Department of Mathematical Sciences

Causal Inference in Dynamical Systems

Convergent Cross Mapping and Alternative Approaches

Master's Defense

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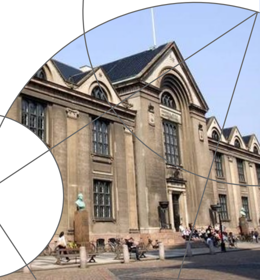
stud.scient.

January 15th, 2024



Program

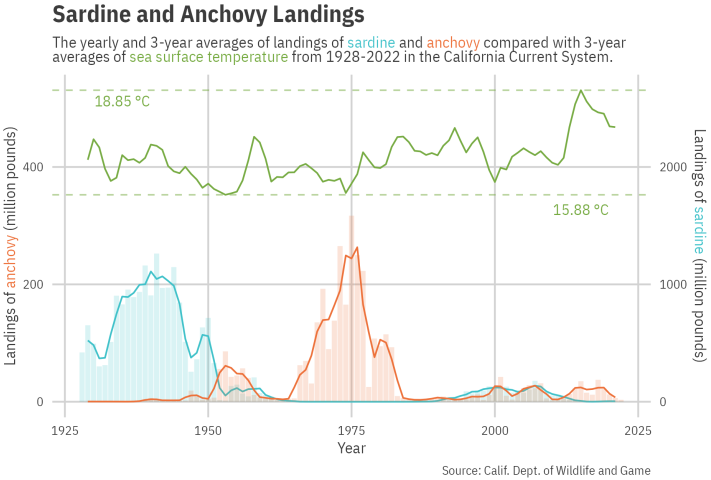
10.30-11.00: A presentation on the following topics:

1. Time Series
2. Stochastic Models
3. Granger Causality
4. Dynamical Models
5. Takens' Theorem
6. Convergent Cross Mapping
7. Extensions

11.00: Questions.

Time Series

Consider the following example:



Time Series Characteristics

We imagine the following generical set-up in this presentation:

$$X_1, \ldots, X_T$$
$$Y_1, \ldots, Y_T$$
$$Z_1, \ldots, Z_T$$

We consider thus the following features of our models:

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$\longrightarrow$ identifiability from lagged information.

Stochastic Models

We get the randomness in stochastic models via noise terms. For each variable, we will model as follows:

$$X_t = f(\underbrace{X_{t-1}}_{\text{last state}}, \underbrace{\ldots\ldots\ldots\ldots\ldots}_{\text{additional information}}, \underbrace{u_t^X}_{\text{noise}})$$

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- Reformulated in terms of Granger causality:

$$X_t \not\perp\!\!\!\perp Z_{t-1} \mid (X_{t-1}, Y_{t-1})$$

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- In this example: Z *causes* X and Y.
- How do we detect it?

Chaotic Attractors

Meteorologist Edward Lorenz at MIT modelled atmospheric convection, and discovered the Lorenz system in 1963:

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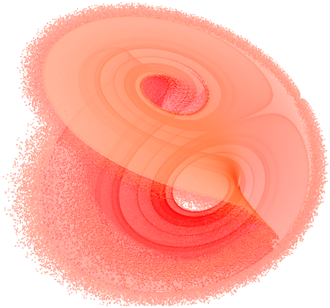
Result: a butterfly!



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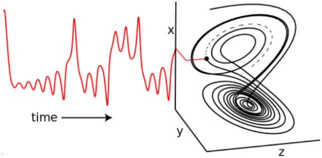

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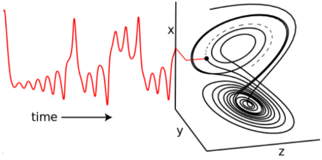

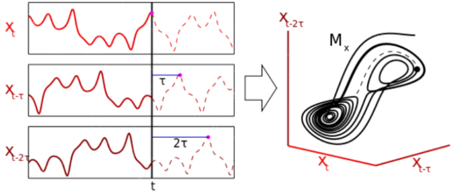

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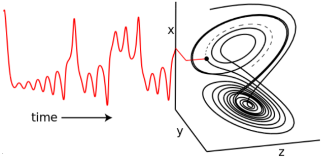

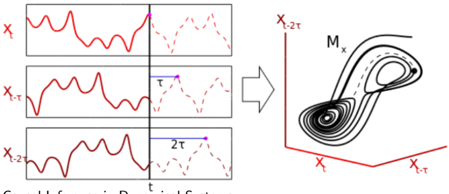

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Takens' theorem:

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Key ingredients:

- Causal model consisting of functional assignments in ODE's.
- Emergence of geometric structure from solutions.
- Decomposition of geometry according to causal structure.

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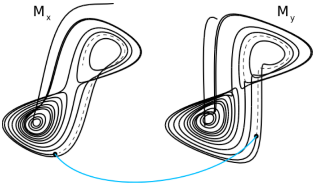
We compute the cross-mapping skill as the correlation between $\hat{Y}_t | M_X$ and Y_t.

Convergent Cross Mapping pt. 2

Visualization:



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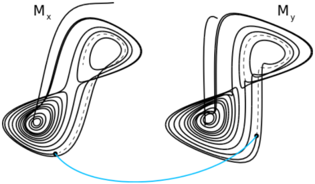
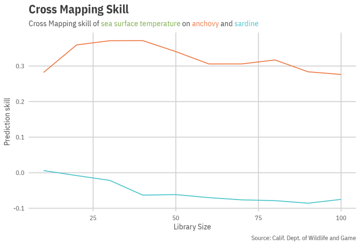
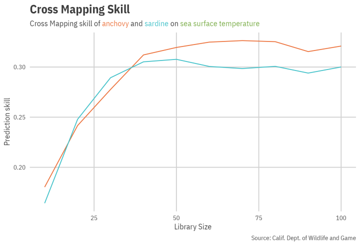


In practice (no transformation!) with $\tau = 1$, $E = 4$:





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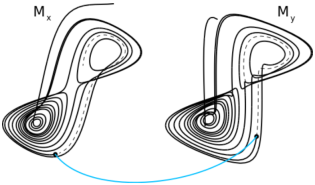
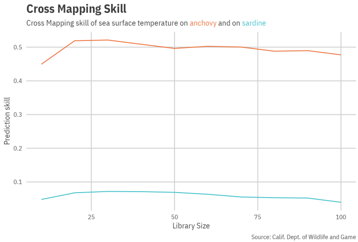
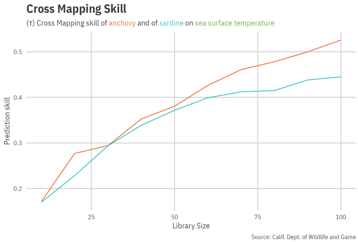


In practice (no transformation!) with $\tau = 2$, $E = 4$:





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- We do not need to observe all variables of interest.
- We do not need to assume much about the functional expressions.*

Disadvantages of convergent cross mapping:

- No model formulated that explicitly satisfies all technical assumptions.
- Problems in face of synchronicity and noise.
- Regime dependence.

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5. The link between limiting distributions and attractor sets.

Questions