Critical exponents of the *N*-vector model

Critical exponents of the N-vector model

R Guida† and J Zinn-Justin‡

CEA-Saclay, Service de Physique Théorique§, F-91191 Gif-sur-Yvette Cedex, France

Abstract.   Recently the series for two renormalization group functions (corresponding to the anomalous dimensions of the fields ϕ and ϕ^2) of the three-dimensional ϕ^4 field theory have been extended to next order (seven loops) by Murray and Nickel. We examine the influence of these additional terms on the estimates of critical exponents of the N-vector model, using some new ideas in the context of the Borel summation techniques. The estimates have slightly changed, but remain within the errors of the previous evaluation. Exponents such as η (related to the field anomalous dimension), which were poorly determined in the previous evaluation of Le Guillou–Zinn-Justin, have seen their apparent errors significantly decrease. More importantly, perhaps, summation errors are better determined.

The change in exponents affects the recently determined ratios of amplitudes and we report the corresponding new values.

Finally, because an error has been discovered in the last order of the published $\varepsilon = 4 - d$ expansions (order ε^5), we have also re-analysed the determination of exponents from the ε-expansion.

The conclusion is that the general agreement between ε-expansion and three-dimensional series has improved with respect to Le Guillou–Zinn-Justin.

1. Introduction and summary of results

Recently the perturbative expansions of the anomalous dimensions of the fields ϕ and ϕ^2 for the O(N) symmetric $(\phi^2)^2_{d=3}$ field theory have been extended to next order (seven loops) in the case $N = 0, \ldots, 3$ by Murray and Nickel [46]. This rather impressive result has led us to re-examine the determinations of the critical exponents for $N = 0$ (polymers), $N = 1$ (Ising-like systems), $N = 2$ (superfluid helium) and $N = 3$ (real ferromagnets). For completeness we have added results (at six loops) for $N = 4$ which correspond to the Higgs sector of the standard model at finite temperature. A limitation of this work is that the series for the renormalization group (RG) β-functions have not been extended (they remain at six loops) and for several exponents this is now the main source of error.

Critical exponents have also been calculated in the form of $\varepsilon = 4 - d$ expansions, up to five loops [11]. Recently, a slight error in the previously published series has been corrected [12], and this has motivated us to also re-examine the corresponding estimates (again adding $N = 4$ results).

For the reader who is not interested in details we summarize our main results for $N = 0, \ldots, 3$ in table 1 ($d = 3$) and in table 2 (ε-expansion) while $N = 4$ results for both methods can be found in table 3. We have chosen central values which satisfy all

† E-mail address: guida@spht.saclay.cea.fr
‡ E-mail address: zinn@spht.saclay.cea.fr
§ Laboratoire de la Direction des Sciences de la Matière du Commissariat à l'Energie Atomique.

Table 1. Critical exponents of the O(N) models from $d = 3$ expansion (present work).

N	0	1	2	3
$\tilde{g}^*_{\mathrm{Ni}}$	1.413 ± 0.006	1.411 ± 0.004	1.403 ± 0.003	1.390 ± 0.004
g^*	26.63 ± 0.11	23.64 ± 0.07	21.16 ± 0.05	19.06 ± 0.05
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
η	0.0284 ± 0.0025	0.0335 ± 0.0025	0.0354 ± 0.0025	0.0355 ± 0.0025
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
α	0.235 ± 0.003	0.109 ± 0.004	-0.011 ± 0.004	-0.122 ± 0.010
ω	0.812 ± 0.016	0.799 ± 0.011	0.789 ± 0.011	0.782 ± 0.0013
$\theta = \omega\nu$	0.478 ± 0.010	0.504 ± 0.008	0.529 ± 0.009	0.553 ± 0.012

Table 2. Critical exponents of the O(N) models from ε-expansion (present work).

N	0	1	2	3
γ (free)	1.1575 ± 0.0060	1.2355 ± 0.0050	1.3110 ± 0.0070	1.3820 ± 0.0090
γ (bc)	1.1571 ± 0.0030	1.2380 ± 0.0050	1.317	1.392
ν (free)	0.5875 ± 0.0025	0.6290 ± 0.0025	0.6680 ± 0.0035	0.7045 ± 0.0055
ν (bc)	0.5878 ± 0.0011	0.6305 ± 0.0025	0.671	0.708
η (free)	0.0300 ± 0.0050	0.0360 ± 0.0050	0.0380 ± 0.0050	0.0375 ± 0.0045
η (bc)	0.0315 ± 0.0035	0.0365 ± 0.0050	0.0370	0.0355
β (free)	0.3025 ± 0.0025	0.3257 ± 0.0025	0.3465 ± 0.0035	0.3655 ± 0.0035
β (bc)	0.3032 ± 0.0014	0.3265 ± 0.0015		
ω	0.828 ± 0.023	0.814 ± 0.018	0.802 ± 0.018	0.794 ± 0.018
θ	0.486 ± 0.016	0.512 ± 0.013	0.536 ± 0.015	0.559 ± 0.017

Table 3. Critical exponents in the O(4) models from $d = 3$ and ε-expansion (present work).

	$d = 3$	ε : free, bc
$\tilde{g}^*_{\mathrm{Ni}}$	1.377 ± 0.005	
g^*	17.30 ± 0.06	
γ	1.456 ± 0.010	$1.448 \pm 0.015, 1.460$
ν	0.741 ± 0.006	$0.737 \pm 0.008, 0.742$
η	0.0350 ± 0.0045	$0.036 \pm 0.004, 0.033$
β	0.3830 ± 0.0045	0.3820 ± 0.0025
α	-0.223 ± 0.018	-0.211 ± 0.024
ω	0.774 ± 0.020	0.795 ± 0.030
θ	0.574 ± 0.020	0.586 ± 0.028

scaling relations, but the apparent errors for γ, ν, β, η in general have been determined independently. For the $d = 3$ IR fixed-point value g^* we give results both in the usual field theory normalization (equations (2.3)) and in the normalization used by Nickel [7],

$$\tilde{g} = \frac{N + 8}{48\pi} g$$

which is such that the fixed-point value is close to 1.

Note that in table 1, in addition to the plain ε-expansion results (denoted as 'free'), we report some additional results denoted as 'bc' (i.e. with a boundary condition) that try to incorporate the knowledge of the exact $d = 2$ values by summing the series

$(f(\varepsilon) - f(2))/(2 - \varepsilon)$, where $f(\varepsilon)$ is an exponent with a known two-dimensional (2D) value. In the case of the exponent ν for $N = 0$ the $d = 1$ value is also known. We have checked that incorporating this additional piece of information has no significant impact on the final result.

For $N \geqslant 2$ the analysis of the series with boundary conditions is quite difficult. Therefore we present here only central values, and no error estimates. Values and errors of the corresponding free estimates give some indication.

Let us finally emphasize that we have no real knowledge about the analytic properties of exponents when d approaches 2. Therefore the bc values could be affected by systematic effects.

This paper is organized as follows. In section 2 we summarize a few ideas about perturbative expansion at fixed $d = 3$ dimension and ε-expansion. In section 3 we briefly recall the Borel summation method based on a conformal mapping of the complex cut plane. Several new variations of the practical implementation of the general method are explained. In section 4 we recall the idea of the pseudo-epsilon expansion and introduce the exponents' correlation analysis, which consists of eliminating the coupling constant between different exponents. Section 5 contains a discussion of the numerical results. Finally, the new values of exponents slightly affect the recently published results [13] for the equation of state of the three-dimensional (3D) Ising model, and we present the new determination in section 6 (as well as a revised version of ε-expansion predictions).

2. Renormalized ϕ^4 field theory: ε-expansion and 3D perturbation series

In this paper the general framework is the $(\phi^2)^2$, O(N) symmetric, quantum field theory whose bare action is:

$$\mathcal{H}(\phi) = \int \{\tfrac{1}{2}[\partial_\mu \phi(x)]^2 + \tfrac{1}{2}\lambda_2 \phi^2(x) + \tfrac{1}{4!}\lambda_4 [\phi^2(x)]^2\} \, \mathrm{d}^d x. \tag{2.1}$$

We recall that near the critical temperature T_c, λ_2 is a linear measure of the temperature. If we denote by λ_{2c} the value for which the theory becomes massless ($T = T_c$) then the parameter t

$$t = \lambda_2 - \lambda_{2c} \propto T - T_c \tag{2.2}$$

characterizes the deviation from the critical temperature.

