Koopman and P-F Operators Examples

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1 Koopman and Perron-Frobenius Operators Basic Theory

Consider a dynamical system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbf{X} \subseteq \mathbb{R}^n. \tag{1}$$

where the vector field is assumed to be $\mathbf{f}(\mathbf{x}) \in \mathcal{C}^1(\mathbf{X}, \mathbb{R}^n)$. Koopman (composition) operator and Perron-Frobenius (transfer) operator are two powerful tools to study the behavior of nonlinear systems by lifting the finite dimensional nonlinear state space to infinite dimensional linear function space [1].

Definition 1.1 (Koopman Operator) $\mathbb{U}_t : \mathcal{L}_\infty(\mathbf{X}) \to \mathcal{L}_\infty(\mathbf{X})$ *for dynamical system* (1) *is defined as*

$$[\mathbb{U}_t \varphi](\mathbf{x}) = \varphi(\mathbf{s}_t(\mathbf{x})). \tag{2}$$

The infinitesimal generator for the Koopman operator is

$$\dot{\varphi} = \lim_{t \to 0} \frac{(\mathbb{U}_t - I)\varphi}{t} = \mathbf{f}(\mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{f}_i \frac{\partial \varphi}{\partial \mathbf{x}_i} =: \mathcal{K}_{\mathbf{f}} \varphi \tag{3}$$

Definition 1.2 (Perron-Frobenius Operator) $\mathbb{P}_t : \mathcal{L}_1(\mathbf{X}) \to \mathcal{L}_1(\mathbf{X})$ *for dynamical system* (1) *is defined as*

$$[\mathbb{P}_t \psi](\mathbf{x}) = \psi(\mathbf{s}_{-t}(\mathbf{x})) \left| \frac{\partial \mathbf{s}_{-t}(\mathbf{x})}{\partial \mathbf{x}} \right|, \tag{4}$$

where $|\cdot|$ *stands for the determinant. The infinitesimal generator for the P-F operator is given by*

$$\dot{\psi} = \lim_{t \to 0} \frac{(\mathbb{P}_t - I)\psi}{t} = -\nabla \cdot (\mathbf{f}(\mathbf{x})\psi(\mathbf{x})) = -\sum_{i=1}^{n} \frac{\partial (\mathbf{f}_i \psi)}{\partial x_i} =: \mathcal{P}_{\mathbf{f}} \psi. \tag{5}$$

Under appropriate conditions the semigroup of Koopman operators is strongly continuous and we can rewrite its relation with the generator as [2]

$$\mathbb{U}_t = e^{\mathcal{K}_{\mathbf{f}} t}. \tag{6}$$

These two operators are dual to each other where the duality is expressed as follows.

$$\int_{\mathbf{X}} [\mathbb{K}_t \varphi](\mathbf{x}) \psi(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{X}} [\mathbb{P}_t \psi](\mathbf{x}) \varphi(\mathbf{x}) d\mathbf{x}.$$

Note that Perron-Frobenius generator can be deduced from Koopman operator using vector calculus identities [3],

$$\mathcal{P}_{\mathbf{f}} \psi = -\nabla \cdot (\mathbf{f}(\mathbf{x}) \psi(\mathbf{x})) = -\mathbf{f} \cdot \nabla \psi - \nabla \cdot \mathbf{f} \psi = -\mathcal{K}_{\mathbf{f}} \psi - \nabla \cdot \mathbf{f} \psi, \tag{7}$$

this equation will be useful for finite approximation of Perron-Frobenius operator.

2 Finite Dimensional Approximation

Usually, the compute of transfer operators and their generators is done with set oriented techniques like Ulam's method, but we will use a data-driven approach [4]. The more famous algorithm to find a finite approximation of Koopman Operator is Extended Dynamic Mode Decomposition (EDMD) [5]. Similarly, the Naturally Structured Dynamic Mode Decomposition (NSDMD) is a variant of EDMD that lead approximate Perron-Frobenius operator and additionally to include natural properties (i.e. Markov and positivity properties) [6]. Finally, for obtaining directly the generators associated with the transfer operators we can use the gEDMD algorithm presented in [7]. In this write-up, we will use the following notation: Finite approximation of Koopman operator ($\mathbb{U}_t \approx \mathbf{K}$). Finite approximation of Perron-Frobenius operator ($\mathbb{P}_t \approx \mathbf{P}$). Finite approximation of Koopman generator ($\mathcal{K}_{\mathbf{f}} \approx \mathbf{L}$). Finite approximation of Perron-Frobenius generator ($\mathcal{P}_{\mathbf{f}} \approx \mathbf{M}$). Dictionary or functions basis are denoted by $\Psi(\mathbf{x}) = [\psi_1(\mathbf{x}), \ldots, \psi_N(\mathbf{x})]^\top$. The set of data is $\{\mathbf{X}, \mathbf{Y}, \dot{\mathbf{X}}\}$ where $\mathbf{X}$ and $\mathbf{Y}$ are time-series data defined by $\{\mathbf{x}_i\}_{i=1}^K$, $\{y_i\}_{i=1}^K$ with $y_i = \mathbf{f}(\mathbf{x}_i)\Delta t + \mathbf{x}_i$, and let $\dot{\mathbf{X}}$ be the derivatives time-series data. Now, we present several alternatives to compute the finite approximation of generators from transfer operators. Finite approximation of Koopman operator $\mathbf{K} \rightarrow EDMD(\mathbf{X}, \mathbf{Y}, \mathbf{\Psi})$, finite approximation of Perron-Frobenius operator $\mathbf{P} \rightarrow NSDMD(\mathbf{X}, \mathbf{Y}, \mathbf{\Psi} > 0)$, and finite approximation of Koopman generator by $\mathbf{L}^\top \rightarrow gEDMD(\mathbf{X}, \dot{\mathbf{X}}, \mathbf{\Psi})$. Similarly, we can use the relation given by generators definition

$$\mathbf{L} = \frac{\mathbf{K} - I}{\Delta t} = \frac{1}{\Delta t} \log \mathbf{K}, \tag{8}$$

$$\mathbf{M} = \frac{\mathbf{P} - I}{\Delta t} = \frac{1}{\Delta t} \log \mathbf{P}, \tag{9}$$

or using the special case when $\mathbf{\Psi}$ is state inclusive (i.e., $\mathbf{x}_i \in \Psi^*$), then the P-F operator can be written as

$$\mathbf{M} = -\mathbf{L} - [\nabla \cdot (C_{\mathbf{x}}^\top \mathbf{L}\Psi^*)]I. \tag{10}$$

3 Koopman Operator Examples

3.1 Discrete-time one dimensional transformation

We approximate the Koopman operator in a subspace using the projection developed in [8].

