Date
Page

2) Given that the data is not standardized and columns have zero mean, obtain the relation between the singular values obtained by applying SVD on the data matrix and the eigenvalues of the covariance matrix

Let's denote matrix as X and covariance matrix as C
Dimensions of $X \rightarrow (m \times n)$
here m is number of data points,
n is number of variables

After performing SVD on data matrix, we get:
$X = U * S * V^T$
$U \Rightarrow$ matrix of left singular vectors
$S \Rightarrow$ diagonal matrix of singular values
$V^T \Rightarrow$ transpose of matrix with right singular vectors

Since data is centred, covariance matrix C is given by

$$C = \frac{1}{m-1} * X^T * X$$

Substituting X in C we get

$$C = \frac{1}{m-1} * (U S V^T)^T * (U S V^T)$$

$$= \frac{1}{m-1} * (V S U^T) * (U S V^T)$$

$$= \frac{1}{m-1} * (V * S^2 * V^T)$$

Thus the value obtained for C is similar to eigen value decomposition of X as $V^* \Lambda^* V^T$

On comparing we see that,

$\left(\frac{1}{m-1} S^2\right)$ is diagonal matrix of eigen values of C

$\therefore$ Eigenvalues of C are proportional to square of singular values of X

Sources / References :—

stacks.stackexchange ⇒ Stack exchange
Relationship between SVD and PCA. How to use SVD to perform PCA.

6) a) $X = U\Sigma V^T = U_1\Sigma_1 V_1^T + U_2\Sigma_2 V_2^T$

$\Sigma_2 \rightarrow$ diagonal matrix with singular values 0

$\therefore \Sigma_2$ is a zero matrix and thus we can approximate the singular value decomposition of X as

$$X = U_1 \Sigma_1 V_1^T$$

From property of SVD, eigenvectors are orthonormal to each other, ie for any eigenvectors corresponding to different eigenvalues U_i and U_j, the inner product is 0.

$\therefore \; u_i^T u_j = 0 \; , \quad \text{if } i = j$

We need to prove that $U_{2,i}^T x = 0$ where $U_{2,i}$ is ith column of U_2. Since x can be represented in terms of M largest singular values, it ~~only~~ a the eigenvector spanning U_1. Also eigen vectors are orthonormal. Thus product of any vector with U_2 with x will be zero

$$U_{2,i}^T x = 0$$

b) Given $\quad X = U \Sigma V^T$

After transformation $\quad Z = \Sigma^{-1/2} U^T X$

Covariance of a matrix, $\quad \mathrm{Cov}(Z) = \dfrac{Z^T Z}{n-1}$

Substituting the transformation in above eqn,

$$\mathrm{Cov}(Z) = \frac{(\Sigma^{-1/2} U^T X)^T (\Sigma^{-1/2} U^T X)}{(n-1)}$$

$$= \frac{X^T U (\Sigma^{-1/2})^T \Sigma^{-1/2} U^T X}{(n-1)}$$

$$= \frac{X^T U U^T X}{(n-1)\Sigma}$$

Since eigen vector are orