Punto 1.

- Lleve a cabo una revisión conceptual del "Teorema de Transporte de Reynolds" (TTR)
- Presente una versión resumida pero clara del desarrollo matemático para la obtención de la expresión final del TTR

There are two ways to apply the governing laws in the fluid mechanics analyses: the system and the control volume approach. The **System** is a collection of matter that remains fixed, it means that the amount of fluid particles is always the same. Additionally, the system can move and interact with its surroundings. On the other hand, the **Control volumen** is a volumen in space that does not depend on mass, and through which the fluid may flow.

The system approach is also known as Lagrangian description, where we follow the fluid and observe its behavior as it move about. On the other hand, the C.V. approach is also know as Eulerian description, where we observe the fluid's behavior at a fixed location. All of the governing equations that describe the fluid motion are stated in terms of a system. For instance, "The energy of the system remains constant" or "The mass of the system remains constant". If we want to change the system approach of the governing laws into a C.V. approach, it is necessary to apply the Reynold Transport Theorem (or RTT). It is the "bridge" that connect the Lagrangian and the Eulerian description through an integral equation.

When would it be useful to apply an Eulerian/Control Volume approach instead of a Lagrangian/System approach? When we
are interested on a particular object/volumen in space with which the fluid is interacting with. For example, when we want to know the force that the fluid is exerting on the airplane's wings. Before deriving RTT, we need to define two properties: B (extensive) and b (intensive). The table below shows the quantities that B may assume depending on the governing law to be applied.

B	$b = B/m$
m	1
$m\mathbf{V}$	$\mathbf{V}$
$\frac{1}{2}mV^2$	$\frac{1}{2}V^2$

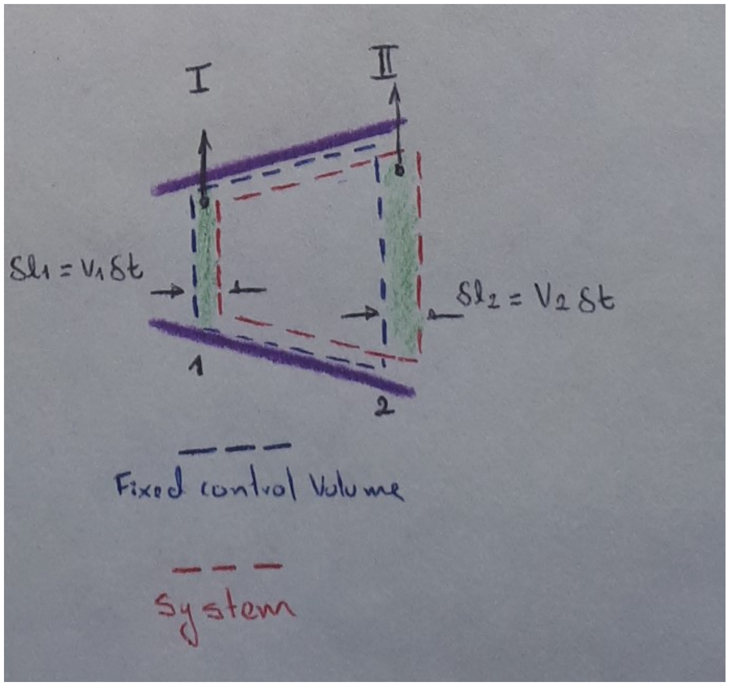
To calculate the amount of an extensive property B in the system at a given moment, we need to add up all the contributions that come from each fluid particle that belongs to the system. This summation has the form of an integration all over the particles in the system when we consider and infinitesimal fluid particle with volume δV and mass $\rho\delta V$:

$$B_{\text{sys}} = \lim_{\delta\forall\to 0}\sum_i b_i\left(\rho_i\delta V_i\right) = \int_{\text{sys}} \rho b d\forall$$

