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On quantum separation of variables

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We present a new approach to construct the separate variables basis leading to the full characterization of the transfer matrix spectrum of quantum integrable lattice models. The basis is generated by the repeated action of the transfer matrix itself on a generically chosen state of the Hilbert space. The fusion relations for the transfer matrix, stemming from the Yang-Baxter algebra properties, provide the necessary closure relations to define the action of the transfer matrix on such a basis in terms of elementary local shifts, leading to a separate transfer matrix spectral problem. Hence our scheme extends to the quantum case a key feature of the Liouville-Arnold classical integrability framework where the complete set of conserved charges defines both the level manifold and the flows on it leading to the construction of action-angle variables. We work in the framework of the quantum inverse scattering method. As a first example of our approach, we give the construction of such a basis for models associated with $Y(gl_n)$ and argue how it extends to their trigonometric and elliptic versions. Then we show how our general scheme applies concretely to fundamental models associated with the $Y(gl_2)$ and $Y(gl_3)$ R -matrices leading to the full characterization of their spectrum. For $Y(gl_2)$ and its trigonometric deformation, a particular case of our method reproduces Sklyanin's construction of separate variables. For $Y(gl_3)$, it gives new results, in particular, through the proper identification of the shifts acting on the separate basis. We stress that our method also leads to the full characterization of the spectrum of other known quantum integrable lattice models, including, in particular, trigonometric and elliptic spin chains, open chains with general integrable boundaries, and further higher rank cases that we will describe in forthcoming publications. *Published by AIP Publishing.* <https://doi.org/10.1063/1.5050989>

Dedicated to the memory of Ludwig Faddeev

I. INTRODUCTION

Despite a huge literature on quantum integrable systems, the very notion of quantum integrability did not reach yet a status that can be put on the same footing as its classical counterpart. In particular, there is no quantum analog of the Liouville-Arnold theorem.¹ Due to the pioneering work of Sklyanin,^{2–7} the situation is much more advanced for the notion of separation of variables (SoV). In the classical case, Hamilton-Jacobi theory and separation of variables simplify drastically the quadratures one has to solve while following the Liouville-Arnold scheme leading to the construction of action-angle variables. In fact, in most cases, the full resolution of a system can be explicitly achieved only when effective separate variables are known. It should be noted however that to construct such separate variables the sole knowledge of the complete set of independent commuting (under Poisson brackets) conserved quantities on the basis of the Liouville theorem is not enough in practice; as a matter of fact, to make their construction explicit, one has to rely in general on some additional (algebraic) structures like the Lax matrix, its associated r -matrix, and the Yang-Baxter algebra they

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obey; see, e.g., Refs. 5 and 6. The real breakthrough achieved by Sklyanin was to realize that in paradigmatic examples, and, in particular, for the integrable models associated with gl_2 algebra, the classical and quantum inverse scattering methods^{8–16} generically provide the necessary ingredients to construct the separate variables. Moreover this scheme leads rather straightforwardly to the spectral curve equation and hence, in the quantum case, to the complete spectrum characterization of the transfer matrix and of the associated Hamiltonian. It should be stressed at that point that such a method provides not only the eigenvalues of the transfer matrix and of the Hamiltonian but also the construction of the complete set of associated eigenstates. This is to be compared with other methods also using the quantum inverse scattering framework and the related Yang-Baxter algebra that in general are not easily shown to reach such a level of spectrum completeness.

Having now the quantum case in mind, the key idea of the Sklyanin approach is to identify the separate variables, say, Y_n , as the operator zeros of some diagonalizable commuting family of operators having simple common spectrum. Within this approach, it is usually given by a distinguished operator in the Yang-Baxter algebra depending on the continuous spectral parameter λ ; let us call it $B(\lambda)$, commuting for different values of λ , and such that $B(Y_n) = 0$. Then the use of the Yang-Baxter algebra permits to construct the local (conjugated) shifts acting on the coordinates Y_n as the properly defined evaluation of another distinguished operator of the Yang-Baxter algebra, say, $A(\lambda)$, at $\lambda = Y_n$. It is then possible to prove, using the Yang-Baxter algebra, that the transfer matrix acts by simple shifts on the B -spectrum and to determine at the same time the quantum spectral separate equations (quantum spectral curve) determining the full set of eigenvalues of the transfer matrix and of the associated Hamiltonian. In particular, for quantum lattice integrable models, the separate basis is identified with the eigenstates basis of $B(\lambda)$. This method works extremely well in numerous examples mainly associated with gl_2 ; see, e.g., Refs. 2–7 and 17–42, in particular, in cases where the algebraic Bethe ansatz fails. It appears however that for higher rank cases some difficulties could arise, see, e.g., Refs. 7 and 43, in particular, due to the fact that the identification of the needed operators $B(\lambda)$ and $A(\lambda)$ becomes more involved, making the construction of the quantum spectral curve rather non-trivial (see comments for the gl_3 case studied in Ref. 7 in the Appendix). These issues reveal that the problem of the identification of a pair of such $A(\lambda)$ and $B(\lambda)$ operators having all required properties is a cornerstone of this approach, questioning the effective applicability of the method for an arbitrary given integrable system.

This situation motivated us to look for the construction of separate basis for generic quantum integrable lattice models that would not rely on the determination of such $B(\lambda)$ and $A(\lambda)$ operators. Moreover, our wish was to construct a basis having the built in property that the action of the transfer matrix on it should be given by simple local shifts, making its wave function and spectral problem separated *per se*. Although this idea could look *a priori* too ambitious, it appears that such a construction is in fact possible for all cases we have been exploring so far and, in particular, for models out of reach of the standard SoV or algebraic Bethe ansatz methods. Moreover, it turns out that it takes a rather simple and universal form as it involves the sole knowledge of the transfer matrix itself and of its fusion properties stemming from the underlying Yang-Baxter algebra.

The aim of the present article is to explain this construction and to show how it works concretely in some paradigmatic examples.

The main idea is that a separate basis can be obtained by the multiple action of the transfer matrix $T(\lambda)$ itself, evaluated in distinguished points ξ_n , on a generically chosen co-vector of the Hilbert space. In most quantum integrable lattice models, it may be given by the following set of co-vectors:

$$\langle h_1, \dots, h_n | = \langle L | \prod_{i=1}^N T(\xi_i)^{h_i}, \quad (1.1)$$

where $i = 1, \dots, N$ and $h_i \in \{0, 1, \dots, d_i - 1\}$, the dimension of the Hilbert space being $d = \prod_{i=1}^N d_i$, the ξ_n are the distinguished values characterizing the representation of the quantum lattice model (in most cases, they will be related to the so-called inhomogeneity parameters) such that all of them are different pairwise (i.e., $\xi_i \neq \xi_j \pm n\eta$ if $i \neq j$ with η being a characteristic constant and n being an integer spanning some model dependent range of relative integers), and $\langle L |$ is a generically chosen co-vector in \mathcal{H}^* that obviously should not be an eigenstate of the transfer matrix. In standard Heisenberg

spin chains, all d_i are equal to some value n and $d = n^N$, with N being the number of lattice sites. Eventually, a slightly more general definition could be necessary,

$$\langle h_1, \dots, h_n | = \langle L | \prod_{i=1}^N \prod_{k_i=1}^{h_i} T(\xi_i^{(k_i)}), \quad (1.2)$$

where $i = 1, \dots, N$ and $h_i \in \{0, 1, \dots, d_i - 1\}$, the different points $\xi_i^{(k_i)}$, $k_i = 1, \dots, d_i - 1$, can be seen as shifted from the first value $\xi_i^{(1)}$ and it is understood that the corresponding factor $T(\xi_i^{(k_i)})$ is absent whenever the corresponding $h_i = 0$. Moreover $\xi_i^{(k_i)} \neq \xi_j^{(k_j)}$ for any choices of k_i and k_j as soon as $i \neq j$.

Obviously (1.2) reduces to (1.1) if all points $\xi_j^{(k_j)}$ for any given j are equal and identified to ξ_j . One could even imagine some more general formula, the key idea being that the basis is generated by the repeated action of a set of conserved charges of the model at hand on a generically chosen co-vector $\langle L |$ that should not be, as is obvious, an eigenstate of the conserved charges $T(\xi_i)$. Let us remark that these formulae are reminiscent of the Frobenius method for generating invariant factors of a matrix; see, e.g., Ref. 44.

The main consequence of the existence of such a basis with (discrete) coordinates h_1, \dots, h_N is that the wave function $\Psi_t(h_1, \dots, h_N)$ in these coordinates of any common eigenvector $|t\rangle$ of the set of conserved charges $T(\xi_i)$ factorizes as a product of N wave functions of one variable, namely,

$$\Psi_t(h_1, \dots, h_N) \equiv \langle h_1, \dots, h_N | t \rangle = \langle L | t \rangle \prod_{i=1}^N t(\xi_i)^{h_i}, \quad (1.3)$$

where $t(\xi_i)$ is the eigenvalue of the operator $T(\xi_i)$ associated with the eigenvector $|t\rangle$. In fact, this remark is at the origin of the idea of considering (1.1) as a possible basis of the Hilbert space together with the fact (see below) that the transfer matrix evaluated in the points ξ_i seems, as needed, to act naturally by local shifts on it. At this point, let us give some general comments about such a simple expression for the separate basis:

- Except if the dimension of the Hilbert space is one, it is obvious that the set (1.1) cannot be a basis if the chosen co-vector $\langle L |$ is an eigenstate of the transfer matrix. Hence $\langle L |$ should be a generic state whose orbit under the action of the conserved charges of the system span indeed a basis of \mathcal{H}^* . In particular, it should be that $\langle L | t \rangle \neq 0$ for any non-zero transfer matrix eigenstate $|t\rangle$.
- We will show in Sec. II that if (1.1) defines a basis it implies that the common spectrum of the set of conserved charges $T(\xi_i)$ is w -simple; i.e., there is only one common eigenvector $|t\rangle$ of the set $T(\xi_i)$ corresponding to a set of given eigenvalues $t(\xi_i)$. It does not mean that $T(\lambda)$ is necessarily diagonalizable as there could be non-trivial Jordan blocks. But these different Jordan blocks are all associated with different eigenvalues.
- Although it could be rather astonishing at first sight that the set given (1.1) defines a basis of the space of states, in most known cases of quantum lattice integrable models it can be proven rather easily that it is indeed the case for generically chosen left state $\langle L |$. Exceptions concern in fact some peculiar situations where the transfer matrix does not span a complete set of conserved charges, like in the periodic XXZ Heisenberg model for which the third component of the spin S_z is a conserved quantity that is not generated by the transfer matrix; other exceptions are some special choices of the co-vector $\langle L |$, like an eigenstate of the transfer matrix or a specially constructed co-vector such that the orbit generated by the transfer matrix action stays in a subspace of \mathcal{H}^* of strictly positive co-dimension. This property relies in fact on rather mild properties of the representation of the quantum space of states carried out by the quantum Lax operator. In Sec. II, we will give an elementary proof of this fact in the example of fundamental gl_n based models.
- The main advantage of a basis such as (1.1) is that the action of the transfer matrix on it is obviously given by elementary shifts as soon as the transfer matrix $T(\lambda)$ can be reconstructed by means of some interpolation formula in terms of its value in the points ξ_i (or $\xi_j^{(k_j)}$) eventually supplemented by the knowledge of some central element describing the asymptotic behavior of

the transfer matrix. This is, in particular, the case as soon as the transfer matrix is a polynomial (or a trigonometric or elliptic polynomial) in λ . In the most simple rational case, this is realized if $T(\lambda)$ is a polynomial of degree N in λ , its asymptotic behavior being given by some central element T_∞ such that

$$T(\lambda) = T_\infty \prod_{i=1}^N (\lambda - \xi_i) + \sum_{i=1}^N T(\xi_i) \prod_{j \neq i, j=1}^N \frac{\lambda - \xi_j}{\xi_i - \xi_j}, \quad (1.4)$$

where by hypothesis in (1.1) all ξ_j are different pairwise. Then for most elements of the set (1.1), the action of $T(\lambda)$ is given by elementary shifts within the same set. However, it appears immediately that one has to take care of what happens at the boundaries of the set (1.1), namely, typically when acting with $T(\xi_j)$ on a co-vector of the set (1.1) for which $h_j = d_j - 1$ already. This is precisely the place where the information coming from the Yang-Baxter algebra about the transfer matrix fusion properties enters by providing the necessary closure relations enabling us to compute this action back in terms of the co-vectors of the set (1.1). In other words, we need to know about the structure constants of the associative and commutative algebra of conserved charges. In Secs. III–V, we will show explicitly how this works for gl_2 and gl_3 based models. In particular, for gl_2 case, we will show that a particular choice of the co-vector $\langle L |$ just reproduces Sklyanin's separate basis. For gl_3 , it leads to a new separate basis and to the full characterization of the spectrum (eigenvalues and eigenvectors). In particular, we will show how these closure relations combined with (1.1) lead to the determination of the quantum spectral curve for these cases.

- Separate basis construction (1.1) is very reminiscent of a key property of the classical Liouville-Arnold theorem for classical integrable systems. Indeed in the classical case the complete set of conserved charges defines a level manifold, and also the tangent vectors associated with any point on it. These tangent vectors might be used to define flows going from a given point to another point on this level manifold. In the definition (1.1), the separate basis is indeed generated by the (here discrete) flows of the conserved quantities $T(\xi_i)$. The construction (1.1) shed some new light on the classical case itself that will be considered in a separate article.
- The separate basis for the transfer matrix spectral problem is generated by the transfer matrix itself, i.e., from the sole knowledge of a complete set of conserved quantities, with the additional necessary input of the closure relations stemming from the Yang-Baxter algebra and R -matrix representations. The additional information determines in fact the transfer matrix spectrum. The construction (1.1) opens the way for a new definition of quantum integrability and of completeness of a given set of conserved charges: it corresponds to cases where the set (1.1) forms a basis of the Hilbert space.

In this article, our aim is to give the general principles of our method and to show how it works concretely for some simple interesting models such as the quasi-periodic XXX and XXZ spin-1/2 chains associated with the 6-vertex R -matrix and then the quasi-periodic model associated with the fundamental representation of the $Y(gl_3)$ R -matrix.

We would like to stress that we have already developed the same SoV program, going from the construction of the SoV basis up to the characterization of the transfer matrix spectrum as solutions to quantum spectral curves (functional equations of difference type), for some other important classes of integrable quantum models. These are the models associated with fundamental representations of the Yang-Baxter and reflection algebra for the Yangian $Y(gl_n)$, the quantum group $U_q(gl_n)$, and the t-J model. In order to show how our new SoV method works for non-fundamental models, we have applied it also to the models associated with cyclic and higher spin representations. All these new results will be soon presented in forthcoming articles.

We are also confident that our approach can be applicable for larger classes of integrable quantum models in the framework of the quantum inverse scattering. This is certainly the case for models like the Izergin-Korepin model and the Hubbard model, for which we have already implemented our basis construction. Models associated with other representations of the Yang-Baxter algebra, like the non-compact or infinite dimensional ones, can be also considered using the concepts and ideas developed in the present article. These more general situations are currently under study.

This article is organized as follows. In Sec. II, we give the general properties of basis of the Hilbert space given by set (1.1) or (1.2). We also investigate on general ground the implications of (1.1) being a Hilbert space basis for the properties of the transfer matrix spectrum. Then considering the example of fundamental model associated with a $Y(gl_n)$ rational R -matrix, we show that the set (1.1) indeed determines a basis of \mathcal{H}^* . We also give simple arguments showing that our proof can be extended straightforwardly to the trigonometric and elliptic cases. In Sec. III, we consider in detail the $Y(gl_2)$ based models and make contact with Sklyanin's construction of the separate basis in this case. In Sec. IV, we show that these features extend to the trigonometric case, so providing new SoV complete characterization of the transfer matrix spectrum. Then in Sec. V, we apply our method to the $Y(gl_3)$ case. There we give the proper identification of the shifts acting on the separate basis and show how to determine the quantum spectral curve. The full characterization of the spectrum is also given. In Sec. VI, we give some conclusions and perspectives. In the Appendix, we discuss similarities and differences with respect to the Sklyanin's approach for the $Y(gl_3)$ case.

II. THE QUANTUM SoV BASIS FROM A COMPLETE SET OF COMMUTING CHARGES

Before going to the proof that the proposal (1.1) provides an SoV basis for models of interest, we need first to show that it indeed defines a basis of the space of states for such models. We will first prove this for fundamental models associated with the $Y(gl_n)$ rational R -matrix. Then we will give arguments showing that (1.1) defines also a basis of the Hilbert space for fundamental trigonometric models. The second purpose of this section is to describe the main consequences of (1.1) being a basis of the space of states with regards to the properties of the transfer matrix spectrum. In particular, we will show that as soon as (1.1) is a basis of the space of states, the common spectrum of the charges $T(\xi_i)$ is w -simple. Here by w -simplicity, we mean that for a given eigenvalue there exists only one eigenvector (up to trivial scalar multiplication). However for $Y(gl_n)$ based fundamental models with quasi-periodic boundary conditions described by some matrix K , we will also show that the corresponding transfer matrix is diagonalizable with simple spectrum as soon as K is diagonalizable with simple spectrum. Let us start by recalling the basic definitions and properties of a separate basis for quantum integrable models that we will use in this article and by reviewing the key features of the Sklyanin approach of this problem in the framework of the quantum inverse scattering method.

A. Quantum separation of variables

Here we introduce a definition of quantum separation of variables, directly in the framework of integrable quantum models on a finite dimensional quantum space \mathcal{H} . Let us consider a quantum system with Hamiltonian $H \in \text{End}(\mathcal{H})$ exhibiting a one parameter family of commuting conserved charge operators $T(\lambda) \in \text{End}(\mathcal{H})$, hence having the two properties,

$$\begin{aligned} (i) \quad & [T(\lambda), T(\lambda')] = 0 \quad \forall \lambda, \lambda' \in \mathbb{C}, \\ (ii) \quad & [T(\lambda), H] = 0 \quad \forall \lambda \in \mathbb{C}, \end{aligned} \quad (2.1)$$

where in fact one asks that the Hamiltonian H can be generated by $T(\lambda)$. Here for simplicity we consider conserved charge operators parametrized by one spectral parameter λ . However, the framework we develop in this article can be easily extended to more general cases where the spectral parameter is not just a complex number. For quantum integrable lattice models, $T(\lambda)$ will be given in general by the transfer matrix. Moreover let us consider the case of a finite dimensional Hilbert space \mathcal{H} realized as the tensor product of N local Hilbert spaces \mathcal{H}_n associated with each lattice site n , namely, $\mathcal{H} = \otimes_{n=1}^N \mathcal{H}_n$, with N being the number of lattice sites. We denote by $\dim(\mathcal{H}_n) = d_n$ and $\dim \mathcal{H} = d$ the finite dimensions of these Hilbert spaces with $d = \prod_{n=1}^N d_n$. Let us introduce a covector basis of \mathcal{H}^* of the form

$$S_L \equiv \{ \langle y_1^{(h_1)}, \dots, y_N^{(h_N)} | \forall h_i \in \{1, \dots, d_i\}, \quad i \in \{1, \dots, N\} \text{ with } \prod_{n=1}^N d_n = d \}, \quad (2.2)$$

where $y_i^{(h_i)} \in \Sigma_i$, some d_i dimensional set of complex number. In this covector basis, we define the following set of N commuting operators $Y_n \in \text{End}(\mathcal{H})$,

$$\langle y_1^{(h_1)}, \dots, y_N^{(h_N)} | Y_n \equiv y_n^{(h_n)} \langle y_1^{(h_1)}, \dots, y_N^{(h_N)} |, \quad (2.3)$$

and the associated N commuting shift operators $\Delta_n \in \text{End}(\mathcal{H})$,

$$\langle y_1^{(h_1)}, \dots, y_n^{(h_n)}, \dots, y_N^{(h_N)} | \Delta_n \equiv \langle y_1^{(h_1)}, \dots, y_n^{(h_n+1-d_n\delta_{h_n,d_n})}, \dots, y_N^{(h_N)} |, \quad (2.4)$$

for cyclic type representations and $2N$ shift operators $\Delta_n^{(\pm)} \in \text{End}(\mathcal{H})$:

$$\langle y_1^{(h_1)}, \dots, y_n^{(h_n)}, \dots, y_N^{(h_N)} | \Delta_n^{(+)} \equiv (1 - \delta_{h_n,d_n}) \langle y_1^{(h_1)}, \dots, y_n^{(h_n+1-d_n\delta_{h_n,d_n})}, \dots, y_N^{(h_N)} |, \quad (2.5)$$

$$\langle y_1^{(h_1)}, \dots, y_n^{(h_n)}, \dots, y_N^{(h_N)} | \Delta_n^{(-)} \equiv (1 - \delta_{h_n,1}) \langle y_1^{(h_1)}, \dots, y_n^{(h_n-1+d_n\delta_{h_n,1})}, \dots, y_N^{(h_N)} |, \quad (2.6)$$

for highest weight type representations. Moreover, let us denote by D_n the coordinate representation of the cyclic shift operator Δ_n ,

$$D_n g(y_n^{(h)}) \equiv g(y_n^{(h+1-d_n\delta_{h,d_n})}), \quad (2.7)$$

and by $D_n^{(\pm)}$ the coordinate representations of the highest weight shift operators $\Delta_n^{(\pm)}$,

$$D_n^{(+)} g(y_n^{(h)}) \equiv (1 - \delta_{h,d_n}) g(y_n^{(h+1-d_n\delta_{h,d_n})}), \quad (2.8)$$

$$D_n^{(-)} g(y_n^{(h)}) \equiv (1 - \delta_{h,1}) g(y_n^{(h-1+d_n\delta_{h,1})}). \quad (2.9)$$

Then we can rephrase the Sklyanin definition as follows:

Definition 2.1. We say that \mathcal{S}_L is the separate variables basis (or equivalently Y_n are a system of quantum separate variables) for the family of commuting conserved charges $T(\lambda)$ if and only if for any T -eigenvalue $t(\lambda)$ and T -eigenstate $|t\rangle$ we have

$$\Psi_t(h_1, \dots, h_N) \equiv \langle y_1^{(h_1)}, \dots, y_N^{(h_N)} | t \rangle = \prod_{n=1}^N Q_t(y_n^{(h_n)}), \quad (2.10)$$

where, for all $n \in \{1, \dots, N\}$, $t(\lambda)$ and $Q_t(\lambda)$ are the solutions of separate equations in the spectrum of the separate variables $y_n^{(h)} \in \Sigma_n$ of the type

$$F_n(D_n, t(y_n^{(h)}), y_n^{(h)}) Q_t(y_n^{(h)}) = 0, \quad (2.11)$$

for the cyclic type representations, or

$$F_n(D_n^{(+)}, D_n^{(-)}, t(y_n^{(h)}), y_n^{(h)}) Q_t(y_n^{(h)}) = 0, \quad (2.12)$$

for the highest weight type representations. Note that here the ordering of objects means that the shifts D_n or $D_n^{(+)}$ and $D_n^{(-)}$ can act on $Q_t(y_n^{(h)})$ and also on the $t(y_n^{(h)})$.

These N quantum separate relations are a natural quantum analog of the classical ones in Hamilton-Jacobi's approach. As already pointed out by Sklyanin, a possible quantum analog definition of *degrees of freedom* for a quantum integrable model is just the number N of quantum separate variables. In the classical case, the separate relations are used to solve the equations of motion, mainly constructing the change to the action-angle variables by quadrature in an additive separate form with respect to the separate variables. In the quantum case, instead, these separate relations are used to solve the spectral problem of the family of commuting conserved charges $T(\lambda)$ by determining its eigenvalues and eigenfunctions in the SoV basis. In particular, these separate relations are N systems of discrete difference equations of maximal order d_n for all $n \in \{1, \dots, N\}$, on the spectrum of the separate variables to be solved for $t(\lambda)$ and $Q_t(\lambda)$ within a given class of functions.

In the quantum case as in the classical case, one important problem to solve given an integrable system is to define its separate variables basis. In the framework of the quantum inverse scattering, Sklyanin has given a procedure to define the separate variables for integrable quantum models associated with Yang-Baxter algebra representations for the rank 1 and 2 cases; for the higher rank

case, see, e.g., Ref. 45. Let us give a short review of it. The key point of the Sklyanin approach is to exhibit a couple of commuting operator families, say, $B(\lambda) \in \text{End}(\mathcal{H})$ and $A(\lambda) \in \text{End}(\mathcal{H})$, written in terms of the generators of the Yang-Baxter algebra (the monodromy matrix elements), such that

- The commutation relations between $A(\lambda)$ and $B(\lambda)$ imply that $A(\lambda)$, in the operator zero Y_n of $B(\lambda)$, is proportional to the shift operator D_n or $D_n^{(+)}$ according to the type of representation.
- $A(\lambda)$, $B(\lambda)$, and the (higher) transfer matrices of the integrable models $T_i(\lambda)$ (the quantum spectral invariants) satisfy a closed difference equation of the generic form,

$$\sum_{j=0}^{r+1} \prod_{a=0}^{j-1} A(\lambda - a\eta) T_{r+1-j}(\lambda) = B(\lambda) \Xi(\lambda), \quad (2.13)$$

where r is the rank of the difference equation, $T_0(\lambda) = 1$, and $\Xi(\lambda)$ is some operator constructed from the associated Yang-Baxter algebra generators, suggesting that a quantum analog of the spectral curve equations is satisfied in the operator zero Y_n of $B(\lambda)$.