The $(\phi^2)^2$ field theory is renormalizable in four dimensions, and to eliminate UV divergences (for $d < 4$ the theory is super-renormalizable) one introduces renormalized correlation functions. This involves choosing a renormalization scheme and then trading the bare parameters λ_2, λ_4 for a (scheme-dependent) renormalized mass m and dimensionless coupling g. The mass parameter m is proportional to the physical mass, or inverse correlation length, of the high-temperature phase. It behaves for $t \propto T - T_c \to 0_+$ as $m \propto t^\nu$, where ν is the correlation length exponent (see [3] for details).

RG arguments state that the long-distance properties of the massless (critical) theory are governed by non-trivial IR fixed points g^*, solution of the equation

$$\beta(g^*) = 0 \qquad \text{with } \beta'(g^*) = \omega > 0.$$

The anomalous dimensions $\eta(g)$ and $\eta_2(g)$, of the renormalized field $\phi_r = \phi/\sqrt{Z}$ and of the renormalized composite operator $[\phi^2]_r = (Z_2/Z)\phi^2$ respectively, evaluated at $g = g^*$ then yield the two independent combinations of critical exponents (e.g. $\eta = \eta(g^*)$). The explicit forms of the RG functions $\beta(g)$, $\eta(g)$, $\eta_2(g)$ depend on the specific renormalization scheme.

The space dimension relevant for statistical physics is $d = 3$ (occasionally $d = 2$). In this case one faces a serious problem: ordinary perturbative expansion in g in the massless theory is IR divergent for any fixed dimension d, $d < 4$. A solution to this problem was first provided by Wilson–Fisher's $\varepsilon = 4-d$ expansion. The idea is to avoid IR problems by expanding in $\varepsilon = 4-d$ as well as in the coupling constant g. IR singularities are then only logarithmic and can be dealt with. The expansion to the highest order presently available have been performed within the minimal subtraction $\overline{\mathrm{MS}}$ scheme, [11, 12]. In this scheme the d-dimensional RG β-function takes the exact form

$$\beta_{(\overline{\mathrm{MS}})}(g_{\overline{\mathrm{MS}}}, \epsilon) = -\varepsilon g_{\overline{\mathrm{MS}}} + f(g_{\overline{\mathrm{MS}}}) = -\varepsilon g_{\overline{\mathrm{MS}}} + \mathrm{O}(g_{\overline{\mathrm{MS}}}^2).$$

The fixed-point equation

$$\beta_{(\overline{\mathrm{MS}})}(g_{\overline{\mathrm{MS}}}^*, \epsilon) = 0$$

can be solved in the form of an ε-expansion. The L-loop expansion of the β-function then yields $g_{\overline{\mathrm{MS}}}^*$ up to order ε^L. By replacing $g_{\overline{\mathrm{MS}}}$ by $g_{\overline{\mathrm{MS}}}^*$ in the perturbative expansion of the anomalous dimensions η, η_2 one finally obtains the ε-expansion of critical exponents. Note that while g^* is scheme dependent, the ε-expansion for universal quantities is scheme independent.

While this method directly yields a formal expansion for exponents a practical problem arises when one wishes to determine exponents for a physical value of ε like $\varepsilon = 1$ ($d = 3$). Indeed the ε-expansion is divergent as first empirically noted in [5] and later confirmed by the large-order behaviour analysis. A summation method is therefore required to obtain accurate results.

Following Parisi's suggestion [6] perturbation series have also been calculated directly in three dimensions in the framework of the massive renormalized theory where correlation functions $\Gamma_{\mathrm{r}}^{(n)}$ of the renormalized field ϕ_{r} are fixed by the normalization conditions

$$\Gamma_{\mathrm{r}}^{(2)}(p; m, g) = m^2 + p^2 + \mathrm{O}(p^4) \tag{2.3a}$$
$$\Gamma_{\mathrm{r}}^{(4)}(p_i = 0; m, g) = mg. \tag{2.3b}$$

One may be surprised by the introduction of coupling and field renormalizations in a super-renormalizable theory. The reasons are simple, the bare coupling constant becomes infinite when the physical mass goes to zero. Simultaneously the field renormalization also diverges (see [3, 4]).

Series up to six loops obtained in this scheme in [7] for $N = 0, \ldots, 3$ have been generalized in [36] to any N. Recently, in [46] the results for η and η_2 (but not for $\beta(g)$) have been extended to seven loops for $N = 0, \ldots, 3$ (see the appendix). One problem here is that the value g^* of the fixed-point coupling is affected by summation errors on the β-function. Errors on g^* then induce systematic errors for all critical exponents (see section 3).

3. Series summation

Perturbative quantum field theory generates divergent series. Summing such series by simply adding successive terms is meaningful provided coupling constants remain small enough (as in quantum electro-dynamics (QED)). Here, however, the expansion parameter, the fixed-point value g^*, is a number of order 1: one therefore faces the problem of evaluating the sum of divergent series in a non-trivial regime.

In this paper the Borel–Leroy transformation has been used, followed by a conformal mapping [33] (a new version of the method developed in [8] for critical exponents) to

sum the series. We recall that the Borel summability of the ϕ^4 theory in two and three dimensions has been established in [32].

Let $\mathcal{S}(z)$ be any (Borel summable) function whose series has to be summed. We transform the series:

$$\mathcal{S}(z) = \sum_{k=0} \mathcal{S}_k z^k \tag{3.1}$$

into

$$\mathcal{S}(z) = \sum_{k=0} B_k(b) \int_0^\infty t^b e^{-t} u^k(zt)\, \mathrm{d}t \tag{3.2}$$

with

$$u(s) = \frac{\sqrt{1+as}-1}{\sqrt{1+as}+1}. \tag{3.3}$$

The coefficients B_k are calculated by expanding in powers of z the r.h.s. of equation 3.2 and identifying with expansion 3.1. The constant a has been determined by the large-order behaviour analysis. The explicit values are

$$a = 0.147\,774\,232 \times \frac{9}{N+8} \tag{3.4}$$

for the perturbative expansion in $d = 3$ dimensions and

$$a = \frac{3}{N+8} \tag{3.5}$$

for the $\varepsilon = 4 - d$ expansion. We map the Borel plane, cut at the instanton singularity $s = -1/a$, onto a circle in the u-plane in such a way to enforce maximal analyticity and thus to optimize the rate of convergence (for details see, e.g. [3]).

3.1. Additional technical details

Following an idea introduced in [8] for the summation of the ε-expansion we have in addition made a homographic transformation on the coupling constant z to displace possible singularities in the complex z-plane:

$$z = z'/(1 + qz'). \tag{3.6}$$

We have looked for values of the parameters q and b for which the results were particularly insensitive to the order k: in practice the absolute differences of results corresponding to three successive orders have been minimized. When several solutions were found the least sensitive solution was chosen. Moreover, the value of b had to stay within a reasonable range around the value predicted by the large-order behaviour.

For each series $\mathcal{S}(z)$ we have applied the summation procedure both to $\mathcal{S}$ and $1/\mathcal{S}$. Finally, we have introduced 'shifts', for each series summing

$$\mathcal{S}_s(z) = \left(\mathcal{S}(z) - \sum_{k=0}^{s-1} \mathcal{S}_k z^k \right)/z^s.$$

In practice only the cases $s = 0$ (no subtractions) and $s = 1, 2$ have proven useful. Thus, for each exponent we have obtained six results whose spread gives an indication of summation errors.
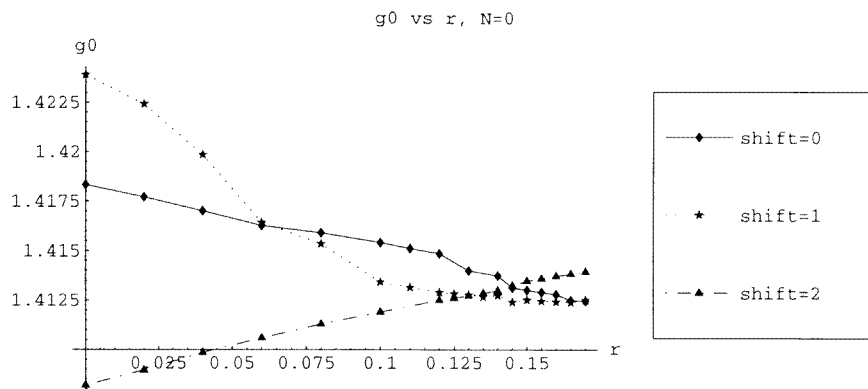
Figure 1. Values of g^*, $N = 0$, as a function of the parameter r for shifts 0, 1, 2.

