

Lecture 2. Conditioning & Bayes' Rule

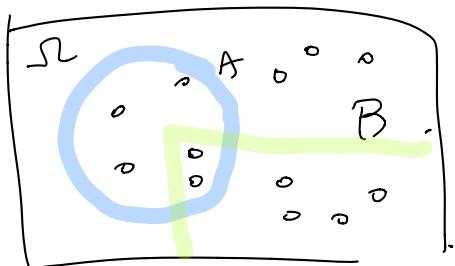
(2.1) Lecture Overview

"New knowledge cause our "beliefs" to change;
original probabilities must be replaced to take
new information into account."

(2.2) Conditional Probabilities

* The idea of conditioning: Use new info to revise a model

Example Assume 12 equally likely outcomes.



$$P(A) = 5/12, \quad P(B) = 6/12.$$

But if told B occurred,

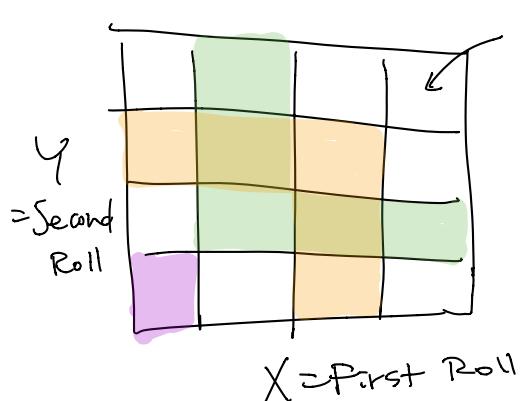
$$P(A|B) = 2/6, \quad P(B|B) = 6/6.$$

Def Conditional Probability.

$P(A|B)$ = "Probability of A, given that B occurred"

$$= P(A \cap B) / P(B) \quad (\text{defined only when } P(B) > 0).$$

(2.3) A Die Roll Example.



Yb. Let B be the event: $\min(X, Y) = 2$.

$$\text{Let } M = \max(X, Y).$$

$$\therefore P(M=1 | B) = 0.$$

$$\therefore P(M=3 | B) = \frac{P(M=3 \text{ and } B)}{P(B)} = \frac{2/16}{5/16} = \frac{2}{5}$$

2.4 Conditional Probabilities obey the same axioms.

i) $P(A|B) \geq 0$. (assuming $P(B) > 0$).

$$\text{ii)} P(S|B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$

$$\text{iii)} P(B|B) = \frac{P(B \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1. \quad "B \text{ as the new sample space}"$$

iv) if $A \cap C = \emptyset$, then $P(A \cup C|B) = P(A|B) + P(C|B)$.

$$= \frac{P((A \cup C) \cap B)}{P(B)} = \frac{P(A \cap B) \cup (C \cap B)}{P(B)} = \frac{P(A \cap B) + P(C \cap B)}{P(B)}$$

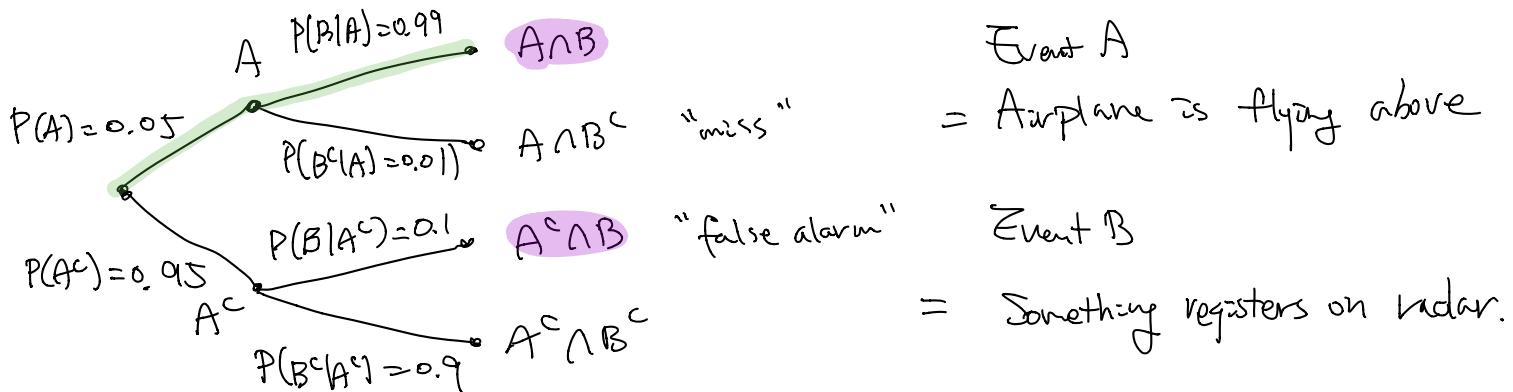
$$A \cap B \cap C \cap B = A \cap C \cap B \cap B = \emptyset \cap B = \emptyset.$$

(Same also for finite or countable additivity).