Then the operator zeros Y_n generate the quantum separate variables. Note that if one can prove that for a given integrable quantum model, the number of separate variables N is constant, this integer defines a natural quantum analog of the number of degrees of freedom, as pointed out by Sklyanin. Then, the separate relations take a form (2.11) or (2.12) in the spectrum of the separate variables under the following conditions:

- (A) $B(\lambda)$ is diagonalizable with simple spectrum.
- (B) $A(\lambda)$ effectively acts as a shift operator on the full $B(\lambda)$ -spectrum—namely, it generates the complete SoV basis starting from one of its vectors,

$$\langle y_1^{(h_1)}, \dots, y_N^{(h_N)} | = \langle y_1^{(1)}, \dots, y_N^{(1)} | \prod_{a=1}^N \prod_{k_a=1}^{h_a-1} A(y_a^{(k_a)}). \quad (2.14)$$

- (C) $\Xi(\lambda)$ is finite in the spectrum of the separate variables $y_n^{(h)} \in \Sigma_n$ for all $n \in \{1, \dots, N\}$.

In the rank 1 case (essentially associated with gl_2 and its quantum deformations), for some Yang-Baxter algebra representations, the operator families $A(\lambda)$ and $B(\lambda)$ just coincide with two elements of the monodromy matrix and the quantum SoV basis can be defined as soon as the property (A) is satisfied; i.e., one can prove that (B) and (C) are satisfied in the given representation. Some modifications or generalizations of Sklyanin's definitions of the operator families $A(\lambda)$ and $B(\lambda)$ have eventually to be introduced to describe more general Yang-Baxter algebra representations, as for the 8-vertex or for the reflection algebra cases. In all these generalizations, however the existence of the SoV basis is reduced to the property (A) for the new $B(\lambda)$. Nevertheless, one has to remark that for the rank 1 case there are still some integrable quantum models for which the SoV approach does not apply even by modification of Sklyanin's definitions. Simple examples are the cases of XXZ spin chains associated with a diagonal twist or to some very special integrable boundary conditions.

In the higher rank case, the situation is even more involved, the expressions given by Sklyanin for the family $A(\lambda)$ and $B(\lambda)$ being eventually not polynomials in the monodromy matrix elements. Hence, while the property (A) can be satisfied, the properties (B) and (C) look non-trivial and need to be proven in the given representations. Until now, it is not known to us if there exists a representation of the Yang-Baxter algebra such that the properties (B) and (C) are satisfied within the Sklyanin approach. More in detail, while we can find simple fundamental representations satisfying the property (A) for the gl_3 case, the properties (B) and (C) are clearly not satisfied for the family $A(\lambda)$ given by Sklyanin in this representation (see the Appendix). This is essentially due to the fact that the operator $\Xi(\lambda)$, which for higher rank cases is no longer a polynomial of the entries of the monodromy matrix, containing inverses of these entries, is not finite over the full spectrum of $B(\lambda)$ in the fundamental representations. The main consequence is that the family $A(\lambda)$ does not realize the shift over the full $B(\lambda)$ -spectrum and hence is not able to generate the $B(\lambda)$ eigenvector basis.

Moreover, in such a situation, Eq. (2.13) cannot lead to the quantum spectral curve equation as its right-hand side does not vanish on the full $B(\lambda)$ -spectrum.

These observations provided us with a strong motivation to look for a different and more universal definition of the separation of variables basis to overcome these problems so far encountered, in particular, for the higher rank case.

B. Toward an SoV basis from transfer matrices: General properties

Although our approach can be formulated in more general terms, for simplicity, in the following, we restrict our analysis to integrable quantum models possessing a one parameter family of commuting conserved charges generated by a transfer matrix $T(\lambda)$, $\lambda \in \mathbb{C}$. Moreover, we will assume that this transfer matrix is a polynomial of finite degree in the variable λ or a polynomial of some simple function of λ (trigonometric or elliptic cases). Our aim in this section is to explore the consequences of (1.1) or (1.2) being a basis have on the transfer matrix spectrum. This will indeed provide us with necessary requirements to ask for the transfer matrix in order to be able to define SoV basis of the type (1.1) or (1.2). For this purpose, let us first define the notion of “basis generating” or “independence property” for the family of conserved charges as follows.

Definition 2.2. A one parameter family of commuting conserved charges $T(\lambda)$, $\lambda \in \mathbb{C}$, acting on the Hilbert space \mathcal{H} of finite dimension d will be said to be “basis generating” or to have the “independence property” if it satisfies the following condition:

(iii) There exist an integer decomposition of the dimension d of \mathcal{H} as $\prod_{n=1}^N d_n = d$, a covector $\langle L|$ in \mathcal{H}^* and N sets of complex numbers $\{y_n^{(1)}, \dots, y_n^{(d_n-1)}\}$, $n = 1, \dots, N$, such that the set of d covectors $\langle h_1, \dots, h_N|$ defined by

$$\langle h_1, \dots, h_N| \equiv \langle L| \prod_{a=1}^N \prod_{k_a=1}^{h_a} T(y_a^{(k_a)}) \text{ for any } \{h_1, \dots, h_N\} \in \otimes_{n=1}^N \{0, \dots, d_n - 1\} \quad (2.15)$$

is a covector basis of \mathcal{H}^* , with the convention that if $h_a = 0$, then the corresponding product $\prod_{k_a=1}^{h_a} T(y_a^{(k_a)})$ is absent and set equal to one in the above formula.

Note that the properties (i) and (ii) in Eq. (2.1) represent a quantum analog of the classical definition of integrals of motion in involution. We refer to the property presented in this definition as “independence condition” as it can be seen as a quantum analog of the independence of the maximal set of integral of motions in involution for a Liouville’s completely integrable classical system. This last property states that for a classical Hamiltonian system with N degrees of freedom the N vector fields associated with these integral of motions that define a level manifold are independent and that they define almost everywhere a tangent basis on this level manifold. Let us remark that whenever a covector $\langle L|$ exists and satisfies this property for a given transfer matrix, it is in general not unique. Indeed, as soon as the operator $T(\lambda)$ is invertible for some value of λ then, using the commutativity of the family $T(\lambda)$, the covector $\langle L|T(\lambda)$ also satisfies the same property. As already noted in the Introduction, it is also obvious that the co-vector $\langle L|$ cannot be an eigenvector of the family $T(\lambda)$ as in that case the set given by (2.15) reduces to the one-dimensional vector space generated by $\langle L|$. For the next considerations about the transfer matrix spectrum, we give now the detailed definition of weak simplicity (or a weakly non-degenerate spectrum).

Definition 2.3. We say that an operator $X \in \text{End}(\mathcal{H})$ is w -simple or has a weakly non-degenerate spectrum if and only if for any X -eigenvalue k there exists one and only one (up to trivial multiplication by a scalar) X -eigenstate $|k\rangle$. For \mathcal{H} finite dimensional, a matrix representing this operator in a basis of \mathcal{H} is called a nonderogatory matrix. A matrix is nonderogatory if and only if its characteristic polynomial is equal to its minimal polynomial. Going to its Jordan form, it means that each eigenvalue is associated with a unique Jordan block. Hence two different Jordan blocks have different eigenvalues.⁴⁶

Let us remark that an operator which is diagonalizable with simple spectrum is w -simple; however, the w -simplicity does not imply that the operator is diagonalizable. Indeed, an operator which

has non-trivial Jordan blocks can still be w -simple if any two different Jordan blocks have different eigenvalues.⁴⁶ Nonderogatory matrices have a very nice property with regards to their characteristic polynomial. Let $P_X(t) = a_0 + ta_1 + t^2a_2 + \dots + t^{d-1}a_{d-1} + t^d$ be the characteristic polynomial of a nonderogatory matrix X , with $d = \dim(\mathcal{H})$. Then the matrix X can be transformed by a similarity transformation into the so-called companion matrix C of its characteristic polynomial—namely, there exists an invertible matrix V_X such that

$$V_X X V_X^{-1} = C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{d-2} & -a_{d-1} \end{pmatrix}. \quad (2.16)$$

Let us consider the canonical covector basis in \mathcal{H}^* denoted by $\langle e_j |$ with $j = 1, \dots, d$ with $d = \dim(\mathcal{H})$. We have that $\langle e_j | C = \langle e_{j+1} |$ for any $j = 1, \dots, d-1$. We denote by $\langle f_j | = \langle e_j | V_X$ the transformed covector basis. Then we have the following property:

Proposition 2.1. *Let us consider any w -simple operator X acting on \mathcal{H} represented by the matrix X in the canonical basis. Then there exists a covector $\langle S |$ such that the set $\langle S | X^{n-1}$, $n = 1, \dots, d$, is a covector basis of \mathcal{H}^* . Moreover let us suppose that the dimension d of \mathcal{H} has the following integer decomposition: $d = \prod_{n=1}^N d_n$ in terms of N integers d_n , then there exist N commuting matrices X_i , $i = 1, \dots, N$, such that the set*

$$\langle h_1, \dots, h_N | = \langle S | \prod_{i=1}^N X_i^{h_i}, \quad (2.17)$$

where $i = 1, \dots, N$ and $h_i \in \{0, 1, \dots, d_i - 1\}$ is a covector basis of \mathcal{H}^* .

Proof. With the above notations, it is enough to set $\langle S | = \langle f_1 |$. Then the set $\langle S | X^{n-1}$, $n = 1, \dots, d$, coincides with the set $\langle f_n |$ which is a covector basis by construction. To prove that there exists at least one N -tuple of commuting matrices satisfying the second part of the proposition, we can proceed as follows. Let us first remark that as soon as $d = \prod_{n=1}^N d_n$ any multiple index (h_1, \dots, h_N) with $h_i \in \{0, 1, \dots, d_i - 1\}$ is uniquely associated with the integer $n_{h_1, \dots, h_N} \in \{0, \dots, d-1\}$ defined as

$$n_{h_1, \dots, h_N} = \sum_{k=1}^N h_k \delta_k, \quad (2.18)$$

where $\delta_k = \prod_{n=1}^{k-1} d_n$ (with the convention that $\delta_1 = 1$) is an increasing sequence of integers determined from the dimensions d_n . The proof of this statement is elementary as it is sufficient to prove that the map is injective, which is rather straightforward. Then, defining the obviously commuting matrices

$$X_j = X^{\delta_j}, \quad (2.19)$$

we immediately get that

$$X^{n_{h_1, \dots, h_N}} = X^{\sum_{k=1}^N h_k \prod_{n=1}^{k-1} d_n} = \prod_{i=1}^N X_i^{h_i}, \quad (2.20)$$

and it just remain to apply this on the chosen covector $\langle S | = \langle f_1 |$ to get the result. \square

Let us remark at this point that the basis we just constructed enables one to obtain an explicit expression of the eigenvectors of the w -simple matrix X from the sole knowledge of its eigenvalues, namely, using the X -spectrum characterization from its characteristic polynomial. In fact, there is the following lemma.

Lemma 2.1. *Let X be a w -simple operator in \mathcal{H} , with the above notations and definitions, the set of covectors $\langle S | X^{n-1} = \langle f_n |$, $n = 1, \dots, d$, $\langle S | = \langle f_1 |$, is a covector basis of \mathcal{H}^* . Then, if λ is an eigenvalue of X , the vector $|\Lambda\rangle$ characterized by its components $\langle f_n | \Lambda \rangle = \alpha \lambda^{n-1}$, with $\alpha = \langle S | \Lambda \rangle \neq 0$, is the unique nonzero eigenvector associated with λ up to trivial multiplication by a scalar.*

Proof. From the w -simple character of the operator X , it follows by definition that for any of its eigenvalue λ there exists one and only one eigenvector $|\Lambda\rangle$, and then, by the definition of the basis, it is immediate to get the following chain of identities:

$$\langle f_n | \Lambda \rangle = \langle S | X^{n-1} | \Lambda \rangle = \lambda^{n-1} \langle S | \Lambda \rangle \quad (2.21)$$

which being $\langle f_n |$ a covector basis implies that it must hold

$$\langle S | \Lambda \rangle \neq 0. \quad (2.22)$$

□

It is also instructive to present a direct proof of the fact that a vector $|\Lambda\rangle$ having components $\langle f_n | \Lambda \rangle = \langle S | \Lambda \rangle \lambda^{n-1}$ on the covector basis is indeed an eigenvector of X as soon as λ is an eigenvalue. In fact, this allows us to present an interesting mechanism similar to the one that will appear in all the following when applying the SoV method. Namely, we want to prove that for any $n = 1, \dots, d$, it holds $\langle f_n | X | \Lambda \rangle = \lambda \langle f_n | \Lambda \rangle$. This is trivial for all values of n , except for $n = d$. Indeed, if $n \leq d-1$, $\langle f_n | X | \Lambda \rangle = \langle f_{n+1} | \Lambda \rangle = \lambda^n \langle S | \Lambda \rangle = \lambda \langle f_n | \Lambda \rangle$. For $n = d$ however, the covector $\langle f_d | X = \langle S | X^d$ is not a member of the basis. Hence this action has to be decomposed back onto the above basis. To achieve this, one has just to use the fact that the characteristic polynomial of X , $P_X(t)$, evaluated on the matrix X vanishes, namely, $P_X(X) = 0$. Hence, $X^d = -a_0 - a_1 X \cdots - a_{d-1} X^{d-1}$. Therefore, one can compute $\langle f_d | X | \Lambda \rangle = \langle S | X^d | \Lambda \rangle$ in terms of the known values $\langle S | X^n | \Lambda \rangle = \lambda^n \langle S | \Lambda \rangle$ for $n = 0, \dots, d-1$. Then, λ itself being an eigenvalue, it verifies also $P_X(\lambda) = 0$. Hence $\lambda^d = -a_0 - a_1 \lambda \cdots - a_{d-1} \lambda^{d-1}$. As a consequence, we have the following chain of equalities:

$$\begin{aligned} \langle f_d | X | \Lambda \rangle &= \langle S | X^d | \Lambda \rangle = -\langle S | \sum_{n=0}^{d-1} a_n X^n | \Lambda \rangle \\ &= -\langle S | \Lambda \rangle \sum_{n=0}^{d-1} a_n \lambda^n = \langle S | \Lambda \rangle \lambda^d = \lambda \langle f_d | \Lambda \rangle, \end{aligned}$$

hence proving that $|\Lambda\rangle$ is an eigenvector of X with eigenvalue λ .

Let us comment that the proof just uses the fact that the secular equation given by the vanishing of the characteristic polynomial holds true both for the operator X and for its eigenvalues. In the SoV framework, the analogous situation follows from the fact that the fusion identities for the transfer matrices holds true both for the transfer matrix as an operator and for its eigenvalues. It makes the above property a precursor of the SoV mechanism that will be used in the Secs. III–V. Let us further notice that if X is w -simple, then any matrix commuting with X is a polynomial in X of at most degree $d-1$.

Let us finally mention another interesting proof of the previous proposition which uses the Jordan block canonical form of X and the computation of the determinant of confluent Vandermonde matrices instead of using as above its companion matrix C .

Proposition 2.2. *Let $X \in \text{End}(\mathcal{H})$ and let us denote by X_J an upper-triangular Jordan form of X obtained through a change of basis induced by the invertible matrix W_X ,*

$$X = W_X X_J W_X^{-1}, \quad (2.23)$$

with the following block form:

$$X_J = \begin{pmatrix} X_J^{(1)} & 0 & \cdots & 0 \\ 0 & X_J^{(2)} & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & X_J^{(M)} \end{pmatrix}, \quad (2.24)$$

where any $X_J^{(a)}$ is an $n_a \times n_a$ upper-triangular Jordan block with eigenvalue k_a , where $\sum_{a=1}^M n_a = d$. Then let us denote by $\langle S |$ the generic covector in \mathcal{H}^ such that*

$$\langle S | W_X = (x_1^{(1)}, \dots, x_{n_1}^{(1)}, x_1^{(2)}, \dots, x_{n_2}^{(2)}, \dots, x_1^{(M)}, \dots, x_{n_M}^{(M)}). \quad (2.25)$$

Then, let $|s_j\rangle = W_X^{-1}|e_j\rangle$ be the canonical vector basis in \mathcal{H} after the change of basis induced by W_X , we have

$$\det_d \left\| \left(\langle S|X^{i-1}|s_j\rangle \right)_{i,j \in \{1, \dots, d\}} \right\| = \prod_{a=1}^M \left(x_1^{(a)} \right)^{n_a} \prod_{1 \leq a < b \leq M} (k_b - k_a)^{n_a n_b}. \quad (2.26)$$

Proof. The proof uses standard techniques of matrix algebra⁴⁶ and we give just a sketch of it. It consists first in computing the action of the powers of the matrix X for each Jordan block. To compute the determinant one first show that by addition and subtractions of lines, each multiplied by adequate coefficients, the result does not depend on the variables $x_j^{(a)}$ for $j \neq 1$ for any a . Then one can perform the computation setting $x_j^{(a)} = \delta_{1,j} x_1^{(a)}$. It is then easy to extract the product of the variables $x_1^{(a)}$ from the determinant and to reduce to the case where for any a , $x_j^{(a)} = \delta_{1,j}$. In such a case, the matrix $\left(\langle S|X^{i-1}|s_j\rangle \right)_{i,j \in \{1, \dots, d\}}$ is a confluent Vandermonde matrix which has the determinant given by the products of all the ordered differences of the Jordan block eigenvalues raised to the power given by the product of the dimensions of the Jordan blocks; see, e.g., Ref. 47. \square

Then we have the following obvious corollary:

Corollary 2.1. Let $X \in \text{End}(\mathcal{H})$ be w -simple, hence its Jordan form is such that its eigenvalues k_j are pairwise distinct. With the notations and definitions of Proposition 2.2, as soon as we take

$$\prod_{a=1}^M x_1^{(a)} \neq 0, \quad (2.27)$$

the determinant (2.26) is non-zero, meaning that the set $\langle S|X^{i-1}$ for $i \in \{1, \dots, d\}$ is a covector basis of \mathcal{H}^* .

Now we can state the main result concerning the w -simplicity of the transfer matrix $T(\lambda)$:

Proposition 2.3. Let us consider an integrable quantum model defined on an Hilbert space of states \mathcal{H} of finite dimension d that admits the integer decomposition $d = \prod_{n=1}^N d_n$ in terms of N integers d_n with a transfer matrix operator $T(\lambda)$ satisfying the properties (i) and (ii) in (2.1).

Then if $T(\lambda)$ satisfies the independence property (iii) in Definition (2.15), the one parameter family of commuting conserved charges $T(\lambda)$ is w -simple; i.e., for any T -eigenvalue $t(\lambda)$, there exists one and only one T -eigenstate $|t\rangle$; moreover, this T -eigenstate is characterized (uniquely up to a normalization) by the following separated wave-function:

$$\Psi_t(h_1, \dots, h_N) \equiv \langle h_1, \dots, h_N | t \rangle = \langle L | t \rangle \prod_{a=1}^N \prod_{k_a=1}^{h_a} t(y_a^{(k_a)}), \quad (2.28)$$

in the basis (2.15).

Conversely, let us assume that the commuting family of conserved charges $T(\lambda)$ satisfying the properties (i) and (ii) is w -simple, then there exists a rearrangement of the conserved charges; that is, there exists a family $\hat{T}(\lambda)$ function of $T(\lambda)$ and conversely $T(\lambda)$ can be reconstructed from $\hat{T}(\lambda)$, and $\hat{T}(\lambda)$ is such that $[\hat{T}(\lambda), T(\mu)] = 0$ and $[\hat{T}(\lambda), \hat{T}(\mu)] = 0$ for all $\lambda, \mu \in \mathbb{C}$, satisfying the properties (i), (ii), and (iii) for some covector $\langle L |$ in \mathcal{H}^* .

Proof. The first part of the Proposition is trivial—namely, the knowledge of the eigenvalue $t(\lambda)$ determines completely the components of the associated eigenvector $|t\rangle$ on the covector basis $\langle h_1, \dots, h_N |$ and hence its unicity. The converse uses the fact that we can fix some value, say, λ_0 such that the transfer matrix $T(\lambda_0) = X$ is w -simple. In this case, any operator commuting with X is a polynomial of X (property of nonderogatory matrices, see Ref. 46). Hence the complete knowledge of conserved charges is contained in X and its successive powers. Let us define the operator $\hat{T}(\lambda)$ as the polynomial of degree $N - 1$ having the value X_j defined as in (2.19) in given points $\xi_j, j = 1, \dots, N$.

It is then enough to apply Proposition (2.1) to obtain the result. Moreover the family $T(\lambda)$ being a polynomial in $X = T(\lambda_0)$ can be reconstructed from the family $\hat{T}(\lambda)$. \square

Note that the optimal decomposition of d is given by its prime decomposition, but here we consider any integer decomposition of d . However, for integrable lattice models of interest that we will consider in Secs. III–V, we will use the prime decomposition of d as it leads to the quantum spectral curve equation of minimal degree. To conclude these general considerations, the results so far obtained show that as soon as the transfer matrix is w -simple there exists a precursor of an SoV basis (2.15). It should be stressed however that to get the full SoV scheme, we also need to have a way to determine the quantum spectral curve associated with it. This in fact amounts to unravel the nature of the commutative algebra of conserved charges that enable to get a characterization of the transfer matrix spectrum in terms of secular equations of degrees d_i , in general much smaller than the dimension d of the Hilbert space. This is somehow analogous to get effectively the canonical rational form⁴⁴ of the transfer matrix written in terms of companion blocks having characteristic polynomials of smaller degree compared to the characteristic polynomial. This is provided as we shall see in concrete examples in Secs. III–V by the fusion relations satisfied by the tower of fused transfer matrices. These fusion relations are in their turn direct consequences of the Yang-Baxter algebra and hence contain the integrability properties of the model at hand.

Let us now give some elementary example of the above basis construction for the quasi-periodic $Y(gl_n)$ fundamental models.

C. The example of the quasi-periodic $Y(gl_n)$ fundamental model

Let us consider the Yangian gl_n R -matrix

$$R_{a,b}(\lambda_a - \lambda_b) = (\lambda_a - \lambda_b)I_{a,b} + \mathbb{P}_{a,b} \in \text{End}(V_a \otimes V_b), \quad \text{with } V_a = \mathbb{C}^n, V_b = \mathbb{C}^n, n \in \mathbb{N}^*, \quad (2.29)$$

where $\mathbb{P}_{a,b}$ is the permutation operator on the tensor product $V_a \otimes V_b$, which is the solution of the Yang-Baxter equation written in $\text{End}(V_a \otimes V_b \otimes V_c)$,

$$R_{a,b}(\lambda_a - \lambda_b)R_{a,c}(\lambda_a - \lambda_c)R_{b,c}(\lambda_b - \lambda_c) = R_{b,c}(\lambda_b - \lambda_c)R_{a,c}(\lambda_a - \lambda_c)R_{a,b}(\lambda_a - \lambda_b), \quad (2.30)$$

and any matrix $K \in \text{End}(\mathbb{C}^n)$ is a scalar solution of the Yang-Baxter equation with respect to it,

$$R_{a,b}(\lambda_a, \lambda_b | \eta) K_a K_b = K_b K_a R_{a,b}(\lambda_a, \lambda_b | \eta) \in \text{End}(V_a \otimes V_b \otimes V_c), \quad (2.31)$$

i.e., it is a symmetry of the considered R -matrix. Then we can define the following monodromy matrix:

$$M_a^{(K)}(\lambda, \{\xi_1, \dots, \xi_N\}) \equiv K_a R_{a,N}(\lambda_a - \xi_N) \cdots R_{a,1}(\lambda_a - \xi_1), \quad (2.32)$$

which satisfies the Yang-Baxter equation

$$R_{a,b}(\lambda_a - \lambda_b) M_a^{(K)}(\lambda_a, \{\xi\}) M_b^{(K)}(\lambda_b, \{\xi\}) = M_b^{(K)}(\lambda_b, \{\xi\}) M_a^{(K)}(\lambda_a, \{\xi\}) R_{a,b}(\lambda_a - \lambda_b) \quad (2.33)$$

in $\text{End}(V_a \otimes V_b \otimes \mathcal{H})$, with $\mathcal{H} \equiv \bigotimes_{l=1}^N V_l$ and its dimension $d = n^N$. Hence it defines a representation of the Yang-Baxter algebra associated with this R -matrix and the following one parameter family of commuting transfer matrices:

$$T^{(K)}(\lambda, \{\xi\}) \equiv \text{tr}_{V_a} M_a^{(K)}(\lambda, \{\xi\}). \quad (2.34)$$

In the above formulae, the complex parameters $\{\xi_1, \dots, \xi_N\}$ are called *inhomogeneity parameters*, and we also assume in the following that they are in generic position such that the above Yang-Baxter algebra representation is irreducible.

The following proposition holds:

Proposition 2.4. The set of covectors

$$\langle h_1, \dots, h_N | \equiv \langle S | \prod_{a=1}^N (T^{(K)}(\xi_a, \{\xi\}))^{h_a}, \quad \forall \{h_1, \dots, h_N\} \in \{0, \dots, n-1\}^{\otimes N}, \quad (2.35)$$

defines a covector basis of \mathcal{H}^ for almost any choice of the covector $\langle S |$ and of the inhomogeneities parameters $\{\xi_1, \dots, \xi_N\}$ under the only condition that the given $K \in \text{End}(\mathbb{C}^n)$ is w -simple on \mathbb{C}^n . In*

particular, for almost all the values of the inhomogeneities parameters $\{\xi_1, \dots, \xi_N\}$, the covector in \mathcal{H}^* of tensor product form,

$$\langle S| \equiv \bigotimes_{a=1}^N \langle S, a|, \quad (2.36)$$

can be chosen as soon as we take for $\langle S, a|$ a local covector in V_a^* such that

$$\langle S, a| K_a^h \text{ with } h \in \{0, \dots, n-1\} \quad (2.37)$$

form a covector basis for V_a for any $a \in \{1, \dots, N\}$, the existence of $\langle S, a|$ being implied by the fact that K is w -simple.

