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Blockspin cluster algorithms for quantum spin systems

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Cluster algorithms are developed for simulating quantum spin systems like the one- and two-dimensional Heisenberg ferro- and anti-ferromagnets. The corresponding two- and three-dimensional classical spin models with four-spin couplings are mapped to blockspin models with two-blockspin interactions. Clusters of blockspins are updated collectively. The efficiency of the method is investigated in detail for one-dimensional spin chains. Then in most cases the new algorithms solve the problems of slowing down from which standard algorithms are suffering.

Two-dimensional quantum spin systems are relevant for the description of the undoped anti-ferromagnetic precursor insulators of high-T_c superconductors. Presumably understanding the physics of the superconductors requires an understanding of their precursor insulators. This already is a nontrivial problem, which most likely does not allow for a complete analytic solution. Therefore it is natural to use a numerical approach to compute the properties of these materials. At present different methods are used in numerical studies of quantum spin systems (for a recent review see for example ref. [1]). Small systems can be solved completely by a direct diagonalization of the Hamiltonian. For larger systems one can use Monte Carlo methods. For this purpose the finite temperature partition function of the d-dimensional quantum spin system is expressed as a path-integral of a $(d+1)$-dimensional classical system of Ising-like spin variables with four-spin couplings. The classical system is then simulated on a Euclidean time lattice with lattice spacing ϵ using importance sampling techniques like the Metropolis algorithm. This approach was pioneered by Suzuki and collaborators [2]. An application to one-dimensional spin chains is described in ref. [3]. For a simulation of the two-dimensional Heisenberg anti-ferromagnet see for example ref. [4]. The standard numerical methods are, however, slowed down in different ways. First of all most local changes of a configuration are forbidden, because many configurations have zero Boltzmann weight. Secondly, a change of the magnetization requires a nonlocal update, which has a very small acceptance rate at low temperatures. Finally, local algorithms are critically slowed down in the continuum limit $\epsilon \to 0$, because then any correlation length in Euclidean time diverges in units of ϵ. The slowing down of an algorithm is characterized by its dynamical critical exponent z. The exponent describes how the Monte Carlo autocorrelation time $\tau \propto 1/\epsilon^z$ (i.e. the time needed to create a new statistically independent spin configuration) behaves in the continuum limit.

For classical spin systems with two-spin couplings – like the Ising model – critical slowing down is almost entirely eliminated by the Swendsen–Wang [5] and Wolff [6] cluster algorithms for which $z \approx 0$. Cluster algorithms are, however, not directly applicable to models with four-spin couplings. In this paper we map the classical spin models with four-spin couplings to blockspin models with two-blockspin interactions, which are then simulated using the Swendsen–Wang or Wolff cluster algorithms. The blockspin cluster algorithm automatically creates allowed spin configurations only. Furthermore, it updates the magnetization efficiently even at low temperatures. The efficiency of the blockspin cluster algorithm is investigated in detail for the one-

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dimensional spin chains. We study the critical slowing down by investigating the ϵ-dependence of the autocorrelation times of different observables. For example, the dynamical exponent of critical slowing down of the algorithm for the one-dimensional Heisenberg anti-ferromagnet is $z=0.0(1)$. The use of improved estimators substantially reduces the variance of measured observables. We should mention that it is presently not clear how efficient the algorithm works for two-dimensional spin systems.

We consider quantum systems of spins $\frac{1}{2}$ with a Hamilton operator

$$H=J\sum_{x,\mu} \boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\hat{\mu}}\,, \tag{1}$$

where $\boldsymbol{S}_x=\frac{1}{2}\boldsymbol{\sigma}_x$ is a spin operator located at the point x of a one-dimensional or two-dimensional quadratic lattice of even length L with periodic boundary conditions and $\boldsymbol{\sigma}_x$ are the Pauli matrices. The coupling is between nearest neighbors ($\hat{\mu}$ is the unit vector in μ-direction) and J is the exchange coupling. For $J<0$ parallel spins are energetically favored and we have a ferromagnet, while $J>0$ corresponds to an anti-ferromagnet.

