Department of Mathematical Sciences

Elliptic Curves and Galois Representations

Bachelor Defense

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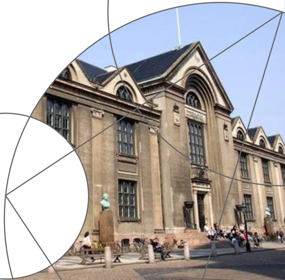
stud.scient.



Program

9.00-9.20: A presentation on the following topics:

1. Central definitions: Elliptic curves and Galois representations
2. Galois representations attached to elliptic curves
3. Elliptic curves with complex multiplication
4. Serre's open image theorem
5. Elliptic curves defined over $\mathbb{Q}$
6. A criterion for surjectivity of adelic Galois representations attached to elliptic curves
7. An example:
   A method to prove surjectivity of some ℓ-adic Galois representations attached to elliptic curves

9.20: Questions.

Elliptic curves

Let K be a perfect field. We define an elliptic curve in the following way:

Definition (Elliptic curve)

Let $f(x,y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 \in K[x,y]$ satisfy that there exists no $P \in \bar{K}^2$, such that $f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = 0$. Then we say f defines the affine elliptic curve E/K by

$$E(\bar{K}) = \{P \in \bar{K}^2 \mid f(P) = 0\}$$

Let $F(X,Y,Z) = Z^3 \cdot f(X/Z, Y/Z) \in K[X,Y,Z]$. Then the set

$$\begin{aligned} &\{P \in \mathbb{P}^2(\bar{K}) \mid F(P) = 0\} \\ =&\{[x_0 : y_0 : 1] \in \mathbb{P}^2(\bar{K}) \mid (x_0, y_0) \in E(\bar{K})\} \cup \{[0:1:0]\} \end{aligned}$$

is the projective elliptic curve E/K. We define $\mathcal{O} := [0:1:0]$

The group law

Then we define the group law on elliptic curves:

Theorem

Let E/K be an elliptic curve, $P = (x_1, y_1), Q = (x_2, y_2) \in E(K)$. If $x_1 \neq x_2$, define $\lambda := \frac{y_2 - y_1}{x_2 - x_1}$ and $\nu := \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$. If $P = Q$ and $\frac{\partial f}{\partial y}(P) \neq 0$, define

$$\lambda := -\frac{\frac{\partial f}{\partial x}(P)}{\frac{\partial f}{\partial y}(P)}, \quad \nu := y_1 - \lambda x_1$$

Then we let:

$$\begin{aligned} P + Q := \big(&\lambda^2 + a_1\lambda - a_2 - 2x_1, \\ &-(\lambda + a_1)(\lambda^2 + a_1\lambda - a_2 - 2x_1) - \nu - a_3\big) \end{aligned}$$

Else let $P + Q = \mathcal{O}$. Then $(E(K) \cup \{\mathcal{O}\}, +)$ is an abelian group with $\mathcal{O}$ as the identity.

Torsion points

Definition

Let E/K be an elliptic curve, and let $n \in \mathbb{N}$. Then for $P \in E(\bar{K})$, we define

$$nP := \underbrace{P + \cdots + P}_{n \text{ times}}$$

and we define

$$E[n] := \{P \in E(\bar{K}) \mid nP = \mathcal{O}\}$$

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Theorem

Let E/K be an elliptic curve, $n \in \mathbb{N}$ coprime with char K *if* char $K \neq 0$. *Then*

$$E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$$

Galois representations

Definition

Let E/F be Galois with Galois group $\mathrm{Gal}(E/F)$, let R be a topological ring and let $n \in \mathbb{N}$. Then a Galois representation is a continuous homomorphism

$$\rho : \mathrm{Gal}(E/F) \to \mathrm{GL}_n(R)$$

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Definition (Serre, McGill: Abelian ℓ-adic Representations and Elliptic curves)

