APPARATUS AND DEMONSTRATION NOTES

Jeffrey S. Dunham, *Editor*

Department of Physics, Middlebury College, Middlebury, Vermont 05753

This department welcomes brief communications reporting new demonstrations, laboratory equipment, techniques, or materials of interest to teachers of physics. Notes on new applications of older apparatus, measurements supplementing data supplied by manufacturers, information which, while not new, is not generally known, procurement information, and news about apparatus under development may be suitable for publication in this section. Neither the *American Journal of Physics* nor the Editors assume responsibility for the correctness of the information presented. Submit materials to Jeffrey S. Dunham, *Editor.*

Mechanics of the slow draining of a large tank under gravity

Josué Njock Libii[a)]

Department of Engineering, Indiana University—Purdue University Fort Wayne, Fort Wayne, Indiana 46805-1499

(Received 20 October 2000; accepted 20 December 2002)

[DOI: 10.1119/1.1545764]

I. INTRODUCTION

It has been our experience that students find mechanics fun and exciting when they are able to use what they have learned to model and to predict the behavior of familiar phenomena. For this reason, we have made considerable efforts to design experiments for the first course in fluid mechanics that are simple and easy to visualize, and that relate theoretical concepts from mechanics directly to the experience that students have with familiar phenomena. These experiments are particularly helpful when one has a class with students who are likely to get lost in the mathematical details that abound in the study of fluid flows. To illustrate this point, we present a draining experiment that is one of many laboratory exercises used to support the first course in fluid mechanics.

The experiment itself consists of draining a large cylindrical tank under the influence of gravity. The tank's axis of revolution is vertical; its top is open to the atmosphere, and it is drained through a small orifice located at the bottom of the tank. We measure both the total time it takes to drain the tank completely and the draining pattern itself, that is, how the volume of liquid in the tank changes with time during the draining process. We model the liquid as an incompressible and inviscid fluid and the flow as quasisteady and irrotational.

Although the actual flow is that of a viscous fluid, the observed behavior is compared with that predicted by the theory of the irrotational draining of an inviscid fluid. When the cross-sectional area of the exit orifice is much smaller than that of the tank, this comparison shows that the inviscid-fluid model approximates the behavior of the real fluid quite well. First, we describe the theory used for modeling, then the experiment itself. Finally, we show experimental results obtained and compare them to theory.

II. THEORY

Consider a cylindrical container of a circular cross section (the tank) that is oriented such that its axis of rotational symmetry is vertical. The tank is open to the atmosphere at the top so that water, or some other liquid, can be poured into the tank easily. The bottom of the tank is capped, but the cap has an orifice placed at its center through which liquid can drain. The tank is equipped with plugs that have openings of various diameters. The plugs can be threaded into the tank so as to change the diameter of the opening. Figure 1 shows the experimental setup.

Let A_t denote the inside cross-sectional area of the tank, A_0 the inside cross-sectional area of the opening of a plug, h_0 the initial elevation of the free surface relative to the bottom of the tank, h the instantaneous elevation of the free surface relative to the bottom of the tank at time t, t the time elapsed since the beginning of the draining process, t_d the total time needed to completely drain the tank of liquid, and g the local acceleration of gravity. Applying the unsteady conservation of energy for open systems to this problem results in the following equation that governs the variation with time of the instantaneous elevation of the free surface, $h(t)$, relative to the bottom of the tank:[1]

$$h\left(g+\frac{d^2h}{dt^2}\right)=\frac{1}{2}\left(\frac{dh}{dt}\right)^2\left[\left(\frac{A_t}{A_0}\right)^2-1\right]. \tag{1}$$

When the tank drains slowly, one can expect the acceleration of the free surface of the liquid to be very small compared to the acceleration of gravity. This means that $g \gg d^2h/dt^2$, and Eq. (1) then becomes

$$hg=\frac{1}{2}\left(\frac{dh}{dt}\right)^2\left[\left(\frac{A_t}{A_0}\right)^2-1\right]. \tag{2}$$

