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Ising-Model Spin Correlations on the Triangular Lattice. IV. Anisotropic Ferromagnetic and Antiferromagnetic Lattices*

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A detailed discussion of pair correlations $\omega_2(\mathbf{r}) = \langle \sigma_0 \sigma_\mathbf{r} \rangle$ between spins at lattice sites **0** and **r** on the *axes* of anisotropic triangular lattices is given. The asymptotic behavior of $\omega_2(\mathbf{r})$ for large spin separation is obtained for ferromagnetic and antiferromagnetic lattices. The axial pair correlation for the ferromagnetic triangular lattice has the same qualitative behavior as that for the ferromagnetic rectangular lattice: There is long-range order below the Curie point T_C and short-range order above. It is shown that correlations on the anisotropic antiferromagnetic triangular lattice must be given separate treatment in *three* different temperature ranges. Below the Néel point T_N (antiferromagnetic critical point), the completely anisotropic lattice exhibits antiferromagnetic long-range order along the two lattice axes with the strongest interactions. Spins along the third axis with the weakest interaction are ordered ferromagnetically. Between T_N and a uniquely located tempetature T_D, there is antiferromagnetic short-range order along the two axes with the strongest interactions, and ferromagnetic short-range order along the other axis. T_D is named the *disorder temperature* because it divides the short-range-order region $T_N < T < T_D$ from the region $T_D < T < \infty$, in which the axial pair correlations have exponential decay with temperature-dependent oscillatory envelope. There is no singularity in the partition function at T_D, so there are only two thermodynamic phases: ordered below the Néel point, and disordered above. Correlations at T_D decay exponentially. Finally, special consideration is given to the anisotropic antiferromagnetic lattice when the two weakest interactions are equal, and $T_N = T_D = 0$. The single disordered phase exhibits exponential correlation decay with oscillatory envelope for $T > 0$. The exact values of the axial pair correlations at $T = 0$ are calculated. For large spin separation r along the strong interaction axis, $\omega_2 = (-1)^r$, and along the weak (equal) interaction axes

$$\omega_2 \sim 2^{\frac{1}{2}} E^2 \cdot r^{-\frac{1}{2}} \cos(\tfrac{1}{2}\pi r)[1 - (8r^2)^{-1} + \cdots],$$

where

$$2^{\frac{1}{2}} E^2 = 0.588352663\cdots,$$

and E is a decay constant relating to pair correlations at the Curie point of a square lattice.

INTRODUCTION

Twenty-five years ago, Onsager[1] published his solution of the Ising problem[2-4] for the thermodynamic properties of the two-dimensional anisotropic rectangular lattice in zero magnetic field. Soon afterwards, Kaufman and Onsager[5] presented an account of the short-range correlations[6] and announced the formula for the long-range order parameter which is equal to the square of the spontaneous magnetization.[7] The first published derivation of the spontaneous magnetization is that of Yang,[8] with a generalization to an anisotropic rectangular lattice by Chang.[9] More recently, Wu[10] has given an account of the way in which the short-range pair correlation along a row of the rectangular lattice behaves for large spin separation at various fixed temperatures. Kadanoff[11] has also calculated the general pair correlation in terms of a parameter linking the spin separation and the deviation of the temperature from the critical point. The results obtained for the quadratic lattice exhibit the following general features[3]:

(i) There is one singular point, or critical point, called the Curie point for a ferromagnet, and the Néel point for an antiferromagnet, below which there is long-range order, and above which there is short-range order.

(ii) The features of anisotropic and isotropic lattices are similar.

(iii) The transformation to an antiferromagnetic is trivial, and no new features in nonmagnetic properties are introduced (apart from an oscillation in sign of some of the pair correlations). In particular, the Néel point (antiferromagnetic critical point) is numerically equal to the Curie point (ferromagnetic critical point).

* Research supported in part by the U.S. Air Force through Grant No. AF-AFOSR-1310-67.

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[7] L. Onsager, Nuovo Cimento Suppl. **6**, 261 (1949); B. Kaufman and L. Onsager, "Long Range Order" (unpublished).
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We may contrast these qualitative features of a quadratic lattice with those of the triangular lattice, for which solutions were obtained in 1950 by a number of authors.[12–14] In particular, we mention the solutions of Houtappel[12] for the general *anisotropic* triangular lattice, and that of Wannier[13] for the isotropic lattice. The results obtained for the ferromagnetic triangular lattice exhibit the same general features as the quadratic lattice. However, for an *antiferromagnetic* lattice the situation is as follows:

(i) There is one singular point which is at zero temperature for the isotropic lattice.

(ii) The features of anisotropic and isotropic lattices are quite different.

(iii) The transformation to an antiferromagnet is not trivial in effect, and the Néel point is numerically lower than the Curie point of the corresponding ferromagnetic lattice.[15] The thermodynamic properties of the antiferromagnetic isotropic triangular lattice were commented on by Wannier.[13] In particular, he emphasized the unphysical appearance of a finite zero-point entropy, and obtained its exact value. Since then the antiferromagnetic triangular lattice has largely been put on one side as a "curiosity."

It is the purpose of this paper to examine carefully the properties of pair correlations $\omega_2(\mathbf{r}) = \langle \sigma_0 \sigma_r \rangle$ on the *axes* of general anisotropic triangular lattices,[16,17] with particular emphasis on the antiferromagnetic case. In Sec. 1, triangular lattices are classified into two types A or B, according to whether they may be transformed to the completely ferromagnetic lattice or to the completely antiferromagnetic lattice, respectively. (The significance of this apparently trivial classification is revealed by the end of Sec. 4.) In Sec. 2, a mathematically necessary classification of the generating functions for the elements of the Toeplitz determinant representing the pair correlation is made [see Eq. (2.9)]. Mathematical arguments in subsequent sections depend on the results of Secs. 1 and 2. In Sec. 3, correlations for the general *ferromagnetic* lattice are discussed in some detail. Much of the work is a generalization of Wu's results for the rectangular lattice,[10] and is needed later. The two temperature ranges $0 < T < T_C$ and $T > T_C$ and the special point $T = T_C$ are considered, where T_C is the Curie point. In the opening paragraphs of Sec. 4, we show that correlations on an *anisotropic antiferromagnetic* lattice must be given separate treatment in *three* different temperature ranges and at *two* special points. Below the Néel point T_N, the completely anisotropic lattice exhibits antiferromagnetic long-range order along the two lattice axes with the two strongest interaction energies. Spins along the third axis with the weakest interaction energy are ordered ferromagnetically in energetically unfavored orientations. A moment's consideration will show that this is also the way in which an antiferromagnetic triangular lattice attains its ground state. Between T_N and a uniquely located temperature T_D (Eq. 4.2), the lattice exhibits antiferromagnetic short-range order along the two axes with the strongest interaction energies, and ferromagnetic short-range order along the other axis. T_D is introduced here as the *disorder temperature* because it divides the short-range-order region $T_N < T < T_D$ just mentioned, from the region $T_D < T < \infty$, in which the pair correlation has exponential decay with a temperature-dependent oscillatory envelope (4.18). Notice that the antiferromagnetic lattice exhibits only two thermodynamic phases: an ordered phase below the Néel point T_N, and a disordered phase above. There is no singularity in the partition function at T_D. Special consideration is given to correlations at T_N and T_D. In fact, the correlations can be evaluated exactly at T_D [Eqs. (4.16) and (4.33)], a result which stems from the special form of the Eq. (4.2) determining T_D and its graphical interpretation. Some special anisotropic antiferromagnetic lattices are mentioned briefly in Sec. 5. Finally, in Sec. 6 the *exact* values of pair correlations at $T = 0$ are calculated for the special anisotropic antiferromagnetic lattice when the two weakest interactions are equal. In this case $T_N = T_D = 0$, so the Toeplitz determinants representing the correlations simplify, and can be evaluated exactly [see Eqs. (6.19) and (6.20)]. Some concluding remarks are made about possible generalizations of this work.