To express the partition function as a path-integral the Hamiltonian of the one-dimensional spin chain is decomposed into $H=H_1+H_2$ with

$$H_1=J\sum_{x=2m} \boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\hat{1}}\,,\qquad H_2=J\sum_{x=2m+1} \boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\hat{1}}\,. \tag{2}$$

Now one uses the Trotter formula for the partition function,

$$Z=\mathrm{Tr}\exp(-\beta H)=\lim_{N\to\infty}\mathrm{Tr}[\exp(-\epsilon\beta H_1)\exp(-\epsilon\beta H_2)]^N\,, \tag{3}$$

where $\epsilon=1/N$ is the lattice spacing in the Euclidean time direction and β is the inverse temperature. The original model is recovered in the continuum limit $\epsilon\to 0$. For the two-dimensional system one decomposes $H=H_1+H_2+H_3+H_4$ with

$$H_1=J\sum_{x=(2m,n)} \boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\hat{1}}\,,\qquad H_2=J\sum_{x=(2m+1,n)} \boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\hat{1}}\,,$$

$$H_3=J\sum_{x=(m,2n)} \boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\hat{2}}\,,\qquad H_4=J\sum_{x=(m,2n+1)} \boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\hat{2}}\,, \tag{4}$$

and one uses the Suzuki formula

$$Z=\lim_{N\to\infty}\mathrm{Tr}[\exp(-\epsilon\beta H_1)\exp(-\epsilon\beta H_2)\exp(-\epsilon\beta H_3)\exp(-\epsilon\beta H_4)]^N\,. \tag{5}$$

In the next step the partition function is expressed as a path-integral of Ising-like variables by inserting complete sets of eigenstates $|1\rangle$ and $|-1\rangle$ of σ_x^3 between the factors $\exp(-\epsilon\beta H_i)$. This gives rise to a $(d+1)$-dimensional classical spin system with periodic boundary conditions in the Euclidean time direction. The H_i are sums of commuting terms. The nonzero elements of the transfer matrix are given by

$$\langle 1\,1|\exp(-\epsilon\beta J\boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\hat{\mu}})|1\,1\rangle=\langle -1\,-1|\exp(-\epsilon\beta J\boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\hat{\mu}})|-1\,-1\rangle=\exp(-\tfrac{1}{4}\epsilon\beta J)\,,$$

$$\langle 1\,-1|\exp(-\epsilon\beta J\boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\hat{\mu}})|1\,-1\rangle=\langle -1\,1|\exp(-\epsilon\beta J\boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\hat{\mu}})|-1\,1\rangle$$

$$=\exp(-\tfrac{1}{4}\epsilon\beta J)\times\tfrac{1}{2}[1+\exp(\epsilon\beta J)]\,,$$

$$\langle 1\,-1|\exp(-\epsilon\beta J\boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\hat{\mu}})|-1\,1\rangle=\langle -1\,1|\exp(-\epsilon\beta J\boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\hat{\mu}})|1\,-1\rangle$$

$$=\exp(-\tfrac{1}{4}\epsilon\beta J)\times\tfrac{1}{2}[1-\exp(\epsilon\beta J)]\,. \tag{6}$$

All other elements are equal to zero. For a ferromagnet ($J<0$) the expressions are positive and can hence be interpreted as Boltzmann factors

$$\langle s_1 s_2|\exp(-\epsilon\beta J\boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\hat{\mu}})|s_3 s_4\rangle=\exp(-S[s_1,s_2,s_3,s_4])\,. \tag{7}$$

of spin configurations $s=\pm 1$ with a classical Euclidean action S. For an anti-ferromagnet, on the other hand, this is not possible because some transfer matrix elements are negative. Therefore in the anti-ferromagnetic case one performs a unitary transformation of the original Hamiltonian by rotating every second spin by an angle π. The nonzero elements of the transfer matrix are then positive for $J>0$. Expressed as a path-integral the partition function takes the form