Example 1 *Consider the dynamical system $x(k+1) = 2x(k) - x^2(k)$ and the projection*

$$\Pi x^j := x^j \ \textit{if}\ j \leq 3 \ \textit{or}\ 0 \ \textit{if}\ j > 3. \tag{11}$$

with the vector of basis $\Psi = [1 \quad x \quad x^2 \quad x^3]^\top$.

The finite approximation the Koopman operator is given by

$$(l_1\psi_1)(x) = \psi_1(2x - x^2) = 1 \tag{12}$$
$$(l_2\psi_2)(x) = \psi_2(2x - x^2) = 2x - x^2 \tag{13}$$
$$(l_3\psi_3)(x) = \psi_3(2x - x^2) = 4x^2 - 4x^3 \tag{14}$$
$$(l_4\psi_4)(x) = \psi_4(2x - x^2) = 8x^3 \tag{15}$$

then, the finite approximation of Koopman operator transpose is given by

$$\mathbf{K}^\top = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

3.2 Discrete-time two dimensional transformation

Example 2 *This example consider a polynomial slow manifold which is presented in [9],*

$$x_1(k+1) = \lambda x_1(k) \tag{16}$$
$$x_1(k+2) = \mu x_2(k) - (\lambda^2 - \mu)x_1^2(k) \tag{17}$$

$$\mathbf{K}^\top = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & (\lambda^2 - \mu) \\ 0 & 0 & \lambda^2 \end{bmatrix}. \tag{18}$$

3.3 Infinitesimal Generator of Koopman operator for a continuous-time ODE

Example 3 *This example is a special case for obtaining the infinitesimal generator transpose (Lifted dynamics) in close form*

$$\dot{x}_1 = x_1 \tag{19}$$

$$\dot{x}_2 = x_2 - x_1^2 \tag{20}$$

and let $\Psi = [1 \quad x_1 \quad x_2 \quad x_1^2]^\top$ *be the dictionary.*

Using this set of basis we can compute a finite dimension close form of the infinitesimal generator as

$$\mathbf{L}^\top = \sum_{i=1}^{n} \mathbf{f}_i(\mathbf{x}) \frac{\partial \varphi(\mathbf{x})}{\partial x_i} \tag{21}$$

$$\begin{aligned}
l1 &= x_1 \frac{\partial 1}{\partial x_1} + (x_2 - x_1^2) \frac{\partial 1}{\partial x_2} \\
&= 0. \\
l2 &= x_1 \frac{\partial x_1}{\partial x_1} + (x_2 - x_1^2) \frac{\partial x_1}{\partial x_2} \\
&= x_1 + 0. \\
l3 &= x_1 \frac{\partial x_2}{\partial x_1} + (x_2 - x_1^2) \frac{\partial x_2}{\partial x_2} \\
&= 0 + x_2 - x_1^2. \\
l4 &= x_1 \frac{\partial x_1^2}{\partial x_1} + (x_2 - x_1^2) \frac{\partial x_1^2}{\partial x_2} \\
&= 2x_1^2 + 0.
\end{aligned}$$

The empirical approximation in compact form is given by

$$\mathbf{L}^\top = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \tag{22}$$

4 Perron-Frobenius Operator Examples

4.1 Discrete-time one dimensional transformation

The simplest and most famous example of a chaotic system is the logistic map and is a few in which is possible computing the PF operator explicitly [10].

To use the definition (5) is necessary that transformation be invertible, but in general it is not the case then we can use the following equation which present the PF operator in terms of each preimage

$$[\mathbb{P}\psi](\mathbf{y}) = \sum_{y:x=\mathbf{s}(y)} \frac{\psi(\mathbf{y})}{|\mathbf{s}'(\mathbf{y})|}. \tag{23}$$

Particularly, the Logistic map $x(k+1) = 4x(k)(1-x(k))$ has two preimages

$$L^{-1}(x) = \frac{1 \pm \sqrt{1-x}}{2}. \tag{24}$$

Therefore, the PF operator for this dynamical systems is

$$[\mathbb{P}\psi](x) = \frac{1}{4\sqrt{1-x}} \left(\psi \left(\frac{1-\sqrt{1-x}}{2} \right) + \psi \left(\frac{1+\sqrt{1-x}}{2} \right) \right). \tag{25}$$

A fixed point for this operator is

$$\psi^*(x) = \frac{1}{\pi\sqrt{x(1-x)}} \tag{26}$$

which check an invariant density.

4.2 Infinitesimal Generator of Perron-Frobenius operator for a continuous-time ODE

Similar to the Koopman case, for this example, we can obtaining the infinitesimal generator corresponding to P-F operator transpose in close form.

Example 4 *Consider again the continuous-time system*

$$\dot{x}_1 = x_1 \tag{27}$$

$$\dot{x}_2 = x_2 - x_1^2 \tag{28}$$

and let $\Psi = [1 \quad x_1 \quad x_2 \quad x_1^2]^\top$ *be the dictionary.*

Using this set of basis we can compute a finite dimension close form of the infinitesimal generator as

$$\mathbf{M} = -\sum_{i=1}^{n} \frac{\partial(\mathbf{f}_i\psi)}{\partial x_i} \tag{29}$$

$$
\begin{aligned}
m_1 &= -\frac{\partial((x_1)1)}{\partial x_1} - \frac{\partial((x_2 - x_1^2)1)}{\partial x_2} \\
&= -1 - 1 = -2 \\
m_2 &= -\frac{\partial((x_1)x_1)}{\partial x_1} - \frac{\partial((x_2 - x_1^2)x_1)}{\partial x_2} \\
&= - - 2x_1 - x_1 = -3x_1 \\
m_3 &= -\frac{\partial((x_2)x_2)}{\partial x_1} - \frac{\partial((x_2 - x_1^2)x_2)}{\partial x_2} \\
&= -x_2 - 2x_2 + x_1^2 = -3x_2 + x_1^2 \\
m_4 &= -\frac{\partial((x_1)x_1^2)}{\partial x_1} - \frac{\partial((x_2 - x_1^2)x_1^2)}{\partial x_2} \\
&= -3x_1^2 - x_1^2 = -4x_1^2
\end{aligned}
$$

