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Quantum Theory of Measurement and Macroscopic Observables[1])

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Abstract. The generation of probabilities from probability amplitudes in a quantum mechanical measurement process is discussed in the framework of infinite quantum systems. In several explicitly soluble models, the measurement leads to macroscopically different 'pointer positions' and to a rigorous 'reduction of the wave packet' with respect to all local observables.

1. Introduction

The interpretation of quantum mechanics is one of those philosophical problems which no physicist can completely avoid. It is naïve to expect that quantum mechanics as a mathematical model will determine its own interpretation. However, there is the inexplicable fact that aspects of our world are explainable in mathematical terms (i.e. that there exists a morphism from mathematical models on experimental data). Hence we have to proceed in the general spirit: 'Erst die Theorie entscheidet, was man beobachten kann' [11].

In our modest contribution, which does not claim any originality, we shall discuss some explicitly soluble dynamical models for measurement processes in which probability amplitudes evolve into probabilities. We shall base our discussion on the quantum theory of systems with infinitely many degrees of freedom, developed during the past decade by Araki, Haag, Kastler, Kadison, Lanford, Robinson, Ruelle, Segal and many others. We believe that our discussion follows closely the pragmatic attitude of an experimental physicist. In fact, we are inspired by the manifestly macroscopic slits, clocks and pointers with which Bohr has so beautifully discussed many of the puzzles of quantum mechanics [2].

The essential points of this paper have been explained to the author with great patience by M. Fierz and R. Jost. To them as well as to S. Coleman, O. Steinmann, A. S. Wightman and M. Winnink we are much indebted for stimulating discussions.

2. Statement of the Problem

In ordinary quantum mechanics, the pure states of a system are the unit rays in a separable Hilbert space and the mixed states or ensembles the density matrices P on $\mathscr{H}$. The observables correspond to bounded hermitean operators A on $\mathscr{H}$, and the time evolution is given by a continuous 2-parameter family of unitary 'propagators' $U(t,s)$.

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[1]) This paper is dedicated to Professor M. Fierz on the occasion of his sixtieth anniversary. It is a personal, but not a comprehensive review of the quantum theory of measurement.

Finally, for every state P and observable A, one defines the expectation value of A in P to be

$$\langle A \rangle_P = \mathrm{Tr}(PA). \tag{2.1}$$

If $A = \sum a_n E_n$ has a spectral decomposition $\{E_n\}$ which is purely discrete, then (2.1) becomes

$$\langle A \rangle_P = \sum a_n p_n$$
$$p_n = \mathrm{Tr}(PE_n) \geqslant 0, \quad \sum p_n = 1. \tag{2.2}$$

The following morphism between this mathematical model and the experimental data of physics is commonly accepted:

A. The measured values of A are the eigenvalues a_n.
B. If the state P is an eigenstate of A with eigenvalue a_n, $P = PE_n$, then A has in P the value a_n.
C. If P is a mixture of eigenstates P_n of A, $P = \sum p_n P_n$, $P_n = P_n E_n$, $p_n \geqslant 0$, $\sum p_n = 1$, then A has in P the value a_n with probability p_n.

From (A), (B) and (C) we shall give strong evidence that (2.2) has the following probability interpretation:

D. There exist quantum mechanical measurement processes (in which the system is coupled to an apparatus with a large number of degrees of freedom) where, when A is measured in a general state P, the resulting state is an incoherent superposition of states, where A has the value a_n, with the probability $\mathrm{Tr}(PE_n)$.

As an abstraction of the Stern–Gerlach experiment (SG), von Neumann [24] has given the following model of a 'measurement of the first kind' for $\mathrm{Tr}(PE_n)$: Let, for simplicity, the state space $\mathscr{H}_S$ of the system be two-dimensional, $\mathscr{H}_S = \mathbb{C}^2$. The states $\psi_+ = \binom{1}{0}$ and $\psi_- = \binom{0}{1}$ are eigenstates for σ^3 with eigenvalues ± 1 and projectors $P_\pm$. By (C), σ^3 has the values ± 1 with probabilities $p_\pm$ in the state $P = p_+ P_+ + p_- P_-$. In the coherent superposition $\psi = c_+ \psi_+ + c_- \psi_-$ with 'probability amplitudes' $c_\pm \in \mathbb{C}$ satisfying $|c_+|^2 + |c_-|^2 = 1$, σ^3 has no definite value. In the measurement the system is coupled to an apparatus with state space $\mathscr{H}_A$. Assume that the latter is initially in the state φ_0 and the combined system in the state $\psi_\pm \otimes \varphi_0 \in \mathscr{H}_S \otimes \mathscr{H}_A$ (in SG, $\psi_\pm$ are spin eigen-states and φ_0 the coordinate wave function). By a well-chosen interaction (in SG the passage through an inhomogeneous magnetic field [8]), the combined system makes the transition

$$\psi_\pm \otimes \varphi_0 \to \psi_\pm \otimes \varphi_\pm. \tag{2.3}$$

Here $\varphi_\pm \in \mathscr{H}_A$ correspond to some big pointer with two well-separated positions (in SG the splitting of the position of a heavy particle). If (2.3) is the effect of some unitary time evolution U, then one arrives at the 'cat paradox' [21] by linearity:

$$U(c_+ \psi_+ + c_- \psi_-) \otimes \varphi_0 \to c_+ \psi_+ \otimes \varphi_+ + c_- \psi_- \otimes \varphi_-. \tag{2.4}$$

