconds, $\mathbf{v}$ is given in $\frac{\mathrm{m}}{\mathrm{s}}$.

(b) At $t = 2.00\,\mathrm{s}$, the value of $\mathbf{v}$ is

$$\mathbf{v}(t = 2.00\,\mathrm{s}) = 3.0\mathbf{i} - (8.0)(2.0)\mathbf{j} = 3.0\mathbf{i} - 16.\mathbf{j}$$

that is, the velocity is $(3.0\mathbf{i} - 16.\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}$.

(c) Using our answer from (b), at $t = 2.00\,\mathrm{s}$ the magnitude of $\mathbf{v}$ is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{(3.00\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (-16.\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (0)^2} = 16.\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

we note that the velocity vector lies in the xy plane (even though this is a three–dimensional problem!) so that we can express its direction with a single angle, the usual angle θ measured anti-clockwise in the xy plane from the x axis. For this angle we get:

$$\tan\theta = \frac{v_y}{v_x} = -5.33 \qquad \Longrightarrow \qquad \theta = \tan^{-1}(-5.33) = -79^\circ \ .$$

When we take the inverse tangent, we should always check and see if we have chosen the right quadrant for θ. In this case -79° *is* correct since v_y is negative and v_x is positive.

Solu.continued

2. A particle moves so that its position as a function of time in SI units is $\mathrm{r} = \mathrm{i} + 4t^2\mathrm{j} + t\mathrm{k}$. Write expressions for (a) its velocity and (b) its acceleration as functions of time. [HRW5 4-11]

Problem

(a) To clarify matters, what we mean here is that when we use the numerical value of t in *seconds*, we will get the values of $\mathbf{r}$ in *meters*. Since the velocity vector is the time–derivative of the position vector $\mathbf{r}$, we have:

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \frac{d}{dt}(\mathbf{i} + 4t^2\mathbf{j} + t\mathbf{k}) \\ &= 0\mathbf{i} + 8t\mathbf{j} + \mathbf{k} \end{aligned}$$

That is, $\mathbf{v} = 8t\mathbf{j} + \mathbf{k}$. Here, we mean that when we use the numerical value of t in seconds, we will get the value of $\mathbf{v}$ in $\frac{\mathrm{m}}{\mathrm{s}}$.

Solution

So $\mathbf{a} = 8\mathbf{j}$, where we mean that the value of $\mathbf{a}$ is in units of $\frac{\mathrm{m}}{\mathrm{s}^2}$. In fact, we should really include the units *here* and write:

$$\mathbf{a} = \left(8\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)\mathbf{j}$$

(b) The acceleration $\mathbf{a}$ is the time–derivative of $\mathbf{v}$, so using our result from part (a) we have:

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= \frac{d}{dt}(8t\mathbf{j} + \mathbf{k}) \\ &= 8\mathbf{j} \end{aligned}$$

Sol.continued

Sample problem 4.1

A rabbit runs across a parking lot on which a set of coordinate axes has, strangely enough, been drawn. The coordinates (meters) of the rabbit's position as functions of time t (seconds) are given by

$$x = -0.31t^2 + 7.2t + 28 \tag{4-5}$$

and

$$y = 0.22t^2 - 9.1t + 30. \tag{4-6}$$

(a) At $t = 15$ s, what is the rabbit's position vector $\vec{r}$ in unit-vector notation and in magnitude-angle notation?

Solution

Calculations: We can write

$$\vec{r}(t) = x(t)\hat{\mathrm{i}} + y(t)\hat{\mathrm{j}}. \qquad (4\text{-}7)$$

(We write $\vec{r}(t)$ rather than $\vec{r}$ because the components are functions of t, and thus $\vec{r}$ is also.)

At $t = 15$ s, the scalar components are

$$x = (-0.31)(15)^2 + (7.2)(15) + 28 = 66 \text{ m}$$

and $$y = (0.22)(15)^2 - (9.1)(15) + 30 = -57 \text{ m},$$

so $$\vec{r} = (66 \text{ m})\hat{\mathrm{i}} - (57 \text{ m})\hat{\mathrm{j}}, \qquad \text{(Answer)}$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(66 \text{ m})^2 + (-57 \text{ m})^2}$$
$$= 87 \text{ m}, \qquad \text{(Answer)}$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-57 \text{ m}}{66 \text{ m}}\right) = -41^\circ. \qquad \text{(Answer)}$$

P15

••15 SSM ILW A particle leaves the origin with an initial velocity $\vec{v} = (3.00\hat{i})$ m/s and a constant acceleration $\vec{a} = (-1.00\hat{i} - 0.500\hat{j})$ m/s^2. When it reaches its maximum x coordinate, what are its (a) velocity and (b) position vector?