$$Z=\prod_{x,t}\sum_{s(x,t)=\pm 1}\exp(-S)\,, \tag{8}$$

with

$$\exp(-S)=\prod_{x=2m,\ t=2p}\exp\{-S[s(x,t),s(x+\hat{1},t),s(x,t+1),s(x+\hat{1},t+1)]\}$$
$$\times\prod_{x=2m+1,\ t=2p+1}\exp\{-S[s(x,t),s(x+\hat{1},t),s(x,t+1),s(x+\hat{1},t+1)]\} \tag{9}$$

in the one-dimensional case and with

$$\exp(-S)=\prod_{x=(2m,n),\ t=4p}\exp\{-S[s(x,t),s(x+\hat{1},t),s(x,t+1),s(x+\hat{1},t+1)]\}$$
$$\times\prod_{x=(2m+1,n),\ t=4p+1}\exp\{-S[s(x,t),s(x+\hat{1},t),s(x,t+1),s(x+\hat{1},t+1)]\}$$
$$\times\prod_{x=(m,2n),\ t=4p+2}\exp\{-S[s(x,t),s(x+\hat{2},t),s(x,t+1),s(x+\hat{2},t+1)]\}$$
$$\times\prod_{x=(m,2n+1),\ t=4p+3}\exp\{-S[s(x,t),s(x+\hat{2},t),s(x,t+1),s(x+\hat{2},t+1)]\} \tag{10}$$

for the two-dimensional system. Note that the classical spins $s(x,t)$ interact with each other via four-spin couplings. Most spin configurations are forbidden (their Boltzmann factor vanishes) because the corresponding elements of the transfer matrix are zero. This is a problem for standard local algorithms because most local changes of a configuration are not allowed. Therefore it is natural to attempt a collective nonlocal update of the spins. For spin models with two-spin couplings this can be done using the Swendsen–Wang [5] or Wolff [6] cluster algorithms, which flip whole clusters of spins simultaneously. The cluster algorithms can, however, not be applied directly to models with four-spin interactions.

To make an application of cluster algorithms possible we map the classical spin models with four-spin couplings to blockspin models with two-blockspin couplings. A blockspin is a collection of a few spins. For the one-dimensional spin chain we define blockspins consisting of four spins located at the corners of a plaquette

$$b(2m,2p)=\{s(x,t),s(x+\hat{1},t),s(x,t+1),s(x+\hat{1},t+1)\}\,, \tag{11}$$

with $x=2m-1$ and $t=2p$. Note that each spin belongs to exactly one blockspin and the blockspins live on a lattice with a doubled lattice spacing. The original action determines the action for the blockspins. For example, the four-spin interactions for $x=2m$, $t=2p$ induce couplings between space-like nearest neighbor blockspins at $(2m, 2p)$ and $(2m+2, 2p)$,

$$S[b(2m,2p),b(2m+2,2p)]=S[s(x,t),s(x+\hat{1},t),s(x,t+1),s(x+\hat{1},t+1)]\,, \tag{12}$$

and the four-spin interactions for $x=2m+1$, $t=2p+1$ induce couplings between time-like nearest neighbor blockspins at $(2m, 2p)$ and $(2m, 2p+2)$,

$$S[b(2m,2p),b(2m,2p+2)]=S[s(x,t),s(x+\hat{1},t),s(x,t+1),s(x+\hat{1},t+1)]\,. \tag{13}$$

The spins can also be arranged to blockspins in another way,

$$\tilde{b}(2m,2p)=\{s(x,t),s(x+\hat{1},t),s(x,t+1),s(x+\hat{1},t+1)\}\,, \tag{14}$$

where now $x=2m$ and $t=2p-1$. Then the four-spin interactions for $x=2m$, $t=2p$ induce two-blockspin couplings between time-like nearest neighbors and the interactions at $x=2m+1$, $t=2p+1$ induce couplings between space-like nearest neighbors. The two blocking schemes of spins into blockspins are illustrated in figs. 1a, 1b. To ensure ergodicity an updating algorithm for the blockspins must alternate between the two blocking schemes.

