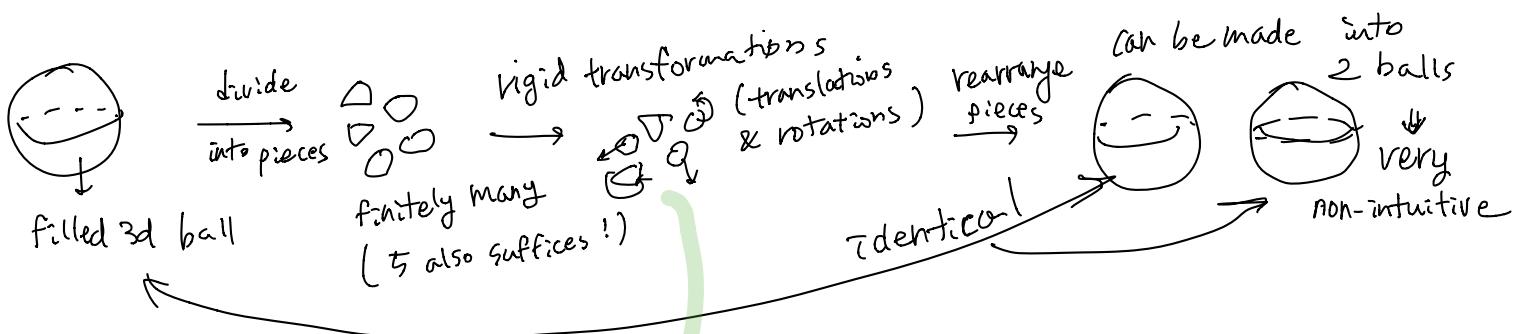


1.1 Measure Theory: Motivation

* The Banach-Tarski Paradox



Assumption

ZFC axioms (Zermelo-Fraenkel set theory with the axiom of choice)

Resolution

- i) Reject axiom of choice
- or ii) Embrace the concept of non-measurable sets.

These individual sets cannot be assigned a measure in any meaningful way.

1.2 Measure Theory: Sigma-algebras.

Def Given a set Ω ,
 a σ -algebra on Ω is a collection $\mathcal{A} \subset 2^\Omega$
 such that \mathcal{A} is non-empty
 and \mathcal{A} is closed under complements ex) $\emptyset \in \mathcal{A} \Rightarrow \Omega \in \mathcal{A}$.
 $(\forall E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A})$

and \mathcal{A} is closed under countable unions

$$\left(E_1, E_2, \dots \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{A} \right)$$

↳ this also covers finite i
 ex) $\forall i \geq 3, E_i = E_1 \Rightarrow \bigcup_{i=1}^3 E_i = E_1 \cup E_2$

- Remarks
- $\Sigma \in A$ since $E \in A$ and $E^c \in A \Rightarrow E \cup E^c = \Sigma \in A$.
 - $\emptyset \in A$ since $\Sigma \in A \Rightarrow \Sigma^c = \emptyset \in A$.
 - A is closed under countable intersections
 pf) suppose $E_1, E_2, \dots \in A$.

$$\bigcap_{i=1}^{\infty} E_i = \bigcap_{i=1}^{\infty} (E_i^c)^c = \left(\bigcup_{i=1}^{\infty} E_i^c \right)^c \in A.$$
DeMorgan's Law

1.3 Measure Theory: Measures

Def Given $C \subset 2^{\Omega}$, the σ -algebra generated C , written $\sigma(C)$, is the "smallest" σ -algebra containing C that is, $\sigma(C) = \bigcap A$

$A \supset C \rightsquigarrow$ every existing σ -algebra A containing C .

Remarks $\sigma(C)$ always exists, because

- 2^{Ω} is a σ -algebra $\rightsquigarrow A$ always exists
- Any intersection of σ -algebras is a σ -algebra.
 $\sigma(C)$ is an intersection of σ -algebras.

Example Examples of σ -algebra.

- $A = \{\emptyset, \Sigma\}$

- $A = \{\emptyset, E, E^c, \Sigma\}$,

- (Def of Borel σ -algebra)

If $\Sigma = \mathbb{R}$, the Borel σ -algebra is

$B = \sigma(\Sigma)$ where $\Sigma = \{\text{open sets of } \mathbb{R}\}$.

Any topological space is fine.

$a < k < b$

$\ni (a, b)$

Def A measure μ on Ω with σ -algebra A

is a function $\mu: A \rightarrow [0, \infty]$

such that i) $\mu(\emptyset) = 0$

and ii) Countable Additivity.