Von Neumann [24] and Wigner [26] have argued that only after the interaction of system and apparatus with the conscious ego the coherent superposition (2.4) acquires a definite pointer position, with relative frequencies $|c_+|^2$ and $|c_-|^2$ in a series of identical experiments. This solipsistic point of view is philosophically tenable and experimentally not refutable. However, the majority of the physicists adhere to the following more pragmatic interpretation of (2.4) (see e.g. [8, 18]): After a measurement of σ^3 in

$c_+\psi_+ + c_-\psi_-$, one arrives at (2.4), which—due to the macroscopic difference between the two pointer states $\varphi_\pm$ with projectors $Q_\pm$—for all feasible experiments on the combined system, can be identified with the mixture

$$|c_+|^2 P_+ \otimes Q_+ + |c_-|^2 P_- \otimes Q_-. \tag{2.5}$$

The replacement of (2.4) by (2.5) will be called the 'reduction of the wave packet'. Since quantum mechanics does not predict in general the outcome of single events, the further reduction from (2.5) to $P_\pm \otimes Q_\pm$, if the pointer position is read in a single experiment, is a pragmatic subjective act. Due to the absence of phase relations in the splitting (2.5), this further reduction is not in conflict with the laws of quantum mechanics, but it will not concern us in the sequel.

In this way, the experimental state-of-the-art defines the admissible pointers for a measurement. If $\mathscr{A} \subset \mathscr{B}(\mathscr{H}_S \times \mathscr{H}_A)$ corresponds to the set of all feasible observations, then (2.3) is a measurement of σ^3, if

$$(\psi_+ \otimes \varphi_+, A\psi_- \otimes \varphi_-) = 0 \tag{2.6}$$

for all $A \in \mathscr{A}$. In other words, (2.4) and (2.5) are equivalent with respect to $\mathscr{A}$ [12]. The following example is due to Jauch:

Example 1. Let $\mathscr{H}_S = \mathscr{H}_A = \mathbb{C}^2$ and $\mathscr{A}_A$ be the diagonal matrices in the basis $\{\psi_\pm\}$. Then $\psi_{\pm S} \otimes \psi_{+A} \to \psi_{\pm S} \otimes \psi_{\pm A}$ is a measurement with respect to $\mathscr{A} = \mathscr{B}(\mathscr{H}_S) \otimes \mathscr{A}_A$.

In order to go beyond formal mathematical arguments, one has to analyse realistic models of measurement and to find a natural set of observables $\mathscr{A}$ and time evolutions, with respect to which an objectification of a microevent is realized through different pointer positions. For this purpose we have first to understand the notion of coherence for large quantum systems.

3. Coherence and Classical Observables

In this section we shall formulate the quantum theory of measurement within the algebraic approach to systems with infinitely many degrees of freedom. We shall rephrase and recapitulate a number of mathematical facts (most of them from [6]) which clarify the notion of coherence of states. It is very satisfactory that in this quantum mechanical description of a microsystem coupled to a macroscopic apparatus, the coherent superposition of states and their incoherent mixture can become equivalent with respect to all quasilocal observables, when some classical observable assumes different values in these states.

General formalism: We assume that the set of observables of the system generates a C^*-algebra $\mathscr{A}$ with unit. The set $S(\mathscr{A})$ of all positive linear functionals ω on $\mathscr{A}$ with $\omega(1) = 1$ contains all states of the system. Every $\omega \in S(\mathscr{A})$ gives rise to a representation π_ω of $\mathscr{A}$ in a Hilbert space $\mathscr{H}_\omega$ with cyclic vector φ_ω.

For $\omega_1,\ \omega_2 \in S(\mathscr{A})$ and $\lambda_1,\ \lambda_2 \geqslant 0,\ \lambda_1 + \lambda_2 = 1$, the incoherent superposition $\omega = \lambda_1\omega_1 + \lambda_2\omega_2 \in S(\mathscr{A})$ is always meaningful and has the interpretation (C). If $\omega_1,\ \omega_2$ are vector states for some representation π of $\mathscr{A}$ ($\omega_i = \omega(\psi_i) \circ \pi,\ \psi_i \in \mathscr{H}_\pi$, $i = 1,2$), then one can form

$$\psi = \psi_1 + \lambda\psi_2/\|\psi_1 + \lambda\psi_2\| \tag{3.1}$$

for all $\lambda \in \mathbb{C}$, for which $\psi_1 + \lambda\psi_2 \neq 0$. However, it is possible that (3.1) as a state on $\mathscr{A}$ is always identical with the incoherent superposition $(1 + |\lambda|^2)^{-1}\omega_1 + |\lambda|^2(1 + |\lambda|^2)^{-1}\omega_2$. This phenomenon occurs, when ω_1 and ω_2 are disjoint, i.e. when no subrepresentation of π_{ω_1} is unitarily equivalent to any subrepresentation of π_{ω_2}:

Lemma 1. *$\omega_1, \omega_2 \in S(\mathscr{A})$ are disjoint, if and only if for every representation π of $\mathscr{A}$ with $\omega_i = \omega(\psi_i) \circ \pi$ for some $\psi_i \in \mathscr{H}_\pi$, $i = 1,2$, one has*

$$(\psi_1, \pi(A)\psi_2) = 0 \text{ for all } A \in \mathscr{A}. \tag{3.2}$$

Proof: If the π_{ω_i} are subrepresentations of π and $E_i \in \pi(\mathscr{A})'$ the projectors on $\pi(\mathscr{A})\psi_i$ with central supports $F_i \in \mathscr{Z}(\pi(\mathscr{A})')$ satisfying $F_i E_i = E_i$, then disjointness is equivalent to $F_1 F_2 = 0$, hence $E_1 E_2 = 0$ and thus (3.2). In the converse case, there exists a representation π with two equivalent subrepresentations acting on the same subspace. Then (3.2) does not hold.

