

# Filling and emptying a tank of liquid

Johann Otto<sup>1</sup>  and Carl E Mungan<sup>2,\*</sup> 

<sup>1</sup> Höhere Technische Bundes-Lehr und Versuchsanstalt, Wiener Neustadt, Austria

<sup>2</sup> Physics Department, US Naval Academy, Annapolis MD, United States of America

E-mail: [mungan@usna.edu](mailto:mungan@usna.edu)

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## Abstract

A right cylindrical tank open to the atmosphere is being filled by a laminar jet of incompressible inviscid liquid falling onto its free surface. At the same time, fluid is escaping through a hole centered in the bottom of the tank. Newton's second law for variable mass and the unsteady Bernoulli equation are combined to find the time dependence of the liquid height in the tank. The level of analysis is suitable for an introductory undergraduate course in fluid dynamics.

Keywords: fluid dynamics, variable mass, unsteady Bernoulli equation, equation of continuity

(Some figures may appear in colour only in the online journal)

## 1. Introduction

A common situation in industry involves the filling and draining of a liquid storage tank. Fluid can be pumped into the tank at a constant rate. At the same time, fluid can leave the tank through an outlet valve. In chemical processes, it is often necessary to keep the liquid level between a defined minimum value, to keep the outlet flow pressure adequate, and maximum value, to avoid the risk of overflow. Modeling the fluid height in the tank as a function of the input flow parameters and output valve opening size can help in regulating the frequency, flow rate, and duration of the liquid transfer operations.

Accordingly, consider the following problem as an instructive classroom exercise in fluid dynamics. An open-topped cylindrical tank is being filled from above by a tap, while liquid is simultaneously flowing out of a hole in the bottom. How does the height  $h$  of the liquid in the tank change with time  $t$ ? In particular, one may be interested in knowing the circumstances under which the tank might overflow, empty out, or reach a steady-state fluid level. The results can be used to establish suitable flow conditions at the inlet and outlet of the tank.

\*Author to whom any correspondence should be addressed.

## 2. General analysis

Assume all flows of liquid into, within, and out of the tank are laminar and have the same constant density  $\rho$  and negligible viscosity. Denote earth's surface gravitational field strength as  $g$ . The tank has constant cross-sectional area  $A$  along its height and the hole has area  $a$ . The bottom of the tank is a fixed distance  $L$  below the tap out of which liquid is issuing with speed  $v_0$  and volumetric flow rate  $Q$  (in  $\text{m}^3 \text{s}^{-1}$ ), both constant. The tap is initially assumed to be off and the hole plugged, and the tank has fluid in it of height  $h_0$  at rest. Then at  $t = 0$  the hole is opened and liquid from the inlet jet simultaneously begins impacting the top surface in the tank.

Separately analyze what is happening at the top and bottom ends of the column of liquid in the tank, and then combine the resulting equations to solve the problem. For this purpose, imagine that a rigid massless piston is located partway up the fluid column dividing it into two portions: a variable upper volume 1 of height  $y$  and mass  $m = \rho Ay$ , and a variable lower volume 2 of height  $x$  and mass  $M = \rho Ax$ , with  $x + y = h$  as sketched in figure 1. The piston exerts (gauge) pressure  $P$  upward on the bottom surface of volume 1, and equal and opposite pressure  $P$  downward on the top surface of volume 2. The liquid jet from the inlet tap has speed  $v_1$  as it contacts the top surface of volume 1, and the exit speed of fluid out of the hole at the bottom surface of volume 2 is  $v_2$ . Choose upward to be the positive direction in the analysis.

Application of the steady Bernoulli equation to the input liquid jet, whose vertical distance of fall is  $L - h$ , results in [1]

$$v_1 = \sqrt{v_0^2 + 2g(L - h)}. \quad (1)$$

This fluid adds to volume 1, increasing it at a rate of  $Q$  so that

$$\dot{y} = \frac{Q}{A}, \quad (2)$$

where overdots denote time derivatives. Thus  $y$  increases linearly with time  $t > 0$ .