Equation (2) can now be integrated to obtain $h(t)$. When this is done, and the quantities involved are made dimensionless through scaling, one finds that the elevation of the free surface, h, varies with time according to[2]

$$\frac{h}{h_0}=\left(1-\frac{t}{t_d}\right)^2, \tag{3a}$$

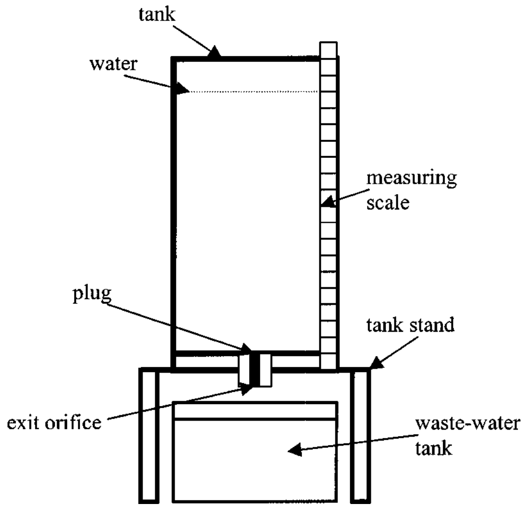

Fig. 1. Diagram of the experimental setup showing the orientation of the tank, the draining orifice fitted with a plug, and the tank stand.

and the velocity and acceleration of the free surface are given, respectively, by

$$\frac{dh}{dt} = -\frac{2h_0}{t_d}\left(1-\frac{t}{t_d}\right) \tag{3b}$$

and

$$\frac{d^2h}{dt^2} = -\frac{2h_0}{t_d^2}, \tag{3c}$$

where the total time necessary to completely drain the tank, t_d, is given by

$$t_d = \left(\frac{2h_0}{g}\right)^{1/2}\left[\left(\frac{A_t}{A_0}\right)^2 - 1\right]^{1/2}. \tag{4}$$

This expression for t_d is reminiscent of the time it takes a free-falling particle to drop through a distance h_0 from rest. Indeed, for a particle that is released from rest at height h_0 above a reference level and falls freely in the absence of air resistance, the instantaneous elevation above that reference level, $h_p(t)$, and the duration of the fall, t_f, are given, respectively, by

$$\frac{h_p}{h_0} = 1 - \left(\frac{t}{t_f}\right)^2 \tag{5}$$

and

$$t_f = \sqrt{\frac{2h_0}{g}}. \tag{6}$$

The instantaneous velocity of the falling particle, $v_p(t)$, is given by

$$v_p(t) = -2\frac{h_0}{t_f}\left(\frac{t}{t_f}\right) \tag{7}$$

and

$$v_p(t_f) = -\sqrt{2gh_0}, \tag{8}$$

where t_f denotes the time it takes a particle to fall freely from rest under gravity from $h_p(0)=h_0$ to $h_p(t_f)=0$, and v_f is the velocity achieved by the particle when $h_p=0$. It is convenient to write the expression for t_d in Eq. (4) in a form that is similar to the expression for t_f in Eq. (6). Doing so leads to

$$t_d = \sqrt{\frac{2h_0}{g_m}}, \tag{9}$$

where

$$g_m = \frac{g}{[(A_t/A_0)^2 - 1]}. \tag{10}$$

The quantity g_m given in Eq. (10) can be considered to be the modified acceleration of gravity resulting from the constriction of the flow at the draining orifice. It indicates that the rate of descent of the free surface during draining will be slower than the velocity of freefall.

III. EXPERIMENT

We performed experiments to gather data that would allow us to compare theory to experiment. A cylindrical shell made of Plexiglas was used as the tank in this experiment. The shell is capped at its lower end to produce a transparent cylinder that can hold water. One or more orifices can be added to the cap. In our case, we used a single orifice but fabricated many threaded plugs. Holes of different diameters were drilled into the threaded plugs. By threading a drilled plug into the orifice, we could change the diameter of the exit orifice. A graduated scale was glued vertically along the length of the tank and it was used to track the position of the free surface of the water in the tank during the draining process. After selecting a given plug and threading it into the orifice, the tank was filled to a specified height and the water was allowed to come to rest. Then, the drain was opened and, using a stopwatch, we observed and recorded the location of the free surface as a function of time during draining. Once data were collected using one plug, the experiment was repeated using another plug. In this way, we collected data for different plugs using the same tank and with the initial height of liquid set to be the same for all trials.

