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Scaling for first-order phase transitions in thermodynamic and finite systems

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Scaling behavior for first-order phase transitions can be derived alternatively but consistently from renormalization-group, phenomenological, or finite-size considerations. A general analysis of densities at a renormalization-group fixed point demonstrates that if the coexistence of p distinct phases is possible, then p distinct eigenvalue exponents must equal the spatial dimensionality. This basic eigenvalue (or scaling) exponent condition can also be derived phenomenologically by various arguments not depending on detailed renormalization-group considerations. A scaling description of first-order phase transitions is presented and extended to finite systems with linear dimensions L, leading to a rounding proportional to L^{-d}, response-function maxima varying as $L^d \propto N$, and boundary-condition-dependent shifts which may be as large as $\sim L^{-1}$.

I. INTRODUCTION

The singular parts of thermodynamic functions at continuous (or "second-order") phase transitions exhibit the scaling behavior of generalized homogeneous functions.[1,2] This observation underlies the modern theory of phase transitions. Its statistical-mechanical explanation is obtained from the renormalization-group approach in terms of the fixed-point mechanism.[3] Specifically, crucial quantitative aspects of the scaling behavior derive from the eigenvalues of the renormalization-group transformation linearized at the fixed point. The fixed point, with its characteristic eigenvalues, controls an entire manifold or phase boundary of critical transitions which is its domain of attraction under the renormalization-group transformation.

The identification of first-order phase transitions, where several phases may coexist, within the context of the renormalization-group approach occurred more recently.[4] According to the widely accepted picture, a first-order phase boundary is also controlled by a fixed point, but one which is situated in a "strong-coupling" region where the standard controlled approximations break down and truncated calculations must be used.[5] The signal for a first-order fixed point is an eigenvalue exponent, $\lambda = \ln \Lambda$, equal to the space dimensionality d of the system.

In this paper we present a general discussion of thermodynamic densities and coexistence at a fixed point (Sec. II). Then, we show the presence of a scaling exponent equal to the dimensionality follows, independently of the renormalization-group approach, from a general consideration of the singularity characterizing a first-order phase transition (Secs. III and IV). This is useful since the rigorous existence of first-order or "discontinuity" fixed points has recently been questioned.[6] A scaling description of thermodynamic functions near a first-order transition is given in Sec. III. In Sec. IV, this scaling theory is extended to systems of finite size.[7] It is seen that the shift in the transition is, in general, larger than its rounding, in contrast to most critical-point transitions.

II. EIGENVALUE EXPONENT FOR FIRST-ORDER TRANSITIONS: FIXED-POINT DESCRIPTION

The operation of the renormalization-group approach[3,5] can be expressed in terms of recursion relations,

$$J'_\alpha = R_\alpha(\{J_\beta\}) , \tag{2.1}$$

where, under a length scale change by a factor b, thermodynamic equivalence is maintained via a set of renormalized interactions $\{J'_\alpha\}$, functions of the original interactions $\{J_\beta\}$ (and of b). The linearized action of the renormalization group is determined by the matrix

$$T_{\alpha\beta}=\frac{\partial J'_\alpha}{\partial J_\beta}\ , \tag{2.2}$$

which features in our subsequent analysis. In almost all nontrivial situations an infinite set of interactions is required to implement the procedure. In practical calculations this infinity is controlled by approximation or truncation, but the issue does not matter for our present purposes. Thus, we adhere to the abstract notation $\{J_\alpha\}$.

A general set of thermodynamic densities may be introduced through the definition

$$M_\alpha=\frac{1}{N}\frac{\partial}{\partial J_\alpha}\ln Z_N\ , \tag{2.3}$$

where Z_N is the partition function for a system of N degrees of freedom. Then, indicating by a prime the system resulting from a single renormalization, we have

$$M_\alpha=\frac{1}{b^dN'}\left[\frac{\partial \ln Z_N}{\partial J'_\beta}\right]\left[\frac{\partial J'_\beta}{\partial J_\alpha}\right]=b^{-d}M'_\beta T_{\beta\alpha}\ , \tag{2.4}$$

where the invariance of the partition function under renormalization, namely $Z_N=Z_{N'}$, has been used and the summation convention on repeated indices is invoked. Repeated application of Eq. (2.4) along a renormalization-group trajectory yields the thermodynamic densities of the initial unrenormalized system.[8]