The same happens when we want to know the amount of an extesnive property B inside the control volume:

$$B_{\text{CV}} = \lim_{\delta\forall\to 0}\sum_i b_i\left(\rho_i\delta V_i\right) = \int_{\text{CV}} \rho b d\forall$$

After these definitions, the process of deriving the RTT will be pretty straigtforward. Will be done for a simple case, with one inlet and one outlet. From this point onwards, it will be easy to derive a more general equation. First a scheme for the case to be analyse:



At time t the control volume and the system coincide, so we have that:

$$B_{sys}(t) = B_{cv}(t) \tag{1}$$

At time $t + \delta t$ the control volume and the system don't coincide, so we have that:

$$B_{\text{sys}}(t+\delta t) = B_{\text{cv}}(t+\delta t) - B_{\text{I}}(t+\delta t) + B_{\text{II}}(t+\delta t)$$

If we want to compute the change of the extensive property B in the system we substract $B_{sys}(t)$ and divide by δt:

$$\frac{\delta B_{\text{sys}}}{\delta t} = \frac{B_{\text{sys}}(t+\delta t) - B_{\text{sys}}(t)}{\delta t} = \frac{B_{\text{cv}}(t+\delta t) - B_{\text{I}}(t+\delta t) + B_{\text{II}}(t+\delta t) - B_{\text{sys}}(t)}{\delta t}$$

But we know from equation 1 that $B_{sys}(t) = B_{cv}(t)$:

$$\frac{\delta B_{\text{sys}}}{\delta t} = \frac{B_{\text{cv}}(t+\delta t) - B_{\text{cv}}(t)}{\delta t} - \frac{B_{\text{I}}(t+\delta t)}{\delta t} + \frac{B_{\text{II}}(t+\delta t)}{\delta t}$$

If we apply the limit when $\delta t \to 0$ in each term, so we have:

$$\lim_{\delta t \to 0} \frac{B_{\text{cv}}(t+\delta t) - B_{\text{cv}}(t)}{\delta t} = \frac{\partial B_{\text{cv}}}{\partial t} = \frac{\partial \left(\int_{\text{cv}} \rho b d\forall\right)}{\partial t}$$

$$\dot{B}_{\text{in}} = \lim_{\delta t \to 0} \frac{B_{\text{I}}(t+\delta t)}{\delta t} = \rho_1 A_1 V_1 b_1$$

$$\dot{B}_{\text{out}} = \lim_{\delta t \to 0} \frac{B_{\text{II}}(t+\delta t)}{\delta t} = \rho_2 A_2 V_2 b_2$$

Grouping all the terms we have:

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial B_{\text{cv}}}{\delta t} + \rho_2 A_2 V_2 b_2 - \rho_1 A_1 V_1 b_1 \tag{2}$$

What if we have tridimensional control volume? We are going to have a control surface through which the fluid may flow, at any infinitesimal area δA. This means that it would be necessary to do a surface integral all over the control surface, in order to capture all the inlets and outlets:

$$\delta B = b\rho\delta\forall = b\rho(V\cos\theta\delta t)\delta A = b\rho(\mathbf{V}\cdot\hat{\mathbf{n}})\delta A$$

Then we apply the limit when $\delta t \to 0$

$$\delta \dot{B}_{\text{out}} = \lim_{\delta t\to 0}\frac{\rho b\delta\forall}{\delta t} = \lim_{\delta t\to 0}\frac{(\rho bV\cos\theta\delta t)\delta A}{\delta t} = \rho bV\cos\theta\delta A$$

Finally we integrate on both sides:

$$\dot{B}_{\text{out}} = \int_{\text{cs}_{\text{out}}}\rho b\mathbf{V}\cdot\hat{\mathbf{n}}dA \tag{3}$$

The same for the inlets:

$$\dot{B}_{\text{in}} = -\int_{\text{cs}_{\text{in}}}\rho b\mathbf{V}\cdot\hat{\mathbf{n}}dA \tag{4}$$

