AN EXACT RESULT FOR THE 1D RANDOM ISING MODEL IN A TRANSVERSE FIELD

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It is shown exactly that for an N-site cyclic chain with hamiltonian $H = -\sum_{i=1}^{N}(\Gamma_i S_i^x + J_i S_i^z S_{i+1}^z)$, the gap in the excitation spectrum goes to zero when $N \to \infty$ at the "critical point" given by the relation $\prod_{i=1}^{N}\Gamma_i = \prod_{i=1}^{N} J_i$.

We consider the 1D random Ising model in a transverse field on a cyclic chain with the hamiltonian

$$H = -\sum_{i=1}^{N}(\Gamma_i S_i^x + J_i S_i^z S_{i+1}^z)\,, \tag{1}$$

where S_i^x, S_i^z are Pauli matrices,

$$S_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and with the cyclic conditions $S_{N+1}^x = S_1^x, S_{N+1}^z = S_1^z$. We shall consider that N is, for instance, even. Using the Jordan–Wigner transformation [1] the spin hamiltonian (1) becomes a fermion system:

$$\begin{aligned} H = -2\sum_{i=1}^{N} (\Gamma_i (c_i^+ c_i - \tfrac{1}{2}) \\ + \tfrac{1}{2} J_i (c_i^+ - c_i)(c_{i+1}^+ + c_{i+1})) \\ - \tfrac{1}{2} J_N ((c_N^+ - c_N)(c_1^+ + c_1))(\exp(\mathrm{i}\pi M) + 1)\,, \end{aligned} \tag{2}$$

where $M = \sum_{i=1}^{N} c_i^+ c_i$ with $c_{N+1} = c_1, c_{N+1}^+ = c_1^+$. We shall first neglect the correction term proportional to $\exp(\mathrm{i}\pi M) + 1$. The hamiltonian (2) without the correction term can be diagonalized using the method developed in the appendix of ref. [2]. The hamiltonian is written

$$\begin{aligned} -\tfrac{1}{2} H &= \sum_{ij} c_i^+ A_{ij} c_j + \tfrac{1}{2}(c_i^+ B_{ij} c_j^+ + \text{h.c.}) \\ &= \sum_{\alpha} \Lambda_\alpha \eta_\alpha^+ \eta_\alpha + \text{constant}. \end{aligned} \tag{3}$$

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The excitation spectrum Λ_α is solution of the secular equation

$$\det((\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) - \Lambda_\alpha^2 \mathbf{1}) = 0\,; \tag{4}$$

the matrices $A - B$ and $A + B$ are the following $N \times N$ matrices:

$$\mathbf{A}-\mathbf{B} = (\mathbf{A}+\mathbf{B})^{\mathrm{t}} = \begin{pmatrix} \Gamma_i & & & J_N \\ J_1 & \Gamma_2 & 0 & \\ & J_2 & \ddots & \\ 0 & & \ddots & \Gamma_N \end{pmatrix}. \tag{5}$$

The secular equation (4) has a solution $\Lambda_\alpha = 0$ if $\det((\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B})) = 0$, but

$$\det(\mathbf{A}-\mathbf{B}) = \det(\mathbf{A}+\mathbf{B}) = \prod_{i=1}^{N}\Gamma_i - \prod_{i=1}^{N} J_i\,.$$

There is an excitation of energy $\Lambda_\alpha = 0$ if the following critical condition is satisfied:

$$\prod_{i=1}^{N}\Gamma_i = \prod_{i=1}^{N} J_i\,. \tag{6}$$

When relation (6) is satisfied the gap in the excitation spectrum is going to zero; this should lead to a singular behavior of the ground state energy. This disappearance of the gap only occurs in the limit of $N \to \infty$ because of the effect of the correction term that we neglected. For the ground state the operator M is odd and thus the correction term is zero and the ground state energy is obtained correctly using eq. (3). For the state with one excitation the operator M is even, the correction term cannot be neglected and the excitation energy Λ_α^2 at the critical condition (6) is zero only to order $1/N$ (due to the correction term): at the "critical point" the gap goes to zero only in the infinite limit for N. The same analysis is easily performed for N odd (in that case we use anticyclic conditions in eq. (2)) leading to the same answer. Identical results can also be obtained for a random chain with free ends.

From eq. (6) we recover the nonrandom case ($\Gamma_i = \Gamma, J_i = J$) $(\Gamma/J)_c = 1$ [3]. Relation (6) obeys the self-duality property for the 1D quantum problem [4] (the dual of the chain is a chain where the new sites are associated with the old bonds and the new bonds with the old sites and where the new interactions are $J_i' = \Gamma_i$ and the new fields $\Gamma_i' = J_i$). When all the J_i are equal to J and when the Γ_i follow a two-delta-functions probability distribution

$$P(\Gamma_i) = p\delta(\Gamma - \Gamma_1) + (1-p)\delta(\Gamma - \Gamma_2)\,,$$

relation (6) becomes

$$J = \Gamma_1^p \Gamma_2^{1-p}\,. \tag{7}$$

The 1D Ising model in a transverse field corresponds [5] to the 2D classical Ising model with horizontal interaction J_1 and vertical interactions J_2 in the limit $J_1 \to 0, J_2 \to \infty$, and

$$e^{-2\beta J_2}/\beta J_1 \to \Gamma/J\,, \qquad \beta = 1/k_T\,. \tag{8}$$

The random 1D Ising model in a transverse field corresponds to the random 2D Ising model considered by McCoy [6] where the N horizontal bonds J_1 ($i = 1, \ldots, N$) in a row and the N vertical bonds J_{2i} attached to the same row are random but repeat exactly from one row to the next. For such a special random model the critical temperature is given by [6,7]

$$\prod_{i=1}^{N} \tanh(\beta J_{1i})^* = \prod_{i=1}^{N} \tanh \beta J_{2i}\,, \tag{9}$$

with $\tanh(\beta J_{1i})^* = \exp(-2\beta J_{1i})$. Relation (6) is well recovered from eqs. (8) and (9).

For the random model (1) it seems difficult to get other exact results. However, for special narrow distributions of Γ_i and J_i similary to those already used by McCoy [6] it is certainly possible to obtain a complete exact solution of (1) as already done for the 1D random spin-$\frac{1}{2}$ XY model [8] in a Z field.

The same method developed in this letter could be used to determine the critical condition for the random anisotropic spin-$\frac{1}{2}$ XY random chain in a Z field.

Another fruitful approach currently under investigation [9] consists in applying a real space renormalization group method (already tested for the nonrandom case [10]) to this random quantum system in 1D and also in 2D.

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