In some examples (in particular g^*) shifts have produced strongly oscillating results. It has appeared that it would be useful to somehow interpolate between shifted series. An idea, new to this work, has been to consider the combination

$$\mathcal{S}' \equiv (1 + rg)\mathcal{S} \tag{3.7}$$

where r has been used as a third variational parameter (dividing of course the final result by the factor $(1+rg)$). The precise value of r has been obtained by minimizing the dependence in b. This new additional parameter has proven quite useful: it has allowed us, as expected, to obtain series with better apparent convergence as well as better general consistency. It has also revealed that in a few cases the apparent convergence at $r = 0$ was deceptive, the results being unstable with respect to a variation of r. These cases have already been singled out by the extreme values of the optimal b, q parameters.

The main consequence of this new approach has been a decrease in the values of g^* although the length of the series has not changed (better agreement between different shifts at non-zero r, see figure 1), and of γ for $N = 0$ and $d = 3$ (best $r = 0$ values of the exponent revealed to be unstable).

3.2. Errors

The summation error for any quantity $\mathcal{S}(z)$ has been estimated by looking at differences between successive orders, sensitivity to the parameters and spread between all results concerning the same exponent (this has also involved checking scaling relations).

In the case of the 3D perturbative expansion, the total error for each exponent $\mathcal{S}$ is the sum of the intrinsic summation error at fixed $\tilde{g}^*$, $\Delta\mathcal{S}$, and the error induced by the error in $\tilde{g}^*$, $\Delta\tilde{g}^*$:

$$\mathcal{S} = \mathcal{S}^* \pm \Delta\mathcal{S} \pm \left(\frac{d\mathcal{S}}{d\tilde{g}}\right)_{\tilde{g}^*} \Delta\tilde{g}^*. \tag{3.8}$$

We thus also give the derivatives of exponents with respect to $\tilde{g}^*$. Two derivatives are displayed in table 4, all of the others can be deduced from scaling relations. The reader can thus infer the sensitivity of exponents to a change in the values and errors of g^*.

Let us stress here that in our tables we quote the total combined error (as in [1, 2]) while only the intrinsic summation error is reported in table 10 for the alternative result of [46].

Table 4. Critical exponents: sensitivity to $\tilde{g}^*$ determination.

N	0	1	2	3	4
$d\gamma/d\tilde{g}^*$	0.10	0.18	0.28	0.39	0.50
$d\nu/d\tilde{g}^*$	0.069	0.11	0.17	0.22	0.29

In view of the ε-expansion, the total error is directly given by the intrinsic summation error of each exponent and the situation is in principle more favourable: the only problem then is that the available series are shorter (they are technically more difficult to obtain) and the summation error is then bigger!

3.3. Remarks.

The comparison between results coming from direct $d = 3$ series and ε-expansion is not only useful to test the accuracy of our numerical methods. Their consistency is also important to test various assumptions or properties.

In the case of the $d = 3$ expansion we assume more analyticity in the Borel plane as has been rigorously proven. Semiclassical instanton analysis indicates that our assumption is quite plausible but this is not a proof. Moreover, several authors (see e.g. [15, 70]) have argued that RG functions are not regular at $g = g^*$. We have of course checked that these singularities, if they exist, are weak. Numerical evidence is that all RG functions are at least differentiable at $g = g^*$ (including $\beta'(g)$ which yields ω). We cannot of course exclude the situation where these singularities are so weak as to escape detection, but strong enough to influence results at the level of accuracy at which exponents are determined. Our apparent errors could then be underestimated. Nevertheless it should be emphasized that if the hypothesis of analyticity in the cut Borel plane holds, the Borel summation should nevertheless converge asymptotically even in the presence of confluent singularities.

In view of the ε-expansion, problems are more serious, since Borel summability has not even be proven. Moreover, there are indications that UV-renormalon singularities could prevent Borel summability, [66]. These singularities are related to the large-momentum behaviour of renormalized perturbation theory (the 'Landau ghost' problem). A plausible conjecture is that quantities related to the massless theory are renormalon-free since they can be calculated in the theory with UV cut-off. This particularly applies to critical exponents. Instead the question remains open for quantities only defined in the massive renormalized theory, such as the fixed-point coupling constant $g^*(\varepsilon)$ defined by (2.3). Note that because the ε series are rather short, empirical evidence is weak.

4. Pseudo-epsilon expansion and exponents' correlation analysis

In [2], a method was introduced to try to circumvent the problem of systematic errors induced by an error in the determination of g^*: the so-called pseudo-epsilon expansion. The idea is to mimic the ε-expansion and introduce a new parameter $\tilde{\varepsilon}$ in terms of which $\tilde{g}^*$ is expanded as well as all critical exponents.

The $d = 3$ β-function in the scheme given by equations (2.3) have the form:

$$\beta(g) = -g + \beta_2(g)$$

where β_2 begins at order g^2 with a positive coefficient of order 1. We then replace the

Table 5. Critical exponents: correlation between exponents at the fixed point.

N	0	1	2	3	4
$d\eta/d\nu$ $(d = 3)$	0 .83	0.59	0.43	0.32	0.27

β-function by a new function $\beta(g, \tilde{\varepsilon})$

$$\beta(g, \tilde{\varepsilon}) = -\tilde{\varepsilon}g + \beta_2(g)$$

and expand $g^*(\tilde{\varepsilon})$, the solution to $\beta(g, \tilde{\varepsilon}) = 0$, in powers of $\tilde{\varepsilon}$. Eventually we have to sum the series for the value of $\tilde{\varepsilon} = 1$ to recover the initial equation.

This method has been systematically used in [2], and this explains why some of the new values of exponents we obtain in this work differ less from the previous values of [2] than the change in g^* would lead to expect.

To apply the same method here, we face the problem that the series for the β-function have not been extended to seven loops, and therefore for the exponents γ or ν, for example, the information of the additional seven loops term cannot be used. However, since $\eta(g)$ starts only at order g^2, to determine η at loop order L, $g^*(\tilde{\varepsilon})$ is required only at loop order $L - 1$. This also applies to the exponent δ which only depends on η, and to which we have equally applied the summation procedures.

It follows that for $N = 0, 1, 2, 3$ the pseudo-epsilon expansion yields genuine seven-loop information on η, δ, with apparent errors much smaller compared with six-loop results.

In this work to try to circumvent the problem of shorter $\beta(g)$ series and g^* determination, we have introduced another idea. We have directly eliminated the coupling constant between a pair of independent exponents. For example we have inverted the relation $g \mapsto 2 - 1/\nu$

$$g(\nu) = \sum g_k (2 - 1/\nu)^k$$

and then expressed other exponents as series in $2 - 1/\nu$. In this way we have obtained correlation curves between exponents, which can all eventually be translated into relations $\eta(\nu)$. We have applied the same idea starting from the exponents γ, β, expanding in powers of $1 - 1/\gamma$ and $4 - 1/\beta$.

With this in mind it is interesting to consider the derivatives $d\eta/d\nu$ at the fixed point, which we thus display in table 5. Other derivatives can be deduced, using scaling relations. The correlation line can be fixed by taking a point from the list of table 1.

Finally, let us note that we can advance this idea to expanding the RG β-function in powers of for example $2 - 1/\nu$ and solving directly the fixed-point equation $\beta(\nu^*) = 0$.

We have tried to implement this idea but the main problem we have faced is that the general structure of series generated by this set of transformations is rather complicated and therefore the apparent errors are quite large (a problem which already limits the accuracy of the pseudo-epsilon expansion). Therefore the method has mainly been used as a check of the consistency among the data generated by more direct summation. It is possible that more accurate constraints could be obtained with more work to better understand the convergence of these new series, but we eventually generated so many data that it became difficult to analyse all of them with the same care.