Consider now the density recursion relation at a fixed point, denoted, as usual, by a star; then, assuming the fixed point and its neighborhood sufficiently regular, we have

$$M^*_\alpha=b^{-d}M^*_\beta T_{\beta\alpha}\ . \tag{2.5}$$

The densities contain, trivially, the expectation value of the unit operator, $M_0=\langle 1\rangle=1$, conjugate to the additive constant NJ_0 in the Hamiltonian. Thus, $\{M_\alpha\}$ cannot be a zero vector. Therefore, the densities $\{M^*_\alpha\}$ at the fixed point constitute a left eigenvector[9] of $T^*_{\beta\alpha}$ with eigenvalue $\Lambda\equiv b^\lambda=b^d$.

Let us next recall the general form of $T_{\beta\alpha}$ which is typically the following:

$$T_{\beta\alpha}=\begin{bmatrix} T_{00}=b^d & \text{even} & 0 \\ & \text{even} & \\ 0 & & \\ & 0 & \text{odd} \\ 0 & & \text{odd}\end{bmatrix}\ . \tag{2.6}$$

The first column follows from the trivial participation of the additive constant into the recursion relations,

$$\begin{aligned} J'_0&=b^dJ_0+\tilde{R}_0(\{J_{\alpha\neq 0}\})\ ,\\ J'_{\alpha\neq 0}&=R_\alpha(\{J_{\beta\neq 0}\})\ .\end{aligned} \tag{2.7}$$

Also exhibited in Eq. (2.6) is a block structure encountered when there is a global reflection symmetry in the problem, which is respected by the fixed point in question. Thus we suppose

$$J_{\text{even}} \Longrightarrow +J_{\text{even}}\ ,\ \ J_{\text{odd}} \Longrightarrow -J_{\text{odd}}\ , \tag{2.8}$$

under the symmetry transformation, as exemplified by temperature and magnetic field, respectively, in a magnetic system with time-reversal symmetry.

From Eq. (2.6) it is clear than an eigenvalue b^d is always trivially present. The leading element $M^*_0=1$ sets the normalization of the corresponding left eigenvector, the succeeding elements of which give the thermodynamic densities at the fixed point.[9]

The representation of two distinct phases at the fixed point requires two *independent* sets of thermodynamic densities; thus, we should expect *two* distinct left eigenvectors both with $\Lambda=b^d$, and, therefore, a second eigenvalue b^d which is now nontrivial. Thus, continuing the previous example, if the odd-odd block had an eigenvalue b^d, the fixed point would represent a first-order transition with a discontinuous order parameter. If the even-even block had such an eigenvalue, there would be a latent heat. Once two degenerate eigenvectors with $\Lambda=b^d$ are located, any linear combination provides an acceptable set of densities describing the overall system and corresponding physically to different ratios of coexisting phases.

Similar considerations clearly apply if three or more phases may coexist as, for example, in Potts models. In general, there will be a *distinct eigenvector with eigenvalue* $\Lambda=b^d$ or exponent $\lambda=d$, *for each distinct phase represented by a fixed point.* Note also that the presence of an exact symmetry, as used above for illustrative purposes, plays *no* role in the argument.

It is worth pointing out that all the features mentioned are nicely illustrated by studies of Blume-Emery-Griffiths,[10(a)] Potts-lattice-gas,[10(b)] and general three-state[10(c)] models. The latter two calculations employed a form of the Migdal-Kadanoff approximate renormalization groups; however, the method is *exact* for a class of hierarchical or fractal pseudolattices.[11,12] In addition to fixed points representing critical, tricritical, and Potts-critical points (or manifolds) in the usual way, there occur[10] fixed points representing two-phase and three-phase coexistence and, also, fixed points representing *critical endpoints,* at which a critical phase (with slowly decaying correlations) can coexist with a noncritical phase. All the fixed points display an eigenvalue $\Lambda = b^d$ for each distinct phase. However, there is an element of arbitrariness in the definition of the dimensionality of a hierarchical or fractal lattice[10(c),12]; what one learns is that the first-order condition should more generally be written $\Lambda = b_N$, where $b_N = N/N'$ is the renormalization-group rescaling factor for the total number of degrees of freedom which can always be defined unambiguously. (For a regular d-dimensional lattice, of course, one has $b_N \equiv b^d$.)