Using equation 3 and 4 instead of $\rho_1 A_1 V_1 b_1$ and $\rho_2 A_2 V_2 b_2$, we obtain the RTT:

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\text{cv}}\rho b d\forall + \int_{\text{cs}}\rho b\mathbf{V}\cdot\hat{\mathbf{n}}dA$$

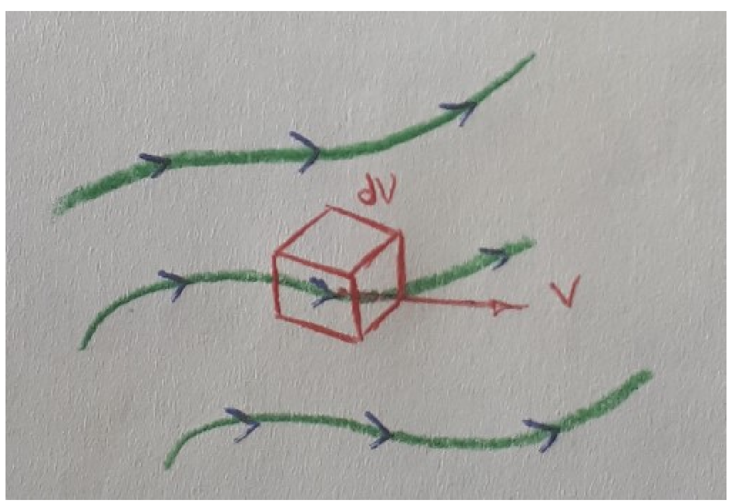
The left side of the equation represents the time rate of change of an extensive property B in the system. This means that we may have time rate of change of mass, momentum, energy... depending on the choice of B.

As the system may move and the control volume is stationary, the rate of change of B inside the CV is not necessarily the same to that of the system. For that reason the first term on the right hand side of the equation is very important. It represents the rate of change of B within the control volume by adding up al the

Punto 2.

Considere los diferentes tipos de modelos matemáticos presentados en clase para expresar la ecuación de continuidad en sus diferentes formas: forma diferencial conservativa, diferencial no conservativa, integral conservativa, e integral no conservativa. Para esta actividad se solicita que usted:

- **Presente la obtención de la forma diferencial no conservativa de la ecuación de conservación de masa (ecuación de continuidad)**



We have an infinitesimally fluid element moving with the flow whose mass remains constant; but its volume an geometry change as it moves downstream:

$$\delta m = \rho\delta\forall$$

As the infinitesimally fluid element is moving in the space, we must apply the substantial derivative to register its mass change. However, as the mass does not change the substantial derivative equals zero:

$$\frac{D(\delta m)}{Dt} = 0$$

Then, we apply the rule of the derivate of a multiplication:

$$\frac{D(\rho\delta\forall)}{Dt} = \delta\forall\frac{D(\rho)}{Dt} + \rho\frac{D(\delta\forall)}{Dt} = 0$$

If we divide all the equation by $\forall$ we have:

$$\frac{D(\rho)}{Dt} + \rho\left[\frac{1}{\delta\forall}\frac{D(\delta\forall)}{Dt}\right] = 0$$

The second term represents the time rate of change of the volume of a moving element, per unit volume and is equal to the divergence of the velocity:

$$\frac{D(\rho)}{Dt} + \rho\nabla\cdot\vec{V} = 0 \tag{1}$$

- **Utilice la expresión anterior y exprésela en forma integral conservativa.**

We can go from equation 1 to the integral conservative form through two ways: D+A or B+C. Here we use the path D+A. First, we poose a volumetric integral, since the control volume is compose of several moving fluid element:

$$\iiint_{\forall}\frac{D(\delta m)}{Dt}d\forall = \iiint_{\forall}\frac{D\rho}{Dt}d\forall + \iiint_{\forall}\rho\nabla\cdot V d\forall = 0$$