For the two-dimensional system a blockspin consists of eight spins located at the corners of a cube,

$$b(2m, 2n, 2p)=\{s(x, t), s(x+\hat{1}, t), s(x+\hat{2}, t),$$
$$s(x+\hat{1}+\hat{2}, t), s(x, t+1), s(x+\hat{1}, t+1), s(x+\hat{2}, t+1), s(x+\hat{1}+\hat{2}, t+1)\}, \quad (15)$$

with $x=(2m-1, 2n-1)$ and $t=2p$. Again the blockspin lattice has a doubled lattice spacing. Another blocking scheme is given by

$$\tilde{b}(2m, 2n, 2p)=\{s(x, t), s(x+\hat{1}, t), s(x+\hat{2}, t), s(x+\hat{1}+\hat{2}, t),$$
$$s(x, t+1), s(x+\hat{1}, t+1), s(x+\hat{2}, t+1), s(x+\hat{1}+\hat{2}, t+1)\}, \quad (16)$$

where now $x=(2m, 2n)$ and $t=2p-1$. Using the action of eq. (10) it is straightforward to show that both schemes induce two-blockspin interactions only. Again, to ensure ergodicity the algorithm should alternate between the two schemes.

The cluster algorithms make use of a flip symmetry of the blockspin model. A blockspin $b=\{s_1, s_2, ..., s_n\}$ is flipped to $-b=\{-s_1, -s_2, ..., -s_n\}$ simply by flipping all spins that belong to b. It is clear that the blockspin action is invariant against flipping all blockspins simply because the original action is invariant against flipping all spins. The cluster algorithm puts bonds between nearest neighbor blockspins b and b' with the probability

$$p=1-\min\{1, \exp(-S[-b, b'])/\exp(-S[b, b'])\}. \quad (17)$$

Two blockspins which are connected by a bond belong to the same cluster. The algorithm flips all blockspins in one cluster simultaneously. In the Swendsen–Wang multi-cluster method each cluster is flipped with probability $\frac{1}{2}$. In the Wolff single-cluster method one blockspin is randomly selected and the cluster to which it belongs is flipped with probability 1. Recently, it has been realized that the single-cluster algorithm can be vectorized [7]. Using $S[b, -b']=S[-b, b']$ and $S[-b, -b']=S[b, b']$ one can show that both algorithms obey detailed balance. The blockspin cluster algorithm automatically generates allowed spin configurations only. If one attempts to flip b when $\exp(-S[-b, b'])=0$, i.e. when the new configuration is forbidden, the algorithm puts a bond with probability $p=1-\min\{1, 0\}=1$. Hence b and b' fall in the same cluster and must be flipped simultaneously. The cluster growth comes to an end only when the new configuration is allowed. When one

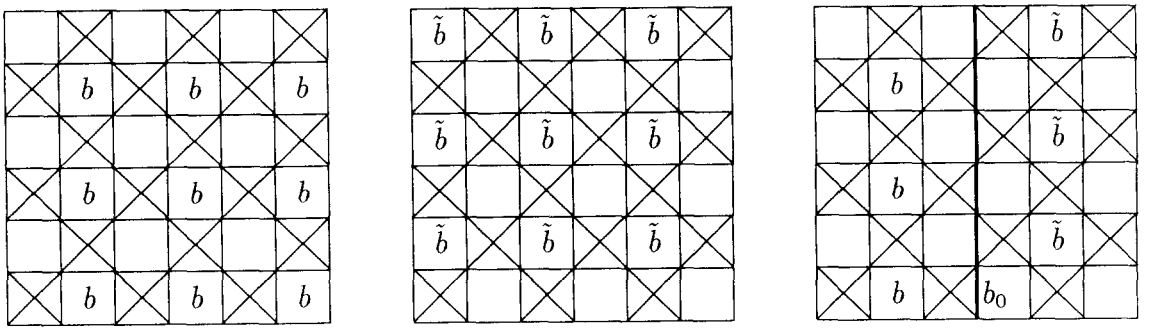

Fig. 1. Three different blocking schemes. The blockspins b and $\tilde{b}$ consist of four spins and live on lattices with a doubled lattice spacing. The crossed plaquettes carry the four-spin interaction which turns into a two-blockspin interaction. In the third scheme one blockspin b_0 consists of all spins with the same space coordinate x_0.

uses the single-cluster method it is guaranteed that one generates a new different configuration in each step.