The above discussion generalizes the original observation made by Nienhuis and Nauenberg.[4] Note that our analysis is formally exact granted the existence of a fixed point representing distinct, potentially coexisting phases.[13]

III. SCALING AND FIRST-ORDER TRANSITIONS

A discussion of first-order transitions on the basis of a scaling hypothesis[2] can be given without making direct appeal to renormalization-group concepts. Consider an ordinary or "simple" first-order transition in which the system, on crossing the phase boundary in the thermodynamic limit, switches from one noncritical phase, with finite correlation length, to another noncritical phase. For the sake of concreteness, consider an Ising-type ferromagnet with (T,H) phase diagram as illustrated in Fig. 1(a), in which the zero-field axis below T_c represents a simple first-order transition.

If one attempts to describe the transition in the standard phenomenological manner used for critical-point transitions, one would postulate the power law

$$M - \overline{M}_t \approx \pm D \,|\, H - H_t \,|^{1/\delta} , \tag{3.1}$$

where H_t is the transition field (equal to zero in the example) while

$$\overline{M}_t = \tfrac{1}{2}[M(H_t+)+M(H_t-)]$$

represents the mean magnetization on the phase boundary (which is also zero, by symmetry, for the Ising-type example). It is then evident that a first-order transition is described simply by $\delta \to \infty$ which leads to a discontinuity in $M(H)$.

Now, if the first-order transition is to be represented by a renormalization-group fixed point, we may anticipate that the field H couples directly to the eigenvector with largest eigenvalue so that the recursion relations for the field and free energy per degree of freedom can be written[3,5,14] asymptotically as

$$H' \approx b^{\lambda} H , \tag{3.2}$$

where $\Lambda = b^{\lambda}$ is the eigenvalue (and for simplicity, we take $H_t = 0$), while

$$f(H) \approx b^{-d} f(H') . \tag{3.3}$$

In addition, one would normally expect the (dominant) correlation length to renormalize as[3,5]

$$\xi(H) \approx b\,\xi(H') . \tag{3.4}$$

Now if, in the standard way, we choose $b = |\, H \,|^{-1/\lambda}$, we obtain, formally,

$$f(H) \approx D_{\pm} \,|\, H \,|^{d/\lambda} \text{ and } \xi(H) \approx a_{\pm} \,|\, H \,|^{-1/\lambda} , \tag{3.5}$$

with $D_{\pm} \simeq f(\pm 1)$ and $a_{\pm} \simeq \xi(\pm 1)$. Since $M \propto (\partial f/\partial H)$, this leads to $d/\lambda = 1 - (1/\delta)$; in the limit $\delta \to \infty$, this in turn reduces to $\lambda = d$ thereby reproducing the original first-order condition.

The argument also suggests that $\xi(H)$ should diverge as $|\, H \,|^{-\nu_H}$ with

$$\nu_H = 1/d . \tag{3.6}$$

This result must, however, be interpreted with some perspicasity; by hypothesis, both thermodynamic phases at the first-order transition are noncritical and hence have perfectly finite correlation lengths which do *not* diverge as $H \to 0$.[15] Formally, this discrepancy can be repaired by noting that the

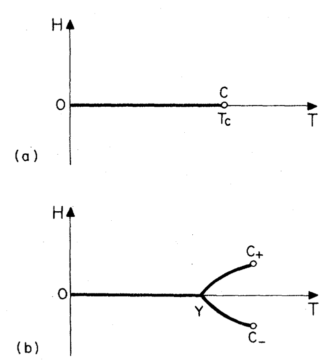

FIG. 1. Phase diagrams illustrating first-order boundaries terminating in critical points C, C_+, and C_-. These critical points may be "first-order critical points" exhibiting discontinuities in the spontaneous magnetization and/or latent heats. On the other hand, the point Y in (b), which has similar features if viewed only in zero field, is in reality a triple point with no critical features.