The tank used in these experiments had an inside diameter of 29.21 cm and a height of 86.40 cm. It was filled to 81.30 cm and then drained. The position of the free surface was recorded at 2.54 cm intervals. The exit orifice diameters used were 0.533, 0.668, 0.945, and 1.087 cm, corresponding to area ratios, A_t/A_0, of 3003, 1912, 955, and 722, respectively. The recorded total drain times, t_d, corresponding to these were 1223, 767, 403, and 288 s, respectively.

IV. RESULTS AND DISCUSSION

When the instantaneous heights of the free surface of water were plotted against time, they yielded the tank's draining pattern for the selected diameter of the exit orifice. The volume of water in the tank at any time is a linear function of the height of water in the tank because its cross-section is constant. Therefore, the volume of liquid remaining in the tank, or that of the liquid flowing out of it, could be calculated from the height of the free surface. Indeed, the ratio of

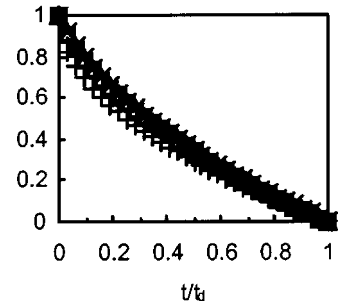

Fig. 2. Fractional height of the free surface of the liquid, h/h_0, as a function of the ratio of elapsed draining time to the emptying time, t/t_d, for values of the area ratio, A_t/A_0, of 3003 (open square), 1912 (open circle), 955 (open triangle), and 722 (solid cross) compared with the theoretical predictions of Eq. (3a) (plus).

heights, h/h_0, is equal to the ratio of the instantaneous volume of fluid that remains in the tank to the original volume of fluid in the tank when draining started.

The expressions derived above were compared to the corresponding results obtained experimentally. The variation of the height of the free surface of liquid with time is predicted to be parabolic in Eq. (3a). This result is compared to our experimental data in Fig. 2. Similarly, when the ratio between the cross-sectional area of the tank and that of the draining orifice is much larger than unity, inviscid-flow theory predicts that draining will be slow and that the time to empty the tank will vary linearly with that ratio. This result is given in Eq. (4) and is compared with experimental data in Fig. 3. In both cases, discrepancies between theory and experimental data were computed at each point and assessed. For the height of the free surface, discrepancies ranged from 0% to 14%, with an average value of 8.2%. For the total draining time, t_d, they ranged from 0.5% to 3% with an average value of 1.5%. It can be seen, therefore, that inviscid theory predicts the slow draining of a large tank reasonably well.

The forms of the expressions for $h(t)$ given in Eq. (3a), and for $h_p(t)$ given in Eq. (5), indicate several contrasting features between the motion of the free surface and that of a free-falling particle. These features are physically instructive because they help clarify the differences between the two behaviors.

(1) Unlike the free-falling particle, which is accelerated uniformly downward by gravity with a constant acceleration g, the falling free surface is uniformly decelerated in its

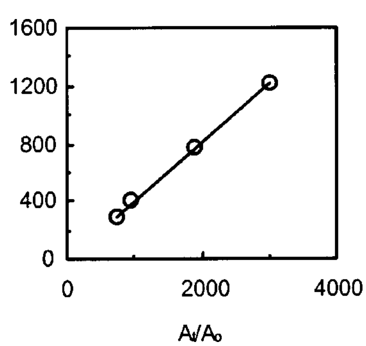

Fig. 3. Experimental values of the total time required to empty the tank, t_d, as a function of the area ratio A_t/A_0 (open circle) compared with the predictions of Eq. (4) (solid line).