As defined up to now the blockspin cluster algorithms are not ergodic because they change the magnetization $M=\frac{1}{2}\sum_x s(x, t_0)$ and the staggered magnetization $M_s=\frac{1}{2}\sum_x(-1)^x s(x, t_0)$ by even numbers only. M and M_s are integers (rather than half-integers) because we work on a lattice with an even number of space points. Note that M is independent of t_0 because the transfer matrix commutes with the total spin. To allow M and M_s to change by odd numbers also, we introduce extra blocking schemes with some block spins $b_0=\{s(x_0, t)$ for all $t\}$ consisting of all spins with the same space coordinates x_0. The other blockspins are taken from the above schemes. A typical example is shown in fig. 1c. Similarly, we include blocking schemes which allow the so-called winding number $N_w=\frac{1}{2}\sum_t s(x_0, t)$ to change by an odd integer.

The Swendsen–Wang and Wolff cluster algorithms eliminate critical slowing down from the Ising model because it is not frustrated. For strongly frustrated models, for example for spin glasses, cluster algorithms do not work efficiently. Therefore the question arises if our blockspin models are frustrated or not. To answer this question for the spin chains one must investigate all allowed configurations of four nearest neighbor blockspins located at the corners of a square of the blockspin lattice. Nearest neighbor pairs of blockspins interact via bonds (the four-spin interactions of the original classical spin model). Now the Boltzmann factor of a bond is compared to the value it has when one of the two blockspins at its ends is flipped. The blockspin configuration with the larger Boltzmann factor is the favored configuration of this bond. If one can flip the four blockspins such that all four bonds between them are in their favored configuration the configuration is not frustrated. We have verified that the blockspin model for the one-dimensional anti-ferromagnet has indeed no frustrated configurations. For the one-dimensional ferromagnet, on the other hand, some configurations are frustrated. In a frustrated configuration one of the four bonds cannot be in its favored configuration. However, one can show that the Boltzmann factor of the disfavored bond is in all cases only a factor $\frac{1}{2}[1+\exp(\epsilon\beta J)]$ smaller than the favored Boltzmann factor. In the continuum limit $\epsilon\to 0$ the factor goes to 1 and the frustration disappears. A priori it is not clear if a weak frustration can slow down the cluster algorithm in the ferromagnetic case. As we will see later, the blockspin cluster algorithm does in fact work perfectly only in the anti-ferromagnetic case. However, also for the ferromagnet the autocorrelation times of the cluster algorithm are much smaller than the ones of the Metropolis algorithm. For the two-dimensional spin systems both for the ferromagnet and for the anti-ferromagnet some blockspin configurations are frustrated. In addition, not all frustrations disappear in the continuum limit. Therefore, one should not expect that the blockspin cluster algorithm for two-dimensional quantum spin systems works as well as in the one-dimensional case. However, we expect that it still decorrelates faster than standard algorithms.

Cluster algorithms offer the possibility to use improved estimators which reduce the variance of different observables. For example, for the single-cluster method the susceptibility can be expressed as

$$\chi=\frac{\beta}{L^d}\langle M^2\rangle=2dN\beta\left\langle\frac{M_{\mathscr{C}}^2}{|\mathscr{C}|}\right\rangle, \tag{18}$$