Remark As conditional probability satisfy all the axioms above, any form or theorem we ever derive for probabilities also remain true for conditional probabilities.

2.5 A Radar example & 3 basic tools.

Example. Airplane registering on radar.



$$\text{: i) } P(A \cap B) = P(A)P(B|A) = 0.05 \cdot 0.99$$

\rightsquigarrow the multiplication rule

$$\text{: ii) } P(B) = P(A \cap B) + P(A^c \cap B) = P(A)P(B|A) + P(A^c)P(B|A^c)$$

$$= 0.05 \cdot 0.99 + 0.95 \cdot 0.1.$$

\rightsquigarrow Total probability theorem

$\text{: iii) } P(A|B) = \text{"Airplane is really there when registered on radar"}$

$$= P(A \cap B)/P(B) \approx 0.34 \text{ } \rightsquigarrow \text{Not really convincing!}$$

\rightsquigarrow The Bayes' rule

[Qb] The multiplication Rule.

$$P(A \cap B) = P(B) \cdot P(A|B)$$

Probability
of two events
to both occur

Probability
of second event
to occur

$$P(A|B)$$

Probability
of first event to occur
given second event occurred.

$$= P(A) \cdot P(B|A)$$

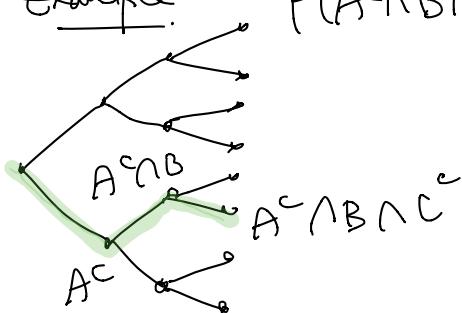
"Free to choose A or B to be the first event".

Example:

$$P(A^c \cap B \cap C^c) = P(A^c \cap B) \cap C^c)$$

$$= P(A^c \cap B) P(C^c | A^c \cap B)$$

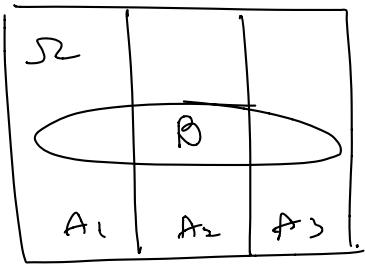
$$= P(A^c) P(B | A^c) P(C^c | A^c \cap B).$$



$$\therefore \text{For } n \text{ events, } P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \prod_{i=2}^n P(A_i | \bigcap_{j=1}^{i-1} A_j)$$

2.7 Total Probability Theorem

Example. Partition of sample space into A_1, A_2, A_3 .



It has $P(A_i)$ and $P(B|A_i)$ for every i .

$$\begin{aligned} P(B) &= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) \\ &= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) \end{aligned}$$

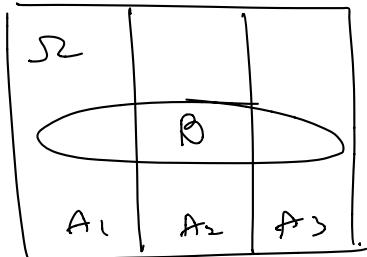
\therefore For n events, $P(B) = \sum_i P(A_i)P(B|A_i)$, if $\sum_i P(A_i) = 1$.

\Rightarrow "Weighted average of $P(B|A_i)$, weighted by $P(A_i)$ "

Remark. This theorem is also true on infinite countable sets.

2.8 Bayes' Rule

Example. Partition of sample space into A_1, A_2, A_3 .



It has $\underbrace{P(A_i)}$ and $\underbrace{P(B|A_i)}$ for every i .

Initial "beliefs" Additional information



How to revise the "beliefs", given that B occurred?

$$\text{Revised "belief"} \quad P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)}.$$

$$\therefore P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_i P(A_i)P(B|A_i)}$$

\rightsquigarrow Systematic approach
for incorporating new evidence

* Bayesian Inference

- Initial beliefs $P(A_i)$ on possible causes of an observed event B .
- Model of the world under each A_i : $P(B|A_i)$.

$$A_i \xrightarrow{\text{model}} B \quad " \text{How likely } B \text{ is going to occur} \\ P(B|A_i) \quad \text{in each particular situation } A_i."$$

- Draw conclusions from causes

$$B \xrightarrow{\text{inference}} A_i \quad " \text{How likely a particular scenario } A_i \text{ is} \\ P(A_i|B) \quad \text{to occur event } B"$$