1. CLASSIFICATION OF ANISOTROPIC LATTICES

In this paper, we consider pair correlations between two spins on the same axis of a general anisotropic triangular lattice, with interaction energies $-J_1$, $-J_2$, $-J_3$ between parallel neighboring spins along row (**1**), column (**2**), and diagonal (**3**) axes, respectively [Fig. 1(a)]. The triangular lattice may alternatively be thought of as a quadratic lattice with rows (**1**) and columns (**2**) plus a single second-neighbor interaction along the diagonal direction (**3**) [Fig. 1(b)]. For a completely *ferromagnetic* lattice the J_l, $l = 1, 2, 3$, are

[12] R. M. F. Houtappel, Physica **16**, 425 (1950).
[13] G. H. Wannier, Phys. Rev. **79**, 357 (1950).
[14] For more detailed references, see Ref. 3.
[15] *Corresponding* ferromagnetic and antiferromagnetic lattices have interactions of equal magnitude and opposite sign.
[16] J. Stephenson, J. Math. Phys. **7**, 1123 (1966).
[17] H. S. Green and C. A. Hurst, *Order Disorder Phenomena* (Interscience Publishers, Inc., New York, 1964).

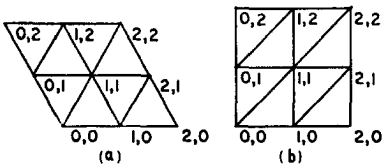

FIG. 1. (a) The triangular lattice and (b) the associated square lattice with a single second-nearest-neighbor bond.

all positive, and parallel spin states are energetically preferred. There is no loss of generality in supposing that $J_1 > J_2 > J_3 > 0$ in this case. For a completely *antiferromagnetic* lattice the J_l, $l = 1, 2, 3$, are all negative, and antiparallel spin states are energetically preferred. There is no loss of generality in supposing that $J_1 < J_2 < J_3 < 0$ in this case, so that the diagonal interaction J_3 is weakest. The pair correlation between two spins at sites **0** and **r** on the same lattice axis may be represented by a Toeplitz determinant[6,16] of order k, where k is a positive integer which is one greater than the number of lattice sites between **0** and **r** on the relevant lattice axis. Thus, for spins along row (**1**), $\mathbf{r} = (k, 0)$ and the separation of the sites **0** and **r** is $r \equiv |\mathbf{r}| = k$, measured in terms of lattice spacings. For spins along the diagonal (**3**), $\mathbf{r} = (k, k)$, and the separation r of the sites **0** and **r** depends on whether the lattice is triangular, as in Fig. 1(a), in which case $r = k$, or whether the lattice is square, as in Fig. 1(b), in which case $r = (2)^{\frac{1}{2}}k$. It is simplest to work in terms of k. The elements of the Toeplitz determinant are then

$$a_{p,q} = a_{q-p}, \quad p, q = 1, \cdots, k, \tag{1.1}$$

and depend only on the difference $(q - p)$ of row and column indices. Explicit formulas for the general element a_n, with $n = q - p$, have been derived elsewhere by the present author[16] and are alternatively available in the monograph by Hurst and Green.[17] If we use the notations

$$K_l = J_l/k_B T, \quad k_B \text{ Boltzmann's constant}, \tag{1.2}$$

$$C_l = \cosh 2K_l \quad \text{and} \quad S_l = \sinh 2K_l, \tag{1.3}$$

$l = 1, 2, 3$, then, for correlations along the (**3**) diagonal axis, we have

$$a_n = a_n(J_1, J_2, J_3)_3 = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega e^{-in\omega} \times \frac{C_1C_2S_3 + S_1S_2C_3 - C_3\cos\omega + i\sin\omega}{|C_1C_2S_3 + S_1S_2C_3 - C_3\cos\omega + i\sin\omega|}. \tag{1.4}$$

The final subscript 3 in $a_n(J_1, J_2, J_3)_3$ refers to the (**3**) axis. Corresponding expressions for the other axes (**1**) and (**2**) are obtained by cyclic permutation of indices. Thus a_n is the coefficient of $e^{-in\omega}$ in the expansion of the generating function

$$A(\omega) = \frac{C_1C_2S_3 + S_1S_2C_3 - C_3\cos\omega + i\sin\omega}{|C_1C_2S_3 + S_1S_2C_3 - C_3\cos\omega + i\sin\omega|}. \tag{1.5}$$

The elements of the Toeplitz determinant have certain symmetry properties which will be useful in classifying the types of lattices and in reducing the number of cases which need be considered. These symmetry properties are, along the (**3**) axis,

$$\begin{aligned} a_n(J_1, J_2, J_3)_3 &= a_n(-J_1, -J_2, J_3)_3 \\ &= (-1)^{n+1}a_n(-J_1, J_2, -J_3)_3 \\ &= (-1)^{n+1}a_n(J_1, -J_2, -J_3)_3. \end{aligned} \tag{1.6}$$

These formulas show that the effect of reversing the signs of any two of the J_l, $l = 1, 2, 3$, may be taken into account quite easily. For example, properties of correlations on a lattice with *two* of the J_l negative may be obtained by transformation using (1.6) from correlations on a completely ferromagnetic lattice with all J_l positive. Lattices which may be transformed to a completely *ferromagnetic* lattice in this way will be called *Class A*. On the other hand, properties of correlations on a lattice with *one* of the J_l negative may be derived using (1.6) from correlations on a completely antiferromagnetic lattice with all J_l negative. Lattices which may be transformed to a completely *antiferromagnetic* lattice in this way will be called *Class B*. It may easily be seen that these cases cannot be transformed into one another, and that they exhaust the possibilities at this level of classification.

2. CLASSIFICATION OF GENERATING FUNCTIONS

Next we classify the types of generating functions by their distinguishing analytic structure. To facilitate the classification we employ the dual and inversion transformations.[3,18] The dual transformation relates a triangular lattice with parameters K_l to a honeycomb-lattice with parameters K_l^* given by

$$e^{-2K_l^*} = \tanh K_l \equiv v_l, \quad l = 1, 2, 3, \tag{2.1}$$

or alternatively,

$$\sinh 2K_l \sinh 2K_l^* = 1 \quad \text{and} \quad \cosh 2K_l^* = \coth 2K_l, \quad l = 1, 2, 3. \tag{2.2}$$

[18] I. Syozi and S. Naya, Progr. Theoret. Phys. (Kyoto) **24**, 829, (1960).

The inversion transformation relates triangular lattices with parameters K_l and K_l^+, $l = 1, 2, 3$, where

$$e^{-4K_3^+} = \frac{(v_1 + v_2v_3)(v_2 + v_1v_3)}{(1 + v_1v_2v_3)(v_3 + v_1v_2)}, \quad \text{and cyclic.} \tag{2.3}$$

Lattices for which K_l and K_l^+ are real exhibit a phase change at a critical temperature given by

$$|K_l| = |K_l^+|. \tag{2.4}$$

Successive application of the inversion and dual transformations (the order is irrelevant) is equivalent to relating a triangular lattice with parameters K_l by a star triangle transformation to a honeycomb lattice with parameters K_l^{+*}. It is easy to confirm that

$$\cosh 2K_3^{+*} = \coth 2K_3^+ = (C_1C_2S_3 + S_1S_2C_3)/S_3, \tag{2.5}$$

and thence rearrange expression (1.4) for the Toeplitz determinant element to

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega e^{-in\omega} \times \frac{(\cosh 2K_3^{+*} \sinh 2K_3 - \cosh 2K_3 \cos\omega + i \sin\omega)}{|\cosh 2K_3^{+*} \sinh 2K_3 - \cosh 2K_3 \cos\omega + i \sin\omega|}. \tag{2.6}$$

Now express $\sinh 2K_3$ in terms of $v_3 = \tanh K_3$, and multiply numerator and denominator in (2.6) by $(1 - v_3^2)$, which is a real positive quantity, to obtain

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega e^{-in\omega}(-e^{-i\omega}) \times \frac{(1 - v_3e^{2K_3^{+*}}e^{i\omega})(1 - v_3e^{-2K_3^{+*}}e^{i\omega})}{|(1 - v_3e^{2K_3^{+*}}e^{i\omega})(1 - v_3e^{-2K_3^{+*}}e^{i\omega})|}. \tag{2.7}$$