5. Numerical results

Let us first consider $d = 3$ results. The values of g^* have been obtained by looking for the zeros of the summed RG function $\beta(g)$. The various methods explained in section 3

have been used, shifts 0, 1, 2, generating three set of values for each N, depending on three parameters b, q, r. Quoted errors for g^* reflect the apparent convergence with the order k ($k \leqslant 7$), the sensitivity of g^* to a variation of the parameters b, q, r around optimal values as well as the spread between different shifts. As additional checks we have looked for the zeros of the function $\nu(g)\beta(g)$ (a rather arbitrary choice with the weak motivation that the derivative yields the exponent $\theta = \omega\nu$) and calculated g^* from the pseudo-expansion (see [2] for details). Final results are reported in table 1 (table 3 for $N = 4$).

In view of what concerns the values of exponents, we have summed the seven (six for $N = 4$) loop series at the values of g^* determined before. We have summed independently the five exponents $\gamma, \beta, \nu, \delta, \eta$ by using the three parameters b, q, r for each exponent (and its inverse) and shifts 0, 1 (shift 2 was considered only as a check). Again, errors have been estimated by decreasing the order and looking at the spread between summation of different equivalent series, as explained in section 3. Additional checks have been derived from pseudo-expansion and exponents' correlation analysis, introduced in section 4. Tables 1 and 3 report the results of the analysis.

In view of the ε-expansion, the procedure is the same as for the $d = 3$ series, apart from the fact that the g^* step is bypassed, the series being summed at $\varepsilon = 1$ for the physical dimension 3.

More precisely we have summed the genuine ε series for the exponents (called 'free' in table 2) and we also summed the modified series

$$\mathcal{S}(d) \equiv \frac{\mathcal{S}(d) - \mathcal{S}(2)}{d - 2}$$

in which for each exponent is imposed the exact value at $d = 2$ (referred to as 'bc', with boundary conditions, in table 2). For $N = 0, 1$ the $d = 2$ exact exponents are obtained from the underlying conformal theories, for $N = 2$ from the identification with the Kosterlitz–Thouless transition, while for $N > 2$ the behaviour near $d = 2$ can be obtained from the O(N) nonlinear σ-model.

The general conclusions are the following. By imposing boundary conditions, we decrease the apparent errors for $N = 0$. For $N = 1$ apparent errors do not change very much, however, the central values are slightly modified. For $N \geqslant 2$ the convergence of the series with boundary condition is worse and in table 2 we report only central values for the exponents for which the convergence seems reasonable. Errors can approximately be inferred from the difference with the free values and the corresponding apparent errors.

6. Updated values for the $N = 1$ equation of state and critical exponents

The new values of the critical exponents obtained in this work directly affect the determination of the scaling equation of state for the $N = 1$, $d = 3$ case, by the method presented in [13]. We thus report here the new estimates. The results for the ε-expansion have also been revised (in particular the errors on amplitude ratios have been reconsidered).

We recall that our starting point was an estimation of the values of coefficients F_k of small magnetization expansion for the derivative of the effective potential V (free energy) with respect to the scaled renormalized field z (magnetization)

$$\frac{\partial V}{\partial z} = z + \frac{1}{6}z^3 + \sum_{k=2} F_{2k+1}(g)z^{2k+1}. \tag{6.1}$$

These coefficients F_k have been summed in [13] by using the available series up to five loops [26–28] and are reported in table 6, compared with results of other techniques (a

Table 6. Equation of state. (HT = high temperature, MC = Monte Carlo, ERG = exact renomalization group.)

	g^*	F_5	$F_7 \times 10^4$	$F_9 \times 10^5$
$\varepsilon-$ exp., this work	23.3	$0.017\,7 \pm 0.001\,0$	4.8 ± 0.6	-3.3 ± 0.3
$\varepsilon-$exp., [70, 38]	23.4 ± 0.1	$0.017\,15 \pm 0.000\,09$	4.9 ± 0.6	-5.5 ± 4
$d = 3$, this work	23.64 ± 0.07	$0.017\,11 \pm 0.000\,07$	4.9 ± 0.5	-7 ± 5
$d = 3$ [39]	23.71	0.017\,03	10	
HT [40]	23.72 ± 1.49	$0.020\,5 \pm 0.005\,2$		
HT [37]	24.45 ± 0.15	$0.017\,974 \pm 0.000\,15$		
HT [41]	23.69 ± 0.10	$0.016\,8 \pm 0.001\,2$	5.4 ± 0.7	-2.3 ± 1.1
MC [42]	23.3 ± 0.5	$0.022\,7 \pm 0.002\,6$		
MC [43]	24.5 ± 0.2	0.027 ± 0.002	23.6 ± 4	
ERG [44]	28.9	0.016	4.3	
ERG [47]	20.72 ± 0.01	$0.017\,19 \pm 0.000\,04$	4.9 ± 0.1	-5.2 ± 0.3

misprint in the last digit of the value of $\tilde{g}^*$ has been corrected). A uniform approximation for the equation of state has then be provided by the determination of the auxiliary function $h(\theta)$ defined by the reparametrization [31, 22, 23]:

$$z = \rho\theta/(1 - \theta^2)^\beta \tag{6.2}$$

$$h(\theta) = \rho^{-1}(1 - \theta^2)^{\beta\delta} F(z(\theta)). \tag{6.3}$$

The order-dependent mapping technique [69] has been used to improve convergence of the small θ-expansion by an optimal choice of the parameter ρ.

The new result coming from the revised values of γ, β is

$$h(\theta)/\theta = 1 - 0.762(3)\,\theta^2 + 0.0082(10)\,\theta^4 \tag{6.4}$$

that is obtained from $\rho^2 = 2.86$. This expression of $h(\theta)$ has a zero at

$$\theta_0^2 = 1.33 \tag{6.5}$$

to which corresponds the value of the complex root z_0 of $F(z)$, $|z_0| = 2.80$ (the phase, given by equation 6.2, is $-i\pi\beta$).

The revised ε-expansion estimations of F_k (in table 6) and of critical amplitudes presented in this paper are obtained using the revised γ, β of table 2 and the following expression of $h(\theta)$ (summed at $\varepsilon = 1$):

$$h(\theta)/\theta = 1 - 0.72(6)\theta^2 + 0.0136(20)\theta^4. \tag{6.6}$$

It should be emphatized that while critical amplitudes and the equation of state are universal quantities, $h(\theta)$ is not a universal function; in particular the variable θ of ε-expansion should not be identified with the corresponding variable of the $d = 3$ analysis, because they are defined from a different mapping in equation (6.2), (different ρ and β). It follows that $h(\theta)$ of the two methods (and their errors) cannot be compared directly.

Our value of g^* from ε-expansion (in table 6) has been obtained from our own analysis of O(ε^4) series of [70]. We also report in table 6 the recent results of [38] obtained by a direct summation of O(ε^3) series for F_k (imposed by imposing boundary conditions at smaller dimensions).

Table 7. Amplitude ratios. (HT = high temperature, MC = Monte Carlo.)

	A^+/A^-	C^+/C^-	R_c	R_χ
$\varepsilon - $ exp., [24, 23]	0.524 ± 0.010	4.9		1.67
$\varepsilon - $ exp., this work	0.527 ± 0.037	4.73 ± 0.16	0.0569 ± 0.0035	1.648 ± 0.036
$d = 3$, [25]	0.541 ± 0.014	4.77 ± 0.30	0.0594 ± 0.001	1.7
$d = 3$, this work	0.537 ± 0.019	4.79 ± 0.10	0.0574 ± 0.0020	1.669 ± 0.018
HT series [32, 33]	0.523 ± 0.009	4.95 ± 0.15	0.0581 ± 0.0010	1.75
MC [68]	0.560 ± 0.010	4.75 ± 0.03		
bin. mix.	0.56 ± 0.02	4.3 ± 0.3	0.050 ± 0.015	1.75 ± 0.30
liqu.-vap.	0.48-0.53	$4.8 - 5.2$	0.047 ± 0.010	1.69 ± 0.14
magn. syst.	$0.49 - 0.54$	4.9 ± 0.5		

Table 8. Other amplitude ratios.