The empirical approximation in compact form is given by

$$
\mathbf{M}^\top = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}. \tag{30}
$$

4.3 PF generator from Koopman Operator

We can use the properties of vector calculus for deriving the PF generator from Koopman generator using the relation (7). In this example, we will calculate $\mathbf{M}^\top$ from example (19) and then, we should obtain (30). Consider the generators relation

$$
\mathbf{M} = -\mathbf{L} - [\nabla \cdot (C_{\mathbf{x}}^\top \mathbf{L} \Psi^*)] I. \tag{31}
$$

with a state inclusive base Ψ denoted by Ψ^*.

$$
\mathbf{M}^\top = - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}^\top - \left[\nabla \cdot \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}^\top \Psi^* \right) \right] I, \tag{32}
$$

$$
\mathbf{M}^\top = - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}^\top - \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \tag{33}
$$

$$\mathbf{M}^\top = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}. \tag{34}$$

5 Koopman Operator with Control Input

5.1 Bilinear structure for afine systems (i.e. $\dot{z} = Az + uBz$)

Example 5 *Close form nonlinear (Uncontrollable linearization)*

$$\dot{x}_1 = -x_1 + 2x_1 u \tag{35}$$

$$\dot{x}_2 = -x_2 + x_1^2 \tag{36}$$

$$y = x_1 \tag{37}$$

First we consider the following basis $\Psi = [1, x_1, x_2, x_1^2]$, Therefor the bilinear with systems $\mathbf{z} = \Psi$ is

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 \\ \dot{x}_1 \\ \dot{x}_2 \\ 2x_1\dot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1 \\ -x_2 + x_1^2 \\ -2x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2x_1 \\ 0 \\ 4x_1^2 \end{bmatrix} u \tag{38}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \mathbf{z} + u \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \mathbf{z} \tag{39}$$

Example 6 *Close form linear (Uncontrollable linearization)*

$$\dot{x}_1 = -x_1 + 2x_1 u \tag{40}$$

$$\dot{x}_2 = x_2 + u \tag{41}$$

Let $\Psi = [1 \quad x_1 \quad x_2 \quad x_1x_2 \quad x_1^2 \quad x_2^2]^\top$ *be the vector of observables.*

$$\dot{\Psi} = \begin{bmatrix} 0 \\ \dot{x}_1 \\ \dot{x}_2 \\ x_1\dot{x}_2 + x_2\dot{x}_1 \\ 2x_1\dot{x}_1 \\ 2x_2\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1 + 2x_1 u \\ x_2 + u \\ x_1 u + 2x_1 x_2 u \\ -2x_1^2 + 4x_1^2 u \\ 2x_2^2 + 2x_2 u \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \Psi + u \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \Psi$$

Example 7

$$\dot{x}_1 = \mu x_1 + u \tag{42}$$
$$\dot{x}_2 = \lambda(x_2 - x_1^2) \tag{43}$$
$$y = x_1 \tag{44}$$

consider the bases $z = [x_1, x_2 - bx_1^2, x_1^2]$ with $b = \frac{\lambda}{\lambda - 2\mu}$

$$\dot{z} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 - 2bx_1\dot{x}_1 \\ 2x_1\dot{x}_1 \end{bmatrix} = \begin{bmatrix} \mu x_1 + u \\ \lambda(x_2 - x_1^2) - 2bx_1(\mu x_1 + u) \\ 2x_1(\mu x_1 + u) \end{bmatrix} = \begin{bmatrix} \mu x_1 + u \\ \lambda(x_2 - x_1^2) - 2\mu b x_1^2 - 2bx_1 u \\ 2\mu x_1^2 + 2x_1 u \end{bmatrix} \tag{45}$$

$$\dot{z} = \begin{bmatrix} \mu x_1 + u \\ \lambda(x_2 - x_1^2) - 2\mu b x_1^2 - 2bx_1 u \\ 2\mu x_1^2 + 2x_1 u \end{bmatrix} \tag{46}$$

$$\dot{z} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \lambda & -2\mu b \\ 0 & 0 & 2\mu \end{bmatrix} z + u \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 1 & 0 & 0 \\ -2b & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} z \tag{47}$$

6 Spectrum of Koopman (eigenvalues and eigenfunctions)

Definition 6.1 (Koopman eigenfunctions) *A function $\varphi(\mathbf{x}) \in \mathcal{C}^1(M)$ is said to be eigenfunction of the Koopman operator associated with value $\lambda \in \mathbb{C}$ if*

$$[\mathbb{U}_t \varphi](\mathbf{x}) = e^{\lambda t}\varphi(\mathbf{x}). \tag{48}$$

The value λ is the associated Koopman eigenvalue and belongs to the point spectrum of the operator. Using the Koopman generator, the (48) can be written as

$$\frac{\partial \varphi}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) = \lambda \varphi(\mathbf{x}). \tag{49}$$

The eigenfunctions associated with the eigenvalues of the A are denominated as *Principal Eigenfunctions* [11]. It is known that Koopman eigenfunctions have an algebraic structure under products, thus the other eigenfunctions of the Koopman operator can be constructed from the principal eigenfunctions.