asymptotic amplitudes, $a_\pm$ in Eq. (3.5) may vanish identically. More physically, however, when considering the behavior of a large finite system, one may interpret the exponent ν_H as relating to what might better be called a *coherence* or *persistence length* for the coexisting phases. Its divergence corresponds to the occurrence of *long-range order* in those correlation functions that distinguish the two phases. For the simple Ising ferromagnet it suffices to consider the standard spin-spin correlation function $\langle s_{\vec{0}} s_{\vec{r}} \rangle$ which approaches the square of the spontaneous magnetization as $|\vec{r}| \to \infty$. (Note that it is the *net* correlation function $\langle s_{\vec{0}} s_{\vec{r}} \rangle - \langle s_{\vec{0}} s_{\vec{\infty}} \rangle$ which determines the finite correlation length in the separate coexisting or "ordered" phases.) This interpretation is also quite consistent with the standard hyperscaling relations[2(c)]

$$2-\eta=d(\delta-1)/(\delta+1)=(2/\nu_H)-d$$

which, when $\delta \to \infty$ or Eq. (3.6) is used, yield the decay exponent $d-2+\eta=0$ corresponding to *no decay*, i.e., to long-range order.

In some models and various real systems one also observes a first-order transition when the *temperature* is increased at fixed, zero ordering field. Specifically, one may find a *discontinuity* in the spontaneous magnetization at $T=T_c$ in place of the normally expected critical-point variation

$$M_0(T) \approx B\,|T-T_c|^{\beta} \text{ as } T \to T_c- , \tag{3.7}$$

with $\beta > 0$. Similarly, one may observe a latent heat at the transition corresponding to a discontinuity in the internal energy, in place of the normal critical behavior

$$U(T)-U_c \approx \pm A_\pm\,|T-T_c|^{1-\alpha} \text{ as } T \to T_c \pm . \tag{3.8}$$

[Note, however, that the absence of a latent heat on varying the magnetic field across a first-order ferromagnetic transition below T_c is merely an accident due to the fact that this phase boundary happens to be parallel to the temperature axis; by contrast, crossing the first-order phase boundary $H_t(T)$ of a metamagnet below its tricritical point *is* associated with a latent heat].

Two cases must now be distinguished: The most common situation in practice is illustrated in Fig. 1(b), where it is seen that the first-order transition observed in zero field, at the point now labeled Y, is properly described as a *triple* point at which three phases, all noncritical, may coexist. This is the standard situation when mean-field or Landau theory yields a first-order transition in zero field. There are now two "nearby" critical points, C_+ and C_-, symmetrically placed in the ferromagnetic example; if, as some parameter changes, both C_+ and C_- approach Y, one obtains a tricritical point when they just merge; if only one of C_+ or C_- meets Y, one obtains instead a critical endpoint.

In the second case, which might be termed a "first-order critical point," the transition in question remains topologically the terminal point on a line of first-order transitions, as in Fig. 1(a). It is intuitively plausible that if the spontaneous magnetization then has a discontinuity at the critical point C, it implies that the susceptibility $\chi=(\partial M/\partial H)_T$ must diverge as C is approached from above in zero field.[16] Thus we may postulate the usual critical-point form

$$\chi \approx C_\pm\,|T-T_c|^{-\gamma} \text{ as } T \to T_c \pm , \tag{3.9}$$

and expect $\gamma > 0$.

An example of such a first-order critical point occurs in a one-dimensional Ising model with long-range ferromagnetic interactions decaying with distance as $1/r^2$, as first suggested by Thouless[17(a)] and later proved for a similar model by Dyson.[17(b)] The critical point at $T=H=0$ in the nearest-neighbor one-dimensional Ising model represents a degenerate case of the same phenomenon.

Now the discontinuous vanishing of $M_0(T)$ at $T=T_c$ may be described by $\beta=0$ in Eq. (3.7). The scaling relation[2] $2-\alpha=\beta(1+\delta)$ then implies $\delta=\infty$ (since $\alpha \le 1$) and similar arguments, as before, indicate a renormalization-group eigenvalue $\Lambda_H=b^d$.

Thermodynamically, a discontinuous spontaneous magnetization does not necessarily imply a latent heat but, via scaling, we would have $\gamma=2-\alpha$. A latent heat can evidently be described in Eq. (3.8) by $\alpha=1$. Parallel arguments then yield a correlation length exponent $\nu=1/d$ and a thermal renormalization-group eigenvalue $\Lambda_T=b^d$, as before.