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

for any E_1, E_2, \dots of pairwise disjoint sets.

This also covers finite i .

ex) $\forall i \geq 4, E_i = \emptyset, \mu(E_i) = 0$.
 $\mu\left(\bigcup_{i=1}^3 E_i\right) = \sum_{i=1}^3 \mu(E_i)$.

Def A probability measure is a measure P

such that $P(\Omega) = 1$.

* All these conditions, which is specified for probability measure,
is called Kolmogorov's Axioms.

(1.4) Measure Theory: Examples of Probability Measures.

i) Uniform Distribution

Finite Set. $\Omega = \{1, 2, \dots, n\}$, $A = 2^{\Omega}$.

$$P(\{k\}) = P(k) = \frac{1}{n} \quad \text{if } k \in \Omega.$$

shorthand notation.

Note that we have to define P for all sets of A , but defining P on each every element is sufficient for inducing in the whole space.

(Claim: There exists a unique probability measure
on all the sets of A that is consistent with the definition.)

(ex) $P(\{1, 2, 4\}) = P(\{1\} \cup \{2\} \cup \{4\}) = P(1) + P(2) + P(4)$

$\nwarrow \quad \swarrow$ pairwise disjoint.
"Decomposed Uniquely".

$$P(\Omega) = P\left(\bigcup_{i=1}^n \{i\}\right) = \sum_{i=1}^n P(\{i\}) = 1. \Rightarrow \text{This is a probability measure.}$$

ii) Geometric Distribution

Countably infinite set $\Omega = \{1, 2, 3, \dots\}$, $A = 2^{\Omega}$.

$P(k)$ = Probability it takes k coinflips to get heads

$$= \alpha(1-\alpha)^{k-1} = 1/2^k \text{ for fair coin.}$$

\hookrightarrow probability of getting heads.

Similar to i), it can be decomposed to single-element sets in a unique way to sum up the probabilities to get the probability of the whole set.

So, P is uniquely defined probability measure on A .

$$P(\Omega) = P\left(\bigcup_{i=1}^{\infty} \{\{i\}\}\right) = \sum_{i=1}^{\infty} P(\{\{i\}\}) = \sum_{i=1}^{\infty} \alpha(1-\alpha)^{i-1} = \frac{\alpha}{1-(1-\alpha)} = 1.$$

iii) Exponential Distribution

Uncountable Set $\Omega = [0, \infty)$, $A = \mathcal{B}([0, \infty))$

$P([0, x]) := \underbrace{1 - e^{-x}}_{\forall x > 0}$ if $x > 0$, it is automatically defined when $x=0$, because $P([0, 0]) = P(\emptyset) = 0$.

Sets of this form also "generates" Borel σ -algebra.

So, in fact, defining a probability measure on $\mathcal{B}([0, \infty))$ alone uniquely induces a probability measure on the whole σ -algebra.

Note that $P(\{x\}) = 0 \quad \forall x \geq 0$.

$$P(\Omega) = P([0, \infty)) = \lim_{x \rightarrow \infty} P([0, x]) = \lim_{x \rightarrow \infty} 1 - e^{-x} = 1.$$

* Lebesgue measure (on \mathbb{R}).

$\Omega = \mathbb{R}$, $A = \mathcal{B}(\mathbb{R})$.

$$\mu((a, b)) = b - a \quad \text{for any } a, b \in \mathbb{R}, a < b.$$

cf. This is used for Lebesgue integral. idea: $dx \rightarrow dm^{m=m}$.
 Riemann Lebesgue.

\Rightarrow This is not a probability measure.
 "length of interval".

"most natural way to define the measure".

1.5-b) Measure Theory: Basic Properties of Measures.

Thm Basic Properties of measures.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space.

i) Monotonicity

If $E, F \in \mathcal{A}$ and $E \subset F$, then $\mu(E) \leq \mu(F)$

pf) $\mu(F) = \mu(E \cup (E^c \cap F)) = \mu(E) + \mu(E^c \cap F) \geq \mu(E)$.
 ↳ measure is non-negative.



ii) Subadditivity

If $E_1, E_2, \dots \in \mathcal{A}$, then $\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$

↳ Arbitrary sets. not necessarily pairwise disjoint.



pf). The disjunctification trick.

Sets F_k defined by $F_1 = E_1$, $F_2 = E_2 - E_1$, $F_3 = E_3 - (E_1 \cup E_2)$...

are disjoint, belong to $\bigcup_{i=1}^{\infty} E_i$, and satisfy $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$.