If ω_1 and ω_2 are not disjoint, they are called coherent. Let π be such that, for $\psi_1, \psi_2 \in \mathscr{H}_\pi$ and $\omega_i = \omega(\psi_i) \circ \pi, \pi(\mathscr{A})\psi_1$ is not orthogonal to $\pi(\mathscr{A})\psi_2$. Then we can form the coherent superposition $\omega(\psi) \circ \pi$ of ω_1 and ω_2 using (3.1). Clearly, every pair $\psi_1, \psi_2 \in \mathscr{H}$ is coherent with respect to $\mathscr{B}(\mathscr{H})$. On the other hand, the states of a living and a dead cat with 10^8 disintegrated neurons should be rightfully described by disjoint states. In the following section we shall construct models for measurement processes, where different pointer positions are disjoint states for a rather big natural algebra of observables. An automorphism $\alpha \in \mathrm{Aut}(\mathscr{A})$ of $\mathscr{A}$ is a 1–1-mapping of $\mathscr{A}$ onto $\mathscr{A}$ which preserves the algebraic structure. The following trivial consequence of Lemma 1 will have far-reaching implications:

Lemma 2. *If $\omega_1, \omega_2 \in S(\mathscr{A})$ are disjoint and $\alpha \in Aut(\mathscr{A})$, then $\omega_1 \circ \alpha$ and $\omega_2 \circ \alpha$ are disjoint.*

While coherence cannot be destroyed by an automorphic time evolution during the measurement process, we shall find in section 4 sequences $\omega_{1,n}$, $\omega_{2,n}$ of coherent states which converge weakly in $S(\mathscr{A})$, $\omega_{i,n} \underset{w}{\to} \omega_i$, towards disjoint states ω_1, ω_2. In this case, all cross-terms (2.6) converge to zero:

Lemma 3. *Consider sequences $\omega_{i,n} \underset{w}{\to} \omega_i$, $i = 1,2$, with ω_1, ω_2 disjoint. Let π_n be representations of $\mathscr{A}$ and $\psi_{i,n} \in \mathscr{H}_{\pi_n}$ with $\omega_{i,n} = \omega(\psi_{i,n}) \circ \pi_n$, $i = 1,2$. Then, for all $A \in \mathscr{A}$*

$$\lim_{n\to\infty} (\psi_{1,n}, \pi_n(A)\psi_{2,n}) = 0. \tag{3.3}$$

Proof: If $\omega_2(A^*A) = 0$, then

$$|(\psi_{1,n}, \pi_n(A)\psi_{2,n})|^2 \leqslant \omega_{2,n}(A^*A) \to 0. \tag{3.4}$$

Otherwise, $\omega_{2,n}(A^*A) \neq 0$ for sufficiently large n, and $\omega_{2,n}^A = \omega_{2,n}(A^*(\cdot)A)/\omega_{2,n}(A^*A) \in S(\mathscr{A})$. Then $\omega_{2,n}^A \underset{w}{\to} \omega_2^A$ and, since ω_1 and ω_2 are disjoint, ω_1 and ω_2^A are disjoint. Hence $\|\omega_1 - \omega_2^A\| = 2$ [7]. By the weak continuity of the norm one obtains

$$4 \leftarrow \|\omega_{1,n} - \omega_{2,n}^A\|^2 \leqslant 4 - 4|(\psi_{1,n}, \pi_n(A)\psi_{2,n})|^2/\|\pi_n(A)\psi_{2,n}\|^2 \leqslant 4 \tag{3.5}$$

and (3.4) holds.

Lemma 4. *Let* $\omega_i = \sum_{n=1}^{\infty} \lambda_{in}\omega_{in}$, $\lambda_{in} \geqslant 0$, $\sum_{n=1}^{\infty} \lambda_{in} = 1$ *for* $i = 1,2$, *and let* ω_{1m} *and* $\omega_{2n} \in S(\mathscr{A})$ *be disjoint for all* m,n. *Then* ω_1 *and* ω_2 *are disjoint.*

Proof: The π_{ω_i} are unitarily equivalent to a subrepresentation of $\oplus\pi_{\omega_{in}}$, and those are disjoint.

An interesting class of states are the primary states ω, where $\pi_\omega(\mathscr{A})' \cap \pi_\omega(\mathscr{A})'' = \{\lambda 1\}$. If $\omega_1, \omega_2 \in S(\mathscr{A})$ are primary, then either ω_1 and ω_2 are disjoint or quasiequivalent [13]. Let ψ_1, ψ_2 be orthogonal vector states in $\mathscr{H}_\omega$ with projectors E_1, E_2 and with ω primary. Then $\omega(\psi_1)$ and $\omega(\psi_2)$ are coherent, and the measurement problem for $A = a_1 E_1 + a_2 E_2$ is well-posed and non-trivial.