where L^d is the spatial volume, $2dN$ is the number of points in the Euclidean time direction, $|\mathscr{C}|=\sum_{(x,t)\in\mathscr{C}}1$ is the cluster size and $M_{\mathscr{C}}=\frac{1}{2}\sum_{(x,t_0)\in\mathscr{C}} s(x, t_0)$ is the cluster magnetization. It is interesting to note that also $M_{\mathscr{C}}$ is independent of t_0. As a consequence, clusters with nonzero magnetization must wrap around the lattice in the Euclidean time direction. Small clusters which do not wrap around the lattice have $M_{\mathscr{C}}=0$. Let $L_{\mathscr{C}}^d$ be the minimal spatial extent of a cluster closed in Euclidean time, i.e. $L_{\mathscr{C}}^d=\min_{t_0}\{\sum_{(x,t_0)\in\mathscr{C}}1\}$. The cluster magnetization is then limited by $|M_{\mathscr{C}}|\leqslant\frac{1}{2}L_{\mathscr{C}}^d$ and the cluster size is restricted by $|\mathscr{C}|\geqslant 2dNL_{\mathscr{C}}^d$ such that

$$\langle L_{\mathscr{C}}^d\rangle\geqslant 4\chi/\beta. \tag{19}$$

The algorithm updates the susceptibility efficiently only if many clusters wrapping around the Euclidean time direction fit into the lattice. This requires $\langle L_{\mathscr{C}}^d\rangle\ll L^d$. Hence we expect that the cluster algorithm works efficiently only if

$$\chi \ll \tfrac{1}{4}\beta L^d . \tag{20}$$

The ferromagnetic systems have degenerate ground states with total spin $\frac{1}{2}L^d$. In the zero temperature limit the susceptibility is then given by $\chi = \frac{1}{12}\beta(L^d+2)$ and the algorithm does not work efficiently. In all other cases, for example at nonzero temperature or for anti-ferromagnets, χ stays finite as the spatial volume goes to infinity and the algorithm should work. One can also define an improved estimator for the staggered susceptibility $\chi_s = \beta\langle M_s^2 \rangle / L^d$. Since the staggered magnetization is t_0-dependent also small clusters contribute to χ_s. Finally, we introduce the internal energy density $e = -(1/L^d)\, \mathrm{d}\ln Z/\mathrm{d}\beta$.

Now let us turn to the numerical results. We have tested the blockspin cluster algorithm in detail for one-dimensional spin chains. To test the efficiency of the blockspin cluster algorithm we compare it to a Metropolis update of the blockspins. In fact, the blockspin Metropolis algorithm is similar to the standard algorithms commonly used. For both algorithms we measure the autocorrelations of the susceptibility

$$C_\chi(\delta\tau) = \overline{\chi(\tau)\chi(\tau+\delta\tau)} \tag{21}$$

as a function of the Monte Carlo time difference $\delta\tau$. From this we obtain the integrated autocorrelation time τ_χ as

$$\exp(-1/\tau_\chi) = \sum_{\delta\tau=1}^{\infty} C_\chi(\delta\tau) \Big/ \sum_{\delta\tau=0}^{\infty} C_\chi(\delta\tau) . \tag{22}$$

Note that $\tau_\chi = \tau_0$ for $C_\chi(\delta\tau) \propto \exp(-\delta\tau/\tau_0)$. In the same way we define the integrated autocorrelation times τ_{χ_s} of the staggered susceptibility and τ_e of the internal energy density. For the cluster algorithm we use the single-cluster method and we measure the autocorrelations based on the improved estimators. Our results are summarized in table 1. The quoted autocorrelation times are in units of Monte Carlo sweeps. A sweep is defined such that it takes about the same CPU time for the cluster and for the Metropolis algorithm. In all cases we have performed a random start followed by 5000 sweeps for thermalization and by 50000 sweeps for measurements.