Proof. Let us define the $n^N \times n^N$ matrix $\mathcal{M}(\langle S|, K, \{\xi\})$ with elements

$$\mathcal{M}_{i,j} \equiv \langle h_1(i), \dots, h_N(i) | e_j \rangle, \quad \forall i, j \in \{1, \dots, n^N\}, \quad (2.38)$$

where we have defined uniquely the N -tuple $(h_1(i), \dots, h_N(i)) \in \{1, \dots, n\}^{\otimes N}$ by

$$1 + \sum_{a=1}^N h_a(i) n^{a-1} = i \in \{1, \dots, n^N\}, \quad (2.39)$$

and $|e_j\rangle \in \mathcal{H}$ is the element $j \in \{1, \dots, n^N\}$ of the elementary basis in \mathcal{H} . Then the condition that the set (2.35) forms a basis of covector in \mathcal{H}^* is equivalent to the condition

$$\det_{n^N} \mathcal{M}(\langle S|, K, \{\xi\}) \neq 0. \quad (2.40)$$

The transfer matrix $T^{(K)}(\lambda, \{\xi\})$ is a polynomial in the parameters of the K matrix and in the inhomogeneity parameters $\{\xi_1, \dots, \xi_N\}$. Then the determinant $\det_{n^N} \mathcal{M}(\langle S|, K, \{\xi\})$ is itself a polynomial of these parameters, and it is moreover a polynomial in the coefficients $\langle S | e_j \rangle$ of the covector $\langle S|$. Taking into account this polynomial dependence in all the variables, it is enough to prove that the condition (2.40) holds for some special limit on the parameters to prove that it is true for almost any value of the parameters, except on the zeros of the corresponding polynomials. The transfer matrix $T^{(K)}(\lambda, \{\xi\})$ satisfies the following identities:

$$T^{(K)}(\xi_l, \{\xi\}) = R_{l,l-1}(\xi_l - \xi_{l-1}) \cdots R_{l,1}(\xi_l - \xi_1) K_l R_{l,N}(\xi_l - \xi_N) \cdots R_{l,l+1}(\xi_l - \xi_{l+1}), \quad (2.41)$$

and then if we chose to impose

$$\xi_a = a\xi, \quad \forall a \in \{1, \dots, N\}, \quad (2.42)$$

it follows that the $T^{(K)}(\xi_l, \{\xi\})$ are polynomials of degree $N-1$ in ξ for all $l \in \{1, \dots, N\}$,

$$T^{(K)}(\xi_l, \{\xi\}) = c_{l,N-1} \xi^{N-1} K_l + \sum_{a=0}^{N-2} c_{l,a} \xi^a T_{l,a}, \quad \text{with } c_{l,N-1} = (-1)^{N-l} (l-1)! (N-l)!, \quad (2.43)$$

and so the same is true for the covectors

$$\langle h_1, \dots, h_N | \equiv \xi^{(N-1) \sum_{a=1}^N h_a} \prod_{a=1}^N c_{a,N-1}^{h_a} \langle S | \prod_{a=1}^N K_a^{h_a} + \sum_{a=0}^{-1+(N-1) \sum_{a=1}^N h_a} \xi^a \langle h_1, \dots, h_N, a |, \quad (2.44)$$

and similarly, $\det_{n^N} \mathcal{M}(\langle S|, K, \{\xi\})$ is a polynomial of degree $(N-1) \sum_{j=1}^{n^N} \sum_{a=1}^N h_a(j)$ with maximal degree coefficient given by

$$\prod_{j=1}^{n^N} \prod_{a=1}^N c_{a,N-1}^{h_a(j)} \det_{n^N} \left\| \left(\langle S | \prod_{a=1}^N K_a^{h_a(i)} | e_j \rangle \right)_{i,j \in \{1, \dots, n^N\}} \right\|. \quad (2.45)$$

If we take $\langle S|$ of the tensor product form (2.36), then it holds, with $|e_j\rangle = \otimes_a |e_j(a)\rangle$,

$$\det_{n^N} \left\| \left(\langle S | \prod_{a=1}^N K_a^{h_a(i)} | e_j \rangle \right)_{i,j \in \{1, \dots, n^N\}} \right\| = \prod_{a=1}^N \det_n \left\| \left(\langle S, a | K_a^{i-1} | e_j(a) \rangle \right)_{i,j \in \{1, \dots, n\}} \right\|. \quad (2.46)$$

Now by hypothesis K is w -simple and this implies the existence of $\langle S, a \rangle$ such that these determinants are all non-zero from Proposition 2.1 or Corollary 2.1. So that we have proven that the leading coefficient of $\det_{\mathcal{M}}(\langle S \rangle, K, \{\xi\})$ is non-zero, so it is non-zero for almost any choice of the parameters. \square

We have already proven that the possibility to introduce such type of basis implies that the transfer matrix spectrum is w -simple, we want to show that in general the transfer matrix is diagonalizable with simple spectrum provided we impose some further requirements on the twist matrix.

Proposition 2.5. *Let us assume that $K \in \text{End}(\mathbb{C}^n)$ is diagonalizable with simple spectrum on \mathbb{C}^n , then, almost for any values of the inhomogeneities, it holds*

$$\langle t|t \rangle \neq 0, \quad (2.47)$$

where $|t\rangle$ and $\langle t|$ are the unique eigenvector and eigencovector associated with $t(\lambda)$ a generic eigenvalue of $T^{(K)}(\lambda, \{\xi\})$ so that $T^{(K)}(\lambda, \{\xi\})$ is diagonalizable with simple spectrum.

Proof. As we have already proven, if we impose

$$\xi_a = a\xi, \forall a \in \{1, \dots, N\}, \quad (2.48)$$

then it follows that the $T^{(K)}(\xi_l, \{\xi\})$ are polynomials of degree $N-1$ in ξ for all $l \in \{1, \dots, N\}$,

$$T^{(K)}(\xi_l, \{\xi\}) = \xi^{N-1} T_{l,N-1}^{(K)} + \sum_{a=0}^{N-2} c_{l,a} \xi^a T_{l,a}, \quad (2.49)$$

with

$$T_{l,N-1}^{(K)} \equiv c_{l,N-1} K_l, \text{ with } c_{l,N-1} = (-1)^{N-l} (l-1)! (N-l)!. \quad (2.50)$$

Under the condition that K is diagonalizable with simple spectrum on \mathbb{C}^n , it follows that the $T_{l,N-1}^{(K)}$ for all $l \in \{1, \dots, N\}$ form a system of N operators simultaneously diagonalizable and with simple spectrum. In particular, let us denote

$$\langle K_a, j | K_a = k_j \langle K_a, j | \text{ and } K_a | K_a, j \rangle = | K_a, j \rangle k_j, \forall (a, j) \in \{1, \dots, N\} \times \{1, \dots, n\}, \quad (2.51)$$

where we can fix their normalization by imposing

$$\langle K_a, r | K_a, s \rangle = \delta_{r,s}, \quad (2.52)$$

and we have that the common left and right eigenbasis of the $T_{l,N-1}^{(K)}$ reads

$$\langle t_{h_1, \dots, h_N} | = \bigotimes_{a=1}^N \langle K_a, h_a |, \quad | t_{h_1, \dots, h_N} \rangle = \bigotimes_{a=1}^N | K_a, h_a \rangle \quad \forall (h_1, \dots, h_N) \in \{1, \dots, n\}^N, \quad (2.53)$$

with

$$\langle t_{h_1, \dots, h_N} | T^{(K)}(\xi_l, \{\xi\}) = \xi^{N-1} c_{l,N-1} k_{h_l} \langle t_{h_1, \dots, h_N} | + O(\xi^{N-2}), \quad (2.54)$$

$$T^{(K)}(\xi_l, \{\xi\}) | t_{h_1, \dots, h_N} \rangle = | t_{h_1, \dots, h_N} \rangle \xi^{N-1} c_{l,N-1} k_{h_l} + O(\xi^{N-2}). \quad (2.55)$$

By using now the following interpolation formula:

$$T^{(K)}(\lambda, \{\xi\}) = \text{tr } K \prod_{a=1}^N (\lambda - \xi_a) + \sum_{a=1}^N \prod_{b \neq a, b=1}^N \frac{\lambda - \xi_b}{\xi_a - \xi_b} T^{(K)}(\xi_a, \{\xi\}), \quad (2.56)$$

we get that for general values of the spectral parameter λ the transfer matrix $T^{(K)}(\lambda, \{\xi\})$ is a polynomial of degree N in ξ with the following expansion:

$$T^{(K)}(\lambda, \{\xi\}) = \xi^N (-1)^N N! \text{tr } K + (-1)^{N-1} \xi^{N-1} \sum_{a=1}^N \prod_{b \neq a, b=1}^N \frac{b}{a-b} T_{a,N-1}^{(K)} + \hat{T}^{(K)}(\lambda, \xi), \quad (2.57)$$

where $\hat{T}^{(K)}(\lambda, \xi)$ is a polynomial in ξ of order $N - 2$. So the left and right states $\langle t_{h_1, \dots, h_N} |$ and $|t_{h_1, \dots, h_N}\rangle$ are left and right eigenstates of the leading terms in ξ of $T^{(K)}(\lambda, \{\xi\})$, respectively. Note that this implies that for any eigenvalue $t(\lambda)$ of $T^{(K)}(\lambda, \{\xi\})$ denoted with $\langle t|$ and $|t\rangle$ the associated eigenvectors and eigenvectors there exists a unique set $(h_1, \dots, h_N) \in \{1, \dots, n\}^N$ such that:

$$\lim_{\xi \rightarrow \infty} \xi^{1-N} \frac{\langle t|}{n_{t,L}(\xi)} T^{(K)}(\xi, \{\xi\}) = c_{l,N-1} k_{h_l} \langle t_{h_1, \dots, h_N} |, \quad (2.58)$$

and for almost any finite λ :

$$\begin{aligned} \lim_{\xi \rightarrow \infty} \xi^{1-N} \langle t| (T^{(K)}(\lambda, \{\xi\}) - \xi^N (-1)^N N! \text{tr } K) = \\ = (-1)^{N-1} \left(\sum_{a=1}^N c_{a,N-1} k_{h_a} \prod_{b \neq a, b=1}^N \frac{b}{a-b} \right) \langle t_{h_1, \dots, h_N} |, \end{aligned} \quad (2.59)$$

once the non-zero normalization $n_{t,L}(\xi)$ of the eigenvector is chosen properly and similar limits for the eigenvectors. So that it has to hold

$$\lim_{\xi \rightarrow \infty} \frac{\langle t|}{n_{t,L}(\xi)} = \langle t_{h_1, \dots, h_N} |, \quad \lim_{\xi \rightarrow \infty} \frac{|t\rangle}{n_{t,R}(\xi)} = |t_{h_1, \dots, h_N}\rangle, \quad (2.60)$$

which, in particular, implies that

$$\lim_{\xi \rightarrow \infty} \frac{\langle t|t\rangle}{n_{t,L}(\xi) n_{t,R}(\xi)} = \langle t_{h_1, \dots, h_N} | t_{h_1, \dots, h_N} \rangle = 1, \quad (2.61)$$

and by the continuity argument, it implies our statement

$$\langle t|t\rangle \neq 0 \quad (2.62)$$

almost for any values of the inhomogeneities. This statement is true for the left and right eigenstates associated with any eigenvalue of the transfer matrix. Together with the already proven w -simplicity of the transfer matrix spectrum, this implies that there is no nontrivial Jordan block associated with any transfer matrix eigenvalue. Indeed, in a nontrivial Jordan block, the right and left eigenvectors are orthogonal. So the transfer matrix is diagonalizable with simple spectrum. \square

D. Further examples as deformation of the $Y(gl_n)$ case

Let us consider an R -matrix,

$$R_{a,b}(\lambda_a, \lambda_b | \eta) \in \text{End}(V_a \otimes V_b), \quad \text{with } V_a = \mathbb{C}^n, V_b = \mathbb{C}^n, n \in \mathbb{N}^*, \quad (2.63)$$

that is regular and continuous $\lambda_a, \lambda_b \in \mathbb{C}, \eta \in \mathbb{C}$ (to be more precise, we consider in particular cases where the R -matrix is indeed a trigonometric or elliptic polynomial of these parameters), solution of the Yang-Baxter equation written in $\text{End}(V_a \otimes V_b \otimes V_c)$,

$$R_{a,b}(\lambda_a, \lambda_b | \eta) R_{a,c}(\lambda_a, \lambda_c | \eta) R_{b,c}(\lambda_b, \lambda_c | \eta) = R_{b,c}(\lambda_b, \lambda_c | \eta) R_{a,c}(\lambda_a, \lambda_c | \eta) R_{a,b}(\lambda_a, \lambda_b | \eta), \quad (2.64)$$

and let us denote by $\text{SYB}_\eta \subset \text{End}(\mathbb{C}^n)$ the set of the scalar solution of the Yang-Baxter equation,

$$R_{a,b}(\lambda_a, \lambda_b | \eta) K_a K_b = K_b K_a R_{a,b}(\lambda_a, \lambda_b | \eta) \in \text{End}(V_a \otimes V_b \otimes V_c), \quad (2.65)$$

for any $K \in \text{SYB}_\eta$, i.e., the set of symmetries of the considered R -matrix. Then we can define the following monodromy matrix:

$$M_a^{(K)}(\lambda, \{\xi_1, \dots, \xi_N\} | \eta) \equiv K_a R_{a,N}(\lambda_a, \xi_N | \eta) \cdots R_{a,1}(\lambda_a, \xi_1 | \eta), \quad (2.66)$$

which satisfies the Yang-Baxter equation,

$$R_{a,b}(\lambda_a, \lambda_b | \eta) M_a^{(K)}(\lambda_a, \{\xi\} | \eta) M_b^{(K)}(\lambda_b, \{\xi\} | \eta) = M_b^{(K)}(\lambda_b, \{\xi\} | \eta) M_a^{(K)}(\lambda_a, \{\xi\} | \eta) R_{a,b}(\lambda_a, \lambda_b | \eta) \quad (2.67)$$

in $\text{End}(V_a \otimes V_b \otimes \mathcal{H})$, with $\mathcal{H} \equiv \otimes_{l=1}^N V_l$ and its dimension $d = n^N$. Hence it defines a representation of the Yang-Baxter algebra associated with this R -matrix and the following one parameter family of commuting transfer matrices:

$$T^{(K)}(\lambda, \{\xi\} | \eta) \equiv \text{tr}_{V_a} M_a^{(K)}(\lambda, \{\xi\} | \eta). \quad (2.68)$$

In the following of this section, we use an upper index Y in the R -matrix and the monodromy matrix and down index in the transfer matrix to evidence that these are those associated with the rational $Y(gl_n)$ case studied in Sec. II C. Then the following proposition holds:

Proposition 2.6. Let us assume that there exists $\eta_0 \in \mathbb{C}$ and continuous functions $f(x, \eta) \in C^0(\mathbb{C}^2)$ and $g(x, \eta) \in C^0(\mathbb{C}^2)$ such that up to a rescaling of the parameters λ_a and λ_b and trivial overall normalization, the R -matrix satisfies the following Yangian R -matrix limit:

$$\lim_{\eta \rightarrow \eta_0} R_{a,b}(f(\lambda_a, \eta), g(\lambda_b, \eta)|\eta) = R_{a,b}^Y(\lambda_a - \lambda_b), \quad (2.69)$$

and then,

$$\langle h_1, \dots, h_N | \equiv \langle S | \prod_{a=1}^N (T^{(K)}(\xi_a, \{\xi\}|\eta))^{h_a}, \quad \forall \{h_1, \dots, h_N\} \in \{0, \dots, n-1\}^{\otimes N}, \quad (2.70)$$

is a covector basis of \mathcal{H}^* for almost any choice of the covector $\langle S |$, of the value of $\eta \in \mathbb{C}$ and of the inhomogeneities parameters under the only condition that the given $K \in SYB_\eta$ is w -simple on \mathbb{C}^n . In particular, for almost all the values of $\eta \in \mathbb{C}$ and of the inhomogeneities parameters, the covector in \mathcal{H}^* of tensor product form,

$$\langle S | \equiv \bigotimes_{a=1}^N \langle S, a |, \quad (2.71)$$

can be chosen as soon as we take for $\langle S, a |$ a local covector in V_a^* such that

$$\langle S, a | K_a^h \text{ with } h \in \{0, \dots, n-1\} \quad (2.72)$$

form a covector basis for V_a for any $a \in \{1, \dots, N\}$, the existence of $\langle S, a |$ being implied by the fact that K is w -simple.

Proof. Let us define the $n^N \times n^N$ matrix $\mathcal{M}(\langle S |, K, \{\xi\}, \eta)$ with elements

$$\mathcal{M}_{i,j} \equiv \langle h_1(i), \dots, h_N(i) | e_j \rangle, \quad \forall i, j \in \{1, \dots, n^N\}, \quad (2.73)$$

where we have defined uniquely the N -tuple $(h_1(i), \dots, h_N(i)) \in \{1, \dots, n\}^{\otimes N}$ by the isomorphism introduced in Sec. II C. Then the condition that the set (2.35) forms a basis of covector in \mathcal{H}^* is equivalent to the condition

$$\det_{n^N} \mathcal{M}(\langle S |, K, \{\xi\}, \eta) \neq 0. \quad (2.74)$$

Here, we assume that the R -matrix and so the transfer matrix are smooth functions of their parameters: the transfer matrix is a polynomial in the parameters of the K matrix and in general a polynomial or a trigonometric or an elliptic polynomial in the parameters $\{\xi_1, \dots, \xi_N\}$ and η . Then the determinant $\det_{n^N} \mathcal{M}(\langle S |, K, \{\xi\}, \eta)$ is itself a smooth function of its parameters (of the same type of the transfer matrix), and it is moreover a polynomial in the coefficients $\langle S | e_j \rangle$ of the covector $\langle S |$. Taking that into account, it is enough to prove that the condition (2.40) holds for some special limit on the parameters to prove that it is true for almost any value of the parameters. Hence, as the following rational limit

$$\mathcal{M}_Y(\langle S |, K, \{\xi\}) = \lim_{\eta \rightarrow \eta_0} \mathcal{M}(\langle S |, K, \{g(\xi, \eta)\}, \eta) \quad (2.75)$$

is satisfied as a consequence of the fact that the transfer matrix reduces to $T_Y^{(K)}(\lambda, \{\xi\})$, the K -twisted rational $gl(n)$ transfer matrix, our current proposition is proven as a consequence of the one shown in Sec. II C for the rational case. \square

We can similarly prove a statement about the diagonalizability and the simple spectrum for these more general transfer matrices once we impose some further requirements on the twist matrix. In fact, the following proposition holds:

Proposition 2.7. Let us assume that there exists $\eta_0 \in \mathbb{C}$, $f(x, \eta) \in C^0(\mathbb{C}^2)$, and $g(x, \eta) \in C^0(\mathbb{C}^2)$ such that the R -matrix satisfies the following Yangian limit:

$$\lim_{\eta \rightarrow \eta_0} R_{a,b}(f(\lambda_a, \eta), g(\lambda_b, \eta)|\eta) = R_{a,b}^Y(\lambda_a - \lambda_b) \quad (2.76)$$

to the rational $gl(n)$ R-matrix, and let us assume that $K \in SYB_\eta$ is diagonalizable with simple spectrum on \mathbb{C}^n , then, almost for any values of the inhomogeneities and η , it holds

$$\langle t|t \rangle \neq 0, \quad (2.77)$$

where $|t\rangle$ and $\langle t|$ are the unique eigenvector and eigencovector associated with $t(\lambda)$ a generic eigenvalue of $T^{(K)}(\lambda, \{\xi\}|\eta)$ so that $T^{(K)}(\lambda, \{\xi\}|\eta)$ is diagonalizable with simple spectrum.

Proof. We have already proven these statements for $T_Y^{(K)}(\lambda, \{\xi\})$, the K -twisted rational $gl(n)$ transfer matrix, the continuity argument implies then that these statements are true also for almost any value of η . \square

Some comments are in order to conclude these general considerations linking the existence of a basis such as (1.1) and the properties of the transfer matrix spectrum. At first, let us stress that the existence of a basis such as (1.1) is not enough to have the full SoV features. One needs in addition to provide the necessary closure relations enabling for the computation of the action of the transfer matrix on it for elements of the basis associated with boundary values for the coordinates h_j , namely, if for some j , $h_j = d_j - 1$. As we will see in Secs. III–V, this information is provided by the fusion relations satisfied by the transfer matrices. They will lead to the spectrum characterization in the form of a quantum spectral curve. This characterization of the spectrum is given in terms of secular equations of much lower degree compared to the characteristic polynomial. In fact, one gains an exponential factor going from the exact diagonalization to the quantum spectral curve. Thus the essence of the integrability properties of a given model is coming from the unraveling of the nontrivial structure constants of the commutative (and associative algebra) of conserved charges.

III. THE QUASI-PERIODIC $Y(g/2)$ FUNDAMENTAL MODEL

The integrable quantum models associated with the fundamental representations of the Yang-Baxter algebra for the rational and trigonometric 6-vertex R-matrix are the first natural models for which it is interesting to make explicit our SoV basis construction. Indeed, on the one hand, in several cases, the SoV approach as proposed by Sklyanin (or some natural generalization of it) applies also for these cases, and therefore, we can make a comparison with the SoV construction that we propose. On the other hand, we can already present cases for which the Sklyanin SoV scheme does not work directly, while our new scheme applies.

A. The $Y(g/2)$ rational Yang-Baxter algebra

Here, we consider the rational 6-vertex R-matrix solution of the Yang-Baxter equation,

$$R_{a,b}(\lambda) \equiv \begin{pmatrix} \lambda + \eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda + \eta \end{pmatrix} \in \text{End}(V_a \otimes V_b), \quad (3.1)$$

and this is just $R_{a,b}^{(Y)}(\lambda)$ of Sec. II in the $n = 2$ case, where we have reintroduced an η parameter for convenience, and V_a, V_b are bidimensional linear spaces. Then any $K \in \text{End}(\mathbb{C}^2)$ is a scalar solution of the Yang-Baxter equation,

$$R_{a,b}(\lambda)K_aK_b = K_bK_aR_{a,b}(\lambda), \quad (3.2)$$

i.e., gives a symmetry of the rational 6-vertex R-matrix. Then we can define the following monodromy matrix:

$$M_a^{(K)}(\lambda, \{\xi_1, \dots, \xi_N\}) \equiv K_a R_{a,N}(\lambda_a - \xi_N) \cdots R_{a,1}(\lambda_a - \xi_1) = \begin{pmatrix} A^{(K)}(\lambda) & B^{(K)}(\lambda) \\ C^{(K)}(\lambda) & D^{(K)}(\lambda) \end{pmatrix}, \quad (3.3)$$

which satisfies the Yang-Baxter equation with the rational 6-vertex R-matrix, so defining a fundamental spin 1/2 representation of the rational 6-vertex Yang-Baxter algebra and the following one parameter family of commuting transfer matrices:

$$T^{(K)}(\lambda, \{\xi\}) \equiv \text{tr}_{V_a} M_a^{(K)}(\lambda, \{\xi\}). \quad (3.4)$$

In the following, we assume that the inhomogeneity condition

$$\xi_a \neq \xi_b + r\eta, \quad \forall a \neq b \in \{1, \dots, N\} \text{ and } r \in \{-1, 0, 1\}, \quad (3.5)$$

is satisfied.

B. Sklyanin's construction of the SoV basis

Let us now observe that Sklyanin's approach to SoV applies with the separate variables generated by the operator zeros of $B^{(K)}(\lambda)$ if and only if the twist matrix satisfies the condition

$$K = \begin{pmatrix} a & b \neq 0 \\ c & d \end{pmatrix}. \quad (3.6)$$

However, let us remark that given a $K \in \text{End}(\mathbb{C}^2)$ such that $K \neq \alpha I$, for any $\alpha \in \mathbb{C}$, either it satisfies this condition directly or it exists a $W^{(K)} \in \text{End}(\mathbb{C}^2)$ such that

$$\bar{K} = \left(W^{(K)}\right)^1 K W^{(K)} = \begin{pmatrix} \bar{a} & \bar{b} \neq 0 \\ \bar{c} & \bar{d} \end{pmatrix}, \quad (3.7)$$

and then we can use $B^{(\bar{K})}(\lambda)$ to generate the SoV variables for

$$T^{(\bar{K})}(\lambda, \{\xi\}) \equiv \text{tr}_{V_a} M_a^{(\bar{K})}(\lambda, \{\xi\}). \quad (3.8)$$

Now from the identity

$$T^{(K)}(\lambda, \{\xi\}) = \mathcal{W}_K T^{(\bar{K})}(\lambda, \{\xi\}) \mathcal{W}_K^{-1}, \quad \text{with } \mathcal{W}_K = \otimes_{a=1}^N W_a^{(K)}, \quad (3.9)$$

it follows that the separate variables for $T^{(K)}(\lambda, \{\xi\})$ are generated by

$$\mathcal{W}_K B^{(\bar{K})}(\lambda) \mathcal{W}_K^{-1} = \text{tr}_{V_a} [W_a^{(K)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_a \left(W^{(K)}\right)_a^1 M_a^{(K)}(\lambda, \{\xi\})]. \quad (3.10)$$

The previous discussion shows that for the fundamental spin 1/2 rational representation of the 6-vertex Yang-Baxter algebra associated with the symmetry matrix $K \in \text{End}(\mathbb{C}^2)$ such that $K \neq \alpha I$, for any $\alpha \in \mathbb{C}$, either one can use directly Sklyanin's definition of SoV or one can easily redefine the SoV generators by using the symmetries of the rational 6-vertex matrix.