	R_0	R_3	C_4^+/C_4^-
high-temperature series [37]	0.1275 ± 0.0003	6.4 ± 0.2	-9.0 ± 0.3
$d = 3$, this work	0.12584 ± 0.00013	6.08 ± 0.06	-9.1 ± 0.6
ε-expansion, this work	0.127 ± 0.002	6.07 ± 0.19	-8.6 ± 1.5

Widom's scaling function $f(x)$ (with $f(-1) = 0$ and $f(0) = 1$) can easily be derived by (numerically) solving the following system:

$$\begin{cases} f(x) = \theta^{-\delta} h(\theta)/h(1) \\ x = \left(-\dfrac{1 - \theta^2}{1 - \theta_0^2}\right) \left(\dfrac{\theta_0}{\theta}\right)^{1/\beta}. \end{cases} \tag{6.7}$$

From equation 6.7 and the revised values of the critical exponents we can calculate various critical amplitude ratios that are reported in tables 7 and 8 and are compared with other theoretical and experimental results (see [34] for a report on the subject). The reader can find all definitions and more details in [13].

7. Conclusions

Before discussing our results, let us review the results for critical exponents obtained by other theoretical methods or experiments.

The previous most accurate determinations of the critical exponents of the O(N) vector model, from quantum field theory and RG, have been reported in [1, 2] and are shown in table 9 (we refer to these results as LeGuillou–Zinn-Justin (LG–ZJ)). In table 10 we report the Murray–Nickel predictions (direct fit of g series) with the authors' preferred choice of $\tilde{g}^*$ (they reported only summation errors; errors from $\tilde{g}^*$ should be added). In table 11 we list some values for $N = 4$ obtained from Padé Borel summation of $d = 3$ series up to six loops (see [39] where results for many values of $N > 3$ are given). An analysis based on order-dependent mapping [67] of $d = 3$ series can be found in [69]. In table 12 we report the previous analysis of ε-expansion while in table 13 we quote some recent results. Other available theoretical predictions come from the analysis of high-temperature series in lattice models, table 14, and Monte Carlo simulations table 15. Finally, in table 16 we report for completeness some estimates from the truncated 'exact renormalization group' approach.

Table 9. Estimates of critical exponents in the O(N) symmetric $(\phi^2)^2_3$ field theory LG–ZJ.

N	0	1	2	3
$\tilde{g}^*$	1.421 ± 0.008	1.416 ± 0.005	1.406 ± 0.004	1.391 ± 0.004
γ	1.1615 ± 0.0020	1.2405 ± 0.0015	1.316 ± 0.0025	1.386 ± 0.0040
ν	0.5880 ± 0.0015	0.6300 ± 0.0015	0.6695 ± 0.0020	0.705 ± 0.0030
η	0.027 ± 0.004	0.032 ± 0.003	0.033 ± 0.004	0.033 ± 0.004
β	0.302 ± 0.0015	0.325 ± 0.0015	0.3455 ± 0.0020	0.3645 ± 0.0025
α	0.236 ± 0.0045	0.110 ± 0.0045	-0.007 ± 0.006	-0.115 ± 0.009
ω	0.80 ± 0.04	0.79 ± 0.03	0.78 ± 0.025	0.78 ± 0.02
θ	0.470 ± 0.025	0.498 ± 0.020	0.522 ± 0.0018	0.550 ± 0.0016

Table 10. Critical exponents: direct fit of $d = 3$ series (error from $\tilde{g}^*$ is not reported).

N	Ref.	$\tilde{g}^*$	γ	ν	η
0	[46]	1.39	1.1569 ± 0.0004	0.5872 ± 0.0004	0.0297 ± 0.0009
1	[46]	1.40	1.2378 ± 0.0006	0.6301 ± 0.0005	0.0355 ± 0.0009
2	[46]	1.40	1.3178 ± 0.0010	0.6715 ± 0.0007	0.0377 ± 0.0006
3	[46]	1.39	1.3926 ± 0.0013	0.7096 ± 0.0008	0.0374 ± 0.0004

Table 11. Critical exponents: results of Padé Borel summation for $N = 4$, [36].

N	Ref.	$\tilde{g}^*$	γ	ν	η
4	[36]	1.369	1.449	0.738	0.036

Table 12. Estimates of critical exponents: previous ε expansion results (LG–ZJ).

N	0	1	2	3
γ	1.157 ± 0.003	1.2390 ± 0.0025	1.315 ± 0.007	1.390 ± 0.010
ν	0.5880 ± 0.0015	0.6310 ± 0.0015	0.671 ± 0.005	0.710 ± 0.007
η	0.0320 ± 0.0025	0.0375 ± 0.0025	0.040 ± 0.003	0.040 ± 0.003
β	0.3035 ± 0.0020	0.3270 ± 0.0015	0.3485 ± 0.0035	0.368 ± 0.004
ω	0.82 ± 0.04	0.81 ± 0.04	0.80 ± 0.04	0.79 ± 0.04

In view of experimental determinations of critical exponents, a few significant results are displayed in table 17.

7.1. 3D series

In general the new estimates displayed in table 1 are more accurate than the previous LG–ZJ results. They are compatible within errors with the previous analysis. A closer inspection shows, however, some significant changes which require discussion.

The main effect comes from the new (and smaller) values of the fixed point coupling constant for $N < 3$. The changes are a direct consequence of the new techniques that we have introduced. In the old calculation LG–ZJ had noticed two puzzling features: the optimal values of the parameter b were somewhat large, compared with what large-order behaviour suggested. Moreover, the three shifts $0, 1, 2$ gave strongly oscillating results.

Table 13. Estimates of critical exponents from ε-expansion [70].

N	0	1	2	3
$\tilde{g}^*(\mathrm{O}(\varepsilon^4))$	1.390 ± 0.017	1.397 ± 0.018	1.413 ± 0.013	1.387 ± 0.007
γ	1.1559 ± 0.0010	1.240 ± 0.005	1.304 ± 0.007	1.372 ± 0.006
ν	0.5882 ± 0.0011	0.631 ± 0.003	0.664 ± 0.003	0.699 ± 0.004

Table 14. Critical exponents for Ising-like systems: high-temperature series.

N	Ref.	γ	ν	α	$\theta = \omega\nu$
0	[49]	1.1595 ± 0.0012	0.588 ± 0.001		
0	[50]	1.16193 ± 0.0001	0.588 ± 0.001		
1	[15]	1.239 ± 0.002	0.631 ± 0.003		
1	[16]	1.2385 ± 0.0025	0.6305 ± 0.0015		0.57 ± 0.07
1	[17]	1.239 ± 0.003	0.631 ± 0.004		
1	[18]	1.2395 ± 0.0004	0.632 ± 0.001	0.105 ± 0.007	0.54 ± 0.05
1	[19]	1.239 ± 0.003	0.632 ± 0.003	0.101 ± 0.004	
1	[20]	1.237 ± 0.002	0.630 ± 0.0015		0.52 ± 0.03
1	[55]			0.104 ± 0.004	
1	[49]	1.2385 ± 0.0005	0.6310 ± 0.0005		
1	[59]	1.237 ± 0.004		0.108 ± 0.005	
2	[49]	1.323 ± 0.003	0.674 ± 0.003		
2	[51]	1.323 ± 0.015	0.670 ± 0.007		
3	[49]	1.402 ± 0.003	0.714 ± 0.002		
3	[52]	1.40 ± 0.03	0.72 ± 0.01		
4	[49]	1.474 ± 0.004	0.750 ± 0.003		

Table 15. Critical exponents: Monte Carlo.

N	Ref.	γ	ν	β	η	$\theta = \omega\nu$
0	[53, 54]	1.1575 ± 0.0006	0.5877 ± 0.0006			
0	[71]		0.58758 ± 0.00007			$0.515^{+0.017}_{-0.007}$
1	[57]		0.631 ± 0.001	0.3269 ± 0.0006	0.038 ± 0.002	
1	[56]		0.6289 ± 0.0008			
1	[58]		0.625 ± 0.001		0.0025 ± 0.006	0.44
1	[62]		0.6294 ± 0.0009		0.0374 ± 0.0014	0.55 ± 0.06
2	[60]	1.324 ± 0.001	0.664 ± 0.006			
2	[61]	1.323 ± 0.002	0.670 ± 0.002			
2	[62]	1.316 ± 0.003	0.6721 ± 0.0013	0.042 ± 0.002	0.54 ± 0.08	
3	[63]	1.3896 ± 0.0070	0.7036 ± 0.0023	0.362 ± 0.004	0.0027 ± 0.002	
3	[62]	1.396 ± 0.003	0.7128 ± 0.0014	0.041 ± 0.002	0.51 ± 0.11	
4	[62]	1.476 ± 0.002	0.7525 ± 0.0010	0.038 ± 0.001		
4	[64]	1.477 ± 0.018	0.748 ± 0.009	0.3836 ± 0.0046		

Introduction of the new parameter r (see equation 3.7) has shown that the old apparent convergence corresponded to an unstable region of parameters. By varying r we find a region where these problems are solved to a large extent: the results are less sensitive, various shifts agree, and all parameters have more reasonable values. Figure 1 exemplifies this situation for the $N = 0$ case.