The substantial derivative can be expressed as:

$$\iiint_{\forall}\frac{D\rho}{Dt}d\forall = \iiint_{\forall}\left[\frac{\delta\rho}{\delta t} + \vec{V}\cdot\nabla\rho\right]d\forall$$

We use the equivalence of the substantial equation:

$$\iiint_{\forall}\left[\frac{\delta\rho}{\delta t} + \vec{V}\cdot\nabla\rho\right]d\forall + \iiint_{\forall}\rho\nabla\cdot\vec{V}d\forall = 0$$

$$\iiint_{\forall}\frac{\delta\rho}{\delta t}d\forall + \iiint_{\forall}\vec{V}\cdot\nabla\rho\, d\forall + \iiint_{\forall}\rho\nabla\cdot\vec{V}d\forall = 0$$

We grouped the last two terms:

$$\iiint_{\forall}\frac{\delta\rho}{\delta t}d\forall + \iiint_{\forall}\left[\vec{V}\cdot\nabla\rho + \rho\nabla\cdot\vec{V}\right]d\forall = 0$$

$$\iiint_{\forall}\frac{\delta\rho}{\delta t}d\forall + \iiint_{\forall}\nabla\cdot(\rho\vec{V})d\forall = 0$$

Finally, we use the Divergence theorem in the last term and take out the partial derivative with respect to time:

$$\frac{\delta}{\delta t}\iiint_{\forall}\rho d\forall + \iint_{S}\rho\vec{V}\cdot\hat{n}\, ds = 0 \tag{2}$$

This is the conservative integral form of the continuity equation.

- **Muestre las conversiones entre los diferentes modelos matemáticos para la ecuación de continuidad.**

To show the conversion between different models, we start from the equation 2. The time derivative can be put inside the volumetric integral becasue the fixed CV's volume remains constant. Thus, the limits of the volumetric integral remains constants and are not affected by time. Additionally, We apply the divergence theorem to the last term:

$$\iiint_{\forall} \frac{\delta \rho}{\delta t} d\forall + \iiint_{\forall} \nabla \cdot (\rho \vec{V}) d\forall = 0$$

The the volumetric integrals are grouped in just one integral. This is possible thanks to the linear nature of the integrals:

$$\iiint_{\forall} \left[\frac{\delta \rho}{\delta t} + \nabla \cdot (\rho \vec{V}) = 0 \right] d\forall$$

The integral's argument must be zero:

$$\frac{\delta \rho}{\delta t} + \nabla \cdot (\rho \vec{V}) = 0 \tag{3}$$

Equation 3 is the differential and conservative form of the continuity equation. If we developed the term $\nabla \cdot (\rho \vec{V})$ we have:

$$\frac{\delta \rho}{\delta t} + \vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V} = 0$$

The first two terms are the substantial derivative:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0 \tag{4}$$

Equation 4 is the differential and Non conservative form of the continuity equation. Then we replace the divergence of the veocity by the time rate of change of the element's volumen per unit volume:

$$\frac{D(\rho)}{Dt} + \rho \left[\frac{l}{\delta \forall} \cdot \frac{D\delta\forall)}{Dt} \right] = 0$$

Reorganizing the expression:

$$\frac{D(\rho)}{Dt} \delta\forall + \rho \frac{D\delta\forall}{Dt} = 0$$

Using the derivative of a multiplication rule:

$$\frac{D(\rho \delta \forall)}{Dt} = 0$$

If we integrate this expression to compute the time rate of change of mass inside the moving control volume we obtain:

$$\frac{D}{Dt} \iiint_{\forall} \rho d\forall = 0 \tag{5}$$

Equation 5 is the integral and non conservative form of the continuity equation. Finally, we can derive the integral and conservative form of the continuity equation as it follows:

$$\iiint_{\forall} \frac{D(\rho)}{Dt} \delta\forall + \iiint_{\forall} \rho \frac{D\delta\forall}{Dt} = 0$$