C. Our approach to the SoV basis

The general property as described in Proposition 2.4 for the SoV basis applies to this special case $n = 2$. Let us see in this framework how it works concretely. We define

$$\langle h_1, \dots, h_N | \equiv \langle S | \prod_{a=1}^N \left(\frac{T^{(K)}(\xi_a, \{\xi\})}{a(\xi_a)} \right)^{h_a} \text{ for any } \{h_1, \dots, h_N\} \in \{0, 1\}^{\otimes N}, \quad (3.11)$$

where we have set

$$a(\lambda - \eta) = d(\lambda) = \prod_{a=1}^N (\lambda - \xi_a) \quad (3.12)$$

to introduce the above normalization for reasons to become clear later on. If for simplicity we take the state $\langle S |$ of the following tensor product form:

$$\langle S | = \bigotimes_{a=1}^N (x, y)_a, \quad (3.13)$$

then for any matrix $K \in \text{End}(\mathbb{C}^2)$ not proportional to the identity matrix, it holds that

$$(x, y) K^{i-1} \text{ for } i = 1, 2 \quad (3.14)$$

form a covector basis for almost any $x, y \in \mathbb{C}$. Indeed, denoting as usual the canonical basis of \mathbb{C}^2 by $|e_j\rangle$ for $j = 1, 2$, we have

$$\det || \left((x, y) K^{i-1} |e_j\rangle \right)_{i,j \in \{1,2\}} || = \det \begin{pmatrix} x & y \\ ax + cy & bx + dy \end{pmatrix} = bx^2 + (d - a)xy + cy^2, \quad (3.15)$$

which under the condition $K \neq \alpha I$, for any $\alpha \in \mathbb{C}$, is non-zero for almost all the values of $x, y \in \mathbb{C}$. This also implies that the above set of covectors is a basis for almost any choice of $x, y \in \mathbb{C}$.

D. Comparison of the two SoV constructions

Here we want to show that under some special choice of the covector $\langle S|$, when the twist matrix $K \in \text{End}(\mathbb{C}^2)$ is not proportional to the identity, our SoV left basis reduces to the SoV basis associated with Sklyanin's construction for the K matrix satisfying (3.6) or otherwise to its generalization described above. Let us start assuming that the K matrix satisfies (3.6), we can write down explicitly the left eigenbasis of $B^{(K)}(\lambda)$, i.e., Sklyanin's SoV basis. Let us remark that it holds

$$A^{(K)}(\lambda) = aA(\lambda) + bC(\lambda), \quad B^{(K)}(\lambda) = aB(\lambda) + bD(\lambda), \quad (3.16)$$

$$C^{(K)}(\lambda) = cA(\lambda) + dC(\lambda), \quad D^{(K)}(\lambda) = cB(\lambda) + dD(\lambda), \quad (3.17)$$

in terms of the elements of the original untwisted monodromy matrix. It is well known that it holds

$$\langle 0|A(\lambda) = a(\lambda)\langle 0|, \quad \langle 0|B(\lambda) = 0, \quad (3.18)$$

$$\langle 0|D(\lambda) = d(\lambda)\langle 0|, \quad \langle 0|C(\lambda) \neq 0, \quad (3.19)$$

where we have defined

$$\langle 0| = \bigotimes_{a=1}^N (1, 0)_a. \quad (3.20)$$

So that it holds

$$\langle 0|B^{(K)}(\lambda) = bd(\lambda)\langle 0| \quad (3.21)$$

and by the Yang-Baxter commutation relations it follows that

$$\langle h_1, \dots, h_N| B^{(K)}(\lambda) \equiv b \prod_{a=1}^N (\lambda - \xi_a + h_a \eta) \langle h_1, \dots, h_N|, \quad (3.22)$$

where we have defined

$$\langle h_1, \dots, h_N| \equiv \langle 0| \prod_{a=1}^N \left(\frac{A^{(K)}(\xi_a, \{\xi\})}{a(\xi_a)} \right)^{h_a} \text{ for any } \{h_1, \dots, h_N\} \in \{0, 1\}^{\otimes N} \quad (3.23)$$

so that $B^{(K)}(\lambda)$ is diagonalizable with simple spectrum. Now, let us prove that in fact our SoV basis coincides with the above basis, i.e., it holds

$$\langle h_1, \dots, h_N| = \langle \underline{h_1, \dots, h_N}| \text{ for any } \{h_1, \dots, h_N\} \in \{0, 1\}^{\otimes N} \quad (3.24)$$

as soon as we take

$$\langle S| = \langle 0|. \quad (3.25)$$

The proof is done by induction just using the identity,

$$\langle 0|D^{(K)}(\xi_a) = 0 \quad \forall a \in \{1, \dots, N\}, \quad (3.26)$$

and the Yang-Baxter commutation relations,

$$A^{(K)}(\mu)D^{(K)}(\lambda) = D^{(K)}(\lambda)A^{(K)}(\mu) + \frac{\eta}{\lambda - \mu} (B^{(K)}(\lambda)C^{(K)}(\mu) - B^{(K)}(\mu)C^{(K)}(\lambda)). \quad (3.27)$$

Let us assume that our statement holds for any state,

$$\langle h_1, \dots, h_N| = \langle \underline{h_1, \dots, h_N}| \text{ with } l = \sum_{a=1}^N h_a \leq N - 1, \quad (3.28)$$

and let us show it for any state with $l + 1$. To this aim, we fix a state in the above set and we denote with π a permutation on the set $\{1, \dots, N\}$ such that

$$h_{\pi(a)} = 1 \text{ for } a \leq l \text{ and } h_{\pi(a)} = 0 \text{ for } l < a \quad (3.29)$$

and let us take $c \in \{\pi(l+1), \dots, \pi(N)\}$ and let us compute

$$\langle h_1, \dots, h_N | T^{(K)}(\xi_c, \{\xi\}) = \langle 0 | \frac{A^{(K)}(\xi_{\pi(1)}, \{\xi\})}{d(\xi_{\pi(1)} - \eta)} \dots \frac{A^{(K)}(\xi_{\pi(l)}, \{\xi\})}{d(\xi_{\pi(l)} - \eta)} (A^{(K)} + D^{(K)})(\xi_c, \{\xi\}) \quad (3.30)$$

so that we have to just prove that

$$\langle 0 | \frac{A^{(K)}(\xi_{\pi(1)}, \{\xi\})}{d(\xi_{\pi(1)} - \eta)} \dots \frac{A^{(K)}(\xi_{\pi(l)}, \{\xi\})}{d(\xi_{\pi(l)} - \eta)} D^{(K)}(\xi_c, \{\xi\}) = 0. \quad (3.31)$$

From the commutation relation (3.27), the above covector can be rewritten as follows:

$$\begin{aligned} \langle 0 | \frac{A^{(K)}(\xi_{\pi(1)}, \{\xi\})}{d(\xi_{\pi(1)} - \eta)} \dots \frac{A^{(K)}(\xi_{\pi(l-1)}, \{\xi\})}{d(\xi_{\pi(l-1)} - \eta)} d^{-1}(\xi_{\pi(l)} - \eta) (D^{(K)}(\xi_c, \{\xi\}) A^{(K)}(\xi_{\pi(l)}, \{\xi\}) \\ + \eta (B^{(K)}(\xi_c, \{\xi\}) C(\xi_{\pi(l)}, \{\xi\}) - B^{(K)}(\xi_{\pi(l)}, \{\xi\}) C(\xi_c, \{\xi\})) / (\xi_c - \xi_{\pi(l)}), \end{aligned} \quad (3.32)$$

which reduces to

$$\langle 0 | \frac{A^{(K)}(\xi_{\pi(1)}, \{\xi\})}{d(\xi_{\pi(1)} - \eta)} \dots \frac{A^{(K)}(\xi_{\pi(l-1)}, \{\xi\})}{d(\xi_{\pi(l-1)} - \eta)} D^{(K)}(\xi_c, \{\xi\}) \frac{A^{(K)}(\xi_{\pi(l)}, \{\xi\})}{d(\xi_{\pi(l)} - \eta)} \quad (3.33)$$

once we observe that the state on the left of $B^{(K)}(\xi_c, \{\xi\})$ and $B^{(K)}(\xi_{\pi(l)}, \{\xi\})$ are left eigenstates of $B^{(K)}(\lambda, \{\xi\})$ with eigenvalue zeros at $\lambda = \xi_{\pi(l)}, \xi_c$. That is we can commute $A^{(K)}(\xi_{\pi(l)}, \{\xi\})$ and $D^{(K)}(\xi_c, \{\xi\})$ in the convector and by the same argument $A^{(K)}(\xi_{\pi(r)}, \{\xi\})$ with $D^{(K)}(\xi_c, \{\xi\})$ with $D^{(K)}(\xi_c, \{\xi\})$ for any $r \leq l-1$ up to bring $D^{(K)}(\xi_c, \{\xi\})$ completely to the left acting on $\langle 0 |$ which proves (3.31) as a consequence of (3.26). Let us also mention that one can prove directly, using the Yang-Baxter commutation relations, that our SoV basis is an eigenstate basis of the operator $B^{(K)}(\lambda)$.

Finally, let us assume that K does not satisfy (3.6) but it is not proportional to the identity. Then we can apply the generalization of Sklyanin's construction with respect to the \bar{K} satisfying (3.6). Now the generalized Sklyanin's left SoV basis is the left eigenbasis of the operator family $\mathcal{W}_K B^{(\bar{K})}(\lambda) \mathcal{W}_K^{-1}$ given by

$$\langle h_1, \dots, h_N | \equiv \langle 0 | \prod_{a=1}^N \left(\frac{A^{(\bar{K})}(\xi_a, \{\xi\})}{a(\xi_a)} \right)^{h_a} \mathcal{W}_K^{-1} \text{ for any } \{h_1, \dots, h_N\} \in \{0, 1\}^{\otimes N}, \quad (3.34)$$

and it holds

$$\langle h_1, \dots, h_N | \mathcal{W}_K B^{(\bar{K})}(\lambda, \{\xi\}) \mathcal{W}_K^{-1} \equiv \bar{b} \prod_{a=1}^N (\lambda - \xi_a + h_a \eta) \langle h_1, \dots, h_N |. \quad (3.35)$$

Repeating the argument proven above then it holds

$$\langle 0 | \prod_{a=1}^N \left(\frac{A^{(\bar{K})}(\xi_a, \{\xi\})}{a(\xi_a)} \right)^{h_a} = \langle 0 | \prod_{a=1}^N \left(\frac{T^{(\bar{K})}(\xi_a, \{\xi\})}{a(\xi_a)} \right)^{h_a}, \quad (3.36)$$

and so from the identity $T^{(K)}(\lambda, \{\xi\}) = \mathcal{W}_K T^{(\bar{K})}(\lambda, \{\xi\}) \mathcal{W}_K^{-1}$ it follows that

$$\langle 0 | \prod_{a=1}^N \left(\frac{A^{(\bar{K})}(\xi_a, \{\xi\})}{a(\xi_a)} \right)^{h_a} \mathcal{W}_K^{-1} = \langle S | \prod_{a=1}^N \left(\frac{T^{(K)}(\xi_a, \{\xi\})}{a(\xi_a)} \right)^{h_a} \quad (3.37)$$

once we fix

$$\langle S | = \langle 0 | \mathcal{W}_K^{-1}, \quad (3.38)$$

i.e., our SoV basis coincides with the generalized Sklyanin's one with a special choice of the $\langle S |$ covector for any $K \neq \alpha I$, for any $\alpha \in \mathbb{C}$, for which both do exist.

E. Transfer matrix spectrum in our SoV scheme

Let us show here how to characterize the transfer matrix spectrum in our new SoV scheme. In Secs. I and II, we have anticipated that in our SoV basis the separate relations are given directly by the particularization of the fusion relations in the spectrum of the separate variables. In the case at hand, these fusion relations just reduce to the following identities:

$$T^{(K)}(\xi_a, \{\xi\})T^{(K)}(\xi_a - \eta, \{\xi\}) = q\text{-det}M^{(K)}(\xi_a, \{\xi\}), \quad \forall a \in \{1, \dots, N\}, \quad (3.39)$$

where

$$q\text{-det}M^{(K)}(\lambda, \{\xi\}) = a(\lambda)d(\lambda - \eta)\det K \quad (3.40)$$

is the quantum determinant given as the quadratic expression,

$$q\text{-det}M^{(K)}(\lambda, \{\xi\}) = A^{(K)}(\lambda, \{\xi\})D^{(K)}(\lambda - \eta, \{\xi\}) - B^{(K)}(\lambda, \{\xi\})C^{(K)}(\lambda - \eta, \{\xi\}), \quad (3.41)$$

and three other equivalent ones. One has to add the knowledge of the analytic properties of the transfer matrix that we can easily derive. In fact, $T^{(K)}(\lambda, \{\xi\})$ is a polynomial of degree 1 in all the ξ_a and of degree N in λ with the following leading central coefficient:

$$\lim_{\lambda \rightarrow +\infty} \lambda^{-N} T^{(K)}(\lambda, \{\xi\}) = \text{tr } K. \quad (3.42)$$

Introducing the notation

$$g_a(\lambda) = \prod_{b \neq a, b=1}^N \frac{\lambda - \xi_b}{\xi_a - \xi_b}, \quad (3.43)$$

the following theorem holds:

Theorem 3.1. *Let us assume that $K \neq \alpha I$, for any $\alpha \in \mathbb{C}$, and that the inhomogeneities $\{\xi_1, \dots, \xi_N\} \in \mathbb{C}^N$ satisfy the condition (3.5), then the spectrum of $T^{(K)}(\lambda, \{\xi\})$ is characterized by*

$$\Sigma_{T^{(K)}} = \left\{ t(\lambda) : t(\lambda) = \text{tr } K \prod_{a=1}^N (\lambda - \xi_a) + \sum_{a=1}^N g_a(\lambda)x_a, \quad \forall \{x_1, \dots, x_N\} \in \Sigma_T \right\}, \quad (3.44)$$

where Σ_T is the set of solutions to the following inhomogeneous system of N quadratic equations:

$$x_n [\text{tr } K \prod_{a=1}^N (\xi_n - \xi_a - \eta) + \sum_{a=1}^N g_a(\xi_n - \eta)x_a] = a(\xi_n)d(\xi_n - \eta)\det K, \quad \forall n \in \{1, \dots, N\}, \quad (3.45)$$

in N unknown $\{x_1, \dots, x_N\}$. Moreover, $T^{(K)}(\lambda, \{\xi\})$ has w -simple spectrum, and for any $t(\lambda) \in \Sigma_{T^{(K)}}$, the associated unique (up-to normalization that we take to be given by $\langle S|t=1 \rangle$ eigenvector $|t\rangle$ has the following factorized wave-function in the left SoV basis:

$$\langle h_1, \dots, h_N | t \rangle = \prod_{n=1}^N \left(\frac{t(\xi_n)}{a(\xi_n)} \right)^{h_n}. \quad (3.46)$$

Proof. Let us start observing that the inhomogeneous system of N quadratic equations (3.45) in N unknown $\{x_1, \dots, x_N\}$ is nothing else but the rewriting of the transfer matrix fusion equations,

$$t(\xi_a)t(\xi_a - \eta) = q\text{-det}M^{(K)}(\xi_a, \{\xi\}), \quad \forall a \in \{1, \dots, N\}, \quad (3.47)$$

for the set of all the polynomials of degree N of the form

$$t(\lambda) = \text{tr } K \lambda^N + \sum_{a=1}^N t_a \lambda^{a-1}, \quad (3.48)$$

where we have used for these functions the interpolation formula in the points $\{\xi_1, \dots, \xi_N\}$. Then, it is clear that any eigenvalue of the transfer matrix $T^{(K)}(\lambda, \{\xi\})$ is the solution of this system, i.e., (3.47), and that the associated right eigenvector $|t\rangle$ admits the characterization (3.46) in the left SoV basis. So we are left with the proof of the reverse statement; i.e., any polynomial $t(\lambda)$ of the above

form satisfying this system is an eigenvalue of the transfer matrix. We will prove this by showing that the vector $|t\rangle$ characterized by (3.46) is a transfer matrix eigenstate; i.e., we have to show

$$\langle h_1, \dots, h_N | T^{(K)}(\lambda, \{\xi\}) | t \rangle = t(\lambda) \langle h_1, \dots, h_N | t \rangle, \quad \forall \{h_1, \dots, h_N\} \in \{0, 1\}^{\otimes N}. \quad (3.49)$$

Let us define

$$\xi_a^{(h)} = \xi_a - h\eta, \quad h \in \{0, 1\}, \quad (3.50)$$

and let us write the following interpolation formula for the transfer matrix:

$$T^{(K)}(\lambda, \{\xi\}) = \text{tr} K \prod_{a=1}^N (\lambda - \xi_a^{(h_a)}) + \sum_{a=1}^N \prod_{b \neq a, b=1}^N \frac{\lambda - \xi_b^{(h_b)}}{\xi_a^{(h_a)} - \xi_b^{(h_b)}} T^{(K)}(\xi_a^{(h_a)}, \{\xi\}) \quad (3.51)$$

and use it to act on the generic element of the left SoV basis. Then, we have

$$\langle h_1, \dots, h_a, \dots, h_N | T^{(K)}(\xi_a^{(h_a)}, \{\xi\}) | t \rangle = \begin{cases} a(\xi_a) \langle h_1, \dots, h'_a = 1, \dots, h_N | t \rangle & \text{if } h_a = 0 \\ q \cdot \det M^{(K)}(\xi_a, \{\xi\}) \frac{\langle h_1, \dots, h'_a = 0, \dots, h_N | t \rangle}{a(\xi_a)} & \text{if } h_a = 1, \end{cases} \quad (3.52)$$

which by the definition of the state $|t\rangle$ can be rewritten as

$$\langle h_1, \dots, h_a, \dots, h_N | T^{(K)}(\xi_a^{(h_a)}, \{\xi\}) | t \rangle = \begin{cases} t(\xi_a) \prod_{n \neq a, n=1}^N \left(\frac{t(\xi_n)}{a(\xi_n)} \right)^{h_n} & \text{if } h_a = 0 \\ \frac{q \cdot \det M^{(K)}(\xi_a, \{\xi\})}{a(\xi_a)} \prod_{n \neq a, n=1}^N \left(\frac{t(\xi_n)}{a(\xi_n)} \right)^{h_n} & \text{if } h_a = 1, \end{cases} \quad (3.53)$$

and finally by Eq. (3.47) reads

$$\langle h_1, \dots, h_a, \dots, h_N | T^{(K)}(\xi_a^{(h_a)}, \{\xi\}) | t \rangle = \begin{cases} t(\xi_a) \prod_{n \neq a, n=1}^N \left(\frac{t(\xi_n)}{a(\xi_n)} \right)^{h_n} & \text{if } h_a = 0 \\ t(\xi_a - \eta) \prod_{n=1}^N \left(\frac{t(\xi_n)}{a(\xi_n)} \right)^{h_n} & \text{if } h_a = 1, \end{cases} \quad (3.54)$$

and so

$$\langle h_1, \dots, h_a, \dots, h_N | T^{(K)}(\xi_a^{(h_a)}, \{\xi\}) | t \rangle = t(\xi_a^{(h_a)}) \langle h_1, \dots, h_a, \dots, h_N | t \rangle, \quad (3.55)$$

from which we have, using the polynomial interpolation,

$$\langle h_1, \dots, h_N | T^{(K)}(\lambda, \{\xi\}) | t \rangle = (\text{tr} K \prod_{a=1}^N (\lambda - \xi_a^{(h_a)}) + \sum_{a=1}^N \prod_{b \neq a, b=1}^N \frac{\lambda - \xi_b^{(h_b)}}{\xi_a^{(h_a)} - \xi_b^{(h_b)}} t(\xi_a^{(h_a)})) \langle h_1, \dots, h_N | t \rangle, \quad (3.56)$$

proving our statement. \square

Let us comment that the same characterization of the transfer matrix eigenvalues and eigenvectors is obtained in the Sklyanin-like SoV representations. This is natural as we have shown that under special choice of the covector $\langle S|$ the two SoV bases coincide. Nevertheless, it is worth remarking the slightly different point of view that we used here. In the Sklyanin-like SoV approach, the fusion relations are derived as compatibility conditions for the existence of nonzero eigenvectors. Here, instead, they are used as the starting point to prove that the vectors $|t\rangle$ of the form (3.46) are indeed eigenvectors of the transfer matrix.

The previous characterization of the spectrum allows us to introduce a functional equation which provides an equivalent characterization of it, by the so-called quantum spectral curve, which in the case at hand is a second order Baxter difference equation.

Theorem 3.2. *Let us assume that $K \neq \alpha I$, for any $\alpha \in \mathbb{C}$, and has at least one non-zero eigenvalue (the case where K is a pure Jordan block with eigenvalue zero is not very interesting as in that situation the transfer matrix is quite degenerated being proportional to the nilpotent operator $B(\lambda)$ or $C(\lambda)$; however, our method would anyway work in those cases too) that the inhomogeneities $\{\xi_1, \dots, \xi_N\} \in \mathbb{C}^N$ satisfy the condition (3.5). Moreover, let us introduce the coefficients*

$$\alpha(\lambda) = \beta(\lambda)\beta(\lambda - \eta), \quad \beta(\lambda) = k_0 a(\lambda), \quad (3.57)$$

where $k_0 \neq 0$ is the solution of the equation

$$k_0^2 - k_0 \text{tr} K + \det K = 0, \quad (3.58)$$

i.e., k_0 is a non-zero eigenvalue of the matrix K , and let $t(\lambda)$ be an entire function of λ , then $t(\lambda)$ is an element of the spectrum of $T^{(K)}(\lambda, \{\xi\})$ if and only if there exists a unique polynomial,

$$Q_t(\lambda) = \prod_{a=1}^M (\lambda - \lambda_a), \text{ with } M \leq N \text{ such that } \lambda_a \neq \xi_b, \quad (3.59)$$

for any $(a, b) \in \{1, \dots, M\} \times \{1, \dots, N\}$ such that $t(\lambda)$ and $Q_t(\lambda)$ are solutions of the following quantum spectral curve functional equation:

$$\alpha(\lambda)Q_t(\lambda - 2\eta) - \beta(\lambda)t(\lambda - \eta)Q_t(\lambda - \eta) + q\text{-det}M^{(K)}(\lambda, \{\xi\})Q_t(\lambda) = 0. \quad (3.60)$$

Moreover, up to a normalization, the associated transfer matrix eigenvector $|t\rangle$ admits the following rewriting in the left SoV basis:

$$\langle h_1, \dots, h_N | t \rangle = k_0^{\sum_{n=1}^N h_n} \prod_{n=1}^N Q_t(\xi_n^{(h_n)}). \quad (3.61)$$

Proof. Let us start assuming the existence of $Q_t(\lambda)$ satisfying with $t(\lambda)$ the functional equation (3.60), and then, it follows that $t(\lambda)$ is a polynomial of degree N with leading coefficient t_{N+1} satisfying the equation

$$k_0^2 - k_0 t_{N+1} + \det K = 0, \quad (3.62)$$

which imposes $t_{N+1} = \text{tr}K = k_0 + k_1$, where k_1 is the second eigenvalue or $k_1 = k_0$ if K has a non-trivial Jordan block. Now from the identities

$$q\text{-det}M^{(K)}(\xi_a + \eta, \{\xi\}) = \alpha(\xi_a) = 0, \quad (3.63)$$

we have that the functional equation reduces to the system of equations,

$$-t(\xi_a - \eta)Q_t(\xi_a - \eta) + k_1 d(\xi_a - \eta)Q_t(\xi_a) = 0, \quad (3.64)$$

$$k_0 a(\xi_a)Q_t(\xi_a - \eta) - t(\xi_a)Q_t(\xi_a) = 0, \quad (3.65)$$

once computed in the points ξ_a and $\xi_a + \eta$, from which it follows that

$$t(\xi_a - \eta) \frac{t(\xi_a)Q_t(\xi_a)}{k_0 a(\xi_a)} = k_1 d(\xi_a - \eta)Q_t(\xi_a) \quad (3.66)$$

which being $Q_t(\xi_a) \neq 0$ implies that $t(\lambda)$ satisfies also the system of equations (3.47), for any $a \in \{1, \dots, N\}$, so that $t(\lambda)$ is a transfer matrix eigenvalue for the previous theorem.