Using this trick, $\mu(\bigcup_{i=1}^{\infty} E_i) = \mu(\bigcup_{i=1}^{\infty} F_i) \underset{F_i \text{ are disjoint.}}{\leq} \sum_{i=1}^{\infty} \mu(F_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$

iii) Continuity from below

If $E_1, E_2, \dots \in \mathcal{A}$ and $E_1 \subset E_2 \subset \dots$, then $\mu(\bigcup_{i=1}^{\infty} E_i) = \lim_{i \rightarrow \infty} \mu(E_i)$.

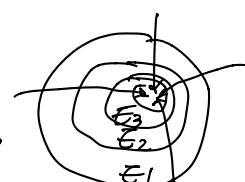


iv) Continuity from above

If $E_1, E_2, \dots \in \mathcal{A}$ and $E_1 \supset E_2 \supset \dots$ and $\mu(E_1) < \infty$,

then $\mu(\bigcap_{i=1}^{\infty} E_i) = \lim_{i \rightarrow \infty} \mu(E_i)$.

Note that it holds for every probability measure.



Ex) Lebesgue. Let $E_i = [i, \infty)$. then $\mu(\bigcap_{i=1}^{\infty} E_i) = 0 \neq \lim_{i \rightarrow \infty} \mu(E_i)$.
 ↳ violates $\mu(E_i) < \infty$.

1.7 Measure Theory: More properties of Probability Measures.

Let (Ω, \mathcal{A}, P) be a probability measure space, with $E, F, E_i \in \mathcal{A}$.

i) $P(E \cup F) = P(E) + P(F)$ if $E \cap F = \emptyset$.

ii) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

iii) $P(E) = 1 - P(E^c)$

iv) $P(E \cap F^c) = P(E) - P(E \cap F)$

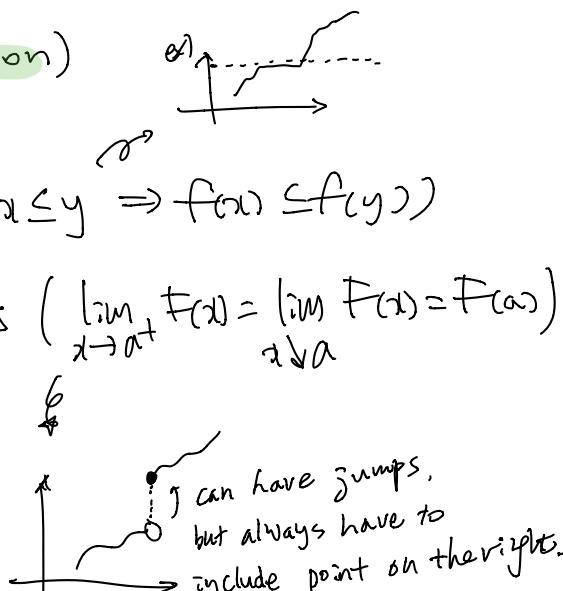
v) Inclusion-Exclusion Formula,

$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_i P(E_i) - \sum_{i < j} P(E_i \cap E_j) + \sum_{i < j < k} P(E_i \cap E_j \cap E_k) - \dots + (-1)^{n+1} P(E_1 \cap E_2 \cap \dots \cap E_n).$$

1.8 Measure Theory: CDFs and Borel Probability Measures

Def. A Borel Measure on \mathbb{R} is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
 (Probability) (Probability)

Def. A CDF (Cumulative Distribution Function) is a function $F: \mathbb{R} \rightarrow \mathbb{R}$ such that i) F is nondecreasing ($x \leq y \Rightarrow F(x) \leq F(y)$)
 ii) F is right-continuous ($\lim_{x \rightarrow a^+} F(x) = \lim_{x \downarrow a} F(x) = F(a)$)
 iii) $\lim_{x \rightarrow \infty} F(x) = 1$.
 iv) $\lim_{x \rightarrow -\infty} F(x) = 0$.



Thm i) If F is a CDF,
then there is a unique Borel probability measure on \mathbb{R}
such that $P((-\infty, x]) = F(x) \quad \forall x \in \mathbb{R}.$

ii) If P is a Borel probability measure on \mathbb{R} ,

then there is a unique CDF F

such that $F(x) = P((-\infty, x]) \quad \forall x \in \mathbb{R}.$

That is, there is an equivalence

between CDF and Borel probability measure.