Another example exhibited a similar instability at $r = 0$: γ, $N = 0$, $d = 3$. This

Table 16. Critical exponents: 'exact renormalization group' estimates.

N	Ref.	$\tilde{g}^*$	γ	ν	η	$\theta = \omega\nu$
1	[44]	1.726	1.247	0.638	0.045	
2	[44]	1.675	1.371	0.700	0.042	
3	[44]	1.619	1.474	0.752	0.038	
4	[44]	1.566	1.556	0.791	0.034	
1	[47]			0.618 ± 0.014	0.054	0.56 ± 0.07
1	[65]			0.6262 ± 0.0013		
1	[72]			0.625 ± 0.007	0.030 ± 0.005	0.48 ± 0.04

Table 17. Critical exponents: selected recent experiments.

N	Ref.	γ	ν	β	α	$\theta = \omega\nu$
0	[48]		0.586 ± 0.004			
1	[73]				0.107 ± 0.006	
1	[74, 75, 80]	1.25 ± 0.01	0.64 ± 0.01		0.109 ± 0.006	0.57 ± 0.09
1	[76]	1.233 ± 0.010		0.327 ± 0.002		0.51 ± 0.03
2	[14, 77]		0.6708 ± 0.0004		$-0.012\,85 \pm 0.000\,38$	
2	[78]		0.6705 ± 0.0006			

(as well as the decrease induced from that of g^*) explains the new different value that we obtained here.

Finally, $N = 3$ values show a consistent effect: the three exponents γ, ν, β increase. This simply suggests that $N = 3$ errors had been underestimated in the LG–ZJ analysis.

Conversely the values of ν, $N = 0, 1, 2$ are quite stable, even though the corresponding values of g^* have changed. This is due to a very good apparent convergence of the pseudo-epsilon expansion (on which previous analysis partially relied) at the previous order.

The apparent errors have generally been reduced, as should be expected, except for $N = 3$ (see the comment above). The improvement is particularly significant for the exponent η that was poorly determined before. With a few exceptions, the general trend for a given exponent is the increase of the direct summation error with N. This effect has a simple explanation: our summation method relies on the large-order behaviour analysis, and the asymptotic regime sets in later when N increases [79]. Perhaps clever use of the knowledge coming from the large N expansion could improve the situation.

Note, finally, that the term added to two of the three RG series has allowed us not only to decrease apparent errors but also to estimate them more reliably.

For several exponents (such as, e.g. $\gamma(N = 1)$) errors are now dominated by errors induced by the determination of g^*. To further improve the situation it will be necessary to also add a new term to the RG β-function.

7.2. (Free) ε-expansion and 3D series

Comparison with the previous LG–ZJ ε estimates shows no striking effect, small deviations are due to the use of corrected series.

For the exponent ν the consistency between 3D and ε results remains very good at all N. The situation has markedly improved for the exponent γ, $N = 0$: there is still a systematic discrepancy in the central values for the exponents (about 0.002), but the difference is

reduced by more than a factor of 2, which is quite encouraging. A similar comment applies to the central values of the exponent η $N = 0, \ldots, 3$, where the discrepancy is also reduced by a factor of 2. In view of the $N = 4$ prediction, the agreement with the corresponding $d = 3$ results is quite satisfactory but apparent errors are large.

One point should, however, be stressed: since the series are shorter it is more difficult to assess apparent errors and the errors we quote are thus less reliable than for the $d = 3$ series.

7.3. Free and boundary condition ε-expansion

For the ε-expansion we report a second set of values, obtained by imposing the exact $d = 2$ values, referred to as bc (i.e. with boundary conditions) in table 2 to distinguish them from the unconstrainted values denoted by free. Some remarks are in order about the bc approach. First we do not know the analytic structure of exponents near $d = 2$. The only piece of evidence comes from using the ε-expansion for $d = 2$. For $N = 0$ the agreement with exact results is quite good [8], suggesting that $d = 2$ is a regular point. For $N = 1$ the agreement is less striking, leaving room for some complex behaviour. For values $N > 2$ there are indications (IR-renormalons of the $d = 2$ nonlinear σ-model) that $d = 2$ corresponds to an essential singularity (probably non-Borel summable). It is then easy to construct examples where fitting leads to worse results.

Second, in the case $N \geqslant 2$ the new series have a more complicated structure which makes summation (which is a form of extrapolating series to higher orders) and even more error estimations difficult.

Although the two analyses give compatible results, it happens that bc has the general tendency to give values of γ, ν for $N \geqslant 2$ larger than free ones and values of η for $N > 2$ smaller than free ones. This point is understood by remembering that at $d = 2$ the corresponding $\gamma, \nu, 1/\eta$ are infinite and thus it is reasonable that imposing this behaviour at $d = 2$ tends to increase the value of the exponent at $d = 3$.
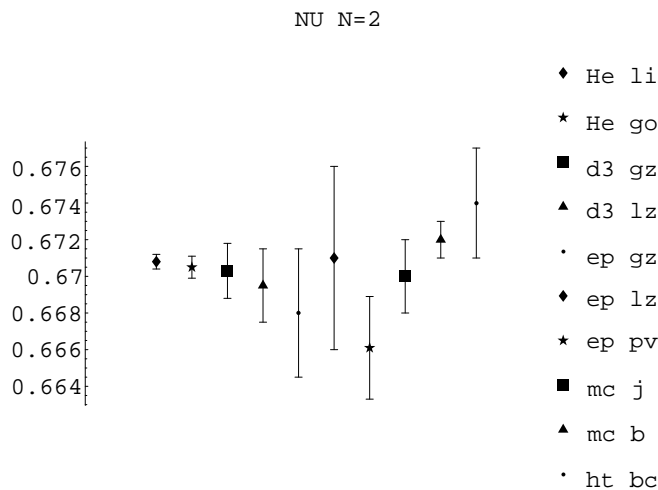
Figure 2. Comparison between various estimates of the exponent ν, $N = 2$: He li from [14, 77], He go from [78], d3 gz from the present work ($d = 3$), d3 lz from LG–ZJ ($d = 3$), ep gz from the present work (ε), ep lz from LG–ZJ (ε), ep pv from [70] (ε), mc j from [61], mc b from [62] and ht bc from [49].

Finally, it is also remarkable that for $N = 0$ the bc errors are smaller than the free ones.

7.4. Other methods

The general agreement between the high-temperature series, the Monte Carlo results, and the new $d = 3$ determinations is in general improved. This in particular applies to the self avoiding walks where recent long simulations provide very accurate estimates.

7.5. Experiments

The improved agreement of these results with the recent measures on superfluid helium systems, $N = 2$, is remarkable and is displayed graphically in figure 2. Despite our efforts, the best experimental value is still much more accurate than the theoretical estimate. In all other cases the agreement with experiments is good, as seen from table 17.