Let us now prove the reverse statement; i.e., we assume that $t(\lambda)$ is a transfer matrix eigenvalue and we want to prove the existence of the polynomial $Q_t(\lambda)$ satisfying the functional equation. The lhs of the equation is a polynomial in λ of maximal degree $2N + M$, with $M \leq N$, so that if we prove that it is zero in $3N + 1$ different points we have proven the functional equation. The leading coefficient of this polynomial is zero thanks to (3.58) once we ask that $t(\lambda)$ is a transfer matrix eigenvalue. It is easy to remark that in the points $\xi_a - \eta$, for any $a \in \{1, \dots, N\}$, the functional equation is directly satisfied. Finally, it is satisfied in the $2N$ points ξ_a and $\xi_a + \eta$, for any $a \in \{1, \dots, N\}$, if the system (3.64) and (3.65) is satisfied. As a consequence of the fact that $t(\lambda)$ satisfies (3.47), this last system reduces, e.g., to the system of the second N equations,

$$k_0 a(\xi_a)Q_t(\xi_a - \eta) = t(\xi_a)Q_t(\xi_a), \quad (3.67)$$

which one can prove to be satisfied by a polynomial $Q_t(\lambda)$ of the form (3.59); see, for example, Ref. 48. Moreover, following the proof of Theorem 2.3 of Ref. 48, one can prove also here that the function $Q_t(\lambda)$ is unique. Finally, from the identities

$$\prod_{n=1}^N Q_t(\xi_n) \prod_{n=1}^N \left(\frac{t(\xi_n)}{a(\xi_n)} \right)^{h_n} = k_0^{\sum_{n=1}^N h_n} \prod_{n=1}^N Q_t(\xi_n^{(h_n)}), \quad (3.68)$$

our statement on the representation of the transfer matrix eigenstate in the left SoV basis follows. \square

IV. THE QUASI-PERIODIC XXZ SPIN-1/2 FUNDAMENTAL MODEL

Let us consider here the trigonometric 6-vertex R-matrix,

$$R_{12}(\lambda) = \begin{pmatrix} \sinh(\lambda + \eta) & 0 & 0 & 0 \\ 0 & \sinh \lambda & \sinh \eta & 0 \\ 0 & \sinh \eta & \sinh \lambda & 0 \\ 0 & 0 & 0 & \sinh(\lambda + \eta) \end{pmatrix}, \quad (4.1)$$

which is a solution of the Yang-Baxter equation,

$$R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu). \quad (4.2)$$

The following family of 2×2 matrices

$$K_0^{(a,\alpha)} = \left[\delta_{0,a} \begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{pmatrix}_0 + \delta_{1,a} \begin{pmatrix} 0 & e^\alpha \\ e^{-\alpha} & 0 \end{pmatrix}_0 \right], \quad \forall (a, \alpha) \in \{0, 1\} \times \mathbb{C}^2, \quad (4.3)$$

characterizes the symmetries of the trigonometric 6-vertex R-matrix,

$$R_{12}(\lambda - \mu)K_1^{(a,\alpha)}K_2^{(a,\alpha)} = K_2^{(a,\alpha)}K_1^{(a,\alpha)}R_{12}(\lambda - \mu), \quad (4.4)$$

i.e., the scalar solutions of the trigonometric 6-vertex Yang-Baxter equation. Let us comment that we can add a normalization factor to this matrix which allows us to describe also the case in which one ($a = 0$ case) or both ($a = 1$ case) the eigenvalues of K are zero while K stays w -simple. While our SoV basis construction applies also for these degenerate cases, we omit them to simplify the notations as these special cases correspond to simple transfer matrices, coinciding with one of the elements of the monodromy matrix and for which direct diagonalization methods exist already. Then by using the R-matrix as the Lax operator and the K-matrix as symmetry twist, we can construct the monodromy matrices (unless they play an explicit role, and to simplify the notations, we omit the upper indices in the matrix K when it is used as an upper index itself for the monodromy and transfer matrices)

$$M_0^{(K)}(\lambda) = K_0^{(a,\alpha)}R_{0N}(\lambda - \xi_N) \dots R_{01}(\lambda - \xi_1) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \quad (4.5)$$

and solutions to the trigonometric 6-vertex Yang-Baxter equation,

$$R_{12}(\lambda - \mu)M_1^{(K)}(\lambda)M_2^{(K)}(\mu) = M_2^{(K)}(\mu)M_1^{(K)}(\lambda)R_{12}(\lambda - \mu), \quad (4.6)$$

in the 2^N -dimensional representation space,

$$\mathcal{H} = \otimes_{n=1}^N \mathcal{H}_n. \quad (4.7)$$

Then the associated transfer matrices

$$T^{(K)}(\lambda) = \text{tr}_0 M_0^{(K)}(\lambda) \in \text{End}(\mathcal{H}) \quad (4.8)$$

define a one parameter family of commuting operators. As in the rational case, in the following we assume that the inhomogeneity condition

$$\xi_a \neq \xi_b + r\eta + im\pi, \quad \forall a \neq b \in \{1, \dots, N\}, \quad r \in \{-1, 0, 1\}, \quad \text{and } m \in \mathbb{Z}, \quad (4.9)$$

is satisfied. The main properties enjoyed by these transfer matrices are collected in the following lemma.

Lemma 4.1. The transfer matrix satisfies the following fusion equations:

$$T^{(K)}(\xi_a)T^{(K)}(\xi_a - \eta) = q\text{-det} M^{(K)}(\xi_a), \quad \forall a \in \{1, \dots, N\}, \quad (4.10)$$

where

$$q\text{-det} M^{(K)}(\lambda) = a(\lambda)d(\lambda - \eta)\det K^{(a,\alpha)} \quad (4.11)$$

with

$$a(\lambda - \eta) = d(\lambda) = \prod_{n=1}^N \sinh(\lambda - \xi_n), \quad \det K^{(a,\alpha)} = (-1)^a \quad (4.12)$$

being the quantum determinant, a one-parameter family of central elements of the Yang-Baxter algebra, which admits the following quadratic form in the generator of the Yang-Baxter algebra:

$$q\text{-det } M^{(K)}(\lambda) = A^{(K)}(\lambda)D^{(K)}(\lambda - \eta) - B^{(K)}(\lambda)C^{(K)}(\lambda - \eta) \quad (4.13)$$

and three other equivalent ones. Moreover, $T^{(K)}(\lambda)$ is a trigonometric polynomial of degree 1 in all the ξ_n and of degree N in λ with the following leading operator coefficients:

$$T_{\pm N}(S_z) \equiv \lim_{\lambda \rightarrow \pm \infty} e^{\mp \lambda N} T^{(K^{(a,\alpha)})}(\lambda) = \delta_{0,a} \frac{(-1)^{(1 \mp 1) \frac{N}{2}} e^{\pm (\frac{\eta N}{2} - \sum_{n=1}^N \xi_n)}}{2^{N-1}} \cosh\left(\frac{\eta}{2} S_z \alpha\right), \quad (4.14)$$

where

$$S_z = \sum_{n=1}^N \sigma_n^z. \quad (4.15)$$

Moreover, it is interesting to present explicitly the interpolation formula for the transfer matrix which follows from the previous lemma:

Lemma 4.2. Let us take the generic $K^{(a,\alpha)} \neq vI$, for any $v \in \mathbb{C}$, and then the transfer matrix,

$$T^{(K^{(a,\alpha)})}(\lambda) = [\delta_{0,a}(e^\alpha A(\lambda) - e^{-\alpha} D(\lambda)) + \delta_{1,a}(e^\alpha C(\lambda) - e^{-\alpha} B(\lambda))], \quad (4.16)$$

for any choice of $\{h_1, \dots, h_N\} \in \{0, 1\}^N$ admits the following interpolation formula:

$$T^{(K^{(a,\alpha)})}(\lambda) = 2i\delta_{0,a} \frac{\cosh \alpha \cosh(\eta S_z/2)}{\sinh t_{h_1, \dots, h_N}} \prod_{n=1}^N \sinh(\lambda - \xi_n^{(h_n)}) + \sum_{n=1}^N \left(\frac{\sinh(t_{h_1, \dots, h_N} + \lambda - \xi_n^{(h_n)})}{\sinh t_{h_1, \dots, h_N}} \right)^{\delta_{0,a}} \prod_{b \neq n, b=1}^N \frac{\sinh(\lambda - \xi_b^{(h_b)})}{\sinh(\xi_n^{(h_n)} - \xi_b^{(h_b)})} T^{(K^{(a,\alpha)})}(\xi_n^{(h_n)}), \quad (4.17)$$

where we have defined

$$t_{h_1, \dots, h_N} = \eta \left(\frac{N}{2} - \sum_{n=1}^N h_n \right) + \frac{i\pi}{2}, \quad (4.18)$$

and moreover, it holds

$$\delta_{0,a} \sum_{\epsilon = \pm 1} \epsilon e^{\epsilon (\frac{\eta N}{2} - \sum_{n=1}^N h_n)} \cosh\left(\frac{\eta}{2} S_z + \epsilon \alpha\right) = \sum_{a=1}^N \frac{T^{(K^{(a,\alpha)})}(\xi_a^{(h_a)})}{\prod_{b \neq a, b=1}^N \sinh(\xi_a^{(h_a)} - \xi_b^{(h_b)})}, \quad (4.19)$$

which in the case $(a=0, \alpha=i\pi/2)$ reads

$$\sinh S_z = \frac{1}{2ik \cosh \eta (\frac{N}{2} - \sum_{n=1}^N h_n)} \sum_{a=1}^N \frac{T^{(K^{(a=0, \alpha=i\pi/2)})}(\xi_a^{(h_a)})}{\prod_{b \neq a, b=1}^N \sinh(\xi_a^{(h_a)} - \xi_b^{(h_b)})}. \quad (4.20)$$

Proof. The proof just follows matching the known leading asymptotic of the transfer matrix $T^{(K^{(a,\alpha)})}(\lambda)$ with that corresponding to the general interpolation formula,

$$T_{h_1, \dots, h_N} \prod_{n=1}^N \sinh(\lambda - \xi_n^{(h_n)}) + \sum_{n=1}^N \left(\frac{\sinh(t_{h_1, \dots, h_N} + \lambda - \xi_a^{(h_a)})}{\sinh t_{h_1, \dots, h_N}} \right)^{\delta_{0,a}} \times \prod_{b \neq n, b=1}^N \frac{\sinh(\lambda - \xi_b^{(h_b)})}{\sinh(\xi_n^{(h_n)} - \xi_b^{(h_b)})} T^{(K^{(a,\alpha)})}(\xi_n^{(h_n)}), \quad (4.21)$$

fixing t_{h_1, \dots, h_N} by (4.18). □

A. Our approach to the SoV basis

The general construction for the SoV basis applies to this special case. Let us see in this framework how it works.

Theorem 4.1. *For almost any choice of $\langle S|$, of $K^{(a,\alpha)} \neq vI$, for any $v \in \mathbb{C}$, and under the condition (4.9), then the following set of covectors*

$$\langle h_1, \dots, h_N | \equiv \langle S | \prod_{n=1}^N \left(\frac{T^{(K)}(\xi_n)}{(e^\alpha)^{\delta_{0,a}} a(\xi_n)} \right)^{h_n} \text{ for any } \{h_1, \dots, h_N\} \in \{0, 1\}^{\otimes N} \quad (4.22)$$

forms a covector basis of \mathcal{H} . In particular, we can take the state $\langle S|$ of the following tensor product form:

$$\langle S | = \bigotimes_{a=1}^N (x, y)_a, \quad (4.23)$$

asking that $xy \neq 0$ in the case $K^{(a=0,\alpha)} \neq vI$, for any $v \in \mathbb{C}$, while asking that $y \neq \pm e^\alpha x$ in the case $K^{(a=1,\alpha)}$.

Proof. The proof proceeds taking the limit of the trigonometric R -matrix toward the rational case as explained in the Subsection II D. In particular, we have a basis choosing the state of the above factorized form once the condition

$$\det \left\| \left((x, y) K^{i-1} e_j(a) \right)_{i,j \in \{1,2\}} \right\| = \det \begin{pmatrix} x & y \\ Ax + Cy & Bx + Dy \end{pmatrix} = Bx^2 + (D - A)xy + Cy^2 \quad (4.24)$$

is satisfied, where we have defined

$$A = \delta_{0,a} e^\alpha, \quad D = \delta_{0,a} e^{-\alpha}, \quad B = \delta_{1,a} e^\alpha, \quad C = \delta_{0,a} e^{-\alpha}, \quad (4.25)$$

which clearly leads to the given requirements on the components $x, y \in \mathbb{C}$ of the two dimensional covector. \square

B. Comparison with Sklyanin's SoV construction

Let us start recalling that Sklyanin's approach for the transfer matrices associated with the trigonometric 6-vertex Yang-Baxter algebra has been developed only for the case of a twist matrix of the form $K^{(a=1,\alpha)}$.^{32-35,38,39,49} In the elliptic 8-vertex Yang-Baxter algebra, it was possible to analyze also transfer matrices associated with twist matrix $K = I$ or σ^z ,^{34,40,50} however, only after the introduction of Baxter's like gauge transformations. It can be interesting to analyze if the trigonometric version of these Baxter's gauge transformations allow us also to describe in the generalized Sklyanin's approach the case $K = I$ or σ^z . We leave this question for further analysis; on the other hand, our approach to SoV applies for any $K^{(a,\alpha)} \neq vI$.

So the comparison can be made only in the case $K^{(a=1,\alpha)}$ for which we have the following lemma:

Lemma 4.3. *Let us fix $a = 1$ so that it holds*

$$A^{(K)}(\lambda) = e^\alpha C(\lambda), \quad B^{(K)}(\lambda) = e^\alpha D(\lambda), \quad (4.26)$$

$$C^{(K)}(\lambda) = e^{-\alpha} A(\lambda), \quad D^{(K)}(\lambda) = e^{-\alpha} B(\lambda), \quad (4.27)$$

and Sklyanin's SoV basis is the covector basis of $B^{(K)}(\lambda)$,

$$\langle h_1, \dots, h_N | B^{(K)}(\lambda) \equiv e^\alpha \prod_{a=1}^N \sinh(\lambda - \xi_a + h_a \eta) \langle h_1, \dots, h_N |, \quad (4.28)$$

where we have defined

$$\langle h_1, \dots, h_N | \equiv \langle 0 | \prod_{a=1}^N \left(\frac{A^{(K)}(\xi_a, \{\xi\})}{ia(\xi_a)} \right)^{h_a} \text{ for any } \{h_1, \dots, h_N\} \in \{0, 1\}^{\otimes N} \quad (4.29)$$

so that $B^{(K)}(\lambda)$ is diagonalizable with simple spectrum. Then our SoV approach reproduces Sklyanin's SoV approach, i.e., it holds

$$\langle h_1, \dots, h_N | = \langle \underline{h_1, \dots, h_N} | \text{ for any } \{h_1, \dots, h_N\} \in \{0, 1\}^{\otimes N} \quad (4.30)$$

as soon as we fix

$$\langle S | = \bigotimes_{a=1}^N (1, 0)_a. \quad (4.31)$$

Proof. The proof is done by induction just by the same steps of the rational 6-vertex case, by using the identity,

$$\langle 0 | D^{(K)}(\xi_a) = 0, \quad (4.32)$$

and the following Yang-Baxter commutation relations:

$$A^{(K)}(\mu)D^{(K)}(\lambda) = D^{(K)}(\lambda)A^{(K)}(\mu) + \frac{\sinh \eta}{\sinh(\lambda - \mu)} (B^{(K)}(\lambda)C^{(K)}(\mu) - B^{(K)}(\mu)C^{(K)}(\lambda)). \quad (4.33)$$

□

C. Transfer matrix spectrum in our SoV scheme

Let us show here how in our SoV scheme the transfer matrix spectrum is characterized. In our introductory Secs. I and II, we have anticipated that in our SoV basis the separate relations are given directly by the particularization of the fusion relations at the spectrum of the separate variables. Let us define

$$g_n^{(a)}(\lambda) = \left(\frac{\cosh(\eta N/2 + \lambda - \xi_n)}{\cosh \eta N/2} \right)^{\delta_{0,a}} \prod_{b \neq n, b=1}^N \frac{\sinh(\lambda - \xi_b)}{\sinh(\xi_n - \xi_b)}, \quad (4.34)$$

then the following theorem holds.

Theorem 4.2. *Let us assume that $K \neq vI$, for any $v \in \mathbb{C}$, and that the inhomogeneities $\{\xi_1, \dots, \xi_N\} \in \mathbb{C}^N$ satisfy the condition (4.9), then the spectrum of $T^{(K)}(\lambda)$ is characterized by*

$$\Sigma_{T^{(K)}} = \bigcup_{l \in \{-N, 2-N, \dots, +N-2, N\}} \Sigma_{T^{(K)}}^{(l)}, \quad (4.35)$$

where we have defined

$$\begin{aligned} \Sigma_{T^{(K)}}^{(l)} = & \left\{ t(\lambda) : t(\lambda) = 2\delta_{0,a} \frac{\cosh \alpha \cosh(\eta l/2)}{\cosh \eta N/2} \prod_{n=1}^N \sinh(\lambda - \xi_n) + \sum_{n=1}^N g_n^{(a)}(\lambda) x_n, \forall \{x_1, \dots, x_N\} \in \Sigma_T^{(l)} \right\}, \\ & (4.36) \end{aligned}$$

where $\Sigma_T^{(l)}$ is the set of solutions to the following inhomogeneous system of N quadratic equations:

$$\begin{aligned} x_n [2\delta_{0,a} \frac{\cosh \alpha \cosh(\eta l/2)}{\cosh \eta N/2} \prod_{b=1}^N \sinh(\xi_n - \xi_b - \eta) + \sum_{b=1}^N g_b^{(a)}(\xi_n - \eta) x_b] = \\ = a(\xi_n) d(\xi_n - \eta) \det K, \end{aligned} \quad (4.37)$$

in N unknown $\{x_1, \dots, x_N\}$, where the integer $l \in \{-N, 2-N, \dots, +N-2, N\}$ is fixed without ambiguity by the following sum rule:

$$\delta_{0,a} \sum_{\epsilon = \pm 1} \epsilon e^{\epsilon \frac{\eta N}{2}} \cosh(\frac{\eta}{2} l + \epsilon \alpha) = \sum_{a=1}^N \frac{x_a}{\prod_{b \neq a, b=1}^N \sinh(\xi_a - \xi_b)}. \quad (4.38)$$

Moreover, $T^{(K)}(\lambda, \{\xi\})$ has w -simple spectrum and for any $t(\lambda) \in \Sigma_{T^{(K)}}$ the associated unique (up-to normalization that we take to be given by $\langle S|t\rangle=1$) eigenvector has the following wave-function in the left SoV basis:

$$\langle h_1, \dots, h_N | t \rangle = \prod_{n=1}^N \left(\frac{t(\xi_n)}{(e^\alpha)^{\delta_{0,a}} a(\xi_n)} \right)^{h_n}. \quad (4.39)$$

Proof. Let us start observing that the inhomogeneous system of N quadratic equations (4.37) in N unknown $\{x_1, \dots, x_N\}$ is nothing else but the rewriting of the transfer matrix fusion equations,

$$t(\xi_a) t(\xi_a - \eta) = q \cdot \det M^{(K)}(\xi_a), \quad \forall a \in \{1, \dots, N\}, \quad (4.40)$$

for the set of all the trigonometric polynomials of degree N with asymptotic terms given by (4.14). Then, it is clear that any eigenvalue of the transfer matrix $T^{(K)}(\lambda, \{\xi\})$ is the solution of this system, satisfying, in particular, the associated sum rule, and that the associated right eigenvector $|t\rangle$ admits the characterization (4.39) in the left SoV basis.

So we are left with the proof of the reverse statement; i.e., any polynomial $t(\lambda)$ of the above form satisfying this system is an eigenvalue of the transfer matrix and this is proven by proving that the vector $|t\rangle$ characterized by (4.39) is a transfer matrix eigenstate; i.e., we have to show

$$\langle h_1, \dots, h_N | T^{(K)}(\lambda, \{\xi\}) | t \rangle = t(\lambda) \langle h_1, \dots, h_N | t \rangle, \quad \forall \{h_1, \dots, h_N\} \in \{0, 1\}^{\otimes N}. \quad (4.41)$$

Let us observe that

$$\langle h_1, \dots, h_n, \dots, h_N | T^{(K)}(\xi_n^{(h_n)}) | t \rangle = \begin{cases} (e^\alpha)^{\delta_{0,a}} a(\xi_n) \langle h_1, \dots, h'_n = 1, \dots, h_N | t \rangle & \text{if } h_n = 0 \\ q \cdot \det M^{(K)}(\xi_n) \frac{\langle h_1, \dots, h'_n = 0, \dots, h_N | t \rangle}{(e^\alpha)^{\delta_{0,a}} a(\xi_n)} & \text{if } h_n = 1, \end{cases} \quad (4.42)$$

which by the definition of the state $|t\rangle$ can be rewritten as

$$\langle h_1, \dots, h_n, \dots, h_N | T^{(K)}(\xi_n^{(h_n)}) | t \rangle = \begin{cases} t(\xi_n) \prod_{m \neq n, m=1}^N \left(\frac{t(\xi_m)}{(e^\alpha)^{\delta_{0,a}} a(\xi_m)} \right)^{h_m} & \text{if } h_n = 0 \\ \frac{q \cdot \det M^{(K)}(\xi_n)}{(e^\alpha)^{\delta_{0,a}} a(\xi_n)} \prod_{m \neq n, m=1}^N \left(\frac{t(\xi_m)}{(e^\alpha)^{\delta_{0,a}} a(\xi_m)} \right)^{h_m} & \text{if } h_n = 1, \end{cases} \quad (4.43)$$

and finally by Eq. (4.40) reads

$$\langle h_1, \dots, h_n, \dots, h_N | T^{(K)}(\xi_n^{(h_n)}) | t \rangle = \begin{cases} t(\xi_n) \prod_{n \neq a, n=1}^N \left(\frac{t(\xi_m)}{(e^\alpha)^{\delta_{0,a}} a(\xi_m)} \right)^{h_m} & \text{if } h_n = 0 \\ t(\xi_n - \eta) \prod_{m=1}^N \left(\frac{t(\xi_m)}{(e^\alpha)^{\delta_{0,a}} a(\xi_m)} \right)^{h_m} & \text{if } h_n = 1, \end{cases} \quad (4.44)$$

and so

$$\langle h_1, \dots, h_n, \dots, h_N | T^{(K)}(\xi_n^{(h_n)}) | t \rangle = t(\xi_n^{(h_n)}) \langle h_1, \dots, h_n, \dots, h_N | t \rangle. \quad (4.45)$$

Let us comment that the sum rule (4.38) implies that the trigonometric polynomial $t(\lambda)$ satisfies the following asymptotics:

$$\lim_{\lambda \rightarrow \pm\infty} e^{\mp \lambda N} t(\lambda) = \frac{(-1)^{(1 \mp 1) \frac{N}{2}} e^{\pm(\frac{\eta N}{2} - \sum_{n=1}^N \xi_n)}}{2^{N-1}} \cosh\left(\frac{\eta}{2} l \pm \alpha\right) \equiv T_{\pm N}(l), \quad (4.46)$$

from which it also follows that the following general sum rule is satisfied:

$$\sum_{a=1}^N \frac{t(\xi_a^{(h_a)})}{\prod_{b \neq a, b=1}^N \sinh(\xi_a^{(h_a)} - \xi_b^{(h_b)})} = \delta_{0,a} \sum_{\epsilon = \pm 1} \epsilon e^{\epsilon(\frac{\eta N}{2} - \sum_{n=1}^N h_n)} \cosh\left(\frac{\eta}{2} l + \epsilon \alpha\right). \quad (4.47)$$

Now by using this identity and the one for the transfer matrix

$$\sum_{a=1}^N \frac{T^{(K(a, \alpha))}(\xi_a^{(h_a)})}{\prod_{b \neq a, b=1}^N \sinh(\xi_a^{(h_a)} - \xi_b^{(h_b)})} = \delta_{0,a} \sum_{\epsilon = \pm 1} \epsilon e^{\epsilon(\frac{\eta N}{2} - \sum_{n=1}^N h_n)} \cosh\left(\frac{\eta}{2} S_z + \epsilon \alpha\right), \quad (4.48)$$

we obtain

$$\langle h_1, \dots, h_N | \left(\delta_{0,a} \sum_{\epsilon=\pm 1} \epsilon e^{\epsilon(\frac{\eta N}{2} - \sum_{n=1}^N h_n)} \cosh(\frac{\eta}{2} S_z + \epsilon \alpha) \right) | t \rangle \quad (4.49)$$

$$= \langle h_1, \dots, h_N | \left(\sum_{a=1}^N \frac{T^{(K^{(a,\alpha)})}(\xi_a^{(h_a)})}{\prod_{b \neq a, b=1}^N \sinh(\xi_a^{(h_a)} - \xi_b^{(h_b)})} \right) | t \rangle \quad (4.50)$$

$$= \left(\sum_{a=1}^N \frac{t(\xi_a^{(h_a)})}{\prod_{b \neq a, b=1}^N \sinh(\xi_a^{(h_a)} - \xi_b^{(h_b)})} \right) \langle h_1, \dots, h_N | t \rangle \quad (4.51)$$

$$= \left(\delta_{0,a} \sum_{\epsilon=\pm 1} \epsilon e^{\epsilon(\frac{\eta N}{2} - \sum_{n=1}^N h_n)} \cosh(\frac{\eta}{2} l + \epsilon \alpha) \right) \langle h_1, \dots, h_N | t \rangle \quad (4.52)$$

that is, $|t\rangle$ is an eigenvector of S_z with eigenvalue l

$$S_z |t\rangle = |t\rangle l, \quad (4.53)$$

for any $t(\lambda) \in \Sigma_{T(K)}^{(l)}$ with $a = 0$ and for any $\alpha \neq i n \pi$ with n integer. So that by using the interpolation formula for both the transfer matrix and the function $t(\lambda) \in \Sigma_{T(K)}^{(l)}$, we prove our theorem. \square

The previous characterization of the spectrum allows us to introduce an equivalent characterization in terms of a functional equation, the so-called quantum spectral curve equation, which in the case at hand is a second order Baxter's difference equation. Here, we present the functional equation reformulation of the SoV spectrum characterization only in the case of diagonal K -twist as this case cannot be directly derived in the standard SoV Sklyanin's approach, while it can be achieved with our new formulation of SoV. For the case of non-diagonal K -twist, the reformulation proven in the standard SoV³⁹ applies as well as, of course, to our present reformulation of the SoV basis.