$$\iiint_{\forall} \frac{D(\rho)}{Dt} \delta\forall + \iiint_{\forall} \rho \frac{1}{\delta\forall} \frac{D\delta\forall}{Dt} \delta\forall = 0$$

$$\iiint_{\forall} \left[\frac{\delta \rho}{\delta t} + \vec{V} \cdot \nabla \rho \right] \delta\forall + \iiint_{\forall} \rho \nabla \cdot \vec{V} \delta\forall = 0$$

$$\iiint_{\forall} \frac{\delta \rho}{\delta t} d\forall + \iiint_{\forall} \left[\vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V} \right] d\forall = 0$$

$$\iiint_{\forall} \frac{\delta \rho}{\delta t} d\forall + \iiint_{\forall} \nabla \cdot (\rho \vec{V}) d\forall = 0$$

$$\frac{\delta}{\delta t} \iiint \rho d\forall + \iint \rho \vec{V} \cdot \hat{n} \, ds = 0$$

Punto 3.

Por favor haga una revisión de los siguientes conceptos, presentando ejemplos de aplicación:

- **Descomposición de un tensor en sus componentes simétrica y anti-simétrica.**

A tensor T can be represented as the sum of its symmetrical S and antisymmetrical A components:

$$T = S + A$$

Where S is defined as:

$$S = \frac{1}{2}[T + T^T]$$

And A is defined as:

$$A = \frac{1}{2}[T - T^T]$$

If we summed up S and A:

$$T = \frac{1}{2}[T + T^T] + \frac{1}{2}[T - T^T] = T + \frac{1}{2}[T^T - T^T] = T$$

For example, we have a tensor of order two call B:

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

The symmetrical component would be:

$$S = \begin{pmatrix} 1 & 1.5 & 2 \\ 1.5 & 2 & 2.5 \\ 2 & 2.5 & 3 \end{pmatrix}$$

The antisymmetrical component would be:

$$A = \begin{pmatrix} 0 & 0.5 & 1 \\ -0.5 & 0 & 0.5 \\ -1 & -0.5 & 0 \end{pmatrix}$$

If we summed up S and A, we would have as a result the tensor B

- **Descomposición de un tensor en sus componentes esférica y deviatórica.**

The spherical and deviatoric components of the stress tensor allows to distinguish and group the stresses by their cause: pression or viscosity. The tensor of order two at the right represents is called the stress tensor. The first one at the left hand side of the equation is called the spherical component and the last one the deviatoric component. For example:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

Spheric component: is the tensor that contains the hydrostatic local pressure acting normal to and towards the center the element. For example:

$$\begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

The components are negative because is a compression stress. Aditionally, when we have incompressible fluid, these componentes are replaced by a mean value:

$$P_m = -\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

deviatoric component: also called as Viscous Stress Tensor, is the tensor that holds the shear stress and the tension stress produced by the viscosity of the fluid and motion of the element.

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

- **Diferencia entre el Tensor de Esfuerzos Viscosos (Viscous Stress Tensor) vs. Tensor Rata de Deformación (Strain Rate Tensor).**

Viscous stress tensor: it is a symmetrical tensor that contains shear and normal stresses due to the viscosity. An example is shown below for a laminar, incompressible and neglecting the second coefficient of viscosity:

$$\tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

$$\tau_{ij} = \begin{pmatrix} 2\mu\frac{\partial u}{\partial x} & \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & 2\mu\frac{\partial v}{\partial y} & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & 2\mu\frac{\partial w}{\partial z} \end{pmatrix}$$

The strain rate tensor: it is a symmetrical tensor commonly used in mechanics of continuums and mechanics of deformable solids to **characterize the change in shape and volume of a body**.

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}$$

$$\varepsilon_{ij} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \end{pmatrix}$$

```
In [ ]:
```