Acknowledgments

Appendix. Series $d = 3$

We report here the seven-loop series for the $O(N)$ symmetric $(\phi^2)^2$ theory, $N = 0, 1, 2, 3$, computed by Murray and Nickel [46]. The functions $\eta(\tilde{g})$, $\eta_2(\tilde{g})$ below are defined by

$$\eta(\tilde{g}) = m\frac{\mathrm{d}\log Z}{\mathrm{d}m} \qquad \eta_2(\tilde{g}) = m\frac{\mathrm{d}\log Z_2}{\mathrm{d}m}$$

where the renormalization constant are defined by $\phi_0 = \sqrt{Z}\phi_r$ and $(\phi^2)_r = \frac{Z_2}{Z}(\phi_0)^2$ (subscript $0, r$ indicate respectively bare and renormalized fields). The critical exponents η, ν can be found by the identification $\eta = \eta(\tilde{g}^*)$ and $\nu = (2 + \eta_2(\tilde{g}^*) - \eta(\tilde{g}^*))^{-1}$. The symmetry number N is reported in square brackets below.

$$\eta[0] = \frac{\tilde{g}^2}{108} + 0.000\,771\,3750\tilde{g}^3 + 0.001\,589\,8706\tilde{g}^4 - 0.000\,660\,6149\tilde{g}^5$$
$$+0.001\,410\,3421\tilde{g}^6 - 0.001\,901\,867\tilde{g}^7$$

$$\eta[1] = \frac{8\tilde{g}^2}{729} + 0.000\,914\,2223\tilde{g}^3 + 0.001\,796\,2229\tilde{g}^4 - 0.000\,653\,6980\tilde{g}^5$$
$$+0.001\,387\,8101\tilde{g}^6 - 0.001\,697\,694\tilde{g}^7$$

$$\eta[2] = \frac{8\tilde{g}^2}{675} + 0.000\,987\,3600\tilde{g}^3 + 0.001\,836\,8107\tilde{g}^4 - 0.000\,586\,3264\tilde{g}^5$$
$$+0.001\,251\,3930\tilde{g}^6 - 0.001\,395\,129\tilde{g}^7$$

$$\eta[3] = \frac{40\tilde{g}^2}{3267} + 0.001\,020\,0000\tilde{g}^3 + 0.001\,791\,9257\tilde{g}^4 - 0.000\,504\,0977\tilde{g}^5$$
$$+0.001\,088\,3237\tilde{g}^6 - 0.001\,111\,499\tilde{g}^7$$

$$\eta_2[0] = \frac{-\tilde{g}}{4} + \frac{\tilde{g}^2}{16} - 0.035\,767\,2729\tilde{g}^3 + 0.034\,374\,8465\tilde{g}^4 - 0.040\,895\,8349\tilde{g}^5$$

$$+0.059\,705\,0472\tilde{g}^6 - 0.099\,284\,87\tilde{g}^7$$

$$\eta_2[1] = \frac{-\tilde{g}}{3} + \frac{2\tilde{g}^2}{27} - 0.044\,310\,2531\tilde{g}^3 + 0.039\,519\,5688\tilde{g}^4 - 0.044\,400\,3474\tilde{g}^5$$
$$+0.060\,363\,4414\tilde{g}^6 - 0.093\,249\,48\tilde{g}^7$$

$$\eta_2[2] = \frac{-2\tilde{g}}{5} + \frac{2\tilde{g}^2}{25} - 0.049\,513\,4446\tilde{g}^3 + 0.040\,788\,1055\tilde{g}^4 - 0.043\,761\,9509\tilde{g}^5$$
$$+0.055\,557\,5703\tilde{g}^6 - 0.080\,413\,36\tilde{g}^7$$

$$\eta_2[3] = \frac{-5\tilde{g}}{11} + \frac{10\tilde{g}^2}{121} - 0.052\,551\,9564\tilde{g}^3 + 0.039\,964\,0005\tilde{g}^4 - 0.041\,321\,9917\tilde{g}^5$$
$$+0.049\,092\,9344\tilde{g}^6 - 0.067\,086\,30\tilde{g}^7.$$

References

[1] Le Guillou J C and Zinn-Justin J 1977 *Phys. Rev. Lett.* **39** 95
[2] Le Guillou J C and Zinn-Justin J 1980 *Phys. Rev.* B **21** 3976
[3] Zinn-Justin J 1989 *Quantum Field Theory and Critical Phenomena* in particular ch 28 of 3rd edn (Oxford: Clarendon) (third edn, 1996)
[4] Lévy M, Le Guillou J C and Zinn-Justin J (ed) 1980 Phase transitions *Proc. Cargèse Summer Institute* vol B72 (New York: Plenum)
[5] Brézin E, Le Guillou J C, Zinn-Justin J and Nickel B G 1973 *Phys. Lett.* **44A** 227
[6] Parisi G 1980 Cargèse lectures 1973 *J. Stat. Phys.* **23** 49
[7] Nickel B G, Meiron D I and Baker G B 1977 *Report* University of Guelph
[8] Le Guillou J C and Zinn-Justin J 1985 *J. Physique Lett.* **46** L137
    Le Guillou J C and Zinn-Justin J 1987 *J. Physique* **48** 19
    Le Guillou J C and Zinn-Justin J 1989 *J. Physique* **50** 1365
[9] Many articles on this topic are reprinted in 1990 Large order behaviour of perturbation theory *Current Physics* vol 7, ed J C Le Guillou and J Zinn-Justin (Amsterdam: North-Holland)
[10] Baker G A, Nickel B G, Green M S and Meiron D I 1976 *Phys. Rev. Lett.* **36** 1351
[11] Vladimirov A A, Kazakov D I and Tarasov O V 1979 *Zh. Eksp. Teor. Fiz.* **77** 1035 (Engl. transl. *Sov. Phys.–JETP* **50** 521)
    Chetyrkin K G, Kataev A L and Tkachov F V 1981 *Phys. Lett.* **99B** 147
    Chetyrkin K G, Kataev A L and Tkachov F V 1981 *Phys. Lett.* **101B** 457
    Chetyrkin K G and Tkachov F V 1981 *Nucl. Phys.* B **192** 159
    Chetyrkin K G, Gorishny S G, Larin S A and Tkachov F V 1983 *Phys. Lett.* **132B** 351
    Kazakov D I 1983 *Phys. Lett.* **133B** 406
    Gorishny S G, Larin S A and Tkachov F V 1984 *Phys. Lett.* **101A** 120
[12] Kleinert H, Neu J, Schulte-Frohlinde V, Chetyrkin K G and Larin S A 1991 *Phys. Lett.* B **272** 39
    Kleinert H, Neu J, Schulte-Frohlinde V, Chetyrkin K G and Larin S A 1993 *Phys. Lett.* B **319** 545 (erratum)
    See also Kleinert H and Schulte-Frohlinde V 1997 *Critical Properties of* Φ^4*-Theories* (Singapore: World Scientific)
[13] Guida R and Zinn-Justin J 1997 *Nucl. Phys.* B **489** 626
[14] Lipa J A, Swanson D R, Nissen J, Chui T C P and Israelson U E 1996 *Phys. Rev. Lett.* **76** 944
[15] Nickel B G 1981 *Physica* **106A** 48
[16] Zinn-Justin J 1981 *J. Physique* **42** 783
[17] Adler J 1983 *J. Phys. A: Math. Gen.* **16** 3585
[18] Chen J H, Fisher M E and Nickel B G 1982 *Phys. Rev. Lett.* **48** 630
    Fisher M E and Chen J H 1985 *J. Physique* **46** 1645
[19] Guttmann A J 1987 *J. Phys. A: Math. Gen.* **20** 1855
    Guttmann A J and Enting I G 1994 *J. Phys. A: Math. Gen.* **2** 8007
[20] Nickel B G and Rehr J J 1990 *J. Stat. Phys.* **61** 1
[21] Avdeeva G M and Migdal A A 1972 *JETP Lett.* **16** 178
    Brézin E, Wallace D J and Wilson K G 1972 *Phys. Rev. Lett.* **29** 591
    Brézin E, Wallace D J and Wilson K G 1973 *Phys. Rev.* B **7** 232
[22] Wallace D J and Zia R P K 1974 *J. Phys. C: Solid State Phys.* **7** 3480