Theorem 4.3. *Let us assume that the twist matrix has the form $K^{(a=0,\alpha)} \neq xI$, for any $x \in \mathbb{C}$, that the inhomogeneities $\{\xi_1, \dots, \xi_N\} \in \mathbb{C}^N$ satisfy the condition (4.9). Moreover, let us introduce the coefficients*

$$\alpha(\lambda) = \beta(\lambda)\beta(\lambda - \eta), \quad \beta(\lambda) = k_0 a(\lambda), \quad (4.54)$$

with k_0 the eigenvalue e^α of K and let $t(\lambda)$ be an entire function of λ , then $t(\lambda) \in \Sigma_{T(K)}^{(N-2M)}$ if and only if there exists a unique polynomial,

$$Q_t(\lambda) = \prod_{n=1}^M \sinh(\lambda - \lambda_n), \quad \text{with } M \leq N \text{ such that } \lambda_n \neq \xi_m + i r \pi, \quad \forall r \in \mathbb{Z}, \quad (4.55)$$

for any $(n, m) \in \{1, \dots, M\} \times \{1, \dots, N\}$, such that $t(\lambda)$ and $Q_t(\lambda)$ are the solutions of the following quantum spectral curve functional equation:

$$\alpha(\lambda)Q_t(\lambda - 2\eta) - \beta(\lambda)t(\lambda - \eta)Q_t(\lambda - \eta) + q\text{-det}M^{(K)}(\lambda, \{\xi\})Q_t(\lambda) = 0. \quad (4.56)$$

Moreover, up to a normalization the associated transfer matrix eigenvector $|t\rangle$ admits the following rewriting in the left SoV basis:

$$\langle h_1, \dots, h_N | t \rangle = \prod_{n=1}^N Q_t(\xi_n^{(h_n)}). \quad (4.57)$$

Proof. Let us start assuming the existence of $Q_t(\lambda)$ satisfying with $t(\lambda)$ the functional equation, which can be rewritten also in a Baxter-like form

$$k_0 a(\lambda)Q_t(\lambda - \eta) - t(\lambda)Q_t(\lambda) + k_1 d(\lambda)Q_t(\lambda + \eta) = 0, \quad (4.58)$$

where $k_1 = e^{-\alpha}$ is the second eigenvalue of K . Then it follows that $t(\lambda)$ is a trigonometric polynomial of degree N with the following leading coefficient:

$$\lim_{\lambda \rightarrow +\infty} e^{\mp \lambda N} t(\lambda) \equiv T_{\pm N}(N - 2M). \quad (4.59)$$

Now from the identities

$$q\text{-det}M^{(K)}(\xi_a + \eta, \{\xi\}) = \alpha(\xi_a) = 0, \quad (4.60)$$

we have that the functional equation reduces to the system of equations,

$$-t(\xi_a - \eta)Q_t(\xi_a - \eta) + k_1 d(\xi_a - \eta)Q_t(\xi_a) = 0, \quad (4.61)$$

$$k_0 a(\xi_a)Q_t(\xi_a - \eta) - t(\xi_a)Q_t(\xi_a) = 0, \quad (4.62)$$

once computed in the points ξ_a and $\xi_a + \eta$, from which it follows

$$t(\xi_a - \eta) \frac{t(\xi_a)Q_t(\xi_a)}{k_0 a(\xi_a)} = k_1 d(\xi_a - \eta)Q_t(\xi_a) \quad (4.63)$$

which being $Q_t(\xi_a) \neq 0$ implies that $t(\lambda)$ satisfies also the system of equations (4.40), for any $a \in \{1, \dots, N\}$, so that for the previous theorem $t(\lambda)$ is a transfer matrix eigenvalue belonging to $\Sigma_{T^{(K)}}^{(a=N-2M)}$.

Let us now prove the reverse statement; i.e., we assume that $t(\lambda)$ is a transfer matrix eigenvalue and we want to prove the existence of the polynomial $Q_t(\lambda)$ satisfying the functional equation. It is easy to remark that in the points $\xi_a - \eta$, for any $a \in \{1, \dots, N\}$, the functional equation is directly satisfied. Moreover, it is satisfied in the $2N$ points ξ_a and $\xi_a + \eta$, for any $a \in \{1, \dots, N\}$, if the system (4.61) and (4.62) is satisfied. This last system reduces to the system of the second N equations,

$$k_0 a(\xi_a)Q_t(\xi_a - \eta) = t(\xi_a)Q_t(\xi_a), \quad (4.64)$$

by the fusion equations satisfied by the transfer matrix eigenvalue $t(\lambda)$. Then, following similar steps to those used in the rational case, one can prove the existence and unicity of a polynomial $Q_t(\lambda)$ of the form

$$Q_t(\lambda) = \prod_{n=1}^R \sinh(\lambda - \lambda_n), \text{ with } R \leq N \text{ such that } \lambda_n \neq \xi_m + i\pi, \forall n, m \text{ and } f \in \mathbb{Z}, \quad (4.65)$$

see, for example, Ref. 39, the solution of the above system of equations. So we are left with the proof of the identity $R = M$ once $t(\lambda) \in \Sigma_{T^{(K)}}^{(N-2M)}$. In order to prove that, let us define

$$F_1(\lambda) = t(\lambda)Q_t(\lambda), \quad F_2(\lambda) = k_0 a(\lambda)Q_t(\lambda - \eta) + k_1 d(\lambda)Q_t(\lambda + \eta), \quad (4.66)$$

and these are two trigonometric polynomial of degree $N + R$ with the following asymptotic:

$$\lim_{\lambda \rightarrow +\infty} e^{\mp \lambda(N+R)} F_h(\lambda) = F_h^\pm \quad (4.67)$$

with $M_h = (\delta_{1,h}M + \delta_{2,h}R)$ and

$$F_h^\pm = \frac{e^{\pm(\frac{\eta N}{2} - \sum_{n=1}^N \xi_n - \sum_{n=1}^R \lambda_n)}}{(-1)^{(1 \mp 1)(N+R)} 2^{(N+R)-1}} \cosh\left(\frac{\eta}{2}(N - 2M_h) \pm \alpha\right). \quad (4.68)$$

Let us now consider the interpolation formulae for the $F_h(\lambda)$ in $N + R$ distinct points $\{x_1, \dots, x_{N+R}\}$ (as already shown in the case of the transfer matrix), then one can prove the following sum rules:

$$\sum_{\epsilon=\pm 1} \frac{\cosh(\frac{\eta}{2}(N - 2M_h) + \epsilon\alpha)}{e^{\epsilon(\sum_{n=1}^N \xi_n + \sum_{n=1}^R \lambda_n - \frac{\eta N}{2} - \sum_{n=1}^{N+R} x_n)}} = \sum_{n=1}^{N+R} \frac{F_h(x_n)}{\prod_{b \neq n, b=1}^N \sinh(x_n - x_b)}. \quad (4.69)$$

So that we obtain the identity

$$\sum_{\epsilon=\pm 1} \frac{\cosh(\frac{\eta}{2}(N - 2M) + \epsilon\alpha)}{e^{\epsilon(\sum_{n=1}^N \xi_n + \sum_{n=1}^R \lambda_n - \frac{\eta N}{2} - \sum_{n=1}^{N+R} x_n)}} = \sum_{\epsilon=\pm 1} \frac{\cosh(\frac{\eta}{2}(N - 2R) + \epsilon\alpha)}{e^{\epsilon(\sum_{n=1}^N \xi_n + \sum_{n=1}^R \lambda_n - \frac{\eta N}{2} - \sum_{n=1}^{N+R} x_n)}} \quad (4.70)$$

for any choice of $\{x_1, \dots, x_{N+R}\} \subset \{\xi_1, \dots, \xi_N, \xi_1 - \eta, \dots, \xi_N - \eta\}$ as the system (4.61) and (4.62) implies, in particular, the identities

$$F_1(x_n) = F_2(x_n), \forall n \in \{1, \dots, N + R\}. \quad (4.71)$$

Then the identity (4.70) implies $R = M$ being η , α , and ξ_n arbitrary.

So that we have shown that the lhs of the quantum spectral curve equation is zero in $3N$ different points plus the points at infinity, and this last statement being true as we have shown $R = M$ for $t(\lambda) \in \Sigma_{T(K)}^{(a=N-2M)}$. Now this lhs being a trigonometric polynomial in λ of maximal degree $2N + M$, with $M \leq N$, we have proven this functional equation.

Finally, the identities

$$\prod_{n=1}^N Q_t(\xi_n) \prod_{n=1}^N \left(\frac{t(\xi_n)}{k_0 a(\xi_n)} \right)^{h_n} = \prod_{n=1}^N Q_t(\xi_n^{(h_n)}) \quad (4.72)$$

imply our statement on the representation of the transfer matrix eigenstate in the left SoV basis. \square

D. Algebraic Bethe ansatz form of separate states: The diagonal case

Let us consider here the case $K^{(a=0,\alpha)} \neq xI$, for any $x \in \mathbb{C}$, which can be described also by algebraic Bethe ansatz and so it allows for a direct comparison with our SoV.

The following identity follows from direct calculation,

$$T^{(K^{(a=0,\alpha)})}(\lambda)|0\rangle = |0\rangle t_0(\lambda) \quad \text{with } t_0(\lambda) = e^\alpha a(\lambda) + e^{-\alpha} d(\lambda) \quad (4.73)$$

so that in our SoV covector basis the eigenstate $|0\rangle$ admits the following representation:

$$\langle h_1, \dots, h_N | t \rangle = 1, \quad (4.74)$$

i.e., the associated Q -function is the identity.

Let us now denote with $\mathbb{B}(\lambda)$ the one parameter family of commuting operators characterized by

$$\langle h_1, \dots, h_N | \mathbb{B}(\lambda) = b_{h_1, \dots, h_N}(\lambda) \langle h_1, \dots, h_N |, \quad (4.75)$$

where we have defined

$$b_{h_1, \dots, h_N}(\lambda) = \prod_{n=1}^N (\lambda - \xi_n^{(h_n)}), \quad (4.76)$$

then the following corollary holds:

Corollary 4.1. *Let us assume that the condition (3.5) is satisfied and that the $K^{(a=0,\alpha)} \neq xI$, for any $x \in \mathbb{C}$, then for any $M \leq N$ taken the generic $t(\lambda) \in \Sigma_{T(K)}^{(N-2M)}$ the unique associated eigenvector $|t\rangle$ admits the following ABA representation:*

$$|t\rangle = (-1)^{NM} \prod_{n=1}^M \mathbb{B}(\lambda_n) |0\rangle, \quad (4.77)$$

where $\{\lambda_1, \dots, \lambda_M\}$ are the Bethe roots, i.e., the zero of the associated Q -function.

Proof. Taken the generic $t(\lambda) \in \Sigma_{T(K)}^{(N-2M)}$, the unique associated eigenvector $|t\rangle$ admits the following SoV representation:

$$\langle h_1, \dots, h_N | t \rangle = \prod_{n=1}^N Q_t(\xi_n^{(h_n)}), \quad (4.78)$$

while by the definition of $\mathbb{B}(\lambda)$ it holds

$$\langle h_1, \dots, h_N | (-1)^{NM} \prod_{n=1}^M \mathbb{B}(\lambda_n) |0\rangle = (-1)^{NM} \prod_{n=1}^M b_{h_1, \dots, h_N}(\lambda_n), \quad (4.79)$$

from which our statement follows being

$$(-1)^{NM} \prod_{n=1}^M b_{h_1, \dots, h_N}(\lambda_n) = \prod_{n=1}^N Q_t(\xi_n^{(h_n)}). \quad (4.80)$$

\square

V. THE QUASI-PERIODIC $Y(\mathfrak{gl}_3)$ FUNDAMENTAL MODEL

We consider now the Yang-Baxter algebra associated with the rational $\mathfrak{gl}(3)$ R-matrix,

$$R_{a,b}(\lambda) = \lambda I_{a,b} + \eta \mathbb{P}_{a,b} = \begin{pmatrix} a_1(\lambda) & b_1 & b_2 \\ c_1 & a_2(\lambda) & b_3 \\ c_2 & c_3 & a_3(\lambda) \end{pmatrix} \in \text{End}(V_a \otimes V_b), \quad (5.1)$$

where $V_a \cong V_b \cong \mathbb{C}^3$ and we have defined

$$\begin{aligned} a_j(\lambda) &= \begin{pmatrix} \lambda + \eta \delta_{j,1} & 0 & 0 \\ 0 & \lambda + \eta \delta_{j,2} & 0 \\ 0 & 0 & \lambda + \eta \delta_{j,3} \end{pmatrix}, \quad \forall j \in \{1, 2, 3\}, \\ b_1 &= \begin{pmatrix} 0 & 0 & 0 \\ \eta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \eta & 0 & 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \eta & 0 \end{pmatrix}, \\ c_1 &= \begin{pmatrix} 0 & \eta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & 0 & \eta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \eta \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (5.2)$$

which satisfies the Yang-Baxter equation,

$$R_{12}(\lambda - \mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda - \mu). \quad (5.3)$$

This R-matrix satisfies the following symmetry properties (scalar Yang-Baxter equation):

$$R_{12}(\lambda) K_1 K_2 = K_2 K_1 R_{12}(\lambda) \in \text{End}(V_1 \otimes V_2), \quad (5.4)$$

where $K \in \text{End}(V)$ is any 3×3 matrix. We can define the following monodromy matrix:

$$M_a^{(K)}(\lambda) = \begin{pmatrix} A_1^{(K)}(\lambda) & B_1^{(K)}(\lambda) & B_2^{(K)}(\lambda) \\ C_1^{(K)}(\lambda) & A_2^{(K)}(\lambda) & B_3^{(K)}(\lambda) \\ C_2^{(K)}(\lambda) & C_3^{(K)}(\lambda) & A_3^{(K)}(\lambda) \end{pmatrix}_a \quad (5.5)$$

$$\equiv K_a R_{a,N}(\lambda - \xi_N) \cdots R_{a,1}(\lambda - \xi_1) \in \text{End}(V_a \otimes \mathcal{H}), \quad (5.6)$$

where $\mathcal{H} = \bigotimes_{n=1}^N V_n$. Moreover, in the following, we will assume that the inhomogeneity parameters ξ_j satisfy the following conditions:

$$\xi_a \neq \xi_b + r\eta, \quad \forall a \neq b \in \{1, \dots, N\} \text{ and } r \in \{-2, -1, 0, 1, 2\}. \quad (5.7)$$

A. First fundamental properties of the transfer matrices

Let us first recall some basic properties of the transfer matrices associated with this higher rank case summarized in the following two propositions (see Ref. 7).

Proposition 5.1. The transfer matrices,

$$T_1^{(K)}(\lambda) \equiv \text{tr}_a M_a^{(K)}(\lambda), \quad T_2^{(K)}(\lambda) \equiv \text{tr}_a U_a^{(K)}(\lambda), \quad (5.8)$$

where

$$U_c^{(K)}(\lambda)^t \equiv 3 \text{tr}_{ab} P_{abc}^- M_a^{(K)}(\lambda) M_b^{(K)}(\lambda + \eta), \quad (5.9)$$

defines two one parameter families of commuting operators,

$$\left[T_1^{(K)}(\lambda), T_1^{(K)}(\mu) \right] = \left[T_1^{(K)}(\lambda), T_2^{(K)}(\mu) \right] = \left[T_2^{(K)}(\lambda), T_2^{(K)}(\mu) \right] = 0. \quad (5.10)$$

Moreover, the quantum determinant,

$$q\text{-det} M^{(K)}(\lambda) \cdot \mathbf{1}_a \equiv \text{tr}_{123} (P_{123}^- M_1^{(K)}(\lambda) M_2^{(K)}(\lambda + \eta) M_3^{(K)}(\lambda + 2\eta)) \cdot \mathbf{1}_a \quad (5.11)$$

$$= \left(M_a^{(K)}(\lambda) \right)^{t_a} U_a^{(K)}(\lambda + \eta) = U_a^{(K)}(\lambda + \eta) \left(M_a^{(K)}(\lambda) \right)^{t_a} \quad (5.12)$$

$$= \left(U_a^{(K)}(\lambda) \right)^{t_a} M_a^{(K)}(\lambda + 2\eta) = M_a^{(K)}(\lambda + 2\eta) \left(U_a^{(K)}(\lambda) \right)^{t_a}, \quad (5.13)$$

is a central element of the algebra, i.e.,

$$[q - \det M^{(K)}(\lambda), M_a^{(K)}(\mu)] = 0. \quad (5.14)$$

Moreover, it holds

Proposition 5.2. The quantum spectral invariants have the following polynomial form:

- (i) $T_1^{(K)}(\lambda)$ is a degree N polynomial in λ with the following central asymptotic:

$$\lim_{\lambda \rightarrow \infty} \lambda^{-N} T_1^{(K)}(\lambda) = \text{tr} K.$$

- (ii) $T_2^{(K)}(\lambda)$ is a degree $2N$ polynomial in λ with the following N central zeros and asymptotic:

$$T_2^{(K)}(\xi_a) = 0, \forall a \in \{1, \dots, N\}, \lim_{\lambda \rightarrow \infty} \lambda^{-2N} T_2^{(K)}(\lambda) = \frac{(\text{tr} K)^2 - \text{tr} K^2}{2}. \quad (5.15)$$

- (iii) The quantum determinant reads

$$q - \det M^{(K)}(\lambda) = \det K \prod_{b=1}^N (\lambda - \xi_b)(\lambda + \eta - \xi_b)(\lambda + 3\eta - \xi_b). \quad (5.16)$$

Moreover, the following fusion identities hold:

$$T_1^{(K)}(\xi_a) T_2^{(K)}(\xi_a - 2\eta) = q - \det M^{(K)}(\xi_a - 2\eta), \quad (5.17)$$

$$T_1^{(K)}(\xi_a - \eta) T_1^{(K)}(\xi_a) = T_2^{(K)}(\xi_a - \eta). \quad (5.18)$$

Proof. These trivial roots of $T_2^{(K)}$ and fusion identities can be obtained from the general fusion properties of the transfer matrices in the various representations^{51,52} when computed in the special points associated with the inhomogeneities and by direct calculation using the reduction of the R -matrix to permutation and projection operators. \square

Let us introduce the functions

$$g_{a,\mathbf{h}}(\lambda) = \prod_{b \neq a, b=1}^N \frac{\lambda - \xi_b^{(h)}}{\xi_a^{(h)} - \xi_b^{(h)}}, \quad \xi_b^{(h)} = \xi_b - h\eta, \quad (5.19)$$

$$f_{a,\mathbf{h}}(\lambda) = g_{a,\mathbf{h}}(\lambda) \prod_{b=1}^N \frac{\lambda - \xi_b}{\xi_a^{(h_a)} - \xi_b}, \quad (5.20)$$

and

$$T_{2,\mathbf{h}}^{(K,\infty)}(\lambda) = \frac{(\text{tr} K)^2 - \text{tr} K^2}{2} \prod_{n=1}^N (\lambda - \xi_n)(\lambda - \xi_n^{(h_n)}), \quad (5.21)$$

then the following corollary holds.

Corollary 5.1. The transfer matrix $T_2^{(K)}(\lambda)$ is completely characterized in terms of $T_1^{(K)}(\lambda)$ by the fusion equations, and the following interpolation formula holds:

$$T_2^{(K)}(\lambda) = T_{2,\mathbf{h}=1}^{(K,\infty)}(\lambda) + \sum_{a=1}^N f_{a,\mathbf{h}=1}(\lambda) T_1^{(K)}(\xi_a - \eta) T_1^{(K)}(\xi_a). \quad (5.22)$$

Proof. The known central zeros and asymptotic behavior imply the above interpolation formula once we use the fusion equations to write $T_2^{(K)}(\xi_a - \eta)$. \square

B. The new SoV covector basis

Our general construction of the SoV covector basis applies, in particular, to the fundamental representation of the $gl(3)$ rational Yang-Baxter algebra.

Let K be a 3×3 w -simple matrix and let us denote by K_J an upper-triangular Jordan form of the matrix K and W the invertible matrix defining the change of basis,

$$K = W_K K_J W_K^{-1} \quad \text{with} \quad K_J = \begin{pmatrix} k_0 & y_1 & 0 \\ 0 & k_1 & y_2 \\ 0 & 0 & k_2 \end{pmatrix}. \quad (5.23)$$

The requirement that K is w -simple implies that we can have only the following three possible cases (up to trivial permutations of the basis vectors):

$$(i) \quad k_i \neq k_j \quad \forall i, j \in \{0, 1, 2\}, \quad y_1 = y_2 = 0, \quad (5.24)$$

$$(ii) \quad k_0 = k_1 \neq k_2, \quad y_1 = 1, \quad y_2 = 0, \quad (5.25)$$

$$(iii) \quad k_0 = k_1 = k_2, \quad y_1 = 1, \quad y_2 = 1, \quad (5.26)$$

then the following theorem holds:

Proposition 5.3. *Let K be a 3×3 w -simple matrix, then for almost any choice of $\langle S|$ and of the inhomogeneities under the condition (5.7), the following set of covectors*

$$\langle h_1, \dots, h_N | \equiv \langle S | \prod_{n=1}^N (T_1^{(K)}(\xi_n))^{h_n} \quad \text{for any } \{h_1, \dots, h_N\} \in \{0, 1, 2\}^{\otimes N} \quad (5.27)$$

forms a covector basis of \mathcal{H}^ . In particular, we can take the state $\langle S|$ of the following tensor product form:*

$$\langle S | = \bigotimes_{a=1}^N (x, y, z)_a \Gamma_W^{-1}, \quad \Gamma_W = \bigotimes_{a=1}^N W_{K,a}, \quad (5.28)$$

simply asking $x y z \neq 0$ in case (i), $x z \neq 0$ in case (ii), and $x \neq 0$ in case (iii).

Proof. As shown in Proposition 2.4, the fact that the set of covectors is a covector basis of \mathcal{H}^* reduces to the requirement that the covectors

$$(x, y, z)_a W^{-1}, (x, y, z)_a W^{-1} K, (x, y, z)_a W^{-1} K^2, \quad (5.29)$$

or equivalently

$$(x, y, z)_a, (x, y, z)_a K_J, (x, y, z)_a K_J^2, \quad (5.30)$$

form a basis in $V_a \cong \mathbb{C}^3$; that is, the following determinant is non-zero:

$$\det || \left((x, y, z) K_J^{i-1} | e_j \right)_{i,j \in \{1,2,3\}} || = \begin{cases} -xyzV(k_0, k_1, k_2) & \text{in case (i)} \\ x^2 z V^2(k_0, k_2) & \text{in case (ii)} \\ x^3 & \text{in case (iii)}, \end{cases} \quad (5.31)$$

where $|e_j\rangle$ is the canonical basis in $V_a \cong \mathbb{C}^3$ after the change of basis induced by W_K . It leads to the given requirements on the components $x, y, z \in \mathbb{C}$ of the three dimensional covector. \square

C. Transfer matrix spectrum in our SoV scheme

The following characterization of the transfer matrix spectrum holds:

Theorem 5.1. *Under the same assumptions ensuring that the set of SoV covector form a basis, then the spectrum of $T_1^{(K)}(\lambda)$ is characterized by*

$$\Sigma_{T^{(K)}} = \left\{ t_1(\lambda) : t_1(\lambda) = \text{tr} K \prod_{a=1}^N (\lambda - \xi_a) + \sum_{a=1}^N g_{a,h=0}(\lambda) x_a, \quad \forall \{x_1, \dots, x_N\} \in \Sigma_T \right\}, \quad (5.32)$$

where Σ_T is the set of solutions to the following inhomogeneous system of N cubic equations:

$$x_a [T_{2,\mathbf{h}=1}^{(K,\infty)}(\xi_a - 2\eta) + \sum_{n=1}^N f_{n,\mathbf{h}=1}(\xi_a - 2\eta) t_1(\xi_n - \eta) x_n] = q \text{-det} M^{(K)}(\xi_a - 2\eta), \quad (5.33)$$

in N unknown $\{x_1, \dots, x_N\}$. Moreover, $T_1^{(K)}(\lambda, \{\xi\})$ is w -simple, and for any $t_1(\lambda) \in \Sigma_{T^{(K)}}$, the associated unique (up-to normalization) eigenvector $|t\rangle$ has the following wave function in the left SoV basis:

$$\langle h_1, \dots, h_N | t \rangle = \prod_{n=1}^N t_1^{h_n}(\xi_n). \quad (5.34)$$

Proof. Let us start observing that the inhomogeneous system of N cubic equations in N unknown $\{x_1, \dots, x_N\}$ is nothing else but the rewriting of the transfer matrix fusion equations,

$$t_1(\xi_a) t_2(\xi_a - 2\eta) = q \text{-det} M^{(K)}(\xi_a - 2\eta), \quad \forall a \in \{1, \dots, N\}, \quad (5.35)$$

for the eigenvalues of the transfer matrices $T_{1,2}^{(K)}(\lambda)$. So that this system has to be satisfied and the associated eigenvector $|t\rangle$ admits the given characterization in the covector SoV basis.