The form of the generating function in (2.7) is convenient for a study of correlations. Sometimes it will be useful to introduce an alternative notation

$$A^{-1} = v_3e^{2K_3^{+*}} \quad \text{and} \quad B = v_3e^{-2K_3^{+*}}, \tag{2.8}$$

so that $B = (v_3)^2A$, with $\operatorname{Re} K_3^{+*} > 0$. Let us observe that $\cosh 2K_3^{+*}$ is real, and distinguish the three cases

$$\begin{aligned} &\text{(a)}\ 1 < \cosh 2K_3^{+*}, \\ &\text{(b)}\ \cosh 2K_3^{+*} < -1, \\ &\text{(c)}\ -1 \leq \cosh 2K_3^{+*} \leq 1. \end{aligned} \tag{2.9}$$

These inequalities will now be re-expressed in terms of v_l variables, since their significance can then be appreciated more easily. In the following inequalities the upper sign refers to the case when $v_3 > 0$, and the lower sign to the case $v_3 < 0$. We first note that

$$(1 + v_1v_2v_3)(v_3 + v_1v_2) \gtrless (v_1 + v_2v_3)(v_2 + v_1v_3). \tag{2.10}$$

Case (a): We may take K_3^{+*} to be real and positive. Then $1 < \cosh 2K_3^{+*}$ if and only if

$$(v_1 + v_2v_3)(v_2 + v_1v_3) \gtrless 0, \tag{2.11}$$

which, in combination with (2.10), yields

$$(1 + v_1v_2v_3)(v_3 + v_1v_2) \gtrless (v_1 + v_2v_3)(v_2 + v_1v_3) \gtrless 0. \tag{2.12}$$

From this inequality and Eq. (2.3) we deduce that $0 < e^{-4K_3^+} < 1$, so K_3^+ is real and positive.

Case (b): We may take $K_3^{+*} = \operatorname{Re}(K_3^{+*}) + \frac{1}{2}i\pi$, with $\operatorname{Re}(K_3^{+*}) > 0$, so

$$\exp(2K_3^{+*}) = (-1)\exp(2 \operatorname{Re} K_3^{+*}).$$

Then $\cosh 2K_3^{+*} < -1$ if and only if

$$(1 + v_1v_2v_3)(v_3 + v_1v_2) \lesssim 0, \tag{2.13}$$

which, in combination with (2.10), yields

$$(v_1 + v_2v_3)(v_2 + v_1v_3) \lesssim (1 + v_1v_2v_3)(v_3 + v_1v_2) \lesssim 0. \tag{2.14}$$

From this inequality and Eq. (2.3) we deduce that $1 < e^{-4K_3^+}$, so K_3^+ is real and negative.

Case (c): We may take $K_3^{+*} = i\frac{1}{2}\theta_3$, where θ_3 is real and positive. Then $-1 \leq \cosh 2K_3^{+*} = \cos\theta_3 \leq 1$ if and only if

$$(v_1 + v_2v_3)(v_2 + v_1v_3) \lesssim 0 \lesssim (1 + v_1v_2v_3)(v_3 + v_1v_2). \tag{2.15}$$

From this inequality and Eq. (2.3) we deduce that $e^{-4K_3^+}$ is negative, so K_3^+ is complex.

To use these inequalities, one determines which of them is satisfied for the lattice direction of interest and the temperature range under consideration, and one calculates K_3^+ from (2.3) and then $e^{2K_3^{+*}}$ from

$$e^{2K_3^{+*}} = \coth K_3^+. \tag{2.16}$$

It is important to note that the above inequalities are unaltered on changing the signs of both J_1 and J_2.

3 CLASS-*A* FERROMAGNETIC LATTICE WITH $J_1 > J_2 > J_3 > 0$

In this section we consider the completely ferromagnetic triangular lattice with $J_1 > J_2 > J_3 > 0$. To determine the analytic structure of the generating function, observe that $1 > v_1 > v_2 > v_3 > 0$, so that the only inequalities which may be satisfied are those

TABLE I.

T_C, Curie Point, J_1, J_2, and $J_3 > 0$	T_N, Néel Point, $J_1 < J_2 < J_3 < 0$	T_D, Disorder Point, $J_1 < J_2 < J_3 < 0$
$1 + v_1v_2v_3 = v_1 + v_2 + v_3 + v_1v_3 + v_2v_3 + v_1v_2$	$1 + v_1v_2v_3 = -v_1 - v_2 + v_3 - v_1v_3 - v_2v_3 + v_1v_2$	$v_3 + v_1v_2 = 0$
$C_1C_2S_3 + S_1S_2C_3 = C_3$ $C_1S_2C_3 + S_1C_2S_3 = C_2$ $S_1C_2C_3 + C_1S_2S_3 = C_1$	$C_1C_2S_3 + S_1S_2C_3 = C_3$ $C_1S_2C_3 + S_1C_2S_3 = -C_2$ $S_1C_2C_3 + C_1S_2S_3 = -C_1$	$C_1C_2S_3 + S_1S_2C_3 = -S_3$ $C_1S_2C_3 + S_1C_2S_3 = S_2$ $S_1C_2C_3 + C_1S_2S_3 = S_1$
$S_1S_2 + S_2S_3 + S_3S_1 = 1$ $z_1z_2 + z_2z_3 + z_3z_1 = 1$	$S_1S_2 - S_2S_3 - S_3S_1 = 1$ $z_1z_2 - z_2z_3 - z_3z_1 = 1$	$C_1C_2 - C_2C_3 - C_3C_1 = -1$ $z_1z_2 - z_1z_3 - z_3z_1 = -1$

of case (a) in (2.12) (upper sign). It follows that correlations along all three axes have the same qualitative features, so, without loss of generality, we consider only correlations in the diagonal (3) direction. Now K_3^+ and K_3^{+*} are real and positive. With notational abbreviations as in Eq. (2.8),

$$A^{-1} = v_3 e^{2K_3^{+*}} \quad \text{and} \quad B = v_3 e^{-2K_3^{+*}}, \tag{3.1}$$

we see that $0 \leq B < 1$, and that the *Curie point* T_C is determined by $A = 1$ or

$$v_3 e^{2K_3^{+*}} = 1. \tag{3.2}$$

In rearranged form

$$e^{-2K_3^+} \equiv \tanh K_3^{+*} = (1 - v_3)/(1 + v_3) \equiv e^{-2K_3}, \tag{3.3}$$

which is equivalent to $K_3^+ = K_3$. Now using the defining equation (2.3) for K_3^+, (3.3) can be written

$$(1 - v_3 - v_1v_2 + v_1v_2v_3)^2 = (v_1 + v_2 + v_2v_3 + v_3v_1)^2. \tag{3.4}$$

Taking the positive square root, we obtain the Curie-point equation[17]

$$1 + v_1v_2v_3 = v_1 + v_2 + v_3 + v_1v_2 + v_2v_3 + v_3v_1, \tag{3.5}$$

which is symmetrical in v_1, v_2, and v_3. Various algebraic rearrangements of this equation are listed in the first column of Table I. For example, in terms of variables

$$z_l = e^{-2K_l}, \quad l = 1, 2, 3, \tag{3.6}$$

the Curie point T_C is determined from[3]

$$z_1z_2 + z_2z_3 + z_3z_1 = 1. \tag{3.7}$$

Graphs displaying the variation of k_BT_C/J_2 with J_3/J_2 for various fixed values of the ratio J_1/J_2 are sketched in the right-hand half of Fig. 2. It is straightforward to verify the following properties of A and B over the temperature intervals indicated:

$$\begin{aligned} &T < T_C\text{: } 0 < B < A < 1, \\ &T = T_C\text{: } A = 1,\ B = (\tanh K_3)^2, \\ &T > T_C\text{: } 0 < B < A^{-1} < 1. \end{aligned} \tag{3.8}$$