Note, however, that in principle one may also

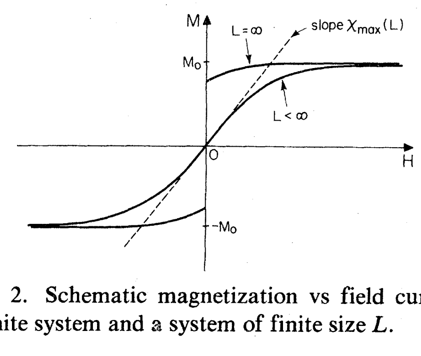

FIG. 2. Schematic magnetization vs field curves for an infinite system and a system of finite size L.

have a latent heat with no discontinuity of the spontaneous magnetization. From this perspective first-order critical points are no more than ordinary critical points in which either one or both of the relevant eigenvalue exponents attains its thermodynamically allowed limiting value $\lambda = d$. If both magnetic and thermal eigenvalues attain this limit, the critical exponents take the values $\alpha=1$, $\beta=0$, $\gamma=1$, $\delta=\infty$, and $\nu=\nu_H=1/d$. In these circumstances, however, it is reasonable to expect that, in general, there will be a standard correlation length $\xi(T,H)$ diverging, as the critical point is approached, with the indicated exponents ν and ν_H.

IV. SYSTEMS OF FINITE SIZE

No true nonanalyticity can appear in the thermodynamic or correlation functions of a finite system, so that all the sharp features discussed above must become rounded off when the largest characteristic linear dimension L is bounded (see Fig. 2 which illustrates magnetization curves). For simplicity, we will suppose that L is the only characteristic dimension so that N varies as L^d (for constant shape).

To describe the growth of a sharp first-order phase transition as $L \to \infty$, let us adopt a finite-size scaling hypothesis[7] for fixed T in terms of $h = H - H_t(T)$. Thus, for the singular part of the free energy density we postulate

$$f(h;L) \approx L^{-\zeta} Z(L/\tilde{\xi}) , \tag{4.1}$$

where, physically, the length $\tilde{\xi}(h) \approx a\,|h|^{-\tilde{\nu}}$ simply embodies the idea that rounding should set in only for $L \sim \tilde{\xi}$. The properties of the scaling function $Z(z)$ and the exponents ζ and $\tilde{\nu}$ may be studied by considering the magnetization and susceptibility given by

$$\Delta M \equiv M(h;L) - M_t = \frac{\partial f}{\partial h} , \qquad \chi(h;L) = \frac{\partial^2 f}{\partial h^2} . \tag{4.2}$$

To that end, however, it is easier to rewrite Eq. (4.1) in the fully equivalent form

$$f(h;L) \approx L^{-\zeta} Y(hL^{\tilde{\theta}}) , \tag{4.3}$$

with $\tilde{\theta} = 1/\tilde{\nu}$, which yields

$$\Delta M \approx L^{\tilde{\theta}-\zeta} Y'(hL^{\tilde{\theta}}) , \tag{4.4}$$

where the prime denotes differentation. Now to describe a first-order transition in the limit $L \to \infty$ this formula must yield $\Delta M = \pm M_{\pm}$ for $h \gtrless 0$. This is possible only if $Y'(y) \to \pm M_{\pm}$ as $y \to \pm\infty$ and if $\zeta = \tilde{\theta} = 1/\tilde{\nu}$.

Accepting this relation, consider the susceptibility, for which one has

$$\chi(h;L) \approx L^{\tilde{\theta}} Y''(hL^{\tilde{\theta}}) . \tag{4.5}$$

It follows that in a finite system the maximum value of the susceptibility $\chi_{\max}(L)$ should vary as $L^{\tilde{\theta}}$ when L or $N \to \infty$. Now, in a magnetic system with spins $s_{\vec{r}}$ at N sites $\vec{r}$, we have, by the standard sum rules,

$$\chi(h;L) = \frac{1}{N} \sum_{\vec{r}} \sum_{\vec{r}\,'} (\langle s_{\vec{r}} s_{\vec{r}\,'} \rangle - \langle s_{\vec{r}} \rangle \langle s_{\vec{r}\,'} \rangle) . \tag{4.6}$$