So we are left with the proof of the reverse statement; i.e., any polynomial $t_1(\lambda)$ of the above form satisfying this system is an eigenvalue of the transfer matrices and this is proven by showing that the vector $|t\rangle$ characterized by (5.34) is a transfer matrix eigenstate; i.e., we have to show

$$\langle h_1, \dots, h_N | T_1^{(K)}(\lambda) | t \rangle = t_1(\lambda) \langle h_1, \dots, h_N | t \rangle, \quad \forall \{h_1, \dots, h_N\} \in \{0, 1, 2\}^{\otimes N}. \quad (5.36)$$

Let us start observing that by using the interpolation formula

$$T_1^{(K)}(\lambda) = T_{1,\mathbf{h}}^{(K,\infty)}(\lambda) + \sum_{a=1}^N g_{a,\mathbf{h}}(\lambda) T_1^{(K)}(\xi_a^{(h_a)}), \quad T_{1,\mathbf{h}}^{(K,\infty)}(\lambda) = \text{tr} K \prod_{a=1}^N (\lambda - \xi_a^{(h_a)}), \quad (5.37)$$

the above statement is proven once we prove the identity in the points $\xi_a^{(h_a)}$ for any $a \in \{1, \dots, N\}$. Let $h_a = 0, 1$ and $h_b \in \{0, 1, 2\}$ for any $b \in \{1, \dots, N\} \setminus a$, then we have the following identities:

$$\begin{aligned} \langle h_1, \dots, h_N | T_1^{(K)}(\xi_a) | t \rangle &= \langle h_1, \dots, h_a + 1, \dots, h_N | t \rangle \\ &= t_1(\xi_a) \langle h_1, \dots, h_a, \dots, h_N | t \rangle, \end{aligned} \quad (5.38)$$

as a direct consequence of the definition of the covector SoV basis and of the state $|t\rangle$. So that we are left with the proof of the statement in the case $h_a = 2$. In this case, we want to prove that it holds

$$\langle h_1, \dots, h_N | T_1^{(K)}(\xi_a - \eta) | t \rangle = t_1(\xi_a - \eta) \langle h_1, \dots, h_a, \dots, h_N | t \rangle, \quad (5.39)$$

and the proof is done by induction on the number R of zeros contained in $\{h_1, \dots, h_N\} \in \{0, 1, 2\}^{\otimes N}$. Let us observe that by the fusion identities it holds

$$\langle h_1, \dots, h_a = 2, \dots, h_N | T_1^{(K)}(\xi_a - \eta) | t \rangle = \langle h_1, \dots, h_a = 1, \dots, h_N | T_2^{(K)}(\xi_a - \eta) | t \rangle. \quad (5.40)$$

Let us start to prove our identity for $R = 0$. We can use the following interpolation formula:

$$T_2^{(K)}(\xi_a - \eta) = T_{2,\mathbf{h}}^{(K,\infty)}(\xi_a - \eta) + \sum_{n=1}^N f_{n,\mathbf{h}=2}(\xi_a - \eta) T_2^{(K)}(\xi_n - 2\eta) \quad (5.41)$$

so that

$$\langle h_1, \dots, h'_a = 1, \dots, h_N | T_2^{(K)}(\xi_a - \eta) | t \rangle = T_{2,\mathbf{h}}^{(K,\infty)}(\xi_a - \eta) \langle h_1, \dots, h'_a, \dots, h_N | t \rangle \quad (5.42)$$

$$+ \sum_{n=1}^N f_{n,\mathbf{h}=2}(\xi_a - \eta) \langle h_1, \dots, h'_a, \dots, h_N | T_2^{(K)}(\xi_n - 2\eta) | t \rangle. \quad (5.43)$$

Then, by the assumption $R = 0$ and by the fusion identity, it follows that

$$\begin{aligned} \langle h_1, \dots, h'_a, \dots, h_N | T_2^{(K)}(\xi_a - \eta) | t \rangle &= T_{2, \mathbf{h}=2}^{(K, \infty)}(\xi_a - \eta) \langle h_1, \dots, h'_a, \dots, h_N | t \rangle \\ &+ \sum_{n=1}^N q\text{-det} M^{(K)}(\xi_n - 2\eta) f_{n, \mathbf{h}=2}(\xi_a - \eta) \langle h_1, \dots, h'_n, \dots, h_N | t \rangle, \end{aligned} \quad (5.44)$$

where $h''_n = h_n - 1$ for $n \neq a$ and $h''_a = h'_a - 1 = 0$. Let us now define the function

$$t_2(\lambda) = T_{2, \mathbf{h}=1}^{(K, \infty)}(\lambda) + \sum_{n=1}^N f_{n, \mathbf{h}=1}(\lambda) t_1(\xi_n - \eta) t_1(\xi_n), \quad (5.45)$$

and then by its definition, it satisfies the equations

$$t_2(\xi_n - \eta) = t_1(\xi_n - \eta) t_1(\xi_n), \quad \forall n \in \{1, \dots, N\}, \quad (5.46)$$

$$t_1(\xi_a) t_2(\xi_a - 2\eta) = q\text{-det} M^{(K)}(\xi_a - 2\eta), \quad \forall n \in \{1, \dots, N\}, \quad (5.47)$$

while the second (quantum determinant) equation is satisfied by the definition of the function $t_1(\lambda)$. Using the function $t_2(\lambda)$ and these identities, we get

$$\begin{aligned} \langle h_1, \dots, h'_a, \dots, h_N | T_2^{(K)}(\xi_a - \eta) | t \rangle &= \\ &= \left(T_{2, \mathbf{h}=2}^{(K, \infty)}(\xi_a - \eta) + \sum_{n=1}^N t_2(\xi_n - 2\eta) f_{n, \mathbf{h}=2}(\xi_a - \eta) \right) \langle h_1, \dots, h'_a, \dots, h_N | t \rangle \end{aligned} \quad (5.48)$$

$$= t_2(\xi_n - \eta) \langle h_1, \dots, h'_a, \dots, h_N | t \rangle \quad (5.49)$$

$$= t_1(\xi_n - \eta) \langle h_1, \dots, h_a = 2, \dots, h_N | t \rangle, \quad (5.50)$$

where we have used the interpolation formula

$$t_2(\xi_a - \eta) = T_{2, \mathbf{h}=2}^{(K, \infty)}(\xi_a - \eta) + \sum_{n=1}^N t_2(\xi_n - 2\eta) f_{n, \mathbf{h}=2}(\xi_a - \eta), \quad (5.51)$$

i.e., we have shown our identity (5.39) for $R = 0$. Let us now make the proof by induction assuming that it holds for generic $\{h_1, \dots, h_N\} \in \{0, 1, 2\}^{\otimes N}$ containing $R - 1$ zeros. Then we have to show the same property for generic $\{h_1, \dots, h_N\} \in \{0, 1, 2\}^{\otimes N}$ containing R zeros. Let us fix the generic $\{h_1, \dots, h_N\} \in \{0, 1, 2\}^{\otimes N}$ with $h_a = 2$ and let us denote with π a permutation of $\{1, \dots, N\}$ such that

$$\begin{aligned} h_{\pi(i)} &= 0, \quad \forall i \in \{1, \dots, R\}, \\ h_{\pi(i)} &= 1, \quad \forall i \in \{R + 1, \dots, R + S\}, \\ h_{\pi(i)} &= 2, \quad \forall i \in \{R + S + 1, \dots, N\}, \end{aligned} \quad (5.52)$$

with $a = \pi(R + S + 1)$. Let us use now the following interpolation formula:

$$T_2^{(K)}(\xi_a - \eta) = T_{2, \mathbf{k}}^{(K, \infty)}(\xi_a - \eta) + \sum_{n=1}^N f_{n, \mathbf{k}}(\xi_a - \eta) T_2^{(K)}(\xi_n^{(k_n)}), \quad (5.53)$$

where we have defined \mathbf{k} by

$$\begin{aligned} k_{\pi(i)} &= 1, \quad \forall i \in \{1, \dots, R\}, \\ k_{\pi(i)} &= 2, \quad \forall i \in \{R + 1, \dots, N\}, \end{aligned} \quad (5.54)$$

and then it holds

$$\begin{aligned} \langle h_1, \dots, h'_a = 1, \dots, h_N | T_2^{(K)}(\xi_a - \eta) | t \rangle &= T_{2, \mathbf{k}}^{(K, \infty)}(\xi_a - \eta) \langle h_1, \dots, h'_a, \dots, h_N | t \rangle \\ &+ \sum_{n=1}^R f_{\pi(n), \mathbf{k}}(\xi_a - \eta) \langle h_1, \dots, h'_a, \dots, h_N | T_2^{(K)}(\xi_{\pi(n)} - \eta) | t \rangle \\ &+ \sum_{n=R+1}^N f_{\pi(n), \mathbf{k}}(\xi_a - \eta) \langle h_1, \dots, h'_a, \dots, h_N | T_2^{(K)}(\xi_{\pi(n)} - 2\eta) | t \rangle. \end{aligned} \quad (5.55)$$

Using the fusion identity, we get

$$\begin{aligned} \langle h_1, \dots, h'_a, \dots, h_N | T_2^{(K)}(\xi_a - \eta) | t \rangle &= T_{2,\mathbf{k}}^{(K,\infty)}(\xi_a - \eta) \langle h_1, \dots, h'_a, \dots, h_N | t \rangle \\ &+ \sum_{n=1}^R f_{\pi(n),\mathbf{k}}(\xi_a - \eta) \langle h_1^{(n)}, \dots, h_N^{(n)} | T_1^{(K)}(\xi_{\pi(n)} - \eta) | t \rangle \\ &+ \sum_{n=R+1}^N q\text{-det} M^{(K)}(\xi_{\pi(n)} - 2\eta) f_{\pi(n),\mathbf{k}}(\xi_a - \eta) \langle h_1^{(n)}, \dots, h_N^{(n)} | t \rangle, \end{aligned} \quad (5.56)$$

where we have defined

$$h_{\pi(m)}^{(n)} = \begin{cases} h_{\pi(m)} + \theta(R-m)\delta_{m,n} & \text{for } n \leq R \\ h_{\pi(m)} - \theta(m-(R+1))\delta_{m,n} - \delta_{m,R+S+1} & \text{for } R+1 \leq n. \end{cases} \quad (5.57)$$

To compute $\langle h_1^{(n)}, \dots, h_N^{(n)} | T_1^{(K)}(\xi_{\pi(n)} - \eta) | t \rangle$ for $n \leq R$, we use the following interpolation formula:

$$T_1^{(K)}(\xi_{\pi(n)} - \eta) = T_{1,\mathbf{k}'}^{(K,\infty)}(\xi_{\pi(n)} - \eta) + \sum_{a=1}^N g_{a,\mathbf{k}'}(\xi_a - \eta) T_1^{(K)}(\xi_a^{(k'_a)}), \quad (5.58)$$

where we have defined

$$k'_{\pi(m)} = \begin{cases} 0 & \text{for } m \leq R+S+1 \\ 1 & \text{for } R+S+2 \leq m, \end{cases} \quad (5.59)$$

which gives

$$\begin{aligned} \langle h_1^{(n)}, \dots, h_N^{(n)} | T_1^{(K)}(\xi_{\pi(n)} - \eta) | t \rangle &= T_{1,\mathbf{k}'}^{(K,\infty)}(\xi_{\pi(n)} - \eta) \langle h_1^{(n)}, \dots, h_N^{(n)} | t \rangle \\ &+ \sum_{a=1}^{R+S+1} g_{\pi(a),\mathbf{k}'}(\xi_{\pi(n)} - \eta) \langle h_1^{(n)}, \dots, h_N^{(n)} | T_1^{(K)}(\xi_{\pi(a)} - \eta) | t \rangle \\ &+ \sum_{n=R+S+2}^N g_{\pi(a),\mathbf{k}'}(\xi_{\pi(n)} - \eta) \langle h_1^{(n)}, \dots, h_N^{(n)} | T_1^{(K)}(\xi_{\pi(a)} - \eta) | t \rangle, \end{aligned} \quad (5.60)$$

which becomes

$$\begin{aligned} \langle h_1^{(n)}, \dots, h_N^{(n)} | T_1^{(K)}(\xi_{\pi(n)} - \eta) | t \rangle &= T_{1,\mathbf{k}'}^{(K,\infty)}(\xi_{\pi(n)} - \eta) \langle h_1^{(n)}, \dots, h_N^{(n)} | t \rangle \\ &+ \sum_{a=1}^{R+S+1} g_{\pi(a),\mathbf{k}'}(\xi_{\pi(n)} - \eta) t_1(\xi_{\pi(a)} - \eta) \langle h_1^{(n)}, \dots, h_N^{(n)} | t \rangle \\ &+ \sum_{n=R+S+2}^N g_{\pi(a),\mathbf{k}'}(\xi_{\pi(n)} - \eta) t_1(\xi_{\pi(a)} - \eta) \langle h_1^{(n)}, \dots, h_N^{(n)} | t \rangle, \end{aligned} \quad (5.61)$$

where in the second line we have used the identity (5.38), while in the third line we have used the identity (5.39), which holds by assumption being $R-1$ the number of zeros in $\{h_1^{(n)}, \dots, h_N^{(n)}\}$. So that we have shown for any $n \leq R$,

$$\langle h_1^{(n)}, \dots, h_N^{(n)} | T_1^{(K)}(\xi_{\pi(n)} - \eta) | t \rangle = t_1(\xi_{\pi(n)} - \eta) \langle h_1^{(n)}, \dots, h_N^{(n)} | t \rangle, \quad (5.62)$$

and substituting it in (5.60), we get

$$\begin{aligned} \langle h_1, \dots, h'_a, \dots, h_N | T_2^{(K)}(\xi_a - \eta) | t \rangle &= T_{2,\mathbf{k}}^{(K,\infty)}(\xi_a - \eta) \langle h_1, \dots, h'_a, \dots, h_N | t \rangle \\ &+ \sum_{n=1}^R t_1(\xi_{\pi(n)} - \eta) f_{\pi(n),\mathbf{k}}(\xi_a - \eta) \langle h_1^{(n)}, \dots, h_N^{(n)} | t \rangle \\ &+ \sum_{n=R+1}^N q\text{-det} M^{(K)}(\xi_{\pi(n)} - 2\eta) f_{\pi(n),\mathbf{k}}(\xi_a - \eta) \langle h_1^{(n)}, \dots, h_N^{(n)} | t \rangle, \end{aligned} \quad (5.63)$$

and so $\langle h_1, \dots, h'_a, \dots, h_N | T_2^{(K)}(\xi_a - \eta) | t \rangle$ reads

$$\begin{aligned} & \left(T_{2,\mathbf{k}}^{(K,\infty)}(\xi_a - \eta) + \sum_{n=1}^R t_1(\xi_{\pi(n)}) t_1(\xi_{\pi(n)} - \eta) f_{\pi(n),\mathbf{k}}(\xi_a - \eta) + \sum_{n=R+1}^N t_2(\xi_{\pi(n)} - 2\eta) f_{\pi(n),\mathbf{k}}(\xi_a - \eta) \right) \\ & \times \langle h_1, \dots, h'_a, \dots, h_N | t \rangle \\ & = t_2(\xi_a - \eta) \langle h_1, \dots, h'_a = 1, \dots, h_N | t \rangle = t_1(\xi_a - \eta) \langle h_1, \dots, h_a = 2, \dots, h_N | t \rangle, \end{aligned} \quad (5.64)$$

i.e., we have proven our formula (5.39). Finally, taking generic $\{h_1, \dots, h_N\} \in \{0, 1, 2\}^{\otimes N}$ with

$$\begin{aligned} h_{\pi(i)} &= 0, \quad \forall i \in \{1, \dots, R\}, \\ h_{\pi(i)} &= 1, \quad \forall i \in \{R+1, \dots, R+S\}, \\ h_{\pi(i)} &= 2, \quad \forall i \in \{R+S+1, \dots, N\} \end{aligned} \quad (5.65)$$

and by using the interpolation formula

$$T_1^{(K)}(\lambda) = T_{1,\mathbf{p}}^{(K,\infty)}(\lambda) + \sum_{n=1}^N g_{n,\mathbf{p}}(\lambda) T_1^{(K)}(\xi_n^{(p_n)}), \quad (5.66)$$

where we have defined \mathbf{p} by

$$\begin{aligned} p_{\pi(i)} &= 0, \quad \forall i \in \{1, \dots, R+S\}, \\ p_{\pi(i)} &= 1, \quad \forall i \in \{R+S+1, \dots, N\}, \end{aligned} \quad (5.67)$$

we get

$$\begin{aligned} \langle h_1, \dots, h_N | T_1^{(K)}(\lambda) | t \rangle &= T_{1,\mathbf{p}}^{(K,\infty)}(\lambda) \langle h_1, \dots, h_N | t \rangle \\ &+ \sum_{n=1}^R g_{\pi(n),\mathbf{p}}(\lambda) \langle h_1, \dots, h_N | T_1^{(K)}(\xi_{\pi(n)}) | t \rangle \\ &+ \sum_{n=R+1}^N g_{\pi(n),\mathbf{p}}(\lambda) \langle h_1, \dots, h_N | T_1^{(K)}(\xi_{\pi(n)} - \eta) | t \rangle. \end{aligned} \quad (5.68)$$

Then, using in the second line the identity (5.38) and (5.39) in the third line, we get

$$\langle h_1, \dots, h_N | T_1^{(K)}(\lambda) | t \rangle = \left(T_{1,\mathbf{p}}^{(K,\infty)}(\lambda) + \sum_{n=1}^N g_{\pi(n),\mathbf{p}}(\lambda) t_1(\xi_{\pi(n)}^{(p_{\pi(n)})} - \eta) \right) \langle h_1, \dots, h_N | t \rangle \quad (5.69)$$

$$= t_1(\lambda) \langle h_1, \dots, h_N | t \rangle, \quad (5.70)$$

which completes the proof of our theorem. Note that clearly, from the following interpolation formula:

$$T_2^{(K)}(\lambda) = T_{2,\mathbf{h}=1}^{(K,\infty)}(\lambda) + \sum_{a=1}^N f_{a,\mathbf{h}=1}(\lambda) T_1^{(K)}(\xi_a - \eta) T_1^{(K)}(\xi_a), \quad (5.71)$$

it also holds

$$\begin{aligned} \langle h_1, \dots, h_N | T_2^{(K)}(\lambda) | t \rangle &= \left(T_{2,\mathbf{h}=1}^{(K,\infty)}(\lambda) + \sum_{a=1}^N f_{a,\mathbf{h}=1}(\lambda) t_1(\xi_a - \eta) t_1(\xi_a) \right) \langle h_1, \dots, h_N | t \rangle \\ &= t_2(\lambda) \langle h_1, \dots, h_N | t \rangle. \end{aligned} \quad (5.72)$$

□

Let us make some elementary remark about this quite lengthy proof. In fact, the main idea behind it is to use the fusion relations until we get the quantum determinant which acts trivially on any covector. While doing so, we use interpolation formulae for the transfer matrix. Then one can note that the same fusion relations and interpolation formulae are true for the eigenvalues of the transfer matrices, hence giving the possibility to reverse the process and to reconstruct it in the necessary points. From the above discrete characterization of the transfer matrix spectrum in our SoV basis, we can prove the following quantum spectral curve functional reformulation.

Theorem 5.2. *Let us assume that the twist matrix K is w -simple and it has at least one nonzero eigenvalues, then the entire function $t_1(\lambda)$ is a $T_1^{(K)}(\lambda)$ transfer matrix eigenvalue if and only if there exists and it is unique the polynomial,*

$$\varphi_t(\lambda) = \prod_{a=1}^M (\lambda - \lambda_a) \quad \text{with } M \leq N \text{ and } \lambda_a \neq \xi_n, \forall (a, n) \in \{1, \dots, M\} \times \{1, \dots, N\}, \quad (5.73)$$

such that $t_1(\lambda)$,

$$t_2(\lambda) = T_{2, \mathbf{h}=1}^{(K, \infty)}(\lambda) + \sum_{n=1}^N f_{n, \mathbf{h}=1}(\lambda) t_1(\xi_n - \eta) t_1(\xi_n), \quad (5.74)$$

and $\varphi_t(\lambda)$ are the solutions of the following quantum spectral curve:

$$\begin{aligned} &\alpha(\lambda) \varphi_t(\lambda - 3\eta) - \beta(\lambda) t_1(\lambda - 2\eta) \varphi_t(\lambda - 2\eta) \\ &+ \gamma(\lambda) t_2(\lambda - 2\eta) \varphi_t(\lambda - \eta) - q - \det M_a^{(K)}(\lambda - 2\eta) \varphi_t(\lambda) = 0, \end{aligned} \quad (5.75)$$

where

$$\alpha(\lambda) = \gamma(\lambda) \gamma(\lambda - \eta) \gamma(\lambda - 2\eta), \quad (5.76)$$

$$\beta(\lambda) = \gamma(\lambda) \gamma(\lambda - \eta), \quad (5.77)$$

$$\gamma(\lambda) = \gamma_0 \prod_{a=1}^N (\lambda + \eta - \xi_a), \quad (5.78)$$

and γ_0 is a nonzero solution of the characteristic equation

$$\gamma_0^3 - \gamma_0^2 \text{tr} K + \gamma_0 \frac{(\text{tr} K)^2 - \text{tr} K^2}{2} = \det K, \quad (5.79)$$

i.e., γ_0 is a nonzero eigenvalue of the matrix K . Moreover, up to a normalization, the common transfer matrix eigenstate $|t\rangle$ admits the following separate representation:

$$\langle h_1, \dots, h_N | t \rangle = \prod_{a=1}^N \gamma^{h_a}(\xi_a) \varphi_t^{h_a}(\xi_a - \eta) \varphi_t^{2-h_a}(\xi_a). \quad (5.80)$$

Proof. Let us assume that the entire function $t_1(\lambda)$ satisfies with the polynomial $t_2(\lambda)$ and $\varphi_t(\lambda)$ the functional equation, then it is a degree N polynomial in λ with leading coefficient $t_{1,N}$ satisfying the equation

$$\gamma_0^3 - \gamma_0^2 t_{1,N} + \gamma_0 \frac{(\text{tr} K)^2 - \text{tr} K^2}{2} = \det K, \quad (5.81)$$

which being γ_0 an eigenvalue of K implies

$$t_{1,N} = \text{tr} K. \quad (5.82)$$

Let us observe that for $\lambda = \xi_a$ it holds

$$\alpha(\xi_a) = \beta(\xi_a) = 0, \quad \gamma(\xi_a) \neq 0, \quad \det K(\xi_a - 2\eta) \neq 0 \quad (5.83)$$

so that the functional equation is reduced in these points to

$$\frac{\gamma(\xi_a) \varphi_t(\xi_a - \eta)}{\varphi_t(\xi_a)} = \frac{\det_q M_a^{(K)}(\xi_a - 2\eta)}{t_2(\xi_a - 2\eta)}, \quad (5.84)$$

while for $\lambda = \xi_a + \eta$ it holds

$$\alpha(\xi_a + \eta) = \det M_a^{(K)}(\xi_a - \eta) = 0, \quad \beta(\xi_a + \eta) \neq 0, \quad \gamma(\xi_a + \eta) \neq 0 \quad (5.85)$$

so that the functional equation is reduced to

$$\frac{\beta(\xi_a + \eta) \varphi_t(\xi_a - \eta)}{\gamma(\xi_a + \eta) \varphi_t(\xi_a)} = \frac{t_2(\xi_a - \eta)}{t_1(\xi_a - \eta)}. \quad (5.86)$$

Finally for $\lambda = \xi_a + 2\eta$, it holds

$$t_2(\xi_a) = \det M_a^{(K)}(\xi_a) = 0, \quad \beta(\xi_a + 2\eta) \neq 0, \quad \alpha(\xi_a + 2\eta) \neq 0, \quad (5.87)$$

so that the functional equation is reduced to

$$\frac{\alpha(\xi_a + 2\eta)\varphi_t(\xi_a - \eta)}{\beta(\xi_a + 2\eta)\varphi_t(\xi_a)} = t_1(\xi_a). \quad (5.88)$$

These identities imply that the following equations are satisfied:

$$t_2(\xi_n - \eta) = t_1(\xi_n - \eta)t_1(\xi_n), \quad \forall n \in \{1, \dots, N\}, \quad (5.89)$$

$$t_1(\xi_a)t_2(\xi_a - 2\eta) = q\text{-}\det M^{(K)}(\xi_a - 2\eta), \quad \forall a \in \{1, \dots, N\} \quad (5.90)$$

so that by our previous theorem we have that $t_1(\lambda)$ and $t_2(\lambda)$ are eigenvalues of the transfer matrices $T_1^{(K)}(\lambda)$ and $T_2^{(K)}(\lambda)$, respectively, associated with the same eigenstate $|t\rangle$.