Let ℓ be a prime. If V is a vector space over $\mathbb{Q}_\ell$ of degree n, and K is a field with separable algebraic closure K_s, then an ℓ-adic representation of $G := \mathrm{Gal}(K_s/K)$ is a continuous homomorphism

$$\rho : G \to \mathrm{Aut}(V)\,(\cong \mathrm{GL}_n(\mathbb{Q}_\ell))$$

Galois representations attached to elliptic curves

Let K be a perfect field, let $G_L = \mathrm{Gal}(\bar{K}/L)$ for L/K an algebraic extension. Let E/K be an elliptic curve and let $n \in \mathbb{N}$ be coprime with char K if char $K \neq 0$.

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Now let $g \in G_K$ act on $P = (x_0, y_0)$ by $gP = (gx_0, gy_0)$, $g\mathcal{O} = \mathcal{O}$. From the group law, we get $g(mP) = mg(P)$, and so we get

$$\rho'_{E,n} : G_K \to \mathrm{Aut}(E[n])$$

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Recall $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$. Hence, we can pick a basis (P, Q) of $E[n]$ and get a mod n Galois representation:

$$\rho_{E,n} : G_K \to \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

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Taking inverse limits and defining $T_\ell[E] = \varprojlim_{n\in\mathbb{N}} E[\ell^n]$, we get a ℓ-adic Galois representation. If $\operatorname{char} K = 0$, then for a compatible system of bases, we may pack the ℓ-adic Galois representations for all primes ℓ into a single adelic Galois representation via $\mathrm{Aut}(\prod_p T_p[E])$:

$$\rho_E : G_K \to \mathrm{GL}_2(\hat{\mathbb{Z}})$$

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These representations are only defined up to conjugation. Also, we can recover the mod m and the ℓ-adic Galois representation via projections $r_m : \mathrm{GL}_2(\hat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$, $\pi_\ell : \mathrm{GL}_2(\hat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}_\ell)$.

Complex multiplication

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Definition

Let E/K be an elliptic curve. Then $\psi : E(\bar{K}) \to E(\bar{K})$ is an endomorphism over K if $\psi = [g_0 : g_1 : g_2]$ with $g_0, g_1, g_2 \in K(E)$ regular (defined at each $P \in E(\bar{K})$) and $\psi(\mathcal{O}) = \mathcal{O}$.

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Any endomorphism ψ of E/K satisfies

$$\psi(P + Q) = \psi(P) + \psi(Q)$$

for all $P, Q \in E(\bar{K})$.

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An endomorphism over K of an elliptic curve E/K induces an endomorphism of $T_\ell[E]$ commuting with the endomorphisms induced by G_K. We have $\operatorname{End}(T_\ell[E]) \cong M_2(\mathbb{Z}_\ell)$, and so multiplication by m maps for $m \in \mathbb{N}$ are endomorphisms with action on $T_\ell[E]$ represented by

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

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Theorem

If E/K has complex multiplication, there exists a non-scalar matrix that commutes with $\rho_{E,\ell^\infty}(G_K)$ and $\rho_{E,\ell^\infty}(G_K)$ is abelian. Hence $\rho_{E,\ell^\infty}(G_K)$ is of infinite index in $GL_2(\mathbb{Z}_\ell)$ and in particular ρ_{E,ℓ^∞} is not surjective.

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If $\operatorname{char} K = 0$, and E/K has complex multiplication over any field L with L/K an algebraic extension, then ρ_E is not surjective. Also, $\rho_E(G_K)$ is not open in $GL_2(\hat{\mathbb{Z}})$

Serre's open image theorem

Let K be a number field, $G_L = \mathrm{Gal}(\bar{K}/L)$ for L/K an algebraic field extension.

Theorem (Serre's open image theorem)

Let E/K be an elliptic curve without complex multiplication. Then the image of ρ_E in $GL_2(\hat{\mathbb{Z}})$ is open.