Quasilocal and classical observables: For infinite systems it has been powerfully argued by Haag and Kastler [9] that the algebra of observables $\mathscr{A}$ has a quasilocal structure in the following sense: There exists a set $\mathscr{D}$ of bounded regions $\Lambda \in \mathbb{R}^3$, such that $\bigcup_{\mathscr{D}} \Lambda = \mathbb{R}^3$; for $\Lambda', \Lambda'' \in \mathscr{D}$ there exists some $\Lambda \in \mathscr{D}$ with $\Lambda \supset \Lambda' \cup \Lambda''$; and for every $\Lambda \in \mathscr{D}$ there exists some $\Lambda' \in \mathscr{D}$ with $\Lambda \cap \Lambda' = \phi$. For every $\Lambda \in \mathscr{D}$ there should exist a C^*-algebra $\mathscr{A}(\Lambda)$ with unit satisfying

$$\begin{aligned} &\mathscr{A}(\Lambda') \subset \mathscr{A}(\Lambda'') \quad \text{for } \Lambda' \subset \Lambda'' \\ &[\mathscr{A}(\Lambda'), \mathscr{A}(\Lambda'')] = 0 \quad \text{for } \Lambda' \cap \Lambda'' = \phi \\ &\textstyle\bigcup_{\mathscr{D}} \mathscr{A}(\Lambda) \quad \text{norm-dense in } \mathscr{A}. \end{aligned} \tag{3.6}$$

The classical observables of the system do not necessarily belong to $\mathscr{A}$. They are supposed to correspond to operations which can be made outside of any bounded set. For $\Lambda \in \mathscr{D}$ let $\tilde{\mathscr{A}}(\Lambda)$ be the C^*-algebra generated by all $\mathscr{A}(\Lambda')$ with $\Lambda' \in \mathscr{D}$ and $\Lambda \cap \Lambda' = \phi$. Let π be a representation of $\mathscr{A}$. Then

$$\mathscr{L}_\pi = \textstyle\bigcap_{\mathscr{D}} \pi(\tilde{\mathscr{A}}(\Lambda))'' \tag{3.7}$$

is called the algebra of observables at infinity in the representation π [15]. Since $\mathscr{L}_\pi$ lies in the center of $\pi(\mathscr{A})'$, it is abelian, which is a necessary prerequisite for a set of classical observables.

Special observables at infinity are the macroscopic observables. For any sequence $\Lambda_n \in \mathscr{D}$ converging to infinity (i.e. almost all Λ_n lie outside of any bounded region), let $A_n \in \mathscr{A}(\Lambda_n)$ with $\|A_n\| \leqslant b$ uniformly in n. Let π be a representation of $\mathscr{A}$. If

$$w\text{-}\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \pi(A_n) = A \tag{3.8}$$

exists, then $A \in \mathscr{L}_\pi$. A state $\omega \in S(\mathscr{A})$ has short range correlations [15], if $\mathscr{L}_\pi = \{\lambda 1\}$ or, equivalently, if for every $A \in \mathscr{A}$ and $\epsilon > 0$ there exists a region $\Lambda \in \mathscr{D}$ such that

$$|\omega(AB) - \omega(A)\omega(B)| \leqslant \|B\|\epsilon \tag{3.9}$$

for all $B \in \tilde{\mathscr{A}}(\Lambda)$. Every primary state has short range correlations.

Lemma 5. *Let* ω *have short range correlations and let* $\{A_n \in \mathscr{A}(\Lambda_n)\}$ *be as in* (3.8). *If*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(A_n) = a \tag{3.10}$$

exists, then one has in $\mathscr{H}_\omega$

$$w\text{-}\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \pi_\omega(A_n) = a. \tag{3.11}$$

Proof: Since $N^{-1}\sum_{n=1}^{N}\pi_\omega(A_n)$ is uniformly bounded, it suffices to prove (3.11) between a dense set of states. We show that for any $\Lambda_1 \in \mathscr{D}$ and any $A \in \mathscr{A}(\Lambda_1)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(A^* A_n A) = a\omega(A^* A). \tag{3.12}$$

Given $A \in \mathscr{A}(\Lambda_1)$ and $\epsilon > 0$, we choose $\Lambda_2 \in \mathscr{D}$ such that (3.9) holds. For some M, $A_n \in \tilde{\mathscr{A}}(\Lambda_2) \cap \mathscr{A}(\Lambda_1)'$, whenever $n > M$. Hence, by locality, for $N > M$:

$$\left|\frac{1}{N}\sum_{n=1}^{N} \omega(A^* A_n A) - a\omega(A^* A)\right| \leqslant \frac{M}{N}|a|\omega(A^* A) + \frac{1}{N}\sum_{n=1}^{M} |\omega(A^* A_n A)|$$
$$+ \left|\frac{1}{N}\sum_{M+1}^{N} \omega(A_n) - a\right| |\omega(A^* A)| + b\epsilon \tag{3.13}$$

