Distribución Temperaturas en una Placa Plana Bi-dimensional

Tips para formulación General

Formulación caso estacionario

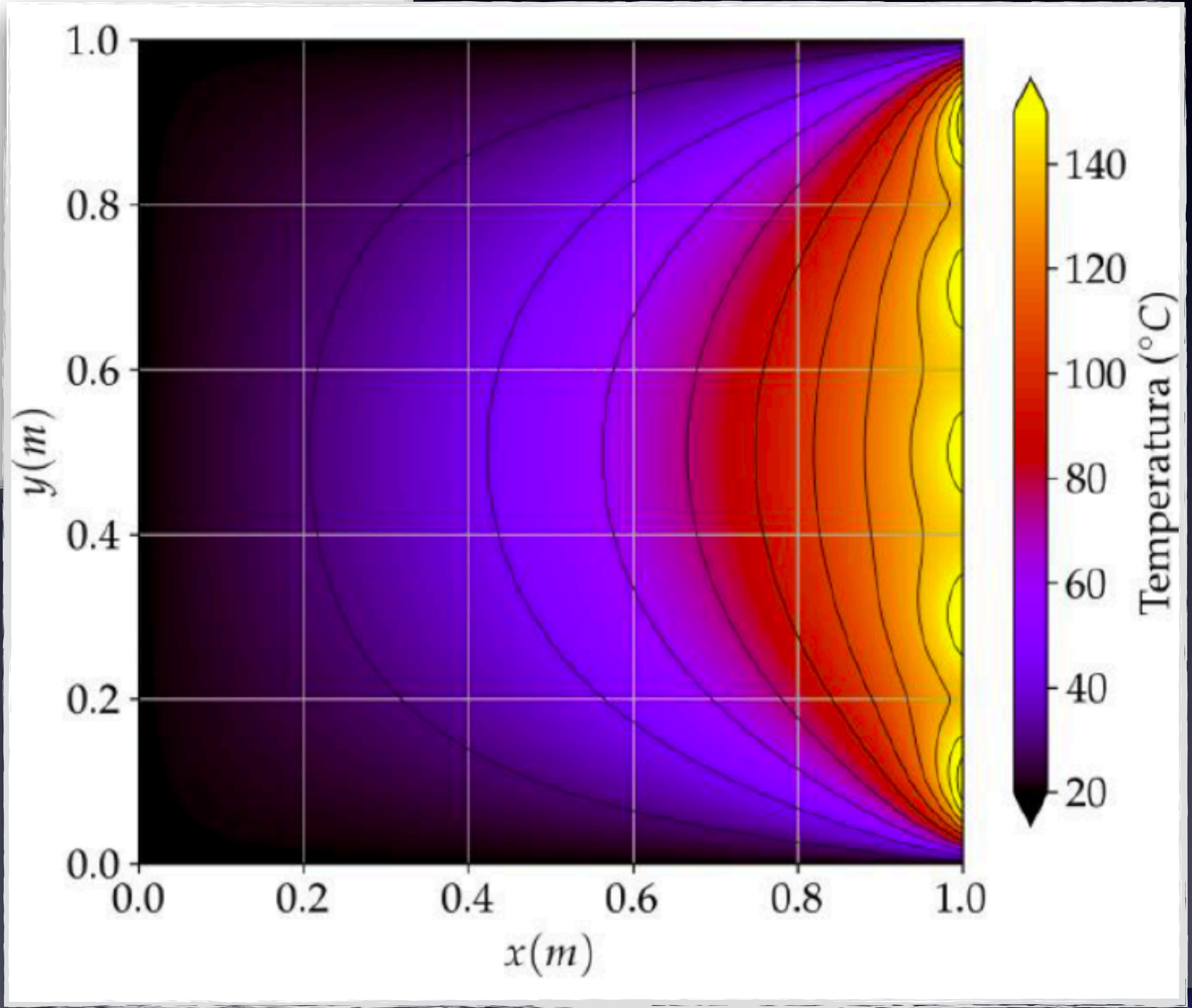
Parte 1. Estado estacionario

Considere un caso de conducción de calor en una placa plana rectangular que es calentada en uno de sus bordes (como se indica en la Figura 1). Es estado estacionario, dicha situación se puede modelar mediante la siguiente ecuación diferencial:

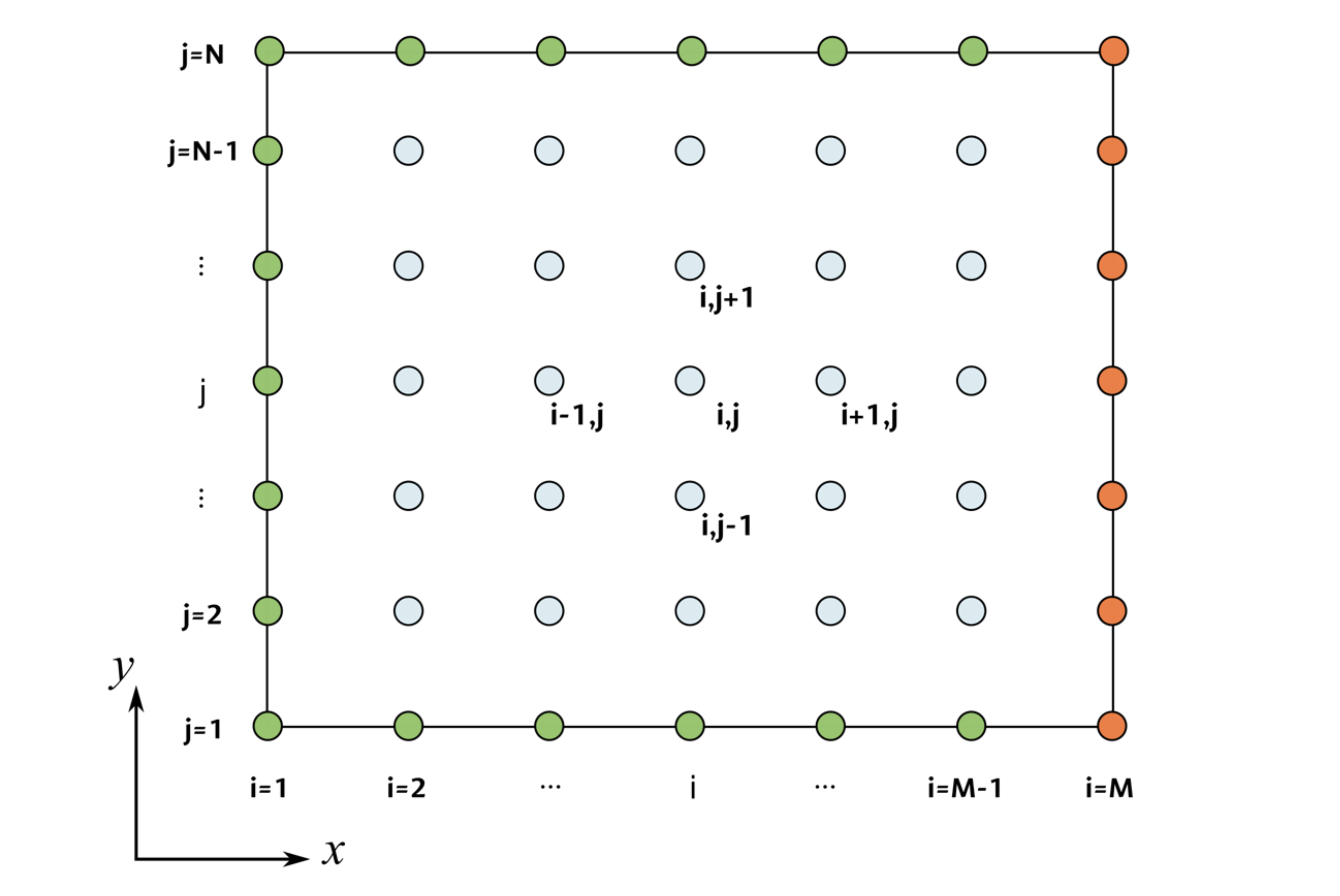
$$\nabla^2 T = 0 \qquad sobre \qquad \Omega = (0,1) \times (0,1)$$