S

15. **THINK** Given the initial velocity and acceleration of a particle, we're interested in finding its velocity and position at a later time.

EXPRESS Since the acceleration, $\vec{a} = a_x\hat{\text{i}} + a_y\hat{\text{j}} = (-1.0\text{ m/s}^2)\hat{\text{i}} + (-0.50\text{ m/s}^2)\hat{\text{j}}$, is constant in both x and y directions, we may use Table 2-1 for the motion along each direction. This can be handled individually (for x and y) or together with the unit-vector notation (for $\Delta\vec{r}$).

Since the particle started at the origin, the coordinates of the particle at any time t are given by $\vec{r} = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$. The velocity of the particle at any time t is given by $\vec{v} = \vec{v}_0 + \vec{a}t$, where $\vec{v}_0$ is the initial velocity and $\vec{a}$ is the (constant) acceleration. Along the x-direction, we have

$$x(t) = v_{0x}t + \frac{1}{2}a_x t^2, \qquad v_x(t) = v_{0x} + a_x t$$

Similarly, along the y-direction, we get

$$y(t) = v_{0y}t + \frac{1}{2}a_y t^2, \qquad v_y(t) = v_{0y} + a_y t\ .$$

Known: $v_{0x} = 3.0\text{ m/s}$, $v_{0y} = 0$, $a_x = -1.0\text{ m/s}^2$, $a_y = -0.5\text{ m/s}^2$.

ANALYZE (a) Substituting the values given, the components of the velocity are

$$v_x(t) = v_{0x} + a_x t = (3.0\text{ m/s}) - (1.0\text{ m/s}^2)t$$
$$v_y(t) = v_{0y} + a_y t = -(0.50\text{ m/s}^2)t$$

When the particle reaches its maximum x coordinate at $t = t_m$, we must have $v_x = 0$. Therefore, $3.0 - 1.0t_m = 0$ or $t_m = 3.0$ s. The y component of the velocity at this time is

$$v_y(t=3.0\,\text{s}) = -(0.50\,\text{m/s}^2)(3.0) = -1.5\,\text{m/s}$$

Thus, $\vec{v}_m = (-1.5\,\text{m/s})\hat{\text{j}}$.

(b) At $t = 3.0$ s , the components of the position are

$$x(t=3.0\,\text{s}) = v_{0x}t + \frac{1}{2}a_x t^2 = (3.0\,\text{m/s})(3.0\,\text{s}) + \frac{1}{2}(-1.0\,\text{m/s}^2)(3.0\,\text{s})^2 = 4.5\ \text{m}$$
$$y(t=3.0\,\text{s}) = v_{0y}t + \frac{1}{2}a_y t^2 = 0 + \frac{1}{2}(-0.5\,\text{m/s}^2)(3.0\,\text{s})^2 = -2.25\ \text{m}$$

Using unit-vector notation, the results can be written as $\vec{r}_m = (4.50\ \text{m})\,\hat{\text{i}} - (2.25\ \text{m})\,\hat{\text{j}}$.

LEARN The motion of the particle in this problem is two-dimensional, and the kinematics in the x- and y-directions can be analyzed separately.

••16 The velocity $\vec{v}$ of a particle moving in the xy plane is given by $\vec{v} = (6.0t - 4.0t^2)\hat{i} + 8.0\hat{j}$, with $\vec{v}$ in meters per second and t (> 0) in seconds. (a) What is the acceleration when $t = 3.0$ s? (b) When (if ever) is the acceleration zero? (c) When (if ever) is the velocity zero? (d) When (if ever) does the speed equal 10 m/s?

(a) The acceleration as a function of time is

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\left(6.0t - 4.0t^2\right)\hat{\mathrm{i}} + 8.0\,\hat{\mathrm{j}}\right) = \left(6.0 - 8.0t\right)\hat{\mathrm{i}}$$

in SI units. Specifically, we find the acceleration vector at $t = 3.0\,\mathrm{s}$ to be $\left(6.0 - 8.0(3.0)\right)\hat{\mathrm{i}} = (-18\ \mathrm{m/s^2})\hat{\mathrm{i}}$.

(b) The equation is $\vec{a} = \left(6.0 - 8.0t\right)\hat{\mathrm{i}} = 0$; we find $t = 0.75$ s.

(c) Since the y component of the velocity, $v_y = 8.0$ m/s, is never zero, the velocity cannot vanish.