Next consider the change of linear momentum over a time interval from  $t$  to  $t + dt$  for a system consisting of volume 1 (of mass  $m$ ) plus the mass  $dm$  of liquid in the jet that will impact this volume during time  $dt$ . The piston is moving downward at velocity  $\dot{x} < 0$ . That is also the velocity of all the fluid inside the tank, except for the added liquid from the tap near the top end and for fluid near the hole at the bottom end where the streamlines have to bend around to pass through it. Thus initially, at time  $t$ , the system consists of mass  $m$  moving with velocity  $\dot{x}$  and a block of fluid of mass  $dm$  with velocity  $-v_1$  so that the initial momentum of the system is

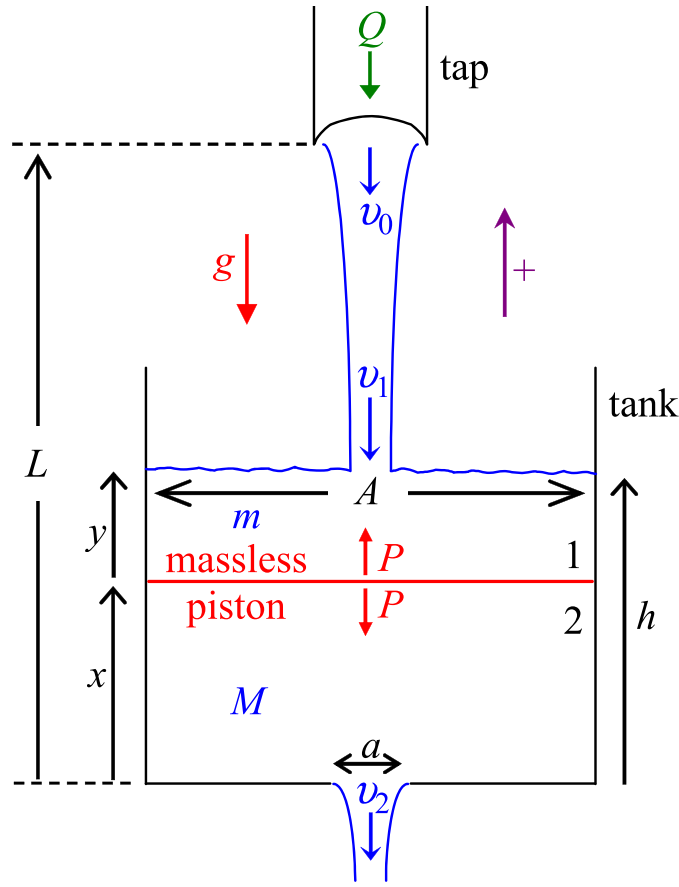
$$p = m\dot{x} - v_1 dm. \quad (3)$$

After the inelastic collision, at time  $t + dt$ , the combined system has mass  $m + dm$  and velocity  $\dot{x} + d\dot{x}$  so that its final momentum is

$$p + dp = (m + dm)(\dot{x} + d\dot{x}). \quad (4)$$

The external forces acting on the system are  $mg$  downward due to gravity (neglecting the infinitesimally small gravitational force on  $dm$ ) and  $PA$  upward due to the piston which together give the rate of change of the momentum of the system,

$$-mg + PA = \frac{dp}{dt}. \quad (5)$$



**Figure 1.** Liquid from a tap falls into a cylindrical tank open to the atmosphere, and then flows out of a hole in its bottom.

Substituting equations (3) and (4) into (5) results in an expression for the piston force of

$$PA = m(\ddot{x} + g) + \dot{m}(\dot{x} + v_1) \quad (6)$$

neglecting the second-order-small term  $dm \, d\dot{x}$ . This approach correctly treats the variable-mass problem [2] of liquid in volume 1; equation (6) would *not* be obtained if one instead wrongly substituted  $p = m\dot{x}$  into equation (5).