For primary states ω_1, ω_2, the existence of a macroscopic observable with different expectation value in ω_1 and ω_2 entails disjointness:

Lemma 6. *Let* $\omega_1, \omega_2 \in S(\mathscr{A})$ *be primary and* $\{A_n \in \mathscr{A}(\Lambda_n)\}$ *be as in* (3.8). *If*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega_i(A_n) = a_i, \quad i = 1,2, \tag{3.14}$$

and $a_1 \neq a_2$, *then* ω_1 *and* ω_2 *are disjoint.*

Proof: By Lemma 5, $N^{-1}\sum_{n=1}^{N}\pi_{\omega_i}(A_n) \underset{w}{\rightarrow} a_i$. If ω_1 and ω_2 are not disjoint, then $\pi_{\omega_1} \leqslant \pi_{\omega_2}$ or $\pi_{\omega_2} \leqslant \pi_{\omega_1}$. In the latter case, there exists a projector $E \in \pi_{\omega_1}(\mathscr{A})'$, such that $\pi_{\omega_2}(A) = \pi_{\omega_1}(A)E$ for all $A \in \mathscr{A}$. Hence $a_2 E = a_1$ and thus $a_2 = a_1$.

We see, how differences in the expectation values of macroscopic pointers lead to disjointness. Only with a measurement apparatus with infinitely many degrees of freedom can one have a non-trivial quasilocal algebra. For infinite systems, there are many other mechanisms which entail disjointness. For instance, two KMS-states ω_1, ω_2 for different temperatures T_1, T_2 are disjoint, if one of them is primary of type III [23]. Again, type III factors only occur in infinite systems.

Product states for quantum spin systems: The above mathematical results will be applied to the quasilocal algebra $\mathscr{A}_{\text{spin}}$ of infinitely many spin 1/2 systems $\mathbb{C}^2$ at lattice sites $n = 1, 2, \ldots$. A pure state $|e\rangle$ in $\mathbb{C}^2$ is characterized by a unit vector $e \in \mathbb{R}^3$ with

$\sigma \cdot e|e) = |e)$. Two product states in $\otimes_{n=1}^N \mathbb{C}_n^2$, $|e^i)_N = \otimes_{n=1}^N |e_n^i)$, have a scalar product [25]

$$|_N(e^1|e^2)_N|^2 = \prod_{n=1}^N (1 + e_n^1 \cdot e_n^2)/2$$

$$\leqslant \exp\left(-\sum_{n=1}^N \|e_n^1 - e_n^2\|^2/4\right) \leqslant \exp(-N^{2\epsilon}\|f_{N\epsilon}^1 - f_{N\epsilon}^2\|^2/4) \tag{3.15}$$

where $f_{N\epsilon} = \sum_{n=1}^N e_n/N^{1/2+\epsilon}$. The estimate (3.15) leads to

Lemma 7. *Let* $|e^i) = \otimes_{n=1}^\infty |e_n^i)$, $i = 1, 2$, *be product states on* $\mathscr{A}_{\text{spin}}$, *such that for some* $\epsilon > 0$

$$\lim_{N\to\infty} (f_{N\epsilon}^1 - f_{N\epsilon}^2) = d \neq 0. \tag{3.16}$$

Then $|e^1)$ and $|e^2)$ are weakly inequivalent and hence disjoint.

For product states we obtain disjointness under weaker assumptions than (3.14). This has been used in [19] for the construction of a pointer. It is reassuring to learn from (3.15) that the finite approximations of the infinite system satisfy

$$|_N(e^1|A|e^2)_N|^2 = 0(\exp - N^{2\epsilon}\|d\|^2/4) \tag{3.17}$$

uniformly for all $A \in \mathscr{B}(\otimes_{n=1}^M \mathbb{C}_n^2)$, $\|A\| \leqslant 1$ and M fixed, if $N \to \infty$. Hence macroscopic differences between product states imply a rapidly decreasing overlap for all finite approximations. The coherence is prohibitively weak, if—as in a laboratory experiment—the number of degrees of freedom is large.

4. Models for Measurement

We turn to the construction of time evolutions which transform a coherent pair of initial states of the combined system into a disjoint pair of final states. The state space of the micro-system will always be a separable Hilbert space $\mathscr{H}_S$ with all hermitean operators in $\mathscr{B}(\mathscr{H}_S)$ as observables. The observable to be measured will always be σ^3. As apparatus we choose either a quantum spin system $\mathscr{A}_{\text{spin}}$ or a continuous fermion system $\mathscr{A}_f$. Let $\mathscr{A}$ be the algebra of observables of the combined system. Then the time evolution can be chosen to be either a continuous 2-parameter family $\alpha_{ts} \in \text{Aut}(\mathscr{A})$, or a non-automorphic strongly continuous family $U(t,s)$ of unitary operators in a fixed representation π, or even a discontinuous unitary family. The more singular time evolutions are admitted, the easier it will be to arrive at disjointness.