The Metropolis algorithm can change the magnetization only by flipping a blockspin b_0 which consists of all spins with the same space coordinate x_0. The flip is allowed only if all spins of b_0 are parallel. Surprisingly, the probability for this situation – and hence the acceptance rate of the global step – is finite in the continuum limit. Therefore the autocorrelation times τ_χ and τ_{χ_s} do not diverge in the continuum limit. However, they are at least an order of magnitude larger than the ones of the cluster algorithm. The autocorrelation time of the internal energy density, on the other hand, diverges as $\tau_e \propto 1/\epsilon^{z_e}$. For the Metropolis algorithm our $\beta = 1$ data both for the ferro- and for the anti-ferromagnet yield a dynamical critical exponent $z_e = 0.8(1)$ while the cluster algorithm has $z_e = 0.0(1)$. Note that the exponent may depend on the observable considered, because it was extracted from the integrated autocorrelation time and not from the exponential decay of an autocorrelation function.

Although the global step of the Metropolis algorithm is not critically slowed down (its acceptance rate is ϵ-independent) it is very severely slowed down at low temperatures (large β). This prevents the application of standard numerical methods to quantum spin systems at low temperatures. Already at $\beta = 8$ the Metropolis algorithm has $\tau_\chi = 3300(200)$ for the ferromagnet and $\tau_\chi = 200(20)$ for the anti-ferromagnet. The cluster algorithm, on the other hand, has autocorrelation times of at most a few sweeps and one can go to lower temperatures like $\beta = 16$ very easily. For the ferromagnet the inequality $\chi \ll \frac{1}{4}\beta L^d$ puts a lower limit on the temperature when the spatial volume is fixed. Our low temperature data for the ferromagnet at $\beta = 16$ show some slowing down but the autocorrelation times are moderate. For the anti-ferromagnet, on the other hand, there is no indication of slowing down when the temperature is lowered or when the continuum limit is approached and $z_\chi = z_{\chi_s} = z_e = 0.0(1)$.

To summarize we have developed blockspin cluster algorithms for one- and two-dimensional quantum spin

Table 1
Results for the one-dimensional Heisenberg spin chains. The blockspin single-cluster algorithm (C) is compared to the Metropolis algorithm (M).