The generating function for the elements of the Toeplitz determinant now has the same form as that considered by Wu,[10] with the notational change $\alpha_2 \to A$, $\alpha_1 \to B$. Below the Curie point the generating function is

$$\frac{(1 - Ae^{-i\omega})(1 - Be^{i\omega})}{|(1 - Ae^{-i\omega})(1 - Be^{i\omega})|}, \tag{3.9}$$

with $0 < B < A < 1$. This form of the generating function gives rise to ferromagnetic long-range order. The limiting value of the correlation is, by application of Szegö's theorem,

$$\begin{aligned} \mathfrak{J}^2 &= \lim_{k \to \infty} \omega_2(k, k) \\ &= (1 - A^2)^{\frac{1}{4}}(1 - B^2)^{\frac{1}{4}}(1 - AB)^{-\frac{1}{2}} \\ &= [1 - (\sinh 2K_3^+/\sinh 2K_3)^2]^{\frac{1}{4}} \\ &= (1 - \mathfrak{K}^2)^{\frac{1}{4}}, \end{aligned} \tag{3.10}$$

where

$$\mathfrak{K}^2 = \frac{[(1 - v_1^2)(1 - v_2^2)(1 - v_3^2)]^2}{16(1 + v_1v_2v_3)(v_1 + v_2v_3)(v_2 + v_3v_1)(v_3 + v_1v_2)}, \tag{3.11}$$

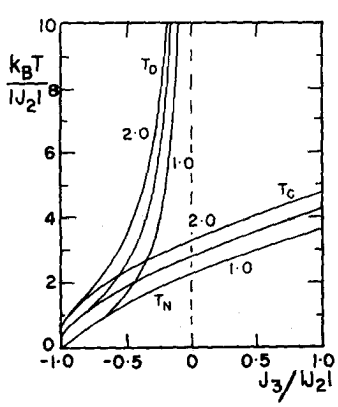

FIG. 2. Graphs of ferromagnetic Curie point T_C, and antiferromagnetic Néel point T_N, and disorder point T_D, in terms of $|J_2|/k_B$ vs $J_3/|J_2|$ for various fixed values of J_1/J_2. Graphs of T_C are in the right-hand half of the figure where $J_3 > 0$. Graphs of T_N are extensions of the T_C graphs into the left-hand half of the figure where $J_3 < 0$. The upper curves in the left-hand half of the figure are T_D curves. Ordering the graphs from the lowest, the values of J_1/J_2 for each graph are 1.0, 1.5, and 2.0.

and is symmetric in v_1, v_2, and v_3. $\mathfrak{J}$ is the spontaneous magnetization.[19,20] The asymptotic approach of the correlation to its limiting value has been studied by Wu,[10] and the results in his paper may be taken over with the appropriate notational changes:

$$\omega_2(k, k) - \mathfrak{J}^2 \sim \mathfrak{J}^2 \frac{A^{2k}}{2\pi k^2}\left[\frac{A}{1 - A^2}\right]^2\left[1 + O\left(\frac{1}{k}\right)\right]. \tag{3.12}$$

Above the critical point the generating function has the form

$$-e^{-i\omega}\frac{(1 - A^{-1}e^{i\omega})(1 - Be^{i\omega})}{|(1 - A^{-1}e^{i\omega})(1 - Be^{i\omega})|}, \tag{3.13}$$

with $0 < B < A^{-1} < 1$, which gives rise to ferromagnetic short-range order. From Wu's paper, the asymptotic form of the correlation is

$$\omega_2(k, k) \sim A^{-k}(\pi k)^{-\frac{1}{2}} \times (1 - A^{-2})^{-\frac{1}{4}}(1 - B^2)^{\frac{1}{4}}(1 - AB)^{-\frac{1}{2}}. \tag{3.14}$$

The decay of the correlation at the critical point[21] has been studied in detail by Wu[10] and by Hartwig and the present author.[22] For completeness, the corresponding result is

$$\omega_2(k, k) \sim \frac{E}{k^{\frac{1}{4}}}\left(\frac{1 + B}{1 - B}\right)^{\frac{1}{4}} \times \left[1 - \frac{1}{8k^2}\left(\frac{1}{8} - \frac{B}{(1 - B)^2}\right) + \cdots\right], \tag{3.15}$$

where

$$E = \exp\left(-\tfrac{1}{4}(1 + \gamma) - \sum_{s=2}^{\infty}\frac{\zeta(2s - 1)}{s4^s}\right) = 0.645002448\cdots, \tag{3.16}$$

and γ is Euler's constant. In particular, for the isotropic ferromagnetic triangular lattice at its Curie point, $A = 1$ and $B = 7 - 4(3)^{\frac{1}{2}}$, so

$$\omega_2(k, k) \sim E_0^T k^{-\frac{1}{4}}[1 + (192k^2)^{-1} + \cdots], \tag{3.17}$$

where

$$E_0^T = 3^{-\frac{1}{8}}2^{\frac{1}{4}}, E = 0.668618986\cdots. \tag{3.18}$$

Since the formulas above are valid for a general anisotropic ferromagnetic lattice, it is clear that, by specialization of the values of A and B, we may derive a variety of results. For example, if we set $J_3 = 0$ and

$$A = (\sinh 2K_1 \sinh 2K_2)^{-1}, \quad B = 0, \tag{3.19}$$

then the above formulas are valid for correlations along the diagonal direction of a quadratic lattice.

In summary, we note that Class A lattices exhibit a Curie point T_C which divides the temperature range $0 < T < \infty$ into two regions. Below T_C there is ferromagnetic long-range order and above T_C there is ferromagnetic short-range order.

4. CLASS-B ANTIFERROMAGNETIC LATTICE WITH $J_1 < J_2 < J_3 < 0$

In this section we consider the completely antiferromagnetic triangular lattice with $J_1 < J_2 < J_3 < 0$. To determine the structure of the generating function, observe that $-1 < v_1 < v_2 < v_3 < 0$, so that only lower inequalities in (2.12), (2.14), and (2.15) need be considered. It is especially important to observe the sign of $e^{-4K_3^+}$ in Eq. (2.3):

$$e^{-4K_3^+} = \frac{(v_1 + v_2v_3)(v_2 + v_1v_3)}{(1 + v_1v_2v_3)(v_3 + v_1v_2)}. \tag{4.1}$$

The choice of $J_1 < J_2 < J_3 < 0$ leads to the following conclusions. The factors $(v_1 + v_2v_3)$ and $(v_2 + v_1v_3)$ are negative at all temperatures. The sign of the factor $(v_3 + v_1v_2)$ depends on temperature, for the equation

$$v_3 + v_1v_2 = 0 \tag{4.2}$$

must be satisfied at some temperature T_D, above which $(v_3 + v_1v_2)$ is negative, and below which it is positive. We shall call T_D the *disorder point* (D ≡ disorder). When $T > T_D$, $e^{-4K_3^+}$ is negative and case (c) in Sec. 2 holds. When $T < T_D$, the lower inequality (2.14) of case (b) holds. Therefore, when $T < T_D$, K_3^+ is real and negative, and $K_3^{+*} = \mathrm{Re}\,(K_3^{+*}) + \frac{1}{2}i\pi$ with $\mathrm{Re}\,(K_3^{+*}) > 0$, so, with notational abbreviations analogous to those of Eq. (3.1),

$$A_3^{-1} = v_3 \exp(2K_3^{+*}) = (-1)v_3 \exp(2\,\mathrm{Re}\,K_3^{+*}) \tag{4.3}$$

and $B_3 = v_3^2A_3$, we have $0 < A_3 < \infty$ and $0 < B_3 < 1$. The *Néel point* T_N (antiferromagnetic critical point) is determined by $A_3 = 1$, or

$$v_3e^{2K_3^{+*}} = 1. \tag{4.4}$$

In rearranged form,

$$e^{-2K_3^+} \equiv \tanh K_3^{+*} = (1 - v_3)/(1 + v_3) \equiv e^{-2K_3}, \tag{4.5}$$

which is equivalent to $K_3^+ = K_3$. Now using the defining equation (2.3) for K_3^+, (4.5) can be written

$$(1 - v_3 - v_1v_2 + v_1v_2v_3)^2 = (v_1 + v_2 + v_2v_3 + v_3v_1)^2, \tag{4.6}$$

[19] H. S. Green, Z. Physik. **171**, 129 (1963).
[20] R. B. Potts, Phys. Rev. **88**, 352 (1952).
[21] M. E. Fisher, Physica **25**, 521 (1959).
[22] R. E. Hartwig and J. Stephenson, J. Math. Phys. **9**, 836 (1968).

or

$$[(1 - v_3)(1 - v_1v_2)]^2 = [(1 + v_3)(v_1 + v_2)]^2. \quad (4.7)$$

Taking the (numerically) positive square root, we obtain the Néel-point equation[17]

$$1 + v_1v_2v_3 = -v_1 - v_2 + v_3 + v_1v_2 - v_2v_3 - v_3v_1, \quad (4.8)$$

which is symmetrical in v_1 and v_2, and may be rearranged in the three forms

$$\begin{aligned} -(v_1 + v_2)(1 + v_3) &= (1 - v_3)(1 - v_1v_2), \\ (v_3 - v_2)(1 - v_1) &= (1 + v_1)(1 + v_2v_3), \\ (v_3 - v_1)(1 - v_2) &= (1 + v_2)(1 + v_1v_3), \end{aligned} \quad (4.9)$$

which are satisfied if and only if J_1, $J_2 < J_3 < 0$. Various other algebraic rearrangements of this equation are listed in the second column of Table I. For example, in terms of z_l variables in Eq. (3.6), the Néel point T_N is determined from

$$z_1z_2 - z_2z_3 - z_3z_1 = 1. \quad (4.10)$$

Equation (4.2) which determines T_D may also be expressed in terms of z_l variables:

$$z_1z_2 - z_2z_3 - z_3z_1 = -1. \quad (4.11)$$

There is a remarkable "similarity" between the equation determining the Curie point T_C of a ferromagnetic lattice (3.7), and Eqs. (4.10) and (4.11) above determining the Néel point T_N and disorder point T_D of an antiferromagnetic lattice. Equation (4.11) for T_D may be rearranged in the three ways

$$\begin{aligned} 1 + z_1z_2 &= z_3(z_1 + z_2), \\ 1 - z_1z_3 &= z_2(z_3 - z_1), \\ 1 - z_2z_3 &= z_1(z_3 - z_2), \end{aligned} \quad (4.12)$$

which have a solution if and only if J_3 is the weakest interaction and is negative. Various other algebraic rearrangements of Eq. (4.11) for T_D are listed in the third column of Table I. Graphs displaying the variation of $k_BT_N/|J_2|$ and $k_BT_D/|J_2|$ with $J_3/|J_2|$ for various fixed values of the ratio J_1/J_2 are sketched in the left-hand half of Fig. 2.

Thus, for the anisotropic antiferromagnetic triangular lattice, we must consider correlations separately in the three temperature regions $0 < T < T_N$, $T_N < T < T_D$, and $T_D < T < \infty$, and at two special points T_N and T_D. However, as we shall see later, the lattice is ordered below T_N and disordered above T_N, so there are only two thermodynamic phases. The equations determining T_N and T_D were obtained using inequalities derived in Sec. 2 for correlations along the diagonal (3) direction of the triangular lattice, but the same equations and the division of the whole temperature range $0 < T < \infty$ into three regions arise also from analysis of correlations along the other lattice axes. In fact, alternative definitions of T_N and T_D can be given independent of the correlations, though this point is not considered further here.

A. Pair Correlation Along Diagonal (3) Axis

The interaction J_3 in the diagonal direction is weakest, and the behavior of the pair correlation in this direction is quite different from that along the other two axes. There are three temperature ranges and two special points T_N and T_D to consider.

Case 1: $0 < T < T_N$. Using the variables A_3 and B_3 defined in Eq. (4.3), we observe, when $0 < T < T_N$, that $0 < B_3 < A_3 < 1$. The generating function (2.7) now has the same form as that in Eq. (3.9), which was derived in Sec. 3 for a ferromagnet below its Curie point. Although J_3 is negative, there is ferromagnetic long-range order below T_N, and the pair correlations along the (3) axis are all positive. The long-range order parameter

$$\mathfrak{R} = \lim_{k \to \infty} \omega_2(k, k) \quad (4.13)$$

is given by Eq. (3.10).

Case 2: $T = T_N$. Now $A_3 = 1$ and $0 < B_3 < 1$, so that Eqs. (3.15) and (3.16) determine the behavior of the correlation. The Néel point may be "low" for an antiferromagnetic lattice (Fig. 2), so the decay amplitude of the critical point correlation in the (3) direction may be "large."

Case 3: $T_N < T < T_D$. In this temperature range, $1 < A_3 < \infty$ and $0 < B_3 < 1$. From Eqs. (3.13) and (3.14), we deduce that there is ferromagnetic short-range order. If the two weakest interactions J_3 and J_2 are comparable in strength, then T_N and T_D are "close" together (Fig. 2), and the short-range order region is "small." Note that, below T_D, the pair correlations along the (3) axis are all positive and decrease monotonically with increasing spin separation. J_3 is the weakest interaction, and below T_D its effect is swamped by the stronger interactions J_1 and J_2. As $T \to T_D$ from below, $(v_3 + v_1v_2) \to 0$ from above, and $K_3^+ \to -\infty$. The inversion lattice, which is antiferromagnetic along its (3) direction, has reached absolute zero! Also $A_3^{-1} = v_3e^{2K_3^{+*}} \to -v_3$ and $B_3 \to -v_3$, so the

asymptotic formula (3.14) for the correlation between T_N and T_D breaks down.

Case 4: $T = T_D$. T_D has been called the *disorder point* because it separates two disordered regions, in contrast to the Néel point which separates regions of long- and short-range order. At T_D the generating function (2.7) now simplifies to

$$-e^{-i\omega}\left(\frac{1 + v_3 e^{i\omega}}{1 + v_3 e^{-i\omega}}\right), \tag{4.14}$$

whence the Toeplitz determinant elements a_n are

$$\begin{aligned} a_n &= 0, \quad n \geq 1, \\ a_0 &= -v_3, \\ a_n &= -(-v_3)^{n-1}(1 - v_3^2), \quad n \leq -1. \end{aligned} \tag{4.15}$$

The Toeplitz determinant is trivial to evaluate at T_D, and the exact value of the correlation is

$$\omega_2(k, k) = (-v_3)^k, \tag{4.16}$$

which represents exponential decay. It is interesting to note that the zero-field pair correlation between two spins separated by a distance k on a linear chain with interaction energy $-J$ is just v^k, where

$$v = \tanh (J/k_B T).^{23}$$

Case 5: $T > T_D$. The factor $(v_3 + v_1 v_2)$ is negative, $e^{-4K_3^+}$ is negative, and case (c) in Sec. 2 holds. Set $K_3^{+*} = i\frac{1}{2}\theta_3$, where θ_3 is real and positive. The generating function (2.7) is now

$$-e^{-i\omega}\frac{(1 - v_3 e^{i\theta_3} e^{i\omega})(1 - v_3 e^{-i\theta_3} e^{i\omega})}{|(1 - v_3 e^{i\theta_3} e^{i\omega})(1 - v_3 e^{-i\theta_3} e^{i\omega})|}, \tag{4.17}$$

and has the same form as that considered by the present author in the preceding paper of this series[24] in a discussion of pair correlations for the isotropic antiferromagnetic triangular lattice. The calculation there is general enough so that it can be taken over to the present case with the notational changes $v \to v_3$ and $\theta \to \theta_3$. The asymptotic form of the correlation is [quoting the leading term only from Eq. (1.31) of Ref. 24]

$$\omega_2(k, k) \sim (\tfrac{1}{2}\pi \sin \theta_3)^{-\frac{1}{2}} v_3^k k^{-\frac{1}{2}} \times \cos (k\theta_3 + \tfrac{1}{2}\theta_3 - \tfrac{1}{4}\pi - \phi_3), \tag{4.18}$$

where

$$\phi_3 = -\tfrac{1}{2} \arg (1 - v_3^2 \cos 2\theta_3 + i v_3^2 \sin 2\theta_3) \tag{4.19}$$

[23] J. S. Marsh, Phys. Rev. **145**, 251 (1966).
[24] J. Stephenson, J. Math. Phys. **11**, 413 (1969) (preceding article).

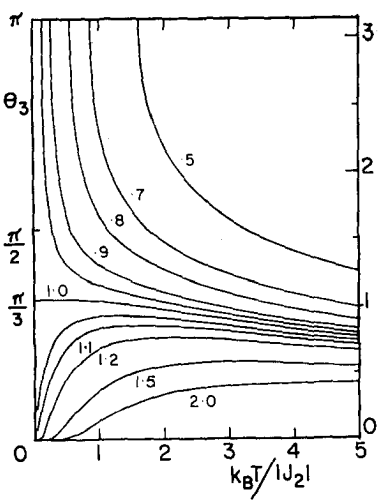

FIG. 3. Graphs of θ_3 vs $k_B T/|J_2|$ for various fixed values of J_3/J_2 in the special case $J_1 = J_2 < 0$. Ordering the graphs from the uppermost, the values of J_3/J_2 for each graph are 0.5, 0.7, 0.8, 0.9, 0.95, 1.0, 1.05, 1.1, 1.2, 1.5, 2.0.

and θ_3 is the real angle lying between 0 and π determined by

$$\cos \theta_3 = (C_1 C_2 S_3 + S_1 S_2 C_3)/S_3. \tag{4.20}$$

The exponential decay with oscillatory envelope of the pair correlation is characteristic of the triangular lattice above its T_D point. Figure 3 displays graphs of θ_3 vs a temperature variable $k_B T/|J_2|$ for various fixed values of J_3/J_2 in the special case when $J_1 = J_2$. Now $\cos \theta_3 = \cosh 2K_3^{+*} = \coth 2K_3^+$, so at T_D, $\theta_3 = \pi$. Then, as T increases from T_D to ∞, θ_3 decreases from π to 0, achieving the value $\frac{1}{2}\pi$ at a temperature $2T_D$. Notice that the nearest-neighbor pair correlation vanishes at $2T_D$, and is positive for lower temperatures.

B. Pair Correlation along Row (1) Axis

While discussing correlations in the (1) direction with the strongest interaction J_1, we shall refer to formulas in Sec. 2 and make the appropriate cyclic permutation of subscripts $3 \to 1$, $2 \to 3$, $1 \to 2$, in them. For the (1) direction

$$e^{-4K_1^+} = \frac{(v_2 + v_3 v_1)(v_3 + v_2 v_1)}{(1 + v_1 v_2 v_3)(v_1 + v_2 v_3)}. \tag{4.21}$$

When $T > T_D$, the factor $(v_3 + v_2 v_1)$ is negative, so $e^{-4K_1^+}$ is negative, and case (c) in Sec. 2 holds because the cyclic permutation of the lower inequality in (2.15) is satisfied:

$$(v_2 + v_3 v_1)(v_3 + v_2 v_1) > 0 > (1 + v_1 v_2 v_3)(v_1 + v_2 v_3). \tag{4.22}$$

When $T < T_D$, the cyclic permutation of the lower inequality in (2.12) is satisfied:

$$(1 + v_1 v_2 v_3)(v_1 + v_2 v_3) < (v_2 + v_3 v_1)(v_3 + v_2 v_1) < 0, \tag{4.23}$$

and we refer to case (a) in Sec. 2, where K_1^{+*} and K_1^+ are real and positive. With notational abbreviations analogous to those of Eq. (3.1),

$$A_1^{-1} = v_1 e^{2K_1{}^{+*}}$$

and

$$B_1 = v_1^2 A_1, \tag{4.24}$$

we have $-\infty < A_1 < 0$ and $-1 < B_1 < 0$. The Néel point T_N is determined by $A_1 = -1$, or

$$v_1 e^{2K_1{}^{+*}} = -1. \tag{4.25}$$

In rearranged form

$$e^{-2K_1{}^+} \equiv \tanh K_1^{+*} = (1 + v_1)/(1 - v_1) \equiv e^{+2K_1}, \tag{4.26}$$

which is equivalent to $K_1^+ = -K_1$. Now using the defining equation (4.21) for K_1^+, we can write (4.26) as

$$(1 + v_1 + v_2v_3 + v_1v_2v_3)^2 = (-v_2 + v_3 - v_3v_1 + v_1v_2)^2, \tag{4.27}$$

or

$$[(1 + v_1)(1 + v_2v_3)]^2 = [(1 - v_1)(v_3 - v_2)]^2. \tag{4.28}$$

Taking the (numerically) positive square root, we re-obtain the Néel-point equation (4.8). This checks the consistency of determining the Néel point from

$$|K_l^+| = |K_l|, \quad l = 1, 2, 3. \tag{4.29}$$

Now let us determine the asymptotic form of the pair correlation in the three temperature ranges $0 < T < T_N$, $T_N < T < T_D$, $T_D < T < \infty$, and at the two special points T_N and T_D. It turns out that correlations in the (**1**) direction, with the strongest interaction J_1, have the expected antiferromagnetic form below T_D.

Case 1: $0 < T < T_N$. Using the variables A_1 and B_1 defined in Eq. (4.24), we observe, when $0 < T < T_N$, that $-1 < A_1 < B_1 < 0$. The generating function in (2.7), with subscript 3 replaced by 1, now has the same form as that in Eq. (3.9), which was derived in Sec. 3 for a ferromagnet below its Curie point, except that A_1 and B_1 are now negative. There is antiferromagnetic long-range order below T_N, and the long-range order parameter

$$\mathfrak{R} = \lim_{k \to \infty} (-1)^k \omega_2(k, 0) \tag{4.30}$$

is given by Eq. (3.10) again, since

$$(-1)\frac{\sinh 2K_1^+}{\sinh 2K_1} = (-1)\frac{\sinh 2K_2^+}{\sinh 2K_2} = \frac{\sinh 2K_3^+}{\sinh 2K_3} = \mathcal{K}, \tag{4.31}$$

where $\mathcal{K}$ is the positive root of Eq. (3.11).

Case 2: $T = T_N$. Now $A_1 = -1$ and $-1 < B_1 < 0$ so that Eqs. (3.15) and (3.16) determine the behavior of the correlation. If the Néel point is "low," then the decay amplitude of the critical point correlation in the (**1**) direction will be "small."

Case 3: $T_N < T < T_D$. In this temperature range, $-\infty < A_1 < 1$ and $-1 < B_1 < 0$. From Eqs. (3.13) and (3.14), we deduce that there is antiferromagnetic short-range order. Note that, below T_D, the pair correlations along the (**1**) axis alternate in sign $(-)^k$ and decrease monotonically in magnitude with increasing spin separation. As $T \to T_D$ from below, $K_1^+ \to \infty$.

Case 4: $T = T_D$. At T_D the generating function for the (**1**) direction now simplifies to

$$-e^{-i\omega}\left(\frac{1 - v_1 e^{i\omega}}{1 - v_1 e^{-i\omega}}\right), \tag{4.32}$$

and the exact formula for the correlation is

$$\omega_2(k, 0) = v_1^k. \tag{4.33}$$

Case 5: $T > T_D$. Above T_D, the factor $(v_3 + v_1v_2)$ is negative, $e^{-4K_1{}^+}$ is negative, and the inequality (4.22) holds. Set $K_1^{+*} = i\frac{1}{2}\theta_1$, where θ_1 is real and positive. The generating function now has the same form as Eq. (4.17), with subscript 3 replaced by 1. Accordingly, the asymptotic form of the pair correlation is given by (4.18) and (4.19) with subscript 3 replaced by 1, and θ_1 is the real angle lying between 0 and π determined by

$$\cos\theta_1 = (C_2C_3S_1 + S_2S_3C_1)/S_1. \tag{4.34}$$

Figure 4 displays graphs of θ_1 vs the temperature variable $k_BT/|J_2|$ for various fixed values of J_3/J_2 in the special case when $J_1 = J_2$. Now $\cos\theta_1 = \cosh 2K_1^{+*} = \coth 2K_1^+$, so at T_D, $\theta_1 = 0$. Then, as T increases from T_D to ∞, θ_1 increases to some maximum value, which must be less than $\frac{1}{2}\pi$, since $e^{-4K_1{}^+}$ is finite, and subsequently decreases to zero again at $T = \infty$.

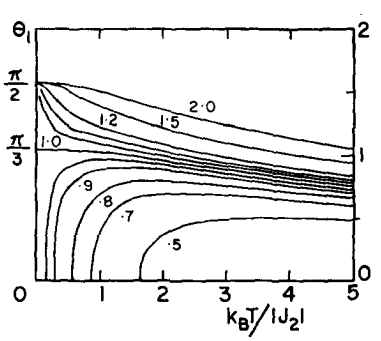

FIG. 4. Graphs of θ_1 vs $k_BT/|J_2|$ for various fixed values of J_3/J_2 in the special case $J_1 = J_2 < 0$. Ordering the graphs from the lowest, the values of J_3/J_2 for each graph are: 0.5, 0.7, 0.8, 0.9, 0.95, 1.0, 1.05, 1.1, 1.2, 1.5, 2.0.

Correlations in the (2) *direction* with intermediate strength interaction J_2 have the same qualitative features as those in the (1) direction. Formulas for this case may be obtained by an appropriate cyclic permutation of subscripts. At T_D, the correlations decay exponentially along the lattice axes, in a form suggesting that the spins along the lattice axes behave like "linear chains."[23]

5. SPECIAL CASES OF ANISOTROPIC ANTIFERROMAGNETIC LATTICES

In special limiting cases, the three-region lattice, with finite T_N and T_D points, may lose one or two of its regions. So far we have retained $J_1 < J_2 < J_3 < 0$. In the trivial special case $J_3 = 0$, we regain the anisotropic antiferromagnetic quadratic lattice ($T_D = \infty$) which has two phases. Now suppose we fix J_1 and J_2, and increase the strength of J_3. While $J_2 < J_3 < 0$, the lattice exhibits three regions. When $J_3 \to J_2 + <0$, $T_D \to 0$. The antiferromagnetic ordered and disordered regions below T_D are eliminated, and only the single triangular lattice type disordered region remains. Trivially $T_N = T_D = 0$, as is shown by Eqs. (4.9) and (4.12). Once $J_3 < J_2 < 0$, the J_2 interaction becomes weakest and the (2) and (3) directions interchange roles. The three region lattice system returns, and T_N and T_D rise again from $T = 0$. Increasing $|J_3|$ further, past $|J_1|$ for example, does not alter the situation in a qualitative manner. In the more special case when $J_1 = J_2 < 0$, a three-region system is obtained when $J_1 < J_3 < 0$. The lattice obtained when $J_3 < J_1 = J_2 < 0$ has one phase of triangular lattice type disorder with $T_N = T_D = 0$, and is considered in more detail in the next section. For details in the case of the completely isotropic lattice, see Ref. 24.

Another case of some interest is when $J_1 > J_2 > 0$ and $J_3 < 0$, which we may regard as that of a ferromagnetic quadratic lattice with a single second-neighbor antiferromagnetic interaction J_3. The formulas of Sec. 4 show that the (3) direction correlation is unaltered by changing the signs of J_1 and J_2. The modification to the discussion of correlations in the (1) and (2) directions consists in deleting the "anti" from antiferromagnetic, which is equivalent to using the transformation properties expressed in Eq. (1.6).

6. CLASS-*B* ANTIFERROMAGNETIC LATTICE WITH $J_3 < J_1 = J_2 < 0$

The distinguishing feature of this special case is that the two weakest interactions J_1 and J_2 are equal. From Eqs. (4.9) and (4.12), the Néel point T_N and disorder point T_D are at zero temperature, and the lattice exhibits one phase of triangular-lattice-type disorder. The asymptotic behavior of the pair correlations along a lattice axis l, $l = 1, 2, 3$, is determined over the whole temperature range $0 < T < \infty$ by

$$\omega_2(\mathbf{r}) \sim (\tfrac{1}{2}\pi \sin\theta_l)^{-\frac{1}{2}} v_l^k k^{-\frac{1}{2}} \cos(k\theta_l + \tfrac{1}{2}\theta_l - \tfrac{1}{4}\pi - \phi_l), \tag{6.1}$$

obtained by generalizing Eq. (4.18) to an arbitrary lattice axis l, where

$$\phi_l = -\tfrac{1}{2} \arg(1 - v_l^2 \cos 2\theta_l + i v_l^2 \sin 2\theta_l), \tag{6.2}$$

and θ_l are real angles for $T > T_D$ defined by

$$\cos\theta_3 = (C_1C_2S_3 + S_1S_2C_3)/S_3, \text{ and cyclic.} \tag{6.3}$$

In the special case under consideration here, $J_3 < J_1 = J_2 < 0$, so

$$\cos\theta_3 = 1 + S_1^2 e^{2K_3}/S_3 \tag{6.4}$$

and

$$\cos\theta_1 = \cos\theta_2 = C_1 e^{2K_3}. \tag{6.5}$$

The formula (6.1) was derived in Sec. 4 (see Ref. 24) under the assumption that $0 < |v_l| < 1$ for $T > 0$, and is invalid at $T = 0$. The remainder of this section is devoted to obtaining the exact values of the pair correlations along the lattice axes at $T = 0$. The (1) and (2) axes are equivalent since $J_1 = J_2$, but the (3) axis must be treated separately.

First we need the properties of θ_3 and $\theta_1 = \theta_2$, which are displayed in Figs. 3 and 4, where graphs of θ_3 and θ_1 as functions of a temperature variable $t = k_B T/|J_2|$ are sketched for various fixed values of J_3/J_2 with $J_1 = J_2$. For the isotropic lattice, set $\theta_1 = \theta_2 = \theta_3 = \theta$. θ has the zero-point value $\frac{1}{3}\pi$, and the curve of θ vs t for the isotropic lattice divides the curves of θ_3 and θ_1 vs t into two classes accordingly as $J_3 \gtrless J_2 = J_1$. The zero-point values of θ_3 and $\theta_1 = \theta_2$ when $J_3 < J_2 = J_1 < 0$ are

$$\lim_{T\to 0} \theta_3 = 0 \tag{6.6}$$

and

$$\lim_{T\to 0} \theta_1 = \tfrac{1}{2}\pi. \tag{6.7}$$

It is important to note that limiting processes $T \to 0$ and $J_3 \to J_2 = J_1$ are *not* interchangeable, and from the results of this section we cannot deduce any corresponding results for the isotropic lattice.[24]

A. (3) Direction at $T = 0$

Let $\theta_3 = 0$ and $v_3 = -1$ in Eq. (4.17) for the elements of the Toeplitz determinant representing the pair correlation. Then

$$a_0 = -1, \quad a_n = 0, \quad n \neq 0, \tag{6.8}$$

so the exact zero-point value of the correlation along the (3) axis is

$$\omega_2(k, k) = (-1)^k, \tag{6.9}$$

corresponding to a rigid arrangement of spins in regular antiferromagnetic order, alternately up and down [compare Eq. (4.16) for the correlation at T_D, and note that the present result (6.9) is *not* obtained by setting $T_D = 0$ in (4.16)]. This arrangement is to be expected at $T = 0$ because, when $J_3 < J_2 = J_1 < 0$, J_3 is the strongest interaction, and the ground state of a triangle of spins has energy $-|J_3|$. That is, all rows of spins in the (3) direction achieve antiferromagnetic order like a set of linear chains. For a lattice of N spins, there will be $O(N^{\frac{1}{2}})$ chains of spins in the (3) direction, which may be arranged in $2^{O(N^{\frac{1}{2}})}$ ways compatible with the ground state, so there is no zero-point entropy per site for an infinite lattice.

B. (1) Direction at $T = 0$

Spins along the (1) and (2) axes have similar properties since $J_1 = J_2$. Set $\theta_1 = \frac{1}{2}\pi$ and $v_1 = -1$ in the generating function for the (1) direction obtained by making the appropriate modification of Eq. (4.17); replace the subscript 3 by 1. Then

$$\begin{aligned} a_n &= (-1)^{\frac{1}{2}(n-1)}[-2/(\pi n)], \quad n \text{ odd}, \\ &= 0, \qquad\qquad\qquad\quad n \text{ even}. \end{aligned} \tag{6.10}$$

Since $a_{p,q} = a_{q-p} = 0$ when p and q are both odd or both even,

$$\omega_2(k, 0) = \det (a_{p,q})_{k\times k} = 0 \quad \text{for odd } k. \tag{6.11}$$

In particular, the nearest-neighbor correlation is zero in the (1) direction. For even spin separation, set $k = 2m$. The $k \times k$ Toeplitz determinant can be rearranged by m^2 row and column transpositions to the form

$$(-1)^m \det \begin{pmatrix} \mathsf{B}_m^T & 0 \\ 0 & \mathsf{B}_m \end{pmatrix} = (-1)^m (\det \mathsf{B}_m)^2, \tag{6.12}$$

where B_m is an $m \times m$ matrix whose (i, j) element b_{ij} is equal to $a_{(2i,2j-1)}$. Explicitly,

$$b_{i,j} = -\frac{2}{\pi}\frac{(-1)^{j-i-1}}{[2(j-i)-1]}, \quad i, j = 1, \cdots, m. \tag{6.13}$$

Now multiply the even numbered rows and columns of B_m by -1, and extract a factor $-2/\pi$ from each element to obtain a new matrix C_m whose determinant $\det \mathsf{C}_m = (-\frac{1}{2}\pi)^m \det \mathsf{B}_m$, and whose elements are

$$c_{i,j} = [2(i-j)+1]^{-1}, \quad i, j = 1, \cdots, m. \tag{6.14}$$

In detail,

$$\mathsf{C}_m = \begin{pmatrix} 1 & -1 & -\frac{1}{3} & -\frac{1}{5} & \cdots \\ \frac{1}{3} & 1 & -1 & -\frac{1}{3} & \cdots \\ \frac{1}{5} & \frac{1}{3} & 1 & -1 & \cdots \\ \frac{1}{7} & \frac{1}{5} & \frac{1}{3} & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & & \end{pmatrix}_{m\times m}. \tag{6.15}$$

But C_m is precisely the Cauchy matrix which appears in the calculation of the pair correlation in the diagonal direction of a ferromagnetic square lattice at its Curie point.[5,10,21,22] Its determinant may be calculated exactly as

$$\det \mathsf{C}_m = \prod_{s=1}^{m} \frac{\Gamma(s)\Gamma(s)}{\Gamma(s-\frac{1}{2})\Gamma(s+\frac{1}{2})}, \tag{6.16}$$

and asymptotically for large m as

$$\det \mathsf{C}_m \sim \left(\frac{\pi}{2}\right)^m \frac{E}{m^{\frac{1}{4}}}\left(1 - \frac{1}{64m^2} + \cdots\right), \tag{6.17}$$

where $E = 0.645002448$ is given by Eq. (3.16). For even spin separation $k = 2m$, we therefore have

$$\begin{aligned} \omega_2(2m, 0) &= (-1)^m\left[\left(\frac{2}{\pi}\right)^m \det \mathsf{C}_m\right]^2 \\ &\sim (-1)^m \frac{E^2}{m^{\frac{1}{2}}}\left[1 - \frac{1}{32m^2} + \cdots\right]. \end{aligned} \tag{6.18}$$

Finally, combining the results for odd and even spin separation k, we have

$$\begin{aligned} \omega_2(k, 0) &= \cos(\tfrac{1}{2}\pi k)\left[\left(\frac{2}{\pi}\right)^{\frac{1}{2}k} \prod_{s=1}^{\frac{1}{2}k} \frac{\Gamma(s)\Gamma(s)}{\Gamma(s-\frac{1}{2})\Gamma(s+\frac{1}{2})}\right]^2 \\ &\sim 2^{\frac{1}{2}} E^2 k^{-\frac{1}{2}} \cos(\tfrac{1}{2}\pi k)[1 - (8k^2)^{-1} + \cdots], \end{aligned} \tag{6.19}$$

where

$$2^{\frac{1}{2}}E^2 = 0.588352663\cdots. \tag{6.20}$$

The above results are exact. Qualitatively, they are just what one gets by setting $\theta_1 = \frac{1}{2}\pi$ in (6.1) and letting $v_1 \to -1$, an invalid procedure! There is a curious similarity of form between Eq. (6.20), obtained here for pair correlations along the (1) direction when $J_3 < J_1 = J_2 < 0$, and that for pair correlations at $T = 0$ on the isotropic antiferromagnetic triangular lattice[24]:

$$\omega_2(k, 0) \sim \epsilon_0 k^{-\frac{1}{2}} \cos(\tfrac{2}{3}\pi k). \tag{6.21}$$

The exact value of ϵ_0 is plausibly conjectured to be

$$\epsilon_0 = 2^{\frac{1}{2}}(E_0^T)^2 = 0.632226080\cdots, \tag{6.22}$$

where E_0^T is the decay amplitude of the pair correlation

at the Curie point of an isotropic ferromagnetic triangular lattice, as in Sec. 3, Eqs. (3.17) and (3.18). In particular, note the appearance in (6.20) and (6.21) of a decay amplitude related to the *square* of a ferromagnetic critical point decay amplitude. This gives one additional confidence in the conjectured value of ϵ_0 in (6.22).

7. CONCLUDING REMARKS

The most important and interesting results obtained in this paper are those for the asymptotic behavior of pair correlations along the axes of a *two-dimensional*, *antiferromagnetic*, *anisotropic*, *triangular*, *Ising* lattice. The italicized words indicate five features which in conjunction determine the correlation properties we have derived here. It would be interesting to know whether the results we have obtained can be generalized to other antiferromagnetic lattices. The occurrence of a *disorder point* T_D, as we have defined it (4.2), is related to the existence of triangles in an anisotropic lattice. For this reason we may expect some of the present considerations to carry over to an antiferromagnetic Kagomé lattice, for example, and possibly also to some three-dimensional lattices, such as the hyper-Kagomé, hypertriangular, and face-centered cubic.

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On Hidden-Variable Theories

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An abstract definition of a general hidden-variables theory is given, and it is shown that such a theory is always possible in the present framework of quantum mechanics and is, in fact, unique in a certain sense. It is noted that the Bohm–Bub hidden-variables example is contained in this theory and an attempt is made to clarify the position of this theory with respect to hidden-variable impossibility proofs. The general definition is used in the consideration of quantum-mechanical ordering and the measurement process.

1. INTRODUCTION

The idea that there is a possibility of a hidden-variables (HV) theory in quantum mechanics is almost as old as quantum mechanics itself. The first mathematical refutation of even the *possibility* of an HV theory in quantum mechanics was given by von Neumann in 1932.[1] However, the proponents of hidden variables dispensed with this refutation for various reasons and continued their insistence. Recently this has spurred a rash of new proofs concerning the impossibility of HV theories.[2–6] Strangely enough, there have also appeared explicit examples of HV theories.[7] Clearly there is something wrong here. One obviously cannot have an HV theory if it is impossible; yet the originators of these impossibility proofs have not explained the reason for this discrepency. The problem, in the author's opinion, is in the definition of an HV theory. The proponents of HV theories have an idea of what these theories should be and have given examples of such theories. The antagonists have a different idea of what an HV theory should be and have proved that such theories are impossible in the present general framework of quantum mechanics. These proofs are irrelevant since they do not refer to the HV theories as formulated by the advocates of these theories, and this point is stressed in a paper by Bub.[8] These proofs put much more stringent requirements on the HV theories than the HV researchers need or want for their theories. The opponents seem to think that the HV researchers advocate a return to classical mechanics or at least an embedding of quantum mechanics into a classical mechanical framework. However, this is not so. As the author sees it, the HV advocates feel that it is within the realm of possibility for determinism to be introduced in quantum mechanics. As a result,

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