The property is true because

$$[\mathbb{U}_t \varphi_{\lambda_1}^{k_1} \varphi_{\lambda_2}^{k_2}] = [\mathbb{U}_t \varphi_{\lambda_1}^{k_1} \mathbb{U}_t \varphi_{\lambda_2}^{k_2}] = [\mathbb{U}_t \varphi_{\lambda_1}]^{k_1} [\mathbb{U}_t \varphi_{\lambda_2}]^{k_2}$$

$$= e^{(k_1\lambda_1 + k_2\lambda_2)t} \varphi_{\lambda_1}^{k_1} \varphi_{\lambda_2}^{k_2}$$

Consider a scalar valued function $g : M \to \mathbb{R}$, and assume that the function g can be expanded in terms of Koopman eigenfunctions as follows.

$$g(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{N}^n}^{N} \bar{g}_{\mathbf{k}} \prod_{i=1}^{n} \phi_i^{k_i}(\mathbf{x}) \tag{50}$$

where $\bar{g}_{\{k_1,\ldots,k_n\}}$ are the Koopman modes and corresponds to the projection of function $g(\mathbf{x})$ on the eigenfunctions, $\varphi_{\lambda_1}^{k_1}(\mathbf{x}), \ldots, \varphi_{\lambda_n}^{k_n}(\mathbf{x})$. The scalar valued function will propagate under system dynamics as follows

$$g(\mathbf{s}_t(\mathbf{x})) = [\mathbb{U}_t g](\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{N}^n}^{N} \bar{g}_{\mathbf{k}} \prod_{i=1}^{n} \phi_i^{k_i}(\mathbf{x}) e^{\lambda_i k_i t} \tag{51}$$

where $\bar{\lambda} = k_1\lambda_1 + \ldots + k_n\lambda_n$

Example 8 *The eigenfunctions for linear systems are the left eigenvalues i.e., $w_j^\top A = \lambda_j w_j^\top$*

$$\dot{x} = Ax$$

$$\varphi(x) = Wx$$

Example 9 ([12]) *The eigenfunctions for polynomial nonlinear dynamics are*

$$\dot{x} = x^2 \quad \varphi(x) = c_0 e^{-\lambda/x}$$

in general

$$\dot{x} = ax^n \quad \varphi(x) = e^{\frac{\lambda}{(1-n)a} x^{1-n}}$$

Example 10 ([8]) *Consider a following dynamical system*

$$\dot{x} = x - x^3$$

For this systems is possible to compute analytically the eigenfunction associate to eigenvalue $\lambda = 1$ as

$$\varphi(x) = \frac{x}{\sqrt{1-x^2}}$$

$$\dot{\varphi} = \frac{\dot{x}}{\sqrt{1-x^2}} + \frac{x}{(1-x^2)^{3/2}} x\dot{x}$$

$$\dot{\varphi} = \frac{x(1-x^2)}{\sqrt{1-x^2}} + \frac{x^3(1-x^2)}{(1-x^2)^{3/2}}$$

$$\dot{\varphi} = x\sqrt{(1-x^2)} + \frac{x^3}{\sqrt{1-x^2}}$$

$$\dot{\varphi} = \frac{x(1-x^2)+x^3}{\sqrt{1-x^2}}$$

$$\dot{\varphi} = \frac{x}{\sqrt{1-x^2}} = \varphi$$

Example 11 *Consider the dynamics*

$$\dot{x}_1 = \mu x_1 \tag{52}$$

$$\dot{x}_2 = \lambda(x_2 - x_1^2) \tag{53}$$

Eigenfunctions are given by

$$\varphi_1 = x_1, \quad \varphi_2 = x_2 + \beta x_1^2$$

$$\dot{\varphi}_1 = \dot{x}_1 = \mu\varphi_1 \tag{54}$$

$$\dot{\varphi}_2 = \dot{x}_2 + 2\beta\dot{x}_1 x_1 = \lambda(x_2 - x_1^2) + 2\beta\mu x_1^2 = \lambda x_2 + x_1^2(2\beta\mu - \lambda) \tag{55}$$

$$= \lambda(x_2 + \frac{2\beta\mu - \lambda}{\lambda} x_1^2) \tag{56}$$

we want
$$\frac{2\beta\mu - \lambda}{\lambda} = \beta \to \beta(2\mu - \lambda) = \lambda \to \beta = \frac{\lambda}{2\mu - \lambda}$$

Then we have

$$\dot{\varphi}_1 = \mu\varphi_1 \tag{57}$$

$$\dot{\varphi}_2 = \lambda\varphi_2 \tag{58}$$

$$\dot{\varphi} = \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} \varphi \tag{59}$$

7 Hamilton and Koopman

Hamiltonian systems

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}; \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

The Hamiltonian is a scalar function $H(p, q, t)$. A particular case is the time-independent Hamiltonian (i.e., $H(p, q, t) = H(p, q)$), that mean that the Energy is equal to the Hamiltonian ($H = E$). Famous examples include the harmonic oscillator, and undamped pendulum.

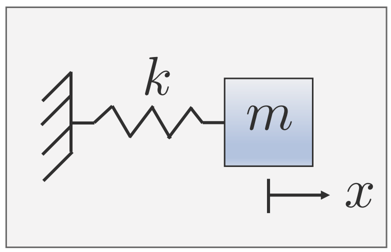

Figure 1: Mass-spring oscillator

Example 12 *Mass-spring oscillator $m\ddot{x} = -kx$, or in stats space*

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} x$$

The solution of this differential equation is given by

$$x(t) = \frac{\dot{x}(0)}{\omega_n} \sin(\omega_n t) + x(0) \cos(\omega_n t) \tag{60}$$

$$\dot{x}(t) = \dot{x}(0) \cos(\omega_n t) - \omega_n x(0) \sin(\omega_n t) \tag{61}$$

where $\omega_n = \sqrt{\frac{k}{m}}$. (e.g., $k = 50N/m$, $m = 1Kg$, $x(0) = 1$, and $\dot{x}(0) = 0$)

$$H(x, \dot{x}) = \frac{1}{2}(\dot{x}^2 + \frac{k}{m}x^2)$$

Note the Koopman generator for the Hamiltonian is equal to zero.

$$\dot{H} = \mathbf{f}(\mathbf{x}) \cdot \nabla H(\mathbf{x}) =: \mathcal{K}_{\mathbf{f}} H = 0$$

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial \dot{x}}\ddot{x} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial \dot{x}} - \frac{\partial H}{\partial \dot{x}}\frac{\partial H}{\partial x} = 0$$

The level set of $H(x, \dot{x}) = c$ is an ellipsoid, so ew have some geometry of the PDE.