A	J	β	L	$2N$	χ	τ_χ	χ_s	τ_{χ_s}	e	τ_e
M	−1	1	32	32	0.352(8)	8.1(5)	0.176(2)	1.8(2)	−0.1347(6)	1.7(2)
C	−1	1	32	32	0.364(2)	0.52(3)	0.1796(6)	0.52(3)	−0.1328(5)	1.4(1)
M	−1	1	32	64	0.378(8)	9.0(5)	0.180(2)	2.4(2)	−0.1335(7)	3.0(2)
C	−1	1	32	64	0.371(2)	0.58(3)	0.1790(6)	0.48(3)	−0.1338(5)	1.5(1)
M	−1	1	32	128	0.348(8)	8.0(3)	0.179(2)	2.6(2)	−0.133(1)	5.6(3)
C	−1	1	32	128	0.369(1)	0.55(3)	0.1790(6)	0.49(2)	−0.1338(5)	1.5(1)
M	−1	1	32	256	0.377(8)	8.5(3)	0.177(2)	3.3(2)	−0.133(1)	10.7(4)
C	−1	1	32	256	0.369(1)	0.56(3)	0.1782(6)	0.54(3)	−0.1339(5)	1.5(1)
M	−1	2	128	16	0.80(7)	94(8)	0.332(1)	1.0(1)	−0.1878(2)	2.4(2)
C	−1	2	128	16	0.935(2)	0.56(2)	0.3304(9)	0.49(2)	−0.1881(2)	1.8(1)
M	−1	4	128	32	2.4(4)	180(15)	0.665(6)	4.9(3)	−0.2212(2)	4.9(3)
C	−1	4	128	32	2.52(1)	1.2(1)	0.664(1)	0.9(1)	−0.2211(1)	1.9(1)
M	−1	8	128	64	8(4)	3300(200)	1.29(1)	22(3)	−0.2379(2)	5.4(6)
C	−1	8	128	64	7.30(8)	3.9(3)	1.331(5)	2.9(2)	−0.23744(9)	1.6(1)
C	−1	16	128	16	19.5(3)	5.1(2)	2.58(1)	3.9(2)	−0.24680(2)	1.4(1)
C	−1	16	128	32	20.6(4)	9.5(8)	2.68(2)	9.9(9)	−0.24550(3)	1.2(1)
C	−1	16	128	64	22.6(6)	18(2)	2.67(2)	17(2)	−0.24482(5)	1.7(1)
C	−1	16	128	128	23.2(7)	17(2)	2.64(3)	19(2)	−0.24475(8)	1.9(2)
M	1	1	32	32	0.134(2)	4.0(2)	0.426(6)	4.3(2)	−0.207(1)	2.8(1)
C	1	1	32	32	0.1354(4)	0.34(3)	0.4275(9)	0.43(4)	−0.2049(8)	1.8(1)
M	1	1	32	64	0.137(2)	3.7(1)	0.429(6)	5.1(2)	−0.205(1)	4.9(2)
C	1	1	32	64	0.1362(4)	0.42(3)	0.426(1)	0.45(3)	−0.2029(8)	2.0(1)
M	1	1	32	128	0.140(2)	3.9(2)	0.422(7)	5.9(2)	−0.203(2)	7.3(3)
C	1	1	32	128	0.1374(4)	0.41(3)	0.4246(9)	0.41(3)	−0.2036(8)	1.9(1)
M	1	1	32	256	0.138(2)	3.4(3)	0.404(7)	5.5(3)	−0.204(2)	13(1)
C	1	1	32	256	0.1362(4)	0.39(2)	0.4246(9)	0.44(3)	−0.2038(8)	1.8(1)
M	1	2	128	16	0.148(5)	17(1)	1.24(1)	1.5(1)	−0.3443(4)	1.9(1)
C	1	2	128	16	0.1435(6)	0.28(2)	1.239(2)	0.41(2)	−0.3443(4)	2.2(1)
M	1	4	128	32	0.122(5)	23(1)	3.34(3)	2.5(1)	−0.4226(2)	2.0(1)
C	1	4	128	32	0.1251(5)	0.37(2)	3.399(7)	0.41(2)	−0.4217(2)	1.6(1)
M	1	8	128	64	0.14(2)	200(20)	8.5(1)	12(1)	−0.4410(2)	2.2(2)
C	1	8	128	64	0.1173(5)	0.42(2)	8.51(2)	0.49(3)	−0.4410(1)	1.3(1)
C	1	16	128	16	0.0714(6)	0.92(5)	47.5(1)	0.70(4)	−0.59088(6)	0.85(5)
C	1	16	128	32	0.0985(6)	0.59(2)	25.9(1)	0.61(2)	−0.48977(4)	0.99(5)
C	1	16	128	64	0.1087(7)	0.58(5)	21.59(5)	0.56(5)	−0.45460(6)	1.00(5)
C	1	16	128	128	0.1120(7)	0.65(5)	20.76(5)	0.61(4)	−0.44512(7)	1.04(9)

systems. Our results for spin chains show that the new algorithms eliminate slowing down for anti-ferromagnets. For ferromagnets slowing down is also practically eliminated as long as the temperature is not too small. One should, however, be careful in generalizing our findings to two-dimensional quantum spin systems. Since the corresponding blockspin models have some frustrated configurations, it is presently not clear how efficient the algorithm works in this case. A detailed analysis of the efficiency of the blockspin cluster algorithm for the two-dimensional Heisenberg anti-ferromagnet is in progress. We like to mention that blockspin cluster algorithms can also be applied to other models. First of all the algorithm also works for spin systems with anisotropic couplings or (with slight modifications) in an external magnetic field. Secondly, it can be applied to models of higher spins ($S=1, \frac{3}{2}, \ldots$) if it is combined with a method that changes the absolute value of the spin projection $|S_x^3|$. The one-dimensional Heisenberg spin chain is equivalent to a six-vertex model which is a special eight-vertex model. General eight-vertex models with a spin flip symmetry can also be treated with

blockspin cluster algorithms. Finally, the algorithm may be useful for simulations of one-dimensional Fermi-systems like the ones described in ref. [8].

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