Since the spins are bounded in magnitude we certainly have $\chi_{\max}(L) < cN \propto L^d$ where c is a constant. The bound $\tilde{\theta} \le d$, or $\tilde{\nu} \ge 1/d$, follows directly by comparison with Eq. (4.5). More specifically, however, consider a symmetric ferromagnetic system with symmetric (e.g., free) boundary conditions or with periodic boundary conditions. Then, as illustrated in Fig. 2, it is reasonable to expect that $\chi(h;L)$ attains its maximum when $h\,(\equiv H) = 0$; at this point $\langle s_{\vec{r}} \rangle$ vanishes identically since the system is finite, so that $N\chi_{\max}$ is simply the double sum over $\langle s_{\vec{r}} s_{\vec{r}\,'} \rangle$. In the thermodynamic limit we have, as already noted, $\langle s_{\vec{r}} s_{\vec{r}\,'} \rangle \to M_0^2$ when $|\vec{r} - \vec{r}\,'| \to \infty$. It is rather plausible that this holds, perhaps up to a constant factor depending on the boundary conditions, also in a finite system when $|\vec{r} - \vec{r}\,'|$ is large. If so, we may conclude $\chi_{\max}(L) \sim M_0^2 N \propto L^d$ from which the *equality*

$$\zeta = \tilde{\theta} = 1/\tilde{\nu} = d \tag{4.7}$$

finally follows; this agrees precisely with the previous renormalization-group-cum scaling suggestion embodied in Eq. (3.6). It should be noted, nevertheless, that in going from the inequality $\tilde{\theta} \le d$, which is rigorously valid, to the equality (4.7), our argument has overlooked the important conceptual difference between the *short* long-range order, which leads to M_0^2 and the *long* long-range order, with $|\vec{r} - \vec{r}\,'|$ of order L, which is what is required for $\chi_{\max}(L)$.[18] An optimistic stance[19] is supported by the exactly known results for the two-dimensional Ising model[18(c)]; the agreement with the earlier analysis is also encouraging. However, in the case of systems, like the Heisenberg model, with a continuous symmetry, it is plausible that $\chi_{\max}(L)$ may increase less rapidly than N because of the large long-wavelength fluctuations[15]; thus some caution is in order.

Of course, if we accept Eq. (3.6) and make the identification $\nu_H \equiv \tilde{\nu}$ directly, finite-size rounding should set in when $\tilde{\xi} \sim |h|^{-\tilde{\nu}} \sim L$; this yields a crossover or rounding field $h_x \sim L^{-d}$. Using the standard critical-point variation $\chi \sim |h|^{-1+(1/\delta)}$ and letting $\delta \rightarrow \infty$ for first-order behavior then indicates a rounded susceptibility with $\chi_{\max} \sim L^d \sim N$. Conversely, if we *accept* $\chi_{\max} \sim N$ on the basis of the sum-rule argument,[19] we can estimate the crossover field from the intersection of the line $M = \chi_{\max} H$ with the limiting magnetization curve, for which $M \simeq M_0$. This yields

$$h_\times \approx M_0 / \chi_{\max} \sim L^{-d} ,$$

and then reversing the line of argument, we can again derive the first-order condition $\lambda = d$.

All these results and identifications are, of course, in accord with the systematic scaling hypothesis (4.3) with (4.7). Similar arguments can be presented for the specific-heat peak which, as $L \rightarrow \infty$, approaches a delta function representing the latent heat; the maximum $C_{\max}(L)$ is again expected to grow like $N \sim L^d$, which once more reflects back to the condition $\lambda = d$ or $\Lambda = b^d$.

Finally, observe that another important effect of finite size is the shifting of the phase-transition region, in addition to its rounding.[7] For the sake of definiteness, consider a discrete spin model at its first-order transition when the field H is scanned at $T=0$. Suppose the boundary spins are all pinned in the direction favored by positive H. Then the two ground-state energies are

$$\begin{aligned} E_+ &= -\tfrac{1}{2} qNJ - NH , \\ E_- &= -\tfrac{1}{2} qNJ + NH + c_s L^{d-1} J_s , \end{aligned} \tag{4.8}$$

where J is the coupling constant and q is the bulk coordination number. In the second equation c_s is a surface geometric factor while J_s is the coupling to the up-pinned spins and would be replaced, in a finite-temperature theory, by the interfacial tension or domain-wall energy. The two energies cross at the shifted field

$$H_s = \tfrac{1}{2} c_s J_s L^{d-1} / N \sim L^{-1} . \tag{4.9}$$

This shift is thus much larger asymptotically than the rounding which should, as seen, vary as $H_x \sim L^{-d}$. Therefore, following general considerations,[7] the field h in the scaling postulates above should be replaced by $\dot{h} = h - h_0(L)$ where $h_0(L)$ measures the shift induced by the "polarized" boundary conditions. Note also that the detailed form of the scaling function will also depend on the boundary conditions. We may mention that in an *XY*- or Heisenberg-type system with an infinitely diffuse Bloch wall, we anticipate[20] that the shift varies as $1/L^2$. Similarly, if the spins are pinned only on $(d-2)$-dimensional "edges" rather than over $(d-1)$-dimensional "faces" the shift should vary again as $1/L^2$; likewise, lower-dimensional pinning would lead to shifts $1/L^{d-d_s}$ with $d_s = 0, 1, 2, \ldots, d-1$.

