RELATION BETWEEN THE SUPERFLUID DENSITY AND ORDER PARAMETER FOR SUPERFLUID He NEAR T_c *

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Relations are derived between the superfluid density, the order parameter, and the specific heat singularity for superfluid helium near the lambda point.

Two different ways have been suggested to define the order parameter in a superfluid. Firstly one may regard $|\Psi|^2$ as the number of helium atoms in the macroscopically occupied zero-momentum state divided by the volume of the system. More generally, the order parameter itself may be defined by the relation $\Psi(r) = \langle\psi(r)\rangle$ where ψ is an annihilation operator for helium atoms, and the average is taken over an ensemble with broken symmetry in which the order parameter has a definite phase. On the other hand, Ginzburg and Pitaevskii [1] define a different quantity, which we shall denote by $\widetilde{\Psi}$, by the equation $\rho_s = m|\widetilde{\Psi}|^2$, where m is the mass of the helium atom and ρ_s is the superfluid density which relates the superfluid current density j_s to the superfluid velocity v_s. The only quantity readily accessible to experiment is the superfluid density ρ_s, but it is $|\Psi|$ which is analogous to quantities which can be measured in other second order phase transitions (such as the magnetization in ferromagnetic transitions and the density in the liquid-gas transition). In analyzing their data on the temperature dependence of ρ_s, Clow and Reppy [2] assumed that Ψ and $\widetilde{\Psi}$ were proportional to each other near T_c, or equivalently $\rho_s \propto |\Psi|^2$. We here examine this assumption and show that it is equivalent to assuming that the order parameter-order parameter correlation function at T_c falls off with distance as $1/r$, as predicted by the Ornstein-Zernicke theory, which is known not to be exact in some cases. We derive a more general relationship between ρ_s and $|\Psi|^2$, and also a relationship between the behaviours of ρ_s and the specific heat in the critical region.

In situations where Ψ varies slowly with position and the departure from equilibrium is small, the free energy of the system is given by an expression of the form

$$F = \int\{f(|\Psi|^2) + \tfrac{1}{2}A_\perp|(\nabla\Psi)_\perp|^2 + \tfrac{1}{2}A_\parallel|(\nabla\Psi)_\parallel|^2\}\mathrm{d}\boldsymbol{r} \tag{1}$$

where f, $A_\perp$ and $A_\parallel$ may depend on temperature and $(\nabla\Psi)_\parallel$ and $(\nabla\Psi)_\perp$ are the components of $\nabla\Psi$ in phase and $\frac{1}{2}\pi$ out of phase with Ψ respectively†.

Putting $\Psi(r) = \Psi_0 \exp i(m\boldsymbol{v}_s\cdot\boldsymbol{r}/\hbar)$, corresponding to uniform superfluid flow with velocity $\boldsymbol{v}_s$, and using the fact that the extra free energy due to the flow is $\frac{1}{2}\rho_s v_s^2$, we deduce

$$\rho_s = A_\perp\,(m/\hbar)^2\,|\Psi|^2 \tag{2}$$

Next we consider a Ψ of the form $\Psi_0 + ia\,V^{-\frac{1}{2}} \times \cos(\boldsymbol{q}\cdot\boldsymbol{r})$, the second term being a small departure from the equilibrium value Ψ_0 and $\frac{1}{2}\pi$ out of phase with it. The quantity $(\partial^2F/\partial a_q^2)$ is the inverse of the susceptibility $\chi_\perp(\boldsymbol{q})$ which determines the response of the system to a perturbation of the type $\delta H = \int\{i\cos(\boldsymbol{q}\cdot\boldsymbol{r})\psi^+(r) + \mathrm{c.c.}\}\mathrm{d}\boldsymbol{r}$. From (1) we see that in the limit $\boldsymbol{q}\to 0$ where (1) is valid

$$\{\chi_\perp(\boldsymbol{q})\}^{-1} = \tfrac{1}{2}A_\perp q^2 \tag{3}$$

(2) and (3) imply the general relation

$$\rho_s \propto |\Psi|^2 \lim_{q\to 0}(q^2\chi_\perp(q))^{-1} \tag{4}$$

Now we examine the behaviour of $\chi_\perp(q)$ at T_c.

† For simplicity we have omitted the dependence of the free energy density on ρ, the total density of the fluid. This neglect is certainly justified for the purposes of this paper, since the only types of disturbance to the system which we consider are ones which from considerations of time-reversal symmetry do not couple directly to density fluctuations. I am indebted to Professor D. Pines for discussion of this point.

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(where the distinction between in phase and $\frac{1}{2}\pi$ out of phase components of $\nabla\Psi$ is absent sine Ψ_0 is zero). In the classical limit $q \rightarrow 0$, $\chi_\perp(q)$ is proportional to the Fourier transform of the equal-time order parameter-order parameter correlation function $C(r) \equiv \langle \frac{1}{2} \{\psi(r)\psi^+(0) + \psi^+(0)\psi(r)\}\rangle$. If for large r $C(r) \propto r^{-1-\eta}$, $\chi_\perp(q) \propto q^{-2+\eta}$ for small q. η represents the departure of the correlation function at the critical point from the prediction of the Ornstein-Zernicke theory $C(r) \propto 1/r$. It is believed to be greater than zero for some systems, in particular the three-dimensional Ising model [3].

Let us now consider the behaviour of $\chi_\perp(q)$ just below T_c. For large q, $\chi_\perp(q)$ will be very close to its value at T_c, but as q is reduced, at some point the behaviour must change from the $q^{-2+\eta}$ dependence on q for large q to the q^{-2} dependence indicated by (3) for small q. This change in behaviour represents the breakdown of (1) due to non-local effects, and is to be expected to occur when q is of the order of the inverse of some coherence length $\xi(T)$. By analogy with other second order phase transitions, ξ is expected to increase as the region where critical fluctuations are important is approached, and to tend to infinity at the critical point [3]. The asymptotic behaviour of $A_\perp$ near T_c may be found by equating the forms obtained for $\chi_\perp(q)$ in the cases $q\xi \ll 1$ and $q\xi \gg 1$ for the value $q = \xi^{-1}$. This gives

$$A_\perp \propto \xi^\eta \tag{5}$$

Defining critical exponents by the relations $|\Psi| \propto (T_c - T)^\beta$, $\xi \propto (T_c - T)^{-\nu'}$, we deduce from (2) and (5) the result

$$\rho_s \propto (T_c - T)^{2\beta - \eta\nu'} \tag{6}$$

According to the scaling laws of Widom [4] and Kadanoff [5], the exponents appearing in (6) can be related to the specific heat exponent α' defined by the formula $c_p \propto (T_c - T)^{-\alpha'}$ for $T < T_c$:

$$2\beta - \eta\nu' = \tfrac{1}{3}(2 - \alpha') \tag{7}$$

Therefore if the specific heat singularity at the Λ-point is logarithmic $(\alpha' = 0)$ [6], ρ_s must be proportional to $(T_c - T)^{\frac{2}{3}}$ near T_c, in agreement with the experiment of Clow and Reppy [2]. We conclude that since η is not known to be zero for helium the measurement of ρ_s gives no definite information concerning the value of β, but that the agreement between the measured temperature dependence of ρ_s and the value predicted from the specific heat data provides some confirmation for the Widom-Kadanoff scaling laws.

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