[23] Nicoll J F and Albright P C 1985 *Phys. Rev.* B **31** 4576
[24] Bervillier C 1986 *Phys. Rev.* B **34** 8141
[25] Bagnuls C, Bervillier C, Meiron D I and Nickel B G 1987 *Phys. Rev.* B **35** 3585
[26] Halfkann F J and Dohm V 1992 *Z. Phys.* B **89** 79
[27] Rajantie A K 1996 *Nucl. Phys.* B **480** 729 (in this article the expansion is given in analytic form up to three loops)
[28] Schofield P 1969 *Phys. Rev. Lett.* **23** 109
     Schofield P, Litster J D and Ho J T 1969 *Phys. Rev. Lett.* **23** 1098
     Josephson B D 1969 *J. Phys. C: Solid State Phys.* **2** 1113
[29] Liu A J and Fisher M E 1989 *Physica* **156A** 35
[30] Aharony A and Hohenberg P C 1976 *Phys. Rev.* B **13** 3081
     Aharony A and Hohenberg P C 1977 *Physica* **86–88B** 611
[31] Privman V, Hohenberg P C and Aharony A 1991 Universal critical point amplitude relations *Phase Transitions and Critical Phenomena* vol 14, ed C Domb and J L Lebowitz (New York: Academic)
[32] Eckmann J P, Magnen J and Sénéor R 1975 *Commun. Math. Phys.* **39** 251
     Feldman J S and Osterwalder K 1976 *Ann. Phys., NY* **97** 80
     Magnen J and Sénéor R 1977 *Commun. Math. Phys.* **56** 237
     Eckmann J-P and Epstein H 1979 *Commun. Math. Phys.* **68** 245
[33] Summation of series by Borel transformation and mapping was proposed by Loeffel J J *Saclay Report* DPh-T/76/20 unpublished
[34] Fisher M E and Tarko H B 1975 *Phys. Rev.* B **11** 1131
[35] Gaunt D S and Domb C 1970 *J. Phys. C: Solid State Phys.* **3** 1442
[36] Antonenko S A and Sokolov A I 1995 *Phys. Rev.* E **51** 1894 (in this article the expressions for the general O(N) theory are reported)
     See also Sokolov A 1998 *Fiz. Tverd. Tela* **7** 40 for related work
[37] Zinn S, Lai S-N and Fisher M E 1996 *Phys. Rev.* E **54** 1176
[38] Pelissetto A and Vicari E 1998 *Nucl. Phys.* B to be published
[39] Sokolov A I, Ul'kov V A and Orlov E V (Renormalization Group 96) 1997 *3rd Int. Conf. (Dubna)* pp 378–83
     Sokolov A I, Ul'kov V A and Orlov E V 1997 *J. Phys. Stud.* **1** 362–5
     Sokolov A I 1996 *Fiz. Tverd. Tela (Solid State Phys.)* **38** 640
     Sokolov A I, Ul'kov V A and Orlov E V 1997 *Phys. Lett.* A **227** 255
[40] Reisz T 1995 *Phys. Lett.* B **360** 77
[41] Butera P and Comi 1997 *Phys. Rev.* E **55** 6391
[42] Tsypin M M 1994 *Phys. Rev. Lett.* **73** 2015
[43] Kim J-K and Landau D P 1997 *Nucl. Phys. Proc. Suppl.* **53** 706
[44] Tetradis N and Wetterich C 1994 *Nucl. Phys.* B **422** 541
[45] Berges J, Tetradis N and Wetterich C 1996 *Phys. Rev. Lett.* **77** 873
[46] Murray D B and Nickel B G unpublished
[47] Morris T 1997 *Nucl. Phys.* B **495** 4777
[48] Cotton J P 1980 *J. Physique Lett.* **41** L231
[49] Butera P and Comi M 1997 *Phys. Rev.* B **56** 8212
[50] MacDonald D, Hunter D L, Kelly K and Jan N 1992 *J. Phys. A: Math. Gen.* **25** 1429
[51] Ferer M, Moore M A and Wortis M 1973 *Phys. Rev.* B **8** 5205
[52] Ritchie D S and Fisher M E 1972 *Phys. Rev.* B **5** 2668
     McKenzie S, Domb C and Hunter D L 1982 *J. Phys. A: Math. Gen.* **15** 3899
     Ferer M and Hamid-Aidinejad A 1986 *Phys. Rev.* B **34** 6481
[53] Caracciolo S, Causo M S and Pelissetto A 1998 *Phys. Rev.* E **57** R1215
[54] Li B, Madras N and Sokal A D 1995 *J. Stat. Phys.* **80** 661
[55] Bhanot G, Creutz M, Glässner U and Schilling K 1994 *Phys. Rev.* B **49** 12 909
[56] Ferremberg A M and Landau D P 1991 *Phys. Rev.* B **44** 5081
[57] Blöte H W J, Heringa J R, Hoogland A, Meyer E W and Smit T S 1996 *Preprint* cond-mat 9602020
     Blöte H W J, Compagner A, Croockewit J H, Fonk Y T J C, Heringa J R, Hoogland A, Smit T S and van Villingen A L 1989 *Physica* **161A** 1
     Talapov A L and Blöte H W J 1996 *J. Phys. A: Math. Gen.* **29** 5727
[58] Baillie C F, Gupta R, Hawick K A and Pawley G S *Phys. Rev.* B **45** 10 438 and references therein
     Gupta R and Tamayo P 1996 Critical exponents of the 3D Ising model *Preprint* cond-mat/9601048, LA UR-96-93
[59] Kolesik M and Suzuki M 1994 *Preprint* cond-mat 9411109

[60] Gottlob A P, Hasenbusch M and Meyer S 1992 *Proc. Int. Symp. on Lattice Field Theory 'Lattice 93'* (*Amsterdam*)
 See also Hasenbusch M and Meyer S 1990 *Phys. Lett.* B **241** 238
 Gottlob A P and Hasenbusch M 1993 *Physica* **201A** 593
[61] Janke W 1990 *Phys. Lett.* A **148** 306
[62] Ballesteros H G, Fernandez L A, Martin-Mayor V and Sudupe A M 1996 *Phys. Lett.* B **387** 125
 Values for $N = 1$ from Fernandez L A Private communication
[63] Holm C and Janke W 1993 *Phys. Rev.* B **48** 936
 Holm C and Janke W 1993 *Phys. Lett.* A **173**
 Chen K, Ferrenberg A M and Landau D P 1993 *Phys. Rev.* B **48** 3249
 Holm C and Janke W 1993 *J. Appl. Phys.* **73** 5488
[64] Kanaya K and Kaya S 1995 *Phys. Rev.* D **51** 2404
[65] Wieczerkowski C and Rolf J 1996 *Preprint* hep-th/9607042, MS-TPI 9610, NBI-HE-96-34
[66] Parisi G 1979 *Cargèse Lectures 1977* vol B39 (New York: Plenum)
[67] Seznec R and Zinn-Justin J 1979 *J. Math. Phys.* **20** 1398
 Le Guillou J C and Zinn-Justin J 1983 *Ann. Phys., NY* **147** 57
 Guida R, Konishi K and Suzuki H 1995 *Ann. Phys., NY* **241** 152
 Guida R, Konishi K and Suzuki H 1996 *Ann. Phys., NY* **249** 109
 For related work see: Kleinert H and Janke W 1995 *Phys. Lett.* A **206** 283
[68] Hasenbusch M and Pinn K 1997 *J. Phys. A: Math. Gen.* submitted
 Caselle M and Hasenbusch M 1997 *J. Phys. A: Math. Gen.* **30** 4963
 Caselle M and Hasenbusch M 1998 *Nucl. Phys. Proc. Suppl.* **63** 613
[69] Kleinert H 1998 *Phys. Rev.* D **57** 2264
[70] Pellisetto A and Vicari E 1998 *Nucl. Phys.* B **519** 626
[71] Belohorec P and Nickel B G 1997 Accurate universal and two-parameter model results from a Monte-Carlo renormalization group study *Preprint* Guelph University
[72] Travesset A 1998 *Nucl. Phys. Proc. Suppl.* B **63**A–C 640
[73] Flewelling A C, Defonseka R J, Khaleeli N, Partee J and Jacobs D T 1996 *J. Chem. Phys.* **104** 8048
[74] Ramos C A, King A R and Jaccarino V 1989 *Phys. Rev.* B **10** 7124
[75] Belanger D P and Yoshizawa H 1987 *Phys. Rev.* B **35** 4823
[76] Pestak M W and Chan H W 1984 *Phys. Rev.* B **30** 274
[77] Swanson D R, Chui T C P and Lipa J A 1992 *Phys. Rev.* B **46** 9043
 Marek D, Lipa J A and Philips D 1988 *Phys. Rev.* B **38** 4465
[78] Goldner L S, Mulders N and Ahlers G 1992 *J. Low Temp. Phys.* **93** 131
[79] Brézin E and Parisi G 1978 *J. Stat. Phys.* **19** 269
[80] Henkel M, Andrieu S, Bauer P and Piecuch M 1998 *Phys. Rev. Lett.* **80** 4783