On the other hand, for volume 2, the equation of continuity relates the fluid velocity in the tank to the exit velocity through the hole as

$$A\dot{x} = -av_2 \quad (7)$$

because either side of this equation gives the rate of change of volume 2. (To be exact,  $a$  is the area of the liquid jet just below the hole, which is smaller than the area of the orifice itself, owing to the *vena contracta* [3]. Specifically, the effective area  $a$  of a sharp-edged round hole is 61% of its geometrical area [4].) However, the fluid velocity at any fixed height inside the tank is time-dependent and so it is described by the unsteady Bernoulli equation [5],

$$\left(P_A + \rho g Z_A + \frac{1}{2} \rho v_A^2\right) - \left(P_B + \rho g Z_B + \frac{1}{2} \rho v_B^2\right) = \frac{d(\rho \dot{x} A)}{dt} \int_A^B \frac{ds}{A} \quad (8)$$

where  $Z$  is the height above the bottom of the tank. This equation would reduce to the conventional (steady) Bernoulli relation if the term on the right-hand side equalled zero, which would be true if the mass flow rate  $\rho \dot{x} A$  (in  $\text{kg s}^{-1}$ ) of liquid flowing vertically across any fixed cross-section of the tank were constant in time. However, in the present problem a horizontal slice of liquid of vertical thickness  $ds$  and thus mass  $\rho A ds$  is accelerating vertically at  $\ddot{x}$ . Hence the net pressure on this slice is the force per unit area,  $(\rho A ds) \ddot{x} / A$ . Integrating it from points A to B will therefore yield the extra pressure differential required to accelerate the fluid in the volume of the tank between those two points.

Choose point A in equation (8) to be just below the hole and point B to be just below the piston to get

$$\left(0 + 0 + \frac{1}{2} \rho v_2^2\right) - \left(P + \rho g x + \frac{1}{2} \rho \dot{x}^2\right) = \frac{d(\rho \dot{x} A)}{dt} \cdot \frac{x}{A} \quad (9)$$

so that

$$\frac{1}{2} \rho A (v_2^2 - \dot{x}^2) - PA - Mg = M \ddot{x}. \quad (10)$$

Substitute into it  $v_2$  from equation (7),  $PA$  from equation (6), and the two masses  $m = \rho A y$  and  $M = \rho A x$  to obtain

$$r \dot{x}^2 - (x + y)(\ddot{x} + g) - \dot{y}(\dot{x} + v_1) = 0, \quad (11)$$

where a dimensionless area ratio has been defined as

$$r \equiv \frac{1}{2} \left( \frac{A^2}{a^2} - 1 \right). \quad (12)$$

Now substitute into equation (11) the expressions  $\dot{y} = Q/A$  from equation (2) and  $x + y = h$  from figure 1, which together imply  $\dot{x} = \dot{h} - Q/A$  and  $\ddot{x} = \ddot{h}$ , to end up with a differential equation for  $h(t)$  in terms of constants

$$r \left( \dot{h} - \frac{Q}{A} \right)^2 - h(\ddot{h} + g) = \frac{Q}{A} \left[ \dot{h} - \frac{Q}{A} + \sqrt{v_0^2 + 2g(L - h)} \right] \quad (13)$$

using equation (1).

To solve equation (13), recast it into dimensionless form by defining the two constants

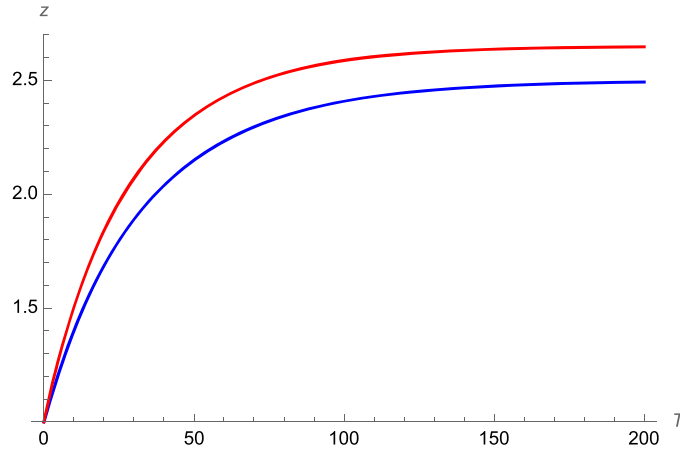
$$\alpha \equiv \frac{Q}{A\sqrt{gh_0}} \quad \text{and} \quad \beta \equiv \frac{v_0^2 + 2gL}{gh_0} \quad (14)$$

along with the normalized height  $z$  and time  $T$  given by

$$z \equiv \frac{h}{h_0} \quad \text{and} \quad T \equiv t \sqrt{\frac{g}{h_0}}. \quad (15)$$