El caso a estudiar presenta condiciones de frontera Dirichlet dadas por las siguientes relaciones:

$$T(0,y) = 20°C \qquad para \qquad 0 \le y \le b$$
$$T(a,y) = 150°C \qquad para \qquad 0 \le y \le b$$
$$T(x,0) = T(x,b) = 20°C \qquad para \qquad 0 \le x \le a$$



Formulación caso estacionario



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$$\nabla^2 T = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta x^2} + \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{\Delta y^2} = 0$$

Formulación caso estacionario

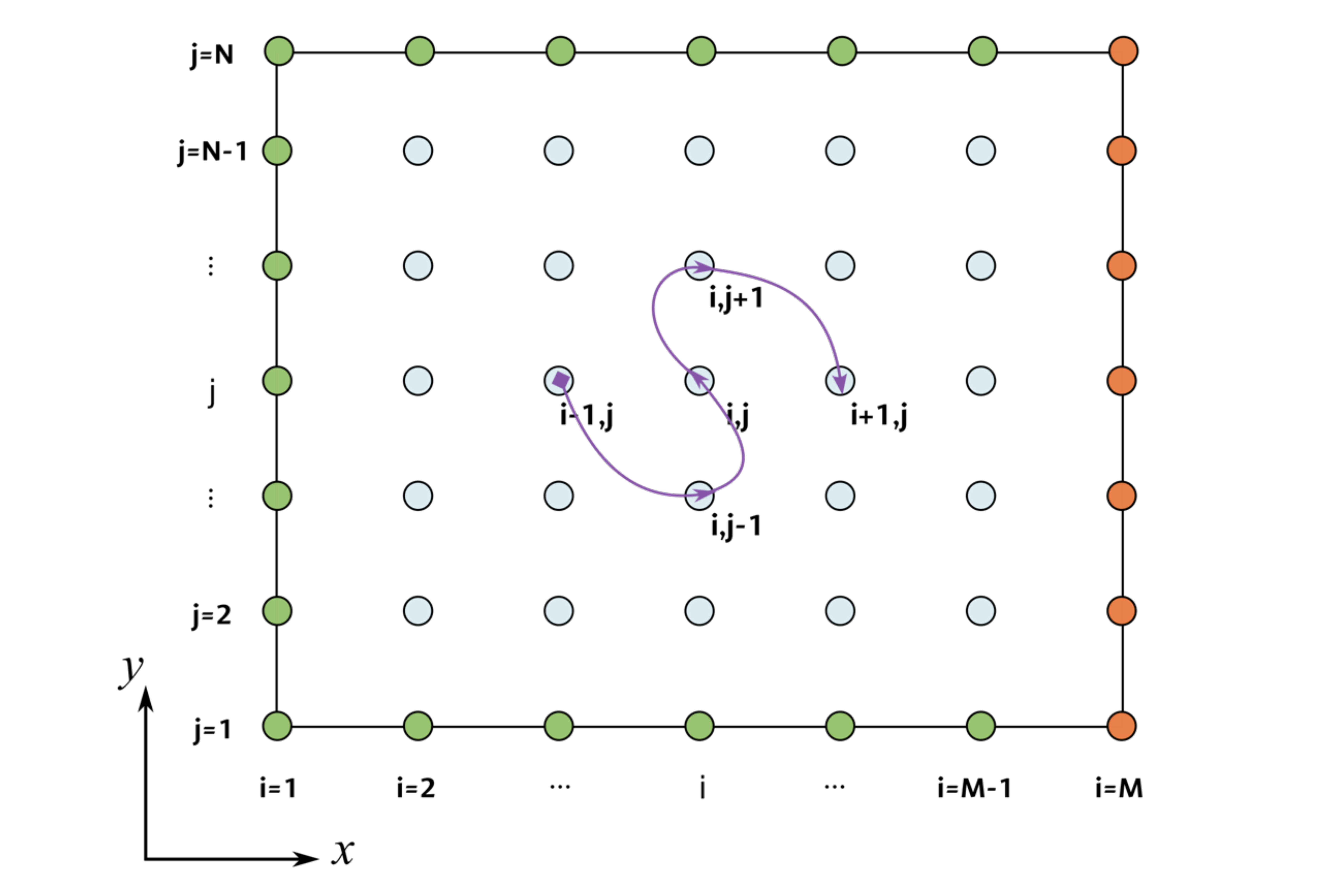
$$\frac{1}{\Delta x^2}T_{i-1,j} + \frac{1}{\Delta y^2}T_{i,j-1} - \left(\frac{2}{\Delta x^2} + \frac{2}{\Delta y^2}\right)T_{i,j} + \frac{1}{\Delta y^2}T_{i,j+1} + \frac{1}{\Delta x^2}T_{i+1,j} = 0$$