Let us now prove the reverse statement; i.e., we assume that $t_1(\lambda)$ is the eigenvalue of the transfer matrix $T_1^{(K)}(\lambda)$ and we want to show that there exists a polynomial $\varphi_t(\lambda)$ which satisfies with $t_1(\lambda)$ and $t_2(\lambda)$ the functional equation. Here, we characterize $\varphi_t(\lambda)$ by imposing that it satisfies the following set of conditions:

$$\gamma(\xi_a)\frac{\varphi_t(\xi_a - \eta)}{\varphi_t(\xi_a)} = t_1(\xi_a). \quad (5.91)$$

The fact that these relations characterize uniquely a polynomial of the form (5.73) can be shown just following the same steps given in the $Y(\mathfrak{gl}(2))$ case. Let us show that this characterization of $\varphi_t(\lambda)$ implies that the functional equation is indeed satisfied. The functional equation is a polynomial in λ with maximal degree $4N$, so to show that it holds we have just to prove that it is satisfied in $4N$ distinct points as the leading coefficient is zero by the choice of γ_0 to be a nonzero eigenvalue of K . We use the following $4N$ points $\xi_a + k_a\eta$, for any $a \in \{1, \dots, N\}$ and $k_a \in \{-1, 0, 1, 2\}$. Indeed, for $\lambda = \xi_a - \eta$, it holds

$$\alpha(\xi_a - \eta) = \beta(\xi_a - \eta) = \gamma(\xi_a - \eta) = \det M_a^{(K)}(\xi_a - 3\eta) = 0, \quad (5.92)$$

from which the functional equation is satisfied for any $a \in \{1, \dots, N\}$ and in the remaining $3N$ points the functional equation reduces to the $3N$ Eqs. (5.84), (5.86), and (5.88) which, thanks to the fusion equations, satisfied by the transfer matrix eigenvalues, are all equivalent to the discrete characterization (5.91) so that our statement holds.

Finally, let us show that the SoV characterization of the transfer matrix eigenvector associated with the eigenvalue $t_1(\lambda)$ is equivalent to the one presented in this theorem. We just have to remark that renormalizing the eigenvector $|t\rangle$ and multiplying it by the non-zero product of the $\varphi_t^2(\xi_a)$ over all the $a \in \{1, \dots, N\}$ we get

$$\prod_{a=1}^N \varphi_t^2(\xi_a) \prod_{a=1}^N t_1^{h_a}(\xi_a) \stackrel{(5.91)}{=} \prod_{a=1}^N \gamma^{h_a}(\xi_a) \varphi_t^{h_a}(\xi_a - \eta) \varphi_t^{2-h_a}(\xi_a). \quad (5.93)$$

□

D. Algebraic Bethe ansatz rewriting of transfer matrix eigenvectors

It is easy to see that the previous SoV representation of the transfer matrix eigenvectors admit an equivalent rewriting of algebraic Bethe Ansatz type. For this, let us first remark that there exists one eigenvector of the transfer matrix $T_1^{(K)}(\lambda)$ and $T_2^{(K)}(\lambda)$, which corresponds to the constant solution of the quantum spectral curve equation.

Lemma 5.1. Let K be a generic 3×3 matrix and let us denote with K_J its Jordan form,

$$K = W_K K_J W_K^{-1} \quad \text{with} \quad K_J = \begin{pmatrix} k_0 & y_1 & 0 \\ 0 & k_1 & y_2 \\ 0 & 0 & k_2 \end{pmatrix}, \quad (5.94)$$

where we assume that $k_0 \neq 0$, and then

$$|t_0\rangle = \Gamma_W \bigotimes_{a=1}^N \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_a \quad \text{with} \quad \Gamma_W = \bigotimes_{a=1}^N W_{K,a} \quad (5.95)$$

is a common eigenstate of the transfer matrices $T_1^{(K)}(\lambda)$ and $T_2^{(K)}(\lambda)$,

$$T_1^{(K)}(\lambda)|t_0\rangle = |t_0\rangle t_{1,0}(\lambda) \quad \text{with} \quad t_{1,0}(\lambda) = k_0 \prod_{a=1}^N (\lambda - \xi_a + \eta) + (k_1 + k_2) \prod_{a=1}^N (\lambda - \xi_a), \quad (5.96)$$

$$T_2^{(K)}(\lambda)|t_0\rangle = |t_0\rangle t_{2,0}(\lambda) \quad \text{with}$$

$$t_{2,0}(\lambda) = \prod_{a=1}^N (\lambda - \xi_a) (k_1 k_2 \prod_{a=1}^N (\lambda - \xi_a + \eta) + (k_1 k_0 + k_2 k_0) \prod_{a=1}^N (\lambda - \xi_a)), \quad (5.97)$$

and $t_{1,0}(\lambda)$ and $t_{2,0}(\lambda)$ satisfy the quantum spectral curve with constant $\varphi_t(\lambda)$,

$$\alpha(\lambda) - \beta(\lambda) t_{1,0}(\lambda - 2\eta) + \gamma(\lambda) t_{2,0}(\lambda - 2\eta) - q \det M_a^{(K)}(\lambda - 2\eta) = 0, \quad (5.98)$$

and with $\gamma_0 = k_0$.

Proof. The proof of the statement is done proving that for the transfer matrices $T_1^{(K)}(\lambda)$ and $T_2^{(K)}(\lambda)$ the vector

$$|0\rangle = \bigotimes_{a=1}^N \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (5.99)$$

is an eigenvector with eigenvalues $t_{1,0}(\lambda)$ and $t_{2,0}(\lambda)$, respectively. The proof of this statement is standard and done proving that it holds

$$A_i^{(I)}(\lambda)|0\rangle = |0\rangle \prod_{a=1}^N (\lambda - \xi_a + \delta_{i,1}\eta), \quad C_i^{(I)}(\lambda)|0\rangle = 0, \quad i \in \{1, 2, 3\}. \quad (5.100)$$

It is interesting to remark that it is simple to verify by direct computation that the $t_{1,0}(\lambda)$ and $t_{2,0}(\lambda)$ satisfy the fusion equations (5.35) and that it holds

$$t_{1,0} \equiv \lim_{\lambda \rightarrow \infty} \lambda^{-N} t_{1,0}(\lambda) = \text{tr } K, \quad (5.101)$$

$$t_{2,0} \equiv \lim_{\lambda \rightarrow \infty} \lambda^{-2N} t_{2,0}(\lambda) = \frac{(\text{tr } K)^2 - \text{tr } K^2}{2} \quad (5.102)$$

so that $t_{1,0}(\lambda)$ satisfies the SoV characterization of the eigenvalues of $T_1^{(K)}(\lambda)$. Observing now that it holds

$$t_{1,0}(\xi_a) = \gamma(\xi_a) \quad \text{for any } a \in \{1, \dots, N\} \quad (5.103)$$

and it follows that the associated $\varphi_t(\lambda)$ satisfies the equations

$$\varphi_t(\xi_a) = \varphi_t(\xi_a - \eta) \quad \text{for any } a \in \{1, \dots, N\} \quad (5.104)$$

and so $\varphi_t(\lambda)$ is constant. Indeed, let us define

$$\bar{\varphi}_t(\lambda) = \varphi_t(\lambda) - \varphi_t(\lambda - \eta), \quad (5.105)$$

and this is a degree $N - 1$ polynomial in λ which is zero in N different points so that it is identically zero. \square

Let us now denote by $\mathbb{B}^{(K)}(\lambda)$ the one parameter family of commuting operators characterized by

$$\langle h_1, \dots, h_N | \mathbb{B}^{(K)}(\lambda) = b_{h_1, \dots, h_N}(\lambda) \langle h_1, \dots, h_N |, \quad (5.106)$$

where we have defined

$$b_{h_1, \dots, h_N}(\lambda) = \prod_{a=1}^N (\lambda - \xi_a)^{2-h_a} (\lambda - \xi_a + \eta)^{h_a}, \quad (5.107)$$

and then, the following corollary holds.

Corollary 5.2. Let $t_1(\lambda) \in \Sigma_{T_1}$ and then the associated eigenvector $|t\rangle$ admits the following algebraic Bethe Ansatz type formulation:

$$|t\rangle = \prod_{a=1}^M \mathbb{B}^{(K)}(\lambda_a) |t_0\rangle \quad \text{with } M \leq N \text{ and } \lambda_a \neq \xi_n, \forall (a, n) \in \{1, \dots, M\} \times \{1, \dots, N\}, \quad (5.108)$$

where the λ_a are the roots of the polynomial $\varphi_t(\lambda)$ which satisfies with $t_1(\lambda)$ the third order Baxter-like functional equation.

Proof. Let us observe that we have

$$\langle h_1, \dots, h_N | \prod_{a=1}^M \mathbb{B}^{(K)}(\lambda_a) | t_0 \rangle = \prod_{j=1}^M b_{h_1, \dots, h_N}(\lambda_j) \langle h_1, \dots, h_N | t_0 \rangle \quad (5.109)$$

$$= \prod_{j=1}^M \prod_{a=1}^N (\lambda_j - \xi_a)^{2-h_a} (\lambda_j - \xi_a + \eta)^{h_a} \prod_{a=1}^N \gamma^{h_a}(\xi_a) \quad (5.110)$$

$$= \prod_{a=1}^N \gamma^{h_a}(\xi_a) \varphi_t(\xi_a)^{2-h_a} \varphi_t(\xi_a - \eta)^{h_a}, \quad (5.111)$$

where we have used that

$$\langle h_1, \dots, h_N | t_0 \rangle = \prod_{a=1}^N \gamma^{h_a}(\xi_a) \quad (5.112)$$

which coincides with the last SoV characterization of the same transfer matrix eigenstate. \square

VI. CONCLUSIONS AND PERSPECTIVES

We have shown that the construction of separate basis for transfer matrices of quantum integrable lattice models can be achieved using the new paradigm given by (1.1). It sheds a completely new light on the notion of quantum integrability itself. Indeed, as soon as the transfer matrix possess the w -simplicity property, which seems to be a quite widely shared property, the construction of a separate basis along these lines can be performed. What remains however nontrivial is the possibility to get close relations that characterize the transfer matrix eigenvalues in the form of a quantum spectral curve equation of minimal degree. In all the examples, we have been able to look at so far, the needed information is provided by the fusion relations among the tower of transfer matrices. These fusion relations are themselves direct consequences of the structure of the R -matrix governing the Yang-Baxter algebra and of the representation theory for it. Looking at this feature from a different point of view should be fruitful for the future investigations of our new separation of variable method. What seems in turn to be of fundamental importance for obtaining constraints that characterize the full spectrum of the transfer matrices is to unravel the structure of the commutative (and associative) algebra of conserved charges, namely, to get the explicit structure constants of such an algebra. The fusion relations are just one way to get this kind of information, but we believe that there should be more algebraic ways to consider this problem; for example, starting directly from general properties of the associated quantum groups. It should be interesting, for example, to test these ideas in the case of the Gaudin models and to understand the relation with the geometrical construction of eigenvectors presented in Ref. 53; see also Refs. 54–57. It should also be interesting to make contact with the approach of Ref. 57. Another important direction is to use the concepts and ideas developed in the present paper for quantum integrable field theories. Let us also make comments

in a quite different direction interesting for statistical physics. The separate basis (1.1) is in our opinion of great interest for the considerations of Quench dynamics, the so-called generalized Gibbs ensembles and generalized hydrodynamics equations in integrable models; see, e.g., Refs. 58–63, and references therein. The first remark is that the existence of a basis (1.1) should give the right criterion to determine which set of conserved charges should be considered. They should be related to the one necessary to construct a basis like (1.1) having the minimal dimension for each h_j , namely, associated with the prime decomposition of the dimension d of the space of states of the model one consider. In the forthcoming articles, we will first explain how our new scheme applies to important classes of integrable quantum models. These are the models associated with the fundamental representations of the Yang-Baxter and reflection algebra for the Yangian $Y(gl_n)$, the quantum group $U_q(gl_n)$ and the t-J model. We have also applied our method successfully to models associated with cyclic and higher spin representations. The next step for these models, beyond the complete resolution of their spectrum concerns their dynamical properties, namely, the computation of their form factors and correlation functions. This amounts first to compute scalar products of states within this new method. We plan to address these important issues in the near future.

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APPENDIX: COMPARISON WITH SKLYANIN SoV FOR THE $Y(gl_3)$ CASE

In this appendix, we compare our construction of the SoV basis with the one proposed by Sklyanin for the quantum model associated with the fundamental representation of $Y(gl_3)$. As this analysis does not play any direct role in our SoV construction and associated results, we restrict here to chains with a small finite number of sites. More precisely, the claims in the following on the $\mathcal{B}^{(K)}$ -Sklyanin and $\mathcal{A}^{(K)}$ -Sklyanin operators are based on direct proofs; i.e., the statements are proven valid for general values of the parameters (inhomogeneities and K -matrix entries) by direct verifications by using Mathematica for chains up to 3 sites. It is natural to believe that these claims should be true for chains of any size, and this can be seen as conjectures whose mathematical proof can be interesting and to which we will come back in the future.

Let us recall that Sklyanin has introduced the following two operators:

$$\mathcal{B}^{(K)}(\lambda) = B_3^{(K)}(\lambda)C_2^{(K)}(\lambda - \eta) - B_2^{(K)}(\lambda)C_3^{(K)}(\lambda - \eta), \quad (A1)$$

$$\mathcal{A}^{(K)}(\lambda) = -[B_3^{(K)}(\lambda - \eta)]^{-1}C_2^{(K)}(\lambda - \eta), \quad (A2)$$

where we use the notations

$$U_a^{(K)}(\lambda) = \begin{pmatrix} A_1^{(K)}(\lambda) & B_1^{(K)}(\lambda) & B_2^{(K)}(\lambda) \\ C_1^{(K)}(\lambda) & A_2^{(K)}(\lambda) & B_3^{(K)}(\lambda) \\ C_2^{(K)}(\lambda) & C_3^{(K)}(\lambda) & A_3^{(K)}(\lambda) \end{pmatrix}_a, \quad (A3)$$

which, respectively, should generate the separate variables for the transfer matrices and the shift operators on the separate variables spectrum. Moreover, the separate relations for the spectral problem of the transfer matrix should be the quantum spectral curve analog computed along the separate variables spectrum.

Indeed, Sklyanin has proven the following identities:

$$(\lambda - \mu)\mathcal{A}^{(K)}(\lambda)\mathcal{B}^{(K)}(\mu) = (\lambda - \mu - \eta)\mathcal{B}^{(K)}(\mu)\mathcal{A}^{(K)}(\lambda) + \mathcal{B}^{(K)}(\lambda)\Xi_1^{(K)}(\lambda, \mu) \quad (A4)$$

for the shift operator and

$$\begin{aligned} & \mathcal{A}^{(K)}(\lambda)\mathcal{A}^{(K)}(\lambda - \eta)\mathcal{A}^{(K)}(\lambda - 2\eta) - \mathcal{A}^{(K)}(\lambda)\mathcal{A}^{(K)}(\lambda - \eta)T_1^{(K)}(\lambda - 2\eta) + \\ & \mathcal{A}^{(K)}(\lambda)T_2^{(K)}(\lambda - 2\eta) - q\text{-det}M^{(K)}(\lambda - 2\eta) = \mathcal{B}^{(K)}(\lambda)\Xi_2^{(K)}(\lambda) \end{aligned} \quad (A5)$$

for the quantum spectral curve; the operators $\Xi_1^{(K)}(\lambda, \mu)$ and $\Xi_1^{(K)}(\lambda, \mu)$ have the following explicit formulae:

$$\Xi_1^{(K)}(\lambda, \mu) = \eta \mathcal{A}^{(K)}(\mu) \left[B_3^{(K)}(\lambda) B_3^{(K)}(\lambda - \eta) \right]^{-1} B_3^{(K)}(\mu - \eta) B_3^{(K)}(\mu), \quad (\text{A6})$$

$$\Xi_2^{(K)}(\lambda) = \left[B_3^{(K)}(\lambda) B_3^{(K)}(\lambda - \eta) B_3^{(K)}(\lambda - 2\eta) \right]^{-1} \quad (\text{A7})$$

$$\times \left[C_1^{(K)}(\lambda - 2\eta) C_3^{(K)}(\lambda - 2\eta) - B_3^{(K)}(\lambda - 2\eta) B_1^{(K)}(\lambda - 2\eta) \right]. \quad (\text{A8})$$

Here, the main problem is that independently from the choice of the matrix K our observation is that some zeros of $\mathcal{B}^{(K)}(\lambda)$ coincide with singularities of the operators $\Xi_1^{(K)}(\lambda, \mu)$ and $\Xi_2^{(K)}(\lambda)$, so that the rhs of the Eqs. (A4) and (A5) are nonzero in some points of the spectrum of the zeros of $\mathcal{B}^{(K)}(\lambda)$. Hence, they do not imply the desired shift and spectral curve equations. So that for the representation under consideration, the operator $\mathcal{A}^{(K)}$ of Sklyanin does not seem to produce the right shift operator on the $\mathcal{B}^{(K)}$ spectrum. Recently however, there appeared an interesting article⁶³ using the same structure for the sl_3 non-compact case. We do not know if a problem similar to the one we discuss here could also be present there, the question being if in the non-compact case the spectrum of the denominators in the above equations have points in common with the $\mathcal{B}^{(K)}$ spectrum. It would be interesting to clarify this point and also to investigate the method we propose here in this non-compact situation. In the fundamental representation we consider here, the following statements hold for chain up to $N = 3$ sites and they have been verified by symbolic computations using Mathematica:

Property A.1. Let us consider a chain with $N \leq 3$ sites, then the following statements hold. Let us write explicitly the matrix

$$K = \begin{pmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \\ k_7 & k_8 & k_9 \end{pmatrix}, \quad (\text{A9})$$

and then, under the condition

$$\kappa_K \equiv k_1 k_3 k_6 - k_3 k_5 k_6 - k_3^2 k_4 + k_2 k_6^2 \neq 0 \quad (\text{A10})$$

and the inhomogeneity condition (5.7), the one-parameter family $\mathcal{B}^{(K)}(\lambda)$ of commuting operators is diagonalizable with simple spectrum. The eigenvalues of $\mathcal{B}^{(K)}(\lambda)$ admit the following representations:

$$b_{h_1, \dots, h_N}^{(K)}(\lambda) = \kappa_K b_0(\lambda) \prod_{a=1}^N (\lambda - \xi_a)^{2-h_a} (\lambda - \xi_a + \eta)^{h_a}, \quad (\text{A11})$$

where for any $i \in \{1, \dots, N\}$ and $h_i \in \{0, 1, 2\}$ we have defined

$$b_0(\lambda) = \prod_{a=1}^N (\lambda - \xi_a - \eta). \quad (\text{A12})$$

The associated covector eigenbasis of $\mathcal{B}^{(K)}(\lambda)$,

$$\langle h_1, \dots, h_N | \mathcal{B}^{(K)}(\lambda) = b_{h_1, \dots, h_N}^{(K)}(\lambda) \langle h_1, \dots, h_N |,$$

coincides with our SoV basis once we fix the form of our basis by imposing

$$\langle h_1, \dots, h_N | = \langle L_1 | \prod_{n=1}^N T_2^{(K)\delta_{h_n,0}}(\xi_n - 2\eta) T_1^{(K)\delta_{h_n,2}}(\xi_n) \forall h_n \in \{0, 1, 2\}, \quad (\text{A13})$$

with

$$\langle h_1 = 1, \dots, h_N = 1 | = \langle L_1 | \equiv \bigotimes_{a=1}^N (-k_6, k_3, 0)_a. \quad (\text{A14})$$

It is interesting to note that it holds

$$\begin{aligned} \langle h_1 = 0, \dots, h_N = 0 | &= 2\eta^{2N} \bigotimes_{a=1}^N (k_6(k_3k_7 + k_6k_8) - k_9(k_3k_4 + k_5k_6), k_9(k_1k_3 + k_2k_6) \\ &\quad - k_3(k_3k_7 + k_6k_8), -\kappa_K)_a \end{aligned} \quad (\text{A15})$$

$$= 2\eta^{2N} \langle h_1 = 1, \dots, h_N = 1 | \bigotimes_{a=1}^N \tilde{K}_a, \quad \text{with } \tilde{K} \text{ the adjoint of } K, \quad (\text{A16})$$

$$\langle h_1 = 2, \dots, h_N = 2 | = \eta^N \bigotimes_{a=1}^N (k_1k_6 - k_3k_4, k_2k_6 - k_3k_5, 0)_a = \eta \langle h_1 = 1, \dots, h_N = 1 | \bigotimes_{a=1}^N K_a. \quad (\text{A17})$$

Moreover, $\mathcal{B}^{(K)}(\lambda)$ has also the following two eigenvectors with tensor product form:

$$|h_1 = 0, \dots, h_N = 0\rangle = \bigotimes_{a=1}^N (0, 0, 1)_a^t, \quad (\text{A18})$$

$$|h_1 = 2, \dots, h_N = 2\rangle = \eta^N \bigotimes_{a=1}^N (k_3, k_6, k_9)_a^t = \eta^N \bigotimes_{a=1}^N K_a R_0. \quad (\text{A19})$$

Finally, by using the operator family $\mathcal{A}^{(K)}(\lambda)$ and $\mathcal{D}^{(K)}(\lambda) \equiv [\mathcal{A}^{(K)}(\lambda)]^{-1}$, we get

$$\lim_{\lambda \rightarrow \xi_n} \langle h_1, \dots, h_n = 0, \dots, h_N | \mathcal{A}^{(K)}(\lambda) = c_{h_1, \dots, h_N}^{(0)} \det K \langle h_1, \dots, h_n = 1, \dots, h_N |, \quad (\text{A20})$$

$$\lim_{\lambda \rightarrow \xi_n} \langle h_1, \dots, h_n = 2, \dots, h_N | \mathcal{D}^{(K)}(\lambda) = c_{h_1, \dots, h_N}^{(2)} \langle h_1, \dots, h_n = 1, \dots, h_N |, \quad (\text{A21})$$

for any $n \in \{1, \dots, N\}$, $h_j \in \{0, 1, 2\}$ for $j \in \{1, \dots, N\} \setminus n$ and with $c_{h_1, \dots, h_N}^{(i)}$ some nonzero finite constants. However, it holds

$$\lim_{\lambda \rightarrow \xi_n, \xi_n \pm \eta} \langle h_1, \dots, h_n = 1, \dots, h_N | \mathcal{A}^{(K)}(\lambda) \neq \bar{c}_{h_1, \dots, h_N}^{(0)} \langle h_1, \dots, h_n = 2, \dots, h_N |, \quad (\text{A22})$$

$$\lim_{\lambda \rightarrow \xi_n, \xi_n \pm \eta} \langle h_1, \dots, h_n = 1, \dots, h_N | \mathcal{D}^{(K)}(\lambda) \neq \bar{c}_{h_1, \dots, h_N}^{(0)} \langle h_1, \dots, h_n = 0, \dots, h_N |, \quad (\text{A23})$$

for any $n \in \{1, \dots, N\}$, $h_j \in \{0, 1, 2\}$ for $j \in \{1, \dots, N\} \setminus n$ and any $\bar{c}_{h_1, \dots, h_N}^{(i)}$ nonzero finite constants.

The above claim about the diagonalizability of the $\mathcal{B}^{(K)}(\lambda)$ operator is restricted to the case $\kappa_K \neq 0$.

Property A.2. Let us consider a chain with $N \leq 3$ sites and let us assume that the inhomogeneity condition (5.7) holds and that the matrix K is w -simple then, we have the following identification:

$$\mathbb{B}^{(K)}(\lambda) = \mathcal{B}^{(K)}(\lambda) / (\kappa_K b_0(\lambda)) \quad \text{if } \kappa_K \neq 0 \quad (\text{A24})$$

once we have fixed our SoV basis imposing (A14). While if $\kappa_K = 0$, then for any \hat{K} similar to K such that $\kappa_{\hat{K}} \neq 0$, we have the identification

$$\mathbb{B}^{(K)}(\lambda) = \Gamma_{W\hat{W}^{-1}} \mathcal{B}^{(\hat{K})}(\lambda) / (\kappa_{\hat{K}} b_0(\lambda)) \Gamma_{W\hat{W}^{-1}}^{-1}, \quad (\text{A25})$$

where

$$\Gamma_{W\hat{W}^{-1}} = \bigotimes_{a=1}^N W_{K,a} \hat{W}_{\hat{K},a}^{-1} \quad (\text{A26})$$

once we have fixed our SoV basis imposing

$$\langle h_1 = 1, \dots, h_N = 1 | = \bigotimes_{a=1}^N (-\hat{k}_6, \hat{k}_3, 0)_a \Gamma_{W\hat{W}^{-1}}^{-1}. \quad (\text{A27})$$

It is a natural conjecture that the Properties A.1 and A.2 hold for chains of any number of sites.

Let us comment that similar statements about the diagonalizability of the Sklyanin's B -operator and of the form of its eigenvalues were previously verified in Ref. 64, always by symbolic computations in Mathematica for chains of small size, for some special class of twist K matrix satisfying the condition (A10). In Ref. 64, it was moreover done the conjecture that transfer matrix eigenvectors have the usual algebraic Bethe ansatz form in terms of this Sklyanin's B -operator. Such conjecture has been recently verified in Ref. 65 algebraically, mainly relying on the use of the Yang-Baxter commutation relations. It is then worth mentioning that if the identity (A24) is proven to hold for chains of any size, then the algebraic Bethe ansatz form of the transfer matrix eigenvectors is derived in just one line proof starting from the SoV representation of these eigenvectors, as described in Sec. V D.

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