Then equation (13) becomes

$$z(z'' + 1) - r(z' - \alpha)^2 + \alpha(z' + \sqrt{\beta - 2z}) = \alpha^2, \quad (16)$$



**Figure 2.** The blue curve is a numerical solution of equation (16) for  $r = 98$ ,  $\beta = 6$ , and  $\alpha = (1 + 991^{1/2})/198$  such that  $z_\infty = 2.5$  according to equation (17). In comparison, the red curve is a parametric plot of the approximate solution given by equation (27) for  $r = 98$  and  $\alpha = (1 + 991^{1/2})/198$  such that  $z_\infty \approx 2.65$  according to equation (26).

where primes denote derivatives with respect to  $T$ . The initial conditions are  $z(0) = 1$  and  $z'(0) = 0$ . Equation (16) has a steady-state solution (i.e.,  $z$  becomes constant so that  $z'$  and  $z''$  are both zero) given by

$$z_\infty = \alpha^2 \left( r - \sqrt{\beta \alpha^{-2} - 2r - 1} \right). \quad (17)$$

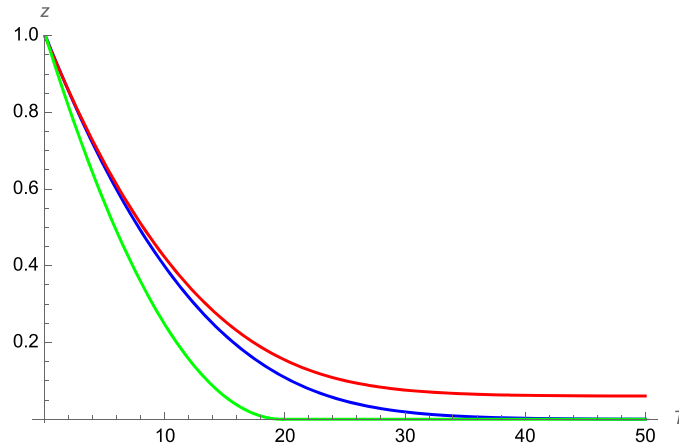
In particular, if  $z_\infty = 1$  then the height of liquid in the tank is  $h_0$  at all times, meaning the inflow from the tap balances the outflow through the hole. For example, if  $r = 98$  (so that the ratio of the diameter of the tank to the diameter of the hole is  $197^{1/4} \approx 3.75$ ) and  $\beta = 6$  (which implies that  $v_1^2 = 4gh_0$  at  $t = 0$ ) then  $z_\infty = 1$  for the critical value  $\alpha_c = 1/9$ . Larger values of  $\alpha$  (corresponding to larger flow rates  $Q$  out of the tap) will mean the liquid level  $h$  in the tank increases with time. For instance, figure 2 uses Mathematica to plot in blue the case of  $\alpha = (1 + 991^{1/2})/198 \approx 1/6$ . As a second example, equation (17) gives  $z_\infty = 0$ , so that the tank asymptotically empties out, for  $\alpha = 6^{1/2}/99 \approx 1/40$  as graphed in blue in figure 3. The blue curves in both figures 2 and 3 start at  $z = 1$  with zero slope (as is visible if the horizontal scale is expanded near  $T = 0$ ) due to the initial conditions.

As a check on these results, the [appendix](#) presents a simpler approximate model describing the time dependence of the fluid height in the tank. Those results are plotted in red in figures 2 and 3.

### 3. Special case of zero inflow

Suppose the tap is not turned on (when the outlet at the bottom of the tank is opened) so that  $Q = 0 \Rightarrow \alpha = 0$ . Then equation (16) reduces to

$$zz'' = rz'^2 - z. \quad (18)$$



**Figure 3.** The blue curve is a numerical solution of equation (16) for  $r = 98$ ,  $\beta = 6$ , and  $\alpha = 6^{1/2}/99$  such that  $z_\infty = 0$  according to equation (17). The red curve is a parametric plot of the approximate solution given by equation (27) for  $r = 98$  and  $\alpha = 6^{1/2}/99$  such that  $z_\infty \approx 0.06$  according to equation (26). The green curve is a parametric plot of the solution for zero inflow given by separating and integrating equation (22) for  $r = 98$ ; it intercepts the horizontal axis at a time determined by equation (23).