y si definimos:

$$\alpha = \frac{1}{\Delta x^2}, \quad \beta = \frac{1}{\Delta y^2}, \quad \gamma = 2\left(\alpha + \beta\right)$$

$$\alpha\, T_{i-1,j} + \beta\, T_{i,j-1} - \gamma\, T_{i,j} + \beta\, T_{i,j+1} + \alpha\, T_{i+1,j} = 0$$

Formulación caso estacionario



Formulación caso estacionario

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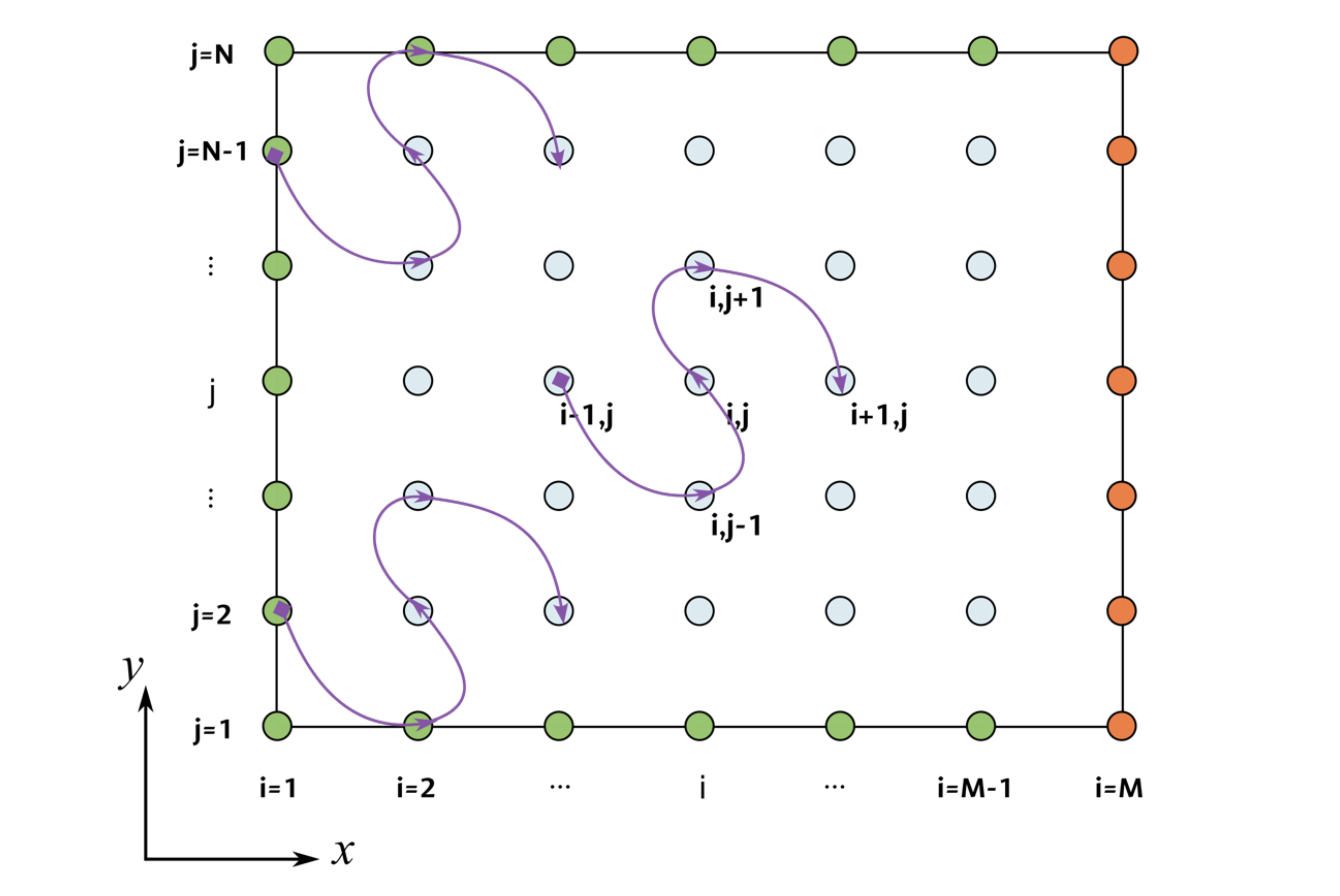
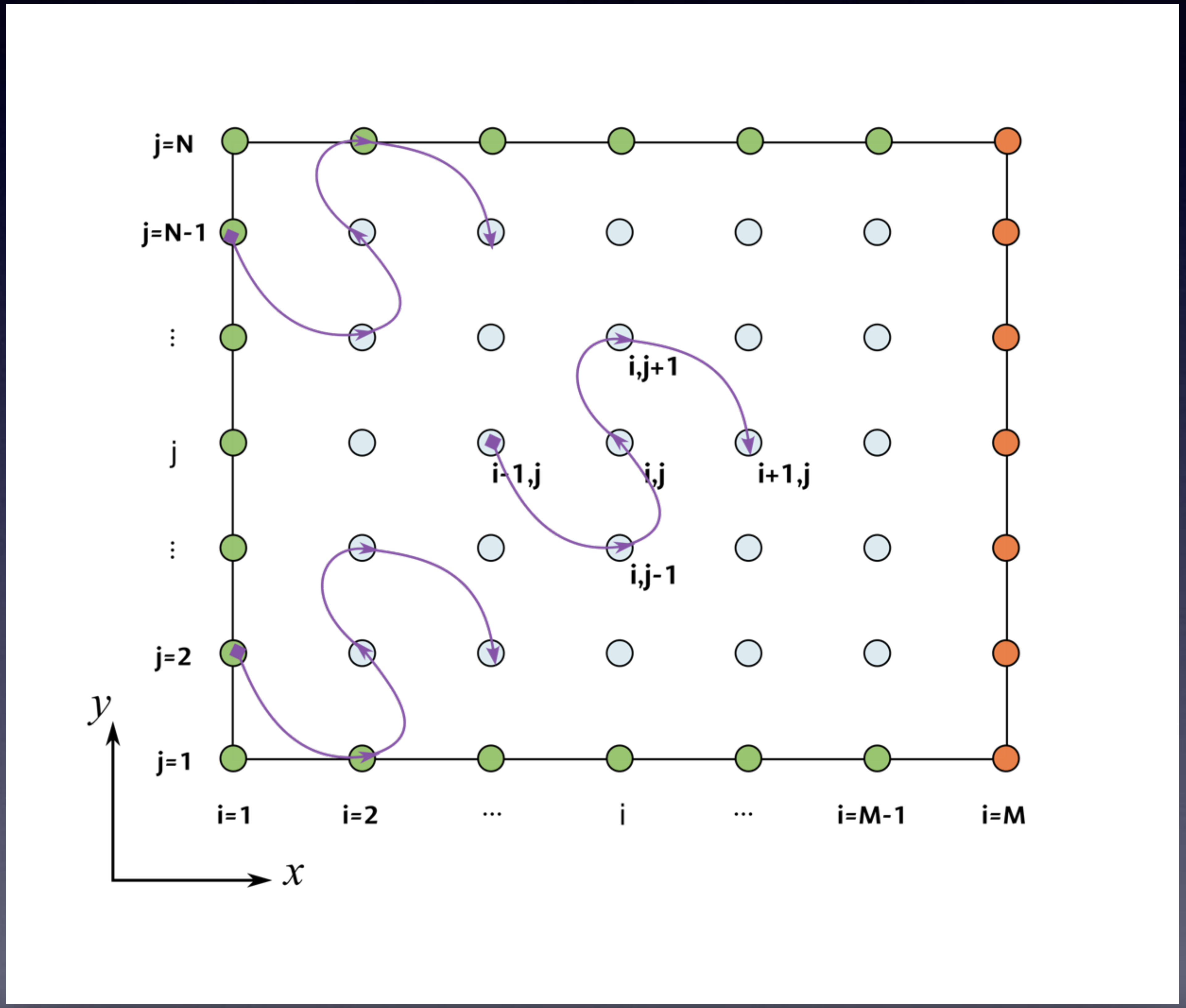