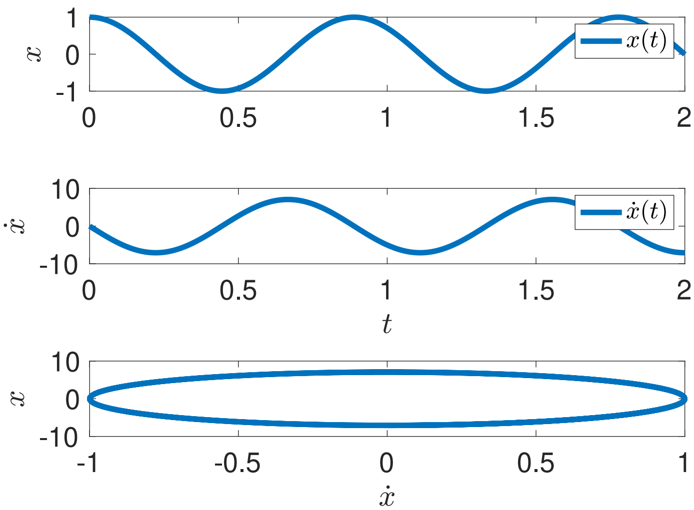

Figure 2: States vs. time and Phase Portrait.

Hamilton Jacobi Equation

$$\frac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) - \frac{1}{2}\frac{\partial V}{\partial \mathbf{x}}\mathbf{R}(\mathbf{x})\frac{\partial V}{\partial \mathbf{x}}^{\top} + q(\mathbf{x}) = 0 \tag{62}$$

Pre-Hamiltonian

$$K(x,p) = p^{\top}\mathbf{f}(\mathbf{x}) - \frac{1}{2}p^{\top}\mathbf{R}(\mathbf{x})p + q(\mathbf{x})$$

Now, we write the Hamiltonian dynamical system of HJ equation using the pre-Hamiltonian.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{R}(\mathbf{x})p \tag{63}$$

$$\dot{p} = -\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}p + \frac{1}{2}\frac{\partial p^{\top}\mathbf{R}(\mathbf{x})p}{\partial \mathbf{x}} - \frac{\partial q}{\partial \mathbf{x}} \tag{64}$$

Optimal Control Linear Case

We consider the infinite horizon optimal control for LTI and single input system described as

$$\min_{u} \int_0^{\infty} x^{\top}Qx + u^{\top}Ru \;\; dt \tag{65}$$

$$\text{s.t. } \dot{x} = Ax + Bu; \;\; x(0) = z, \tag{66}$$

$$\min_u \int_0^\infty q(\mathbf{x}) + Ru^2 \;\; dt \tag{67}$$

$$\text{s.t. } \dot{x} = Ax + Bu; \;\; x(0) = z, \tag{68}$$

where $Q = Q^\top \geq 0$ and $R = R^\top > 0$.

Remark. this is an infinite-dimensional problem: (trajectory $u : [0, \infty] \to \mathbb{R}^m$ is the variable.

pre-Hamiltonian

$$p^\top (Ax + Bu) + x^\top Qx + u^\top Ru$$

derivation w.r.t, to u and equaling to zero we have

$$p^\top B + 2Ru = 0$$

$$u^* = -\frac{1}{2} R^{-1} p^\top B$$

To find the Hamiltonian we will plug the optimal u (i.e.u^*) in the pre-Hamiltonian.

$$p^\top (Ax + B(-\frac{1}{2}R^{-1}p^\top B)) + x^\top Qx + (-\frac{1}{2}R^{-1}p^\top B)^\top R(-\frac{1}{2}R^{-1}p^\top B)$$

$$p^\top Ax + p^\top B(-\frac{1}{2}R^{-1}p^\top B)) + x^\top Qx + (\frac{1}{2}R^{-1}p^\top B)^\top R(\frac{1}{2}R^{-1}p^\top B) = 0$$

$$p^\top Ax + p^\top B(-\frac{1}{2}R^{-1}p^\top B)) + x^\top Qx + (\frac{1}{4}R^{-1}p^\top B)^\top (p^\top B = 0$$

$$p^\top Ax + p^\top B(-\frac{1}{2}R^{-1}p^\top B)) + x^\top Qx + (\frac{1}{4}R^{-1}p^\top B)^\top (p^\top B) = 0$$

$$p^\top Ax - \frac{1}{4}p^\top BR^{-1}B^\top p + x^\top Qx = 0$$

If we take $p^\top = \frac{\partial V}{\partial x}$ then we have the Hamilton Jacobi equation

$$\frac{\partial V}{\partial x}Ax - \frac{1}{4}\frac{\partial V}{\partial x}BR^{-1}B^\top \frac{\partial V}{\partial x}^\top + x^\top Qx = 0$$

In particular, with help of the following parameterization $V = \frac{1}{2}x^\top Px$ and $\frac{\partial V}{\partial x} = Px$ we can transform the above HJ equation in an equivalent t algebraic Ricatti equation as follows

$$PxAx - \frac{1}{4}PxBR^{-1}B^\top (Px)^\top + x^\top Qx = 0$$

Consider $P = P^\top$ and $P = (P + P^\top)/2$ hence

$$\frac{1}{2}x(PA + A^\top P)x^\top - \frac{1}{4}PxBR^{-1}B^\top (Px)^\top + x^\top Qx = 0$$

Very nice because transform an infinite-dimensional problem in a finite dimension and algebraic rewritten in an ARE as

$$PA + A^\top P - PBR^{-1}B^\top P + Q = 0.$$

Change of Coordinates Original system

$$\dot{x} = \mathbf{f}(x) + g(x)u \tag{69}$$

Consider n-eigenfunctions denoted by $\phi(x)$, so we can write the new system in Koopman eigenfunctions as

$$\dot{z} = \Lambda z + \frac{\partial z}{\partial x} g(x)u \tag{70}$$

Procedure 1 Let $\phi(x)$ the principal eigenfunctions of unforced system, then we can to do a change of coordinates as

$$\dot{x} = f(x) \to \dot{\phi} = \Lambda\phi$$

If we consider the storage function as

$$V = \phi(x)^\top P\phi(x)$$

we want to solve for P

$$\phi(x)(\Lambda^\top P + P\Lambda - PR(x)P + \bar{Q})\phi(x) = 0$$

alternative

$$\Lambda^\top P + P\Lambda - PR(x)P + Q(x) = 0$$

Then, we can transform the HJ equation to an infinite-dimensional algebraic equation. Moreover, under specific assumptions like use $R(x)|_0 = BB^\top$ or sampling methods, we can write this equality as a convex optimization problem.
Procedure 2