Change variables to  $u = z'^2$  with

$$\frac{du}{dz} = 2z' \frac{dz'}{dz} = 2z' \frac{dz'}{dT} \frac{dT}{dz} = 2z'' \quad (19)$$

which implies that equation (18) can be rewritten as

$$z \frac{du}{dz} = 2ru - 2z. \quad (20)$$

The homogeneous equation is separable in the form

$$\frac{du}{u} = 2r \frac{dz}{z} \Rightarrow u = Cz^{2r}, \quad (21)$$

where  $C$  is a constant of integration. A particular solution of equation (20) can be found using the trial form  $u = Bz$  which results in  $B = 2/(2r - 1)$  assuming  $r > 1/2$ . Adding together the homogeneous and particular solutions to obtain the general solution for  $u$  and imposing the initial conditions  $z(0) = 1$  and  $z'(0) = 0$  shows that  $C = -B$ . In this way the first integral of equation (18) is found to be

$$z' = -\sqrt{\frac{2}{2r-1}(z - z^{2r})}, \quad (22)$$

where the negative square root was taken because  $z$  decreases with increasing  $T$  as the liquid drains out of the tank.

The time required for the tank to fully empty can then be expressed in terms of a ratio of gamma functions as [6]

$$t_{\text{empty}} = \sqrt{\frac{(2r-1)h_0}{2g}} \int_0^1 \frac{dz}{\sqrt{z-z^{2r}}} = \sqrt{\frac{\pi h_0}{g(4r-2)}} \Gamma\left(\frac{1}{4r-2}\right) / \Gamma\left(\frac{r}{2r-1}\right). \quad (23)$$

For example, if  $r = 98$  then  $T_{\text{empty}} \approx 19.8$ . More generally, equation (22) can be separated and integrated; the result is plotted parametrically as the green curve in figure 3.

#### 4. Concluding comments

Inserting an imaginary piston into the tank, as has been done in figure 1, is reasonable only if the fluid pressure in the tank is constant across some intermediate cross section of the liquid column. Accordingly, the analysis does not precisely describe the final stages of draining of liquid from the tank, once the level  $h$  becomes small compared to the radius  $(A/\pi)^{1/2}$  of the tank. In addition, the model proposed above assumes that the speeds of inflow  $v_1$  and outflow  $v_2$  are associated with small enough Reynolds numbers that the flow does not become turbulent.

Following D'Alessio [7], it might be interesting to extend the present results to liquid tanks with shapes other than right cylinders. Another extension of the present work would be to add a long vertical tube to the bottom of the exit hole in the tank to get Poiseuille flow through it [8].

#### Appendix

If the rate of change of the height  $h$  of fluid in the tank is small compared to  $v_2$  then that speed of outflow can be approximated using Torricelli's theorem [9] so that

$$A\dot{h} \cong Q - a\sqrt{2gh} \quad (24)$$

which can be rewritten in dimensionless form as

$$z' = \alpha - \sqrt{\frac{2z}{2r+1}}. \quad (25)$$

This result implies that the initial condition  $z'(0)$  is no longer a free parameter [10] and is not in general equal to 0. The steady-state solution of equation (25) is

$$z_{\infty} = \alpha^2(r+0.5) \quad (26)$$

to be compared to equation (17). In particular, it is now impossible to fully empty the tank unless the tap is shut off so that  $\alpha = 0$ . For example, the red curve in figure 3 levels off when the liquid height  $h$  results in a pressure at the bottom of the tank such that the rate of fluid flow out of the hole balances that into the tank from the tap. Equation (25) can be separated and integrated to find the time it takes the liquid level in the tank to reach any given height,

$$T(z) = \sqrt{4r+2}(1-\sqrt{z}) + \alpha(2r+1) \ln \frac{\alpha\sqrt{2r+1}-\sqrt{2}}{\alpha\sqrt{2r+1}-\sqrt{2z}}. \quad (27)$$

## ORCID iDs

Johann Otto  <https://orcid.org/0000-0001-9983-9479>

Carl E Mungan  <https://orcid.org/0000-0001-7084-5402>

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