Formulación caso estacionario



Probemos formulando algunas de las expresiones específicas…

asumamos que M=6 y N=6

$$
\begin{bmatrix}
T_{22} & T_{32} & T_{42} & T_{52} & T_{23} & T_{33} & T_{43} & T_{53} & T_{24} & T_{34} & T_{44} & T_{54} & T_{25} & T_{35} & T_{45} & T_{55} \\
-\gamma & \alpha & & & \beta & & & & & & & & & & & \\
\alpha & -\gamma & \alpha & & & \beta & & & & & & & & & & \\
 & \alpha & -\gamma & \alpha & & & \beta & & & & & & & & & \\
 & & \alpha & -\gamma & & & & \beta & & & & & & & & \\
\beta & & & & -\gamma & \alpha & & & \beta & & & & & & & \\
 & \beta & & & \alpha & -\gamma & \alpha & & & \beta & & & & & & \\
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 & & & \beta & & & \alpha & -\gamma & & & & \beta & & & & \\
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\end{bmatrix}
\begin{bmatrix}
T_{\text{nodo}} \\ T_{22} \\ T_{32} \\ T_{42} \\ T_{52} \\ T_{23} \\ T_{33} \\ T_{43} \\ T_{53} \\ T_{24} \\ T_{34} \\ T_{44} \\ T_{54} \\ T_{25} \\ T_{35} \\ T_{45} \\ T_{55}
\end{bmatrix}
+\alpha
\begin{bmatrix}
T_{\text{bc}} \\ T_{12} \\ \\ \\ T_{62} \\ T_{13} \\ \\ \\ T_{63} \\ T_{14} \\ \\ \\ T_{64} \\ T_{15} \\ \\ \\ T_{65}
\end{bmatrix}
+\beta
\begin{bmatrix}
T_{\text{bc}} \\ T_{21} \\ T_{31} \\ T_{41} \\ T_{51} \\ \\ \\ \\ \\ \\ \\ \\ \\ T_{26} \\ T_{36} \\ T_{46} \\ T_{56}
\end{bmatrix}
= 0
$$