Automorphic time evolutions: Here the main obstacle is Lemma 2, which is often circumvented by a time average. For instance, let $\mathscr{A} = \mathscr{B}(\mathbb{C}^2)$ and $\alpha_t = \exp(i\sigma^3 t)(.)\exp(-i\sigma^3 t) \in \text{Aut}(\mathscr{A})$. If $\omega_\pm = \omega(\psi_\pm)$ and $\omega = \omega(\psi)$, $\psi = c_+\psi_+ + c_-\psi_-$, then $(2\pi)^{-1}\int_0^{2\pi} dt\,\omega \circ \alpha_t = |c_+|^2\omega_+ + |c|^2\omega_-$. We do not, however, accept the ergodic mean [5, 16] as a fundamental solution to the problem of the reduction of wave packets. The first and trivial solution in our spirit are time evolutions $\alpha_{ts} \in \text{Aut}(\mathscr{A})$, such that for $i = 1, 2$ $1\omega_i \circ \alpha_{ts} \underset{w}{\to} \overline{\omega}_i$ for $t \to \infty$, where ω_1 and ω_2 are coherent and $\overline{\omega}_1$ and $\overline{\omega}_2$ disjoint. For non-conservative forces this can be easily achieved:

Example 2 (*development of a photo-emulsion*). Let $\mathscr{H}_S = \mathbb{C}_0^2$, $\mathscr{H}_A = \otimes_{n=1}^{\infty} \mathbb{C}_n^2$ and $\mathscr{A} = \mathscr{A}_{\text{spin}}$ on all sites $n = 0, 1, 2, \ldots$. The state ψ_n^+ could represent an AgBr molecule at the site n and ψ_n^- an Ag atom. The 'development' at n should only act for $2n\pi \leqslant t \leqslant (2n + 1/2)\pi$, $n = 1, 2, \ldots$, and only under the catalytic action of a 'germ' ψ_0^- at 0. This is described by the time-dependent Hamiltonian

$$H(t) = \begin{cases} P_0^- \sigma_n^1, & 2n\pi \leqslant t \leqslant (2n+1/2)\pi \quad n = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

(4.1) has the propagator $U(t,s) = \exp i \int_s^t dr H(r) \in \mathscr{A}$ and leads to the automorphism $\alpha_{ts} = U(t,s)(.)U(s,t)$. Furthermore,

$$U(0,t)\psi_0^+ \bigotimes_{n=1}^{\infty} \psi_n^+ \underset{w}{\rightarrow} \psi_0^+ \bigotimes_{n=1}^{\infty} \psi_n^+ = \psi^+$$

$$U(0,t)\psi_0^- \bigotimes_{n=1}^{\infty} \psi_n^+ \underset{w}{\rightarrow} \psi_0^- \bigotimes_{n=1}^{\infty} \psi_n^- = \psi^- \tag{4.2}$$

for $t \rightarrow \infty$, and $\omega(\psi^+)$ and $\omega(\psi^-)$ are pure and macroscopically different, hence disjoint. By (3.17), a developed silver grain with 10^{10} atoms is an excellent approximation to an infinite pointer.

For conservative forces it is less interesting to study the weak convergence of $\omega \circ \alpha_t$ for $t \rightarrow \infty$. More physical, in the spirit of scattering, is the convergence of the interaction picture time evolution $\omega \circ \alpha_t \circ \alpha_{-t}^0$. Here α_t is the evolution of the interacting micro- and macro-system and α_t^0 the free evolution. The following model has found some interest in solid state physics [17]:

Example 3 (*X-ray edge*). Let $\mathscr{H}_S = \mathbb{C}^2$ and $\mathscr{A}_S = \mathscr{B}(\mathbb{C}^2)$. Let $\mathscr{A}_A = \mathscr{A}_f$ be the C^*-algebra of the canonical anticommutation relations over $L^2(\mathbb{R}^3)$ [20]. Let $\mu > 0$ and $\alpha_t^0 \in \text{Aut}(\mathscr{A})$ be the automorphism generated by

$$H_0 = \int d^3k\, (k^2 - \mu) a^*(k) a(k) \tag{4.3}$$

in the Fock representation. Let $\alpha_t \in \text{Aut}(\mathscr{A})$ be similarly generated by

$$H = H_0 + P^- V$$

$$V = a^*(g)a(g) \in \mathscr{A}_A \tag{4.4}$$

where $g \in \mathscr{S}(\mathbb{R}^3)$ and $P^- \in \mathscr{B}(\mathbb{C}^2)$ is the projector on ψ_-. As the initial state of the apparatus we choose the ground state ω_0 of H_0 (minus an infinite self-energy):

$$\omega_0(a^*(f_m) \ldots a^*(f_1)a(g_1) \ldots a(g_n)) = \delta_{mn} \det[(f_i, Ag_j)] \tag{4.5}$$

where $A : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ is an integral operator with the kernel

$$A(p,q) = \theta(\mu - p^2)\delta(p - q).$$

Let $\omega^{\pm} = P^{\pm} \otimes \omega_0$. Then $\omega^+ \circ \alpha_t \circ \alpha_{-t}^0 = \omega^+$, while $\omega^- \circ \alpha_t \circ \alpha_{-t}^0(A)$ can be computed from (4.5) and the time evolution

$$a(f) \rightarrow a(e^{-iht} e^{ih_0 t} f) \rightarrow \alpha(\Omega_+ f) \quad \text{for } t \rightarrow \infty. \tag{4.6}$$