(d) Since speed is the magnitude of the velocity, we have

$$v = |\vec{v}| = \sqrt{\left(6.0t - 4.0t^2\right)^2 + \left(8.0\right)^2} = 10$$

in SI units (m/s). To solve for t, we first square both sides of the above equation, followed by some rearrangement:

$$\left(6.0t - 4.0t^2\right)^2 + 64 = 100 \;\Rightarrow\; \left(6.0t - 4.0t^2\right)^2 = 36$$

Taking the square root of the new expression and making further simplification lead to

$$6.0t - 4.0t^2 = \pm 6.0 \;\Rightarrow\; 4.0t^2 - 6.0t \pm 6.0 = 0$$

Finally, using the quadratic formula, we obtain

$$t = \frac{6.0 \pm \sqrt{36 - 4(4.0)(\pm 6.0)}}{2(8.0)}$$

where the requirement of a real positive result leads to the unique answer: $t = 2.2$ s.

••17 A cart is propelled over an xy plane with acceleration components $a_x = 4.0\ \text{m/s}^2$ and $a_y = -2.0\ \text{m/s}^2$. Its initial velocity has components $v_{0x} = 8.0$ m/s and $v_{0y} = 12$ m/s. In unit-vector notation, what is the velocity of the cart when it reaches its greatest y coordinate?

17. We find t by applying Eq. 2-11 to motion along the y axis (with $v_y = 0$ characterizing $y = y_{max}$):

$$0 = (12\ \text{m/s}) + (-2.0\ \text{m/s}^2)t \;\Rightarrow\; t = 6.0\ \text{s}.$$

Then, Eq. 2-11 applies to motion along the x axis to determine the answer:

$$v_x = (8.0\ \text{m/s}) + (4.0\ \text{m/s}^2)(6.0\ \text{s}) = 32\ \text{m/s}.$$

Therefore, the velocity of the cart, when it reaches $y = y_{max}$, is $(32\ \text{m/s})\hat{\text{i}}$.

•••19 The acceleration of a particle moving only on a horizontal xy plane is given by $\vec{a} = 3t\hat{i} + 4t\hat{j}$, where $\vec{a}$ is in meters per second-squared and t is in seconds. At $t = 0$, the position vector $\vec{r} = (20.0\text{ m})\hat{i} + (40.0\text{ m})\hat{j}$ locates the particle, which then has the velocity vector $\vec{v} = (5.00\text{ m/s})\hat{i} + (2.00\text{ m/s})\hat{j}$. At $t = 4.00$ s, what are (a) its position vector in unit-vector notation and (b) the angle between its direction of travel and the positive direction of the x axis?

PROBLEM

••17 A cart is propelled over an xy plane with acceleration components $a_x = 4.0\ \text{m/s}^2$ and $a_y = -2.0\ \text{m/s}^2$. Its initial velocity has components $v_{0x} = 8.0\ \text{m/s}$ and $v_{0y} = 12\ \text{m/s}$. In unit-vector notation, what is the velocity of the cart when it reaches its greatest y coordinate?

Using $\vec{a} = 3t\hat{\mathrm{i}} + 4t\hat{\mathrm{j}}$, we have (in m/s)

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}\, dt = (5.00\hat{\mathrm{i}} + 2.00\hat{\mathrm{j}}) + \int_0^t (3t\hat{\mathrm{i}} + 4t\hat{\mathrm{j}})\, dt = \left(5.00 + 3t^2/2\right)\hat{\mathrm{i}} + \left(2.00 + 2t^2\right)\hat{\mathrm{j}}$$

Integrating using Eq. 4-10 then yields (in meters)

SOLUTION

$$\vec{r}(t)=\vec{r}_0+\int_0^t \vec{v}\,dt=(20.0\hat{\mathrm{i}}+40.0\hat{\mathrm{j}})+\int_0^t[(5.00+3t^2/2)\hat{\mathrm{i}}+(2.00+2t^2)\hat{\mathrm{j}}]dt$$
$$=(20.0\hat{\mathrm{i}}+40.0\hat{\mathrm{j}})+(5.00t+t^3/2)\hat{\mathrm{i}}+(2.00t+2t^3/3)\hat{\mathrm{j}}$$
$$=(20.0+5.00t+t^3/2)\hat{\mathrm{i}}+(40.0+2.00t+2t^3/3)\hat{\mathrm{j}}$$

(a) At $t=4.00$ s, we have $\vec{r}(t=4.00\,\text{s})=(72.0\text{ m})\hat{\mathrm{i}}+(90.7\text{ m})\hat{\mathrm{j}}$.

(b) $\vec{v}(t=4.00\,\text{s})=(29.0\text{ m/s})\hat{\mathrm{i}}+(34.0\text{ m/s})\hat{\mathrm{j}}$. Thus, the angle between the direction of travel and $+x$, measured counterclockwise, is $\theta=\tan^{-1}[(34.0\,\text{m/s})/(29.0\,\text{m/s})]=49.5°$.

SOLUTION CONTINUE

THANK YOU