$$
\begin{array}{c}
\begin{array}{*{16}{c}}
T_{22} & T_{32} & T_{42} & T_{52} & T_{23} & T_{33} & T_{43} & T_{53} & T_{24} & T_{34} & T_{44} & T_{54} & T_{25} & T_{35} & T_{45} & T_{55}
\end{array} \\
\left[\begin{array}{*{16}{c}}
-\gamma & \alpha & & & \beta & & & & & & & & & & & \\
\alpha & -\gamma & \alpha & & & \beta & & & & & & & & & & \\
& \alpha & -\gamma & \alpha & & & \beta & & & & & & & & & \\
& & \alpha & -\gamma & & & & \beta & & & & & & & & \\
\beta & & & & -\gamma & \alpha & & & \beta & & & & & & & \\
& \beta & & & \alpha & -\gamma & \alpha & & & \beta & & & & & & \\
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& & & & & & & & \beta & & & & -\gamma & \alpha & & \\
& & & & & & & & & \beta & & & \alpha & -\gamma & \alpha & \\
& & & & & & & & & & \beta & & & \alpha & -\gamma & \alpha \\
& & & & & & & & & & & \beta & & & \alpha & -\gamma
\end{array}\right]
\end{array}
\begin{array}{c}
T_{\text{nodo}} \\
\left[\begin{array}{c}
T_{22} \\ T_{32} \\ T_{42} \\ T_{52} \\ T_{23} \\ T_{33} \\ T_{43} \\ T_{53} \\ T_{24} \\ T_{34} \\ T_{44} \\ T_{54} \\ T_{25} \\ T_{35} \\ T_{45} \\ T_{55}
\end{array}\right]
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+\alpha
\begin{array}{c}
T_{\text{bc}} \\
\left[\begin{array}{c}
T_{12} \\ \\ \\ T_{62} \\ T_{13} \\ \\ \\ T_{63} \\ T_{14} \\ \\ \\ T_{64} \\ T_{15} \\ \\ \\ T_{65}
\end{array}\right]
\end{array}
+\beta
\begin{array}{c}
T_{\text{bc}} \\
\left[\begin{array}{c}
T_{21} \\ T_{31} \\ T_{41} \\ T_{51} \\ \\ \\ \\ \\ \\ \\ \\ \\ T_{26} \\ T_{36} \\ T_{46} \\ T_{56}
\end{array}\right]
\end{array}
= 0
$$

Formulación caso transitorio

$$\frac{\partial T}{\partial t} = \nabla^2 T$$

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \frac{1}{2}\left[\frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta x^2} + \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{\Delta y^2}\right]^{n+1} + \frac{1}{2}\left[\frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta x^2} + \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{\Delta y^2}\right]^{n}$$

Formulación caso transitorio

$$\alpha = \frac{\Delta t}{2\,\Delta x^2},\ \beta = \frac{\Delta t}{2\,\Delta y^2},\ \gamma = 2(\alpha + \beta)$$

$$T_{i,j}^{n+1} - T_{i,j}^{n} = \left[\alpha\left(T_{i-1,j} - 2T_{i,j} + T_{i+1,j}\right) + \beta\left(T_{i,j-1} - 2T_{i,j} + T_{i,j+1}\right)\right]^{n+1} + \left[\alpha\left(T_{i-1,j} - 2T_{i,j} + T_{i+1,j}\right) + \beta\left(T_{i,j-1} - 2T_{i,j} + T_{i,j+1}\right)\right]^{n}$$

$$-\alpha\,T_{i-1,j}^{n+1} - \beta\,T_{i,j-1}^{n+1} + (1+\gamma)\,T_{i,j}^{n+1} - \beta\,T_{i,j+1}^{n+1} - \alpha\,T_{i+1,j}^{n+1} = \alpha\,T_{i-1,j}^{n} + \beta\,T_{i,j-1}^{n} + (1-\gamma)\,T_{i,j}^{n} + \beta\,T_{i,j+1}^{n} + \alpha\,T_{i+1,j}^{n}$$

$$\underline{\underline{\mathbf{A}}}\,\underline{\mathbf{T}}^{n+1} = \underline{\underline{\mathbf{B}}}\,\underline{\mathbf{T}}^{n} + \underline{\mathbf{B.C.s}}$$