Analytical Examples

Optimal control for scalar system

$$\min_u \int_0^\infty \frac{1}{2}(x^2 + u^2)dt \tag{71}$$

$$\dot{x} = -x + xu; \quad x(0) = z. \tag{72}$$

Analytical solution for the HJB Equation
pre-Hamiltonian

$$K(x, p, u) = p(-x + xu) + \frac{1}{2}(x^2 + u^2)$$

derivation w.r.t. u and equalization to zero

$$px + u = 0$$

$$u^* = -px$$

Hamiltonian

$$H(x,p) = p(-x + x(-px)) + \frac{1}{2}(x^2 + (-px)^2)$$

$$-xp - x^2p^2 + \frac{1}{2}x^2 + \frac{1}{2}p^2x^2$$

$$-xp - \frac{1}{2}p^2x^2 + \frac{1}{2}x^2$$

using cuadratic formnula

$$p = \frac{1 \pm \sqrt{1+x^2}}{x}$$

in order to stabilize

$$p = \frac{1 + \sqrt{1+x^2}}{x}$$

$$u^* = -1 - \sqrt{1+x^2}$$

Closed-loop

$$\dot{x} = -x + x(-1 - \sqrt{1+x^2})$$

$$\dot{x} = -2x - \sqrt{1+x^2})$$

Procedure 1

We pick-up this example because the principal eigenfunction of unforced part of the system is known (linear) and global.

$$\lambda = -1; \quad \phi_\lambda = x; \quad \frac{\partial \phi_\lambda}{\partial x} = 1$$

$$\phi(x)(\Lambda^\top P + P\Lambda - PR(x)P + \bar{Q})\phi(x) = 0$$

alternative

$$R(x) = \frac{\partial \phi}{\partial x} R(x) \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x} \frac{1}{2} g(x) g(x)^\top \frac{\partial \phi}{\partial x}; \quad \longrightarrow R(x) = \frac{1}{2}x^2$$

$$q(x) = \phi(x)^\top \bar{Q} \phi(x); \quad \longrightarrow \bar{Q} = 1$$

$$\Lambda^\top P + P\Lambda - PR(x)P + \bar{Q} = 0$$

$$-2p - \frac{1}{2}p^2x^2 + 1 = 0$$

This no match with analytical solution of HJB Equaiton.

Procedure 2

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \mathbf{R}(\mathbf{x})p \tag{73}$$

$$\dot{p} = -\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}p + \frac{1}{2}\frac{\partial p^\top \mathbf{R}(\mathbf{x})p}{\partial \mathbf{x}} - \frac{\partial q}{\partial \mathbf{x}} \tag{74}$$

$$\min_u \int_0^\infty \frac{1}{2}(x^2 + u^2)dt \tag{75}$$

$$\dot{x} = -x + xu; \quad x(0) = z. \tag{76}$$

$$\mathbf{f}(\mathbf{x}) = -x; \quad \mathbf{R}(\mathbf{x}) = \mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x}) = x^2; \quad q(\mathbf{x}) = \frac{1}{2}x^2.$$

Pre-Hamiltonian:

$$K(x,p) = p^\top \mathbf{f}(\mathbf{x}) - \frac{1}{2}p^\top \mathbf{R}(\mathbf{x})p + q(\mathbf{x})$$

$$H(\mathbf{x},p) = p(-x) - \frac{1}{2}p^2x^2 + \frac{1}{2}x^2 = 0$$

.

Its stabilizing solution is $u^* = -1 - \sqrt{1+x^2}$.
Hamiltonian dynamical system

$$\begin{aligned} \dot{x} &= x^3 - x - p \\ \dot{p} &= -(3x^2 - 1)p - x \end{aligned} \tag{77}$$