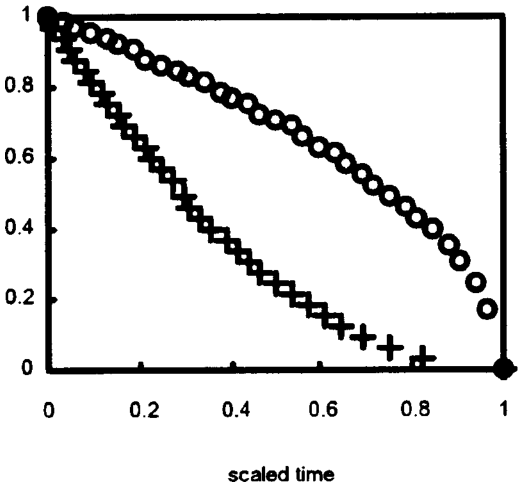

Fig. 4. Calculated scaled height, h/h_0, as a function of the scaled time, t/t_0, for the free surface of the liquid (plus), and calculated scaled height, h_p/h_0, as a function of scaled time, t/t_f, for a free-falling particle (open circle).

downward movement. The magnitude of the deceleration is given by Eq. (10); it is a small fraction of the acceleration of gravity, the fraction being determined by the area ratio, A_t/A_0. Figure 4 compares the motion of the free-falling particle in the absence of air resistance, as given in Eq. (5), to the downward movement of the free surface of a large tank that is being emptied slowly, as given by Eq. (3a).

(2) Unlike Eq. (7) for the free-falling particle that is released from rest, Eq. (3b) indicates that the initial velocity of the free surface is not zero. Indeed, by combining Eqs. (3b) and (4), it can be seen that the velocity of the free surface at $t=0$ is given by

$$\nu(0)=\frac{\nu_f}{\sqrt{(A_t/A_0)^2-1]}}. \tag{11}$$

Equation (11) predicts a velocity that cannot be zero, while Eq. (3b) makes it possible to determine that it is twice as large as the average velocity of the draining process. This implies a rapid change in the velocity of the free surface, and, perhaps, even a sudden jump at the beginning of draining. Mathematically, this comes from the nature of Eq. (2), the simplified equation that was used to determine the solution in this application. It is of first order, and therefore the first derivative, which represents the velocity in this case, cannot be specified as an initial condition. Although surprising at first, the result indicated by Eq. (11) is consistent with the derivation of Eq. (1), which assumes that the tank is draining when analysis starts. Since the fluid is incompressible, the conservation of mass implies that the free surface of the liquid must be moving downward while draining is in effect. Accordingly, an impulsive start of the draining process is not modeled by Eq. (1).

(3) The downward motion of the free surface is much slower than that of a free-falling particle. One expects the area ratio, A_t/A_0, to be much larger than unity because the exit area, A_0, is ordinarily much less than the cross-sectional area of the tank, A_t. This ratio introduces an apparent acceleration of gravity g_m, given by Eq. (10), that is considerably smaller than the actual acceleration of gravity. Consequently, the time to drain the tank is much larger than the time it

would take a particle to fall freely from rest through a vertical distance h_0 equal to the original height of fluid at the start of the draining process.

(4) The distribution of mechanical energy during motion can also be used to explain the differences in the behaviors. In the case of a particle that is falling freely in the absence of air resistance, gravity is the only force that acts on it; because gravity is a conservative force, the total mechanical energy of the particle is conserved at all times. Since it consists of only kinetic energy and gravitational potential energy, the total mechanical energy is distributed between these two forms during motion. Taking the ratio between the instantaneous kinetic and gravitational potential energies for the falling particle, one obtains

$$R_f=\left(\frac{E_k}{E_p}\right)_f=\frac{(t/t_f)^2}{1-(t/t_f)^2}, \tag{12}$$

where E_k represents the kinetic energy, E_p the potential energy, and R_f their ratio. This ratio varies with time and, as expected, it increases during the fall because potential energy is continually converted into kinetic energy.

For a fluid particle on the free surface of the draining liquid, the situation is quite different. Gravity acts on it, but it is also in contact with the adjacent fluid. Its mechanical energy can be stored in three distinct forms: pressure, kinetic energy, and gravitational potential energy. The pressure that acts on it is the ambient atmospheric pressure because the tank is open. It is conventional to use the energy associated with atmospheric pressure as a reference. In that case, the total mechanical energy of this particle consists only of the sum of its kinetic and gravitational potential energies; however, when one computes the ratio between the two, as was done in Eq. (12) for the falling particle, one finds that the energy ratio for a particle on the liquid surface, R_s, is independent of time and given by