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- There exists $m \in \mathbb{N}$, such that

$$\rho_E(G_K) = r_m^{-1}(\rho_{E,m}(G_K))$$

Surjectivity of adelic Galois representations attached to elliptic cruves

We can prove the following statements:

Theorem

Let $E/\mathbb{Q}$ be an elliptic curve. Then ρ_E is not surjective.

And the more general version:

Theorem (Greicius 2010)

Let E/K be an elliptic curve over a number field K. Let $\Delta \in K^\times$ be the discriminant of any Weierstrass model of E/K. Then ρ_E is surjective if and only if

1. *the ℓ-adic Galois representation $\rho_{\ell^\infty} : G_K \to GL_2(\mathbb{Z}_\ell)$ is surjective for all ℓ,*
2. $K \cap \mathbb{Q}(\zeta_\infty) = \mathbb{Q}$ *and*
3. $\sqrt{\Delta} \notin K(\zeta_\infty)$

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Uniformity conjecture (Serre): *For every number field K, there exists some prime p, such that for every elliptic curve E/K and prime $\ell > p$, ρ_{E,ℓ^∞} is surjective.*

For $K = \mathbb{Q}$, $p = 37$ is conjectured to be such a number.

Surjectivity of ℓ-adic Galois representations attached to elliptic curves

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Theorem

Assume $\ell \geq 5$ is a prime, E/K an elliptic curve and $\det : \rho_{E,\ell^\infty}(G_K) \to \mathbb{Z}_\ell^\times$ is surjective. Then $\rho_{E,\ell}$ is surjective if and only if ρ_{E,ℓ^∞} is surjective.
Further, $\rho_{E,8}$ surjective if and only if $\rho_{E,2^\infty}$ is surjective and $\rho_{E,9}$ is surjective if and only if $\rho_{E,3^\infty}$ is surjective.

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Since the subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$ are classified stemming back to a book by Dickson from 1901, we may actually derive a sufficient condition for primes $p \geq 5$:

Theorem

Let $\ell \geq 5$ and suppose $H \leq GL_2(\mathbb{F}_\ell)$ contains

1. *s such that $\mathrm{Tr}(s)^2 - 4\det s$ is a non-zero square in $\mathbb{F}_\ell$ and so that $\mathrm{Tr}(s) \neq 0$*
2. *s' such that $\mathrm{Tr}(s')^2 - 4\det s'$ is not a square in $\mathbb{F}_\ell$ and so that $\mathrm{Tr}(s') \neq 0$*
3. *s'' such that $u = \mathrm{Tr}(s'')^2/\det(s'') \neq 0, 1, 2, 4$ and such that $u^2 - 3u + 1 \neq 0$*

Then H contains $SL_2(\mathbb{F}_\ell)$. If further $\det : H \to \mathbb{F}_\ell^\times$ is surjective, $H = GL_2(\mathbb{F}_\ell)$.

Choose a model of E with coefficients in $\mathcal{O}_K$.

Theorem

Let $\mathfrak{p} \nmid \ell$ be a prime ideal in $\mathcal{O}_K$ with $\Delta_E \not\equiv 0 \pmod{\mathfrak{p}}$. Let $t_\mathfrak{p}$ be the trace of the Frobenius map of $\tilde{E}_\mathfrak{p}$ (the reduction of E modulo $\mathfrak{p}$) and $q_\mathfrak{p}$ the determinant. There exists $g \in G_K$ such that
$$t_\mathfrak{p} \equiv \mathrm{Tr}\,\rho_{E,\ell}(g) \pmod{\ell}, \quad q_\mathfrak{p} \equiv \det \rho_{E,\ell}(g) \pmod{\ell}$$

An example

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3. In the case $\ell=31$, we can find elements with traces and determinants satisfying the conditions listed needed to prove $\rho_{E,31}=\mathrm{GL}_2(\mathbb{F}_31)$ as outlined in the previous slide. This can be done by using the method involving Frobenius maps at the primes (7) and $(\alpha-2)$.

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   Similar but more technical arguments can be used to deal with the cases $\ell = 2, 3$.

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