$$\dot{x} = -x - x^2p \tag{78}$$

$$\dot{p} = p + xp^2 - x \tag{79}$$

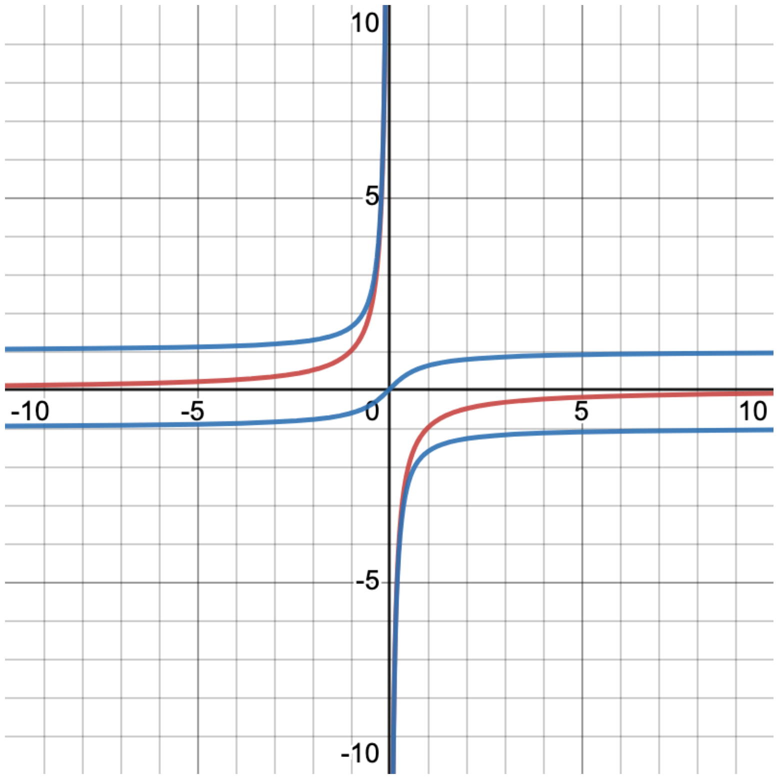

Figure 3: Fixed points of scalar bilinear system

Analytical Examples 2

Optimal control for scalar system

$$\min_u \int_0^\infty \frac{1}{2}(x^2 + u^2)dt \tag{80}$$

$$\dot{x} = x + xu + u; \quad x(0) = z. \tag{81}$$

Analytical solution for the HJB Equation
pre-Hamiltonian

$$K(x, p, u) = p(x + xu + u) + \frac{1}{2}(x^2 + u^2)$$

derivation w.r.t. u and equalization to zero

$$px + p + u = 0$$

$$u^* = -px - p$$

Hamiltonian

$$H(x,p) = p(x + x(-px - p) - px - p) + \frac{1}{2}(x^2 + (-px - p)^2)$$

$$px - x^2p^2 - p^2 - p^2x - p^2 + \frac{1}{2}x^2 + \frac{1}{2}(x^2p^2 + 2xp^2 + p^2)$$

$$(-x^2 - 1 - x - 1 + \frac{1}{2}x^2 + 2x + 1)p^2 + px + \frac{1}{2}x^2$$

$$(-\frac{1}{2}x^2 + x - 1)p^2 + xp + \frac{1}{2}x^2$$

using quadratic formula

$$p = \frac{-x \pm \sqrt{x^2 - 2(-\frac{1}{2}x^2 + x - 1)(x^2)}}{-x^2 + 2x - 2}$$

$$p = \frac{-x \pm \sqrt{x^2 + x^4 - 2x^3 + 2x^2}}{-x^2 + 2x - 2}$$

$$p = \frac{-x \pm \sqrt{x^4 - 2x^3 + 3x^2}}{-x^2 + 2x - 2}$$

in order to stabilize

$$p = \frac{1 + \sqrt{1 + x^2}}{x}$$

$$u^* = -px - p$$

Closed-loop

$$\dot{x} = -x + x(-1 - \sqrt{1 + x^2})$$

$$\dot{x} = -2x - \sqrt{1 + x^2})$$

8 Path Integral

Consider

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{f}_n(\mathbf{x})$$

where $\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0)$ and $\mathbf{f}_n = \mathbf{f}(\mathbf{x}) - \mathbf{A}\mathbf{x}$.

Definition of Koopman eigenfunction

$$[\mathbb{U}_t \varphi](\mathbf{x}) = e^{\lambda t}\varphi(\mathbf{x}).$$

or equivalent

$$\varphi(s_t(\mathbf{x})) = e^{\lambda t}\varphi(\mathbf{x})$$

taking the derivation w.r.t to time we have

$$\frac{\partial}{\partial t}\varphi(s_t(\mathbf{x})) = \frac{\partial}{\partial t}e^{\lambda t}\varphi(\mathbf{x}) = \lambda e^{\lambda t}\varphi(\mathbf{x}) = \lambda\varphi(s_t(\mathbf{x}))$$

$$\frac{\partial}{\partial t}\varphi(\mathbf{x}) = \lambda\varphi(\mathbf{x})$$

Koopman generator

$$\frac{d}{dt}\varphi(\mathbf{x}) = \frac{\partial\varphi}{\partial\mathbf{x}}\mathbf{f}(\mathbf{x})$$

Koopman generator on Eigenfunction

$$\frac{\partial\varphi}{\partial\mathbf{x}}\mathbf{f}(\mathbf{x}) = \lambda\varphi(\mathbf{x})$$

Koopman eigenfunction with particular structure

$$\varphi(\mathbf{x}) = \mathbf{w}^\top\mathbf{x} + h(\mathbf{x})$$

where $\mathbf{w}^\top\mathbf{A} = \lambda\mathbf{w}^\top$. Tasking derivation w.r.t. $\mathbf{x}$

$$\frac{\partial\varphi(\mathbf{x})}{\partial\mathbf{x}} = \mathbf{w}^\top + \frac{\partial h(\mathbf{x})}{\partial\mathbf{x}}$$

Using generator on eigenfunction

$$(\mathbf{w}^\top + \frac{\partial h(\mathbf{x})}{\partial\mathbf{x}})\mathbf{f}(\mathbf{x}) = \lambda(\mathbf{w}^\top\mathbf{x} + h(\mathbf{x}))$$

$$\frac{\partial h(\mathbf{x})}{\partial\mathbf{x}}\mathbf{f}(\mathbf{x}) - \lambda h(\mathbf{x}) = -\mathbf{w}^\top\mathbf{f}(\mathbf{x}) + \lambda\mathbf{w}^\top\mathbf{x}$$

$$\frac{\partial h(\mathbf{x})}{\partial\mathbf{x}}\mathbf{f}(\mathbf{x}) - \lambda h(\mathbf{x}) = -\mathbf{w}^\top(\mathbf{A}\mathbf{x} + \mathbf{f}_n(\mathbf{x})) + \lambda\mathbf{w}^\top\mathbf{x}$$

The nonlinear part of the eigenfunction satisfies the following linear PDE

$$\frac{\partial h(\mathbf{x})}{\partial\mathbf{x}}\mathbf{f}(\mathbf{x}) - \lambda h(\mathbf{x}) + \mathbf{w}^\top\mathbf{f}_n(\mathbf{x}) = 0 \tag{82}$$

Lemma 1 *The linear PDE 82 admits a solution at the form*

$$h(\mathbf{x}) = \int_0^\infty e^{-\lambda t}g(s_t(\mathbf{x}))dt = \int_0^\infty e^{-\lambda t}[\mathbb{U}_t g]((\mathbf{x}))dt$$

where $g(\mathbf{x}) = \mathbf{w}^\top\mathbf{f}_n(\mathbf{x})$ *and* $s_t(\mathbf{x})$ *is the solution of the system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

Example 13

$$\dot{x}_1 = \mu x_1 \tag{83}$$

$$\dot{x}_2 = \lambda(x_2 - x_1^2) \tag{84}$$

$$\varphi(\mathbf{x}) = \mathbf{w}^\top + h(\mathbf{x})$$

where $\mathbf{A} = \begin{bmatrix} \mu & 0 \\ 0 & \lambda \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, *and* $h(\mathbf{x}) = \begin{bmatrix} 0 \\ \beta x_1^2 \end{bmatrix}$

Individual eigenfunctions are given by

$$\varphi_1 = x_1, \quad \varphi_2 = x_2 + \beta x_1^2$$

where $\beta = \frac{\lambda}{2\mu - \lambda}$.

$$\frac{\partial h(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) - \lambda h(\mathbf{x}) + \mathbf{w}^\top \mathbf{f}_n(\mathbf{x}) = 0$$

$$\begin{bmatrix} 0 & 0 \\ 2\beta x_1 & 0 \end{bmatrix} \begin{bmatrix} \mu x_1 \\ \lambda(x_2 - x_1^2) \end{bmatrix} - \begin{bmatrix} \mu & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} 0 \\ \beta x_1^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\lambda x_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ (\frac{2\mu\lambda - \lambda^2 - 2\mu\lambda + \lambda^2}{2\mu - \lambda}) x_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Assume $x_1(0) = 1$ *and* $x_2(0) = 1$, *so the solution for this example is given by*

$$x_1(t) = s_t^1 = x_{10} e^{\mu t}$$

Path integral

$$h_2(\mathbf{x}) = \int_0^\infty e^{-\lambda t} g_2(s_t(\mathbf{x})) dt = \int_0^\infty e^{-\lambda t} - \lambda (s_t^1(\mathbf{x}))^2 dt = -\lambda x_{10}^2 \int_0^\infty e^{(2\mu - \lambda)t} dt$$

where $\beta = \int_0^\infty e^{(2\mu - \lambda)t} dt$

9 Quadratic Change of Coordinates

Example 14

$$\dot{x}_1 = \mu x_1 \tag{85}$$
$$\dot{x}_2 = \lambda(x_2 - x_1^2 - x_1^3) \tag{86}$$

where $\mathbf{A} = \begin{bmatrix} \mu & 0 \\ 0 & \lambda \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, *and* $h(\mathbf{x}) = \begin{bmatrix} 0 \\ \lambda(x_1^2 + x_1^3) \end{bmatrix}$

Individual eigenfunctions are given by

$$\varphi_1 = x_1, \quad \varphi_2 = x_2 + \beta_1 x_1^2 + \beta_2 x_1^2$$

where $\beta_1 = \frac{\lambda}{2\mu - \lambda}$ *and* $\beta_2 = \frac{\lambda}{3\mu - \lambda}$.

10 Path Integral Mixing with Basis

Scalar Example Consider the following optimal control

$$\dot{x} = -x + x^3 + u; \qquad \int_0^\infty \frac{1}{2}(x^2 + u^2)dt$$

$$\mathbf{f}(\mathbf{x}) = x^3 - x; \quad \mathbf{R}(\mathbf{x}) = \mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x}) = 1; \quad q(\mathbf{x}) = \frac{1}{2}x^2.$$

Pre-Hamiltonian: $H(\mathbf{x}, p) = p(x^3 - x) - \frac{1}{2}p^2 + \frac{1}{2}x^2 = 0$.
Its stabilizing solution is $p = x^3 - x + x\sqrt{x^4 - 2x^2 + 2}$.
Hamiltonian dynamical system

$$\begin{aligned} \dot{x} &= x^3 - x - p \\ \dot{p} &= -(3x^2 - 1)p - x \end{aligned} \tag{87}$$

Hamilton dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \mathbf{R}(\mathbf{x})p \tag{88}$$

$$\dot{p} = -\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}p + \frac{1}{2}\frac{\partial p^\top \mathbf{R}(\mathbf{x})p}{\partial \mathbf{x}} - \frac{\partial q}{\partial \mathbf{x}} \tag{89}$$

Split the Hamiltonian in linear and nonlinear. Consider $f(x) = Ax + f_n(x)$ and $\mathbf{R}(x)$ constant (i.e., R).

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} Ax - Rp \\ -A^\top p - Qx \end{pmatrix} + \begin{pmatrix} f_n \\ \left(\frac{\partial f_n}{\partial x}^\top - A\right)p - \left(\frac{\partial q}{\partial x} - Qx\right) \end{pmatrix}$$

11 Analytical examples Optimal control Proc. 1 and Proc.2

11.1 Procedure 1

In this section we present analytical examples using procedure 1. Consider first the scalar case. The eigenfunction equation is given by

$$\frac{\partial \phi}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) = \lambda\phi$$

For the scalar case we can get the vector field of the system as

$$\mathbf{f}(\mathbf{x}) = \left(\frac{\partial \phi}{\partial \mathbf{x}}\right)^{-1}\lambda\phi$$

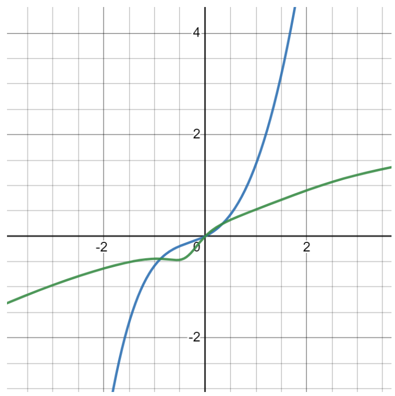

Figure 4: Vector field and eigenfunction. https://www.desmos.com/calculator/9erwdnqt6p

Example 15 *Consider the values of $\lambda = 0.5$, $\alpha = 1$ and $\beta = 1$*

$$\phi(\mathbf{x}) = \lambda(\mathbf{x} + \alpha\mathbf{x}^3 + \beta\mathbf{x}\sin(\mathbf{x})) \tag{90}$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \frac{\lambda(\mathbf{x} + \alpha\mathbf{x}^3 + \beta\mathbf{x}\sin(\mathbf{x}))}{1 + 3\alpha\mathbf{x} + \beta\sin\mathbf{x} + \beta\mathbf{x}\cos(\mathbf{x})} \tag{91}$$

The procedure 1 can be link with procedure 2, one perfect example is the analytical struxtur [13]

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