$$R_s=\left(\frac{E_k}{E_p}\right)_s=\frac{1}{(A_t/A_0)^2-1}. \tag{13}$$

Indeed, it is the same as the ratio obtained by dividing the acceleration of the free surface, given in Eq. (10), by the local acceleration of gravity. In the conventional terminology of fluid mechanics, the energy ratio expressed in Eq. (13) is proportional to the square of the Froude number. The slow draining of a large tank under gravity is, therefore, a process that maintains the Froude number[5] constant. The fact that this ratio does not vary with time can be explained in the following way. Draining causes the free surface to fall and, as before, potential energy is continually converted into kinetic energy. However, in this case, the kinetic energy generated by this conversion does not stay in the tank; it is carried out by the exiting mass of fluid. Therefore, both forms of energy decrease continually within the tank. If the draining process is slow enough, the free surface will decelerate slowly and uniformly, as described by Eq. (3c), and, as expected, Eq. (13) indicates that the kinetic energy of particles on the free surface will be very small compared to their potential energy.

V. CONCLUSION

The lab exercise discussed here considers the slow draining of a large tank under gravity. When the draining is modeled assuming an inviscid fluid in irrotational motion, theory predicts draining patterns and total draining times that are in good agreement with experiment. However, in our observations, the experiment is successful only when the exit orifices are small. Exit orifices for which data are reported here are such that ratios of the area of the tank to the area of the orifice were greater than or equal to 722. For much smaller area ratios, the acceleration of the fluid particles can no longer be assumed constant and negligible compared to that of gravity, and the quasisteady approximation is no longer valid. Indeed, the rate of fall of the free surface is so rapid that it is impractical to keep track of its location and to record the elapsed time with a stopwatch for more than one or two data points. In that case, computer data-acquisition systems have to be used. Although our experiments could not confirm it, it is important, nevertheless, to note that a theoretical estimate found in the literature indicates that the quasisteady approximation would hold for area ratios as low as 100.[3]

Although the slow draining of a large tank is a common problem in textbooks of fluid mechanics,[3–6] the author is not aware of experiments that show students that approximations used to solve it yield realistic results when tested in the laboratory. This exercise was used in our lab to fill this void. The apparatus that is needed is easy to build, and the experimental procedure is quite simple. The experiment can be used in introductory mechanics classes to illustrate uniformly decelerated motion that is caused by gravity. It can also be used to demonstrate that viscosity can be neglected in special circumstances, thereby justifying the use of inviscid flow models to approximate the behavior of real flows without recourse to advanced mathematical arguments about the behavior of viscous boundary layers.[4] The experiment can also be used to illustrate the difference between the downward motion of the free surface of a liquid during draining and the free fall of a solid particle, thereby motivating a discussion of the differences between the mechanics of fluids and the mechanics of solids. We have used this simple experiment for all these purposes over the years and our students have enjoyed learning mechanics from it.

ACKNOWLEDGMENT

The author thanks Professor Jeffrey S. Dunham of Middlebury College for his guidance in preparing this note for publication.

[a]Electronic mail: libii@engr.ipfw.edu

[1]W. R. Debler, *Fluid Mechanics Fundamentals* (Prentice–Hall, Englewood Cliffs, NJ, 1990), pp. 225–227.

[2]J. Njock Libii and S. O. Faseyitan, "Data acquisition systems in the fluid mechanics laboratory: Draining of a tank," *Proceedings of the National Conference and Exposition of the American Society for Engineering Education (ASEE)*, Milwaukee, WI, Session 1426, June 1997.

[3]R. H. Sabersky, A. J. Acosta, and E. G. Hauptmann, *Fluid Flow: A First Course in Fluid Mechanics*, 5th ed. (Macmillan, New York, 1971), pp. 68–72.

[4]G. K. Batchelor, *An Introduction to Fluid Mechanics* (Cambridge University Press, Cambridge, 1994), pp. 378–392 and 511–517.

[5]T. E. Faber, *Fluid Dynamics for Physicists* (Cambridge University Press, Cambridge, 1995), pp. 37–46, 150–155, 186–188, and 222–225.

[6]J. W. Daily and D. R. F. Harleman, *Fluid Dynamics* (Addison-Wesley, Reading, MA, 1966), pp. 129–133.