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[1] B. Widom, J. Chem. Phys. 43, 3898 (1965); L. P. Kadanoff, Physics 2, 263 (1966).

[2] (a) M. E. Fisher, Rep. Prog. Phys. 30, 615 (1967); (b) L. P. Kadanoff *et al.*, Rev. Mod. Phys. 39, 395 (1967); (c) M. E. Fisher, in *Collective Properties of Physical Systems, Proceedings of the 24th Nobel Symposium 1973*, edited by B. Lundqvist and S. Lundqvist (Academic, New York, 1974), p. 16.

[3] K. G. Wilson, Phys. Rev. B 4, 3174 (1971); 4, 3184 (1971); K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972).

[4] B. Nienhuis and M. Nauenberg, Phys. Rev. Lett. 35, 477 (1975).

[5] See, for example, the review articles in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 6.

[6] R. B. Griffiths and P. A. Pearce, Phys. Rev. Lett. 41, 917 (1978); J. Stat. Phys. 20, 499 (1979).

[7] M. E. Fisher, in *Critical Phenomena, Proceedings of the 51st "Enrico Fermi" Summer School, Varenna, Italy*, edited by M. S. Green (Academic, New York, 1971), p. 1; J. Vac. Sci. Technol. 10, 665 (1973).

[8] M. Nauenberg and B. Nienhuis, Phys. Rev. Lett. 33, 944 (1975).

[9] L. P. Kadanoff, A. Houghton, and M. C. Yalabik, J. Stat. Phys. 14, 171 (1976).

[10](a) A. N. Berker and M. Wortis, Phys. Rev. B 14, 4946 (1976); (b) A. N. Berker, S. Ostlund, and F. A. Putnam, *ibid.* 17, 3650 (1978); S. Ostlund and A. N. Berker, *ibid.* 21, 5410 (1980); (c) M. Kaufman, R. B. Griffiths, J. M. Yeomans, and M. E. Fisher, *ibid.* 23, 3448 (1981); J. M. Yeomans and M. E. Fisher, *ibid.* 24, 2925 (1981).

[11]A. N. Berker and S. Ostlund, J. Phys. C 12, 4961 (1979).

[12]M. Kaufman and R. B. Griffiths, Phys. Rev. B 24, 496 (1981).

[13]Note that we say "potentially coexisting" since the actual coexistence of phases in a physical system depends on boundary conditions and/or the precise fashion in which the phase boundary is approached by varying the J_α (e.g., the magnetic fields) in the thermodynamic limit or while the thermodynamic limit is being taken. This is clear by considering the "polarized" boundary conditions discussed in Sec. IV.

[14]For simplicity of presentation we neglect the mapping along the trajectory to the fixed point which should be handled properly along the lines of the previous section.

[15]Note we have thus excluded Heisenberg-type or XY-like ferromagnets where, owing to the spin waves or Goldstone modes, the correlation length *does* diverge when $H \rightarrow 0$ yielding net correlation functions with slow power-law decays (even for $d > 2$).

[16]One may, however, construct thermodynamic potentials for which this will not be true. However, χ will necessarily diverge on some paths approaching C.

[17](a) D. J. Thouless, Phys. Rev. B 187, 732 (1969); (b) F. J. Dyson, Commun. Math. Phys. 21, 269 (1971).

[18](a) M. E. Fisher, in *Essays in Physics,* edited by G. K. T. Conn and G. N. Fowler (Academic, London, 1972), Vol. 4, p. 43; (b) D. Jasnow and M. E. Fisher, Phys. Rev. B 3, 895 (1971); (c) T. D. Schultz, D. C. Mattis, and E. H. Lieb, Rev. Mod. Phys. 36, 856 (1964).

[19]P. Kleban and C.-K. Hu (unpublished).

[20]M. E. Fisher, M. N. Barber, and D. Jasnow, Phys. Rev. A 8, 1111 (1973).