Here h and h_0 are the 1-body operators corresponding to $H_0 + V$ and H_0, and Ω_+ is the unitary wave operator in $L^2(\mathbb{R}^3)$ with the kernel

$$\Omega_+(p,q) = \delta(p-q) - \frac{g(p)^* g(q)}{p^2 - q^2 + i0} h(q^2)$$

$$h(q^2) = \left(1 - \int \frac{d^3 p |g(p)|^2}{q^2 - p^2 - i0}\right)^{-1}. \tag{4.7}$$

Hence $\omega^- \circ \alpha_t \circ \alpha^0_{-t} \underset{w}{\rightarrow} P^- \otimes \omega$, where ω is the ground state of $H_0 + V$ (minus an infinite self-energy) and has the form (4.5) with A replaced by $\Omega_+^* A \Omega_+$. By a theorem of Powers and Størmer [20], the pure states ω_0 and ω (and hence $P^+ \otimes \omega_0$ and $P^- \otimes \omega$) are unitarily equivalent, if and only if $A - \Omega_+^* A \Omega_+$ (or $\Omega_+ A - A\Omega_+$) is a Hilbert-Schmidt operator. The kernel of $\Omega_+ A - A\Omega_+$ is

$$(\theta(\mu - p^2) - \theta(\mu - q^2)) \frac{g(p)^* g(q) h(q^2)}{p^2 - q^2 + i0}. \tag{4.8}$$

The square integral of (4.8) diverges. Hence ω_0 and ω are disjoint, and the X-ray edge acts as a measuring apparatus. The initial state $P^+ \otimes \omega_0$ can be viewed as the equilibrium state at $T = 0$ of a non-interacting electron gas in a conduction band with an occupied impurity level. In $P^- \otimes \omega_0$ the impurity is ejected (by an X-ray), and here all electrons have a 1-body interaction with the hole. In the latter situation, $P^- \otimes \omega$ is the equilibrium state, which by (4.6) is the weak limit for $t \to +\infty$ of $P^- \otimes \omega_0 \circ \alpha_t \circ \alpha^0_{-t}$. ω differs from ω_0 by an excitation of infinitely many particles and holes with probability one [3].

This is an infrared divergence, without a macroscopic difference in the sense of section 3. We remark that by Lemma 3 all the cross-terms converge to zero. An explicit proof of this fact is quite difficult: While the Heisenberg picture time evolution $\alpha_t \circ \alpha^0_{-t}$ operates as a 1-body operator on the test functions, the evolution $\exp(iHt)\exp(-iH_0 t)$ is a many-body operator in the ω_0-representation. A similar apparatus using bosons can be constructed with the Blanchard model [1].

Example 4 (*Coleman model*). One can also obtain the transition (4.2) by a time-translation invariant automorphism $\alpha_{ts} = \alpha_{t-s}$: Consider on $L^2(\mathbb{R}^1) \otimes \otimes_{n=1}^{\infty} \mathbb{C}_n^2$ the operators

$$H_0 = p, \quad H = H_0 + V, \quad V = \sum_{n=1}^{\infty} V(x-n)\sigma_n^1. \tag{4.9}$$

Here V is real, continuous, of compact support with $\int dx\, V(x) = \pi/2$. The Dyson equation for $U(t) = \exp(iH_0 t)\exp(-iHt)$ is

$$U(t) = 1 - i \int_0^t ds V(s) U(s)$$

$$V(s) = \exp(iH_0 s) V \exp(-iH_0 s) = \sum_{n=1}^{\infty} V(x+s-n)\sigma_n^1. \tag{4.10}$$

(4.10) has the solution

$$U(t) = \exp\left(-i\int_0^t ds \sum_{n=1}^{\infty} V(x+s-n)\sigma_n^1\right). \tag{4.11}$$

One sees that (4.11) leaves $L^2(I) \otimes \otimes_{n=1}^{\infty} \mathbb{C}_n^2$ invariant, if I is any bounded open interval in $\mathbb{R}^1$. Let $\mathscr{H}_S = L^2(I) \otimes \mathbb{C}^2$ with projectors $P^{\pm}$ on $\psi_{\pm} \in \mathbb{C}^2$, and let $\mathscr{A}_A = \mathscr{A}_{\text{spin}}$. The interaction picture evolution $\alpha_t \circ \alpha_{-t}^0 = W(t)(.)W(t)^*$,

$$W(t) = P^+ + P^- U(t) \in \mathscr{A} \tag{4.12}$$

is automorphic for all t. As initial states we choose $\chi \otimes \psi_{\pm} \otimes \varphi_+$, where $\chi \in L^2(I)$, $\sigma^3 \psi_{\pm} = \pm\psi_{\pm}$ and where the apparatus state $\varphi_{\pm}$ has all spins up or down. Since I and supp(V) are bounded, there exists some N and some unitary $U \in \mathscr{A}$ such that

$$U(\infty) = U \prod_{n=1}^{\infty} \sigma_n^1$$

$$U = \left(\prod_{n=1}^{N} \sigma_n^1\right) \exp\left(-i\int_0^{\infty} ds \sum_{n=1}^{N} V(x+s-n)\sigma_n^1\right) \tag{4.13}$$

Hence as states on $\mathscr{A}$ one obtains essentially (4.2):

$$\begin{aligned} &W(t)\chi \otimes \psi_+ \otimes \varphi_+ = \chi \otimes \psi_+ \otimes \varphi_+ \\ &W(t)\chi \otimes \psi_- \otimes \varphi_+ \underset{w}{\to} U(\chi \otimes \psi_- \otimes \varphi_-). \end{aligned} \tag{4.14}$$

By Lemma 4, it is not necessary to start with the pointer in a pure state. In the incomplete tensor product to φ_+ the states $\varphi_n = U_n\varphi_+$ with local unitary $U_n \in \mathscr{A}_{\text{spin}}$ are total. Instead of starting from $\omega(\varphi_+)$, we could take $\omega = \sum p_m \omega(\varphi_m)$. Then $\omega(\chi \otimes \psi_-) \otimes \omega$ would converge weakly to $\omega(\chi \otimes \psi_-) \otimes \sum p_m \omega(V_m \varphi_-)$, with local unitary $V_m \in \mathscr{A}_{\text{spin}}$. By Lemma 4, this pointer position would be disjoint from $\omega(\chi \otimes \psi_+) \otimes \omega$.

The Coleman model can be considered as a caricature of an electron in one-dimensional motion, whose spin is measured by the result of a local interaction with an infinite spin array.

Non-automorphic time evolutions: In Example 1 the evolution is not an automorphism of the algebra of observables. However, it can be accomplished by a strongly continuous group of unitary time translations in $\mathscr{H}_S \otimes \mathscr{H}_A$. If one wants a measurement which leads to macroscopically different pointer positions in finite times, then the unitary time evolution is so discontinuous that it has no Hamiltonian:

Example 5 (*big bang*). Let $\mathscr{H}_S = \mathbb{C}_0^2$ and $\mathscr{H}_A = \otimes_{n=1}^{\infty} \mathbb{C}_n^2$ with $\mathscr{A}_A = \mathscr{A}_{\text{spin}}$. Let $U(t)$: $\mathscr{H}_A \to \mathscr{H}_A$ be defined by linear extension of

$$U(t) \bigotimes_{n=1}^{\infty} \psi_n = \bigotimes_{n=1}^{\infty} (\exp(i\sigma_n^1 t)\psi_n). \tag{4.15}$$

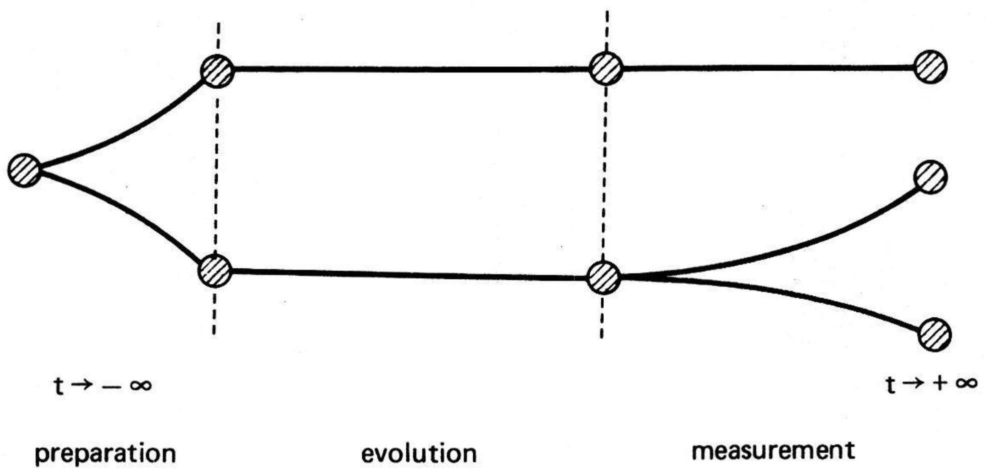
Then $W(t) = P^+ + P^- U(t)$ is unitary on $\mathscr{H}$. If one takes as initial states $\psi_{\pm} \otimes \varphi_+$ with all apparatus spins pointing up, then the pure states $W(t)\psi_{\pm} \otimes \varphi_+$ are macroscopically different in arbitrarily short times earlier and later. A boson model of somewhat similar structure has been discussed by Primas [19].

5. Conclusion

The solution of the problem of measurement is closely connected with the yet unknown correct description of irreversibility in quantum mechanics. In the framework of infinite systems we were able to analyse some models for measuring processes, which are not manifestly in contradiction with physical and mathematical common sense.

We have shown, without using any averaging procedure, that coherent states of a quantum system coupled to a quantum apparatus can split into disjoint states. For non-catastrophic time evolutions one has to wait infinitely long in order to arrive at those macroscopic changes which destroy all phase relations for local observables. The introduction of an asymptotic condition into measurement theory is as natural as elsewhere in microphysics, where S-matrix theory is sometimes considered as the ultimate receptable of all physics.

It is gratifying that time is two-sidedly unbounded for $t \to \pm\infty$. Hence the same mechanism which we have employed for measurement can also be used for the preparation of the system, in the sense of the following diagram [14]:



For practical purposes it is not necessary to pass to infinite systems and times. However, one has to establish the existence of the limits $N \to \infty$ and $t \to \infty$ and the disjointness of the resulting states of the system and apparatus, in order to be sure that in the finite approximations the error can be made arbitrarily small for sufficiently large N and t. We also note the direction of time in the above diagram: while the automorphic evolution between finite times is reversible, it is precisely the irreversibility in the limit of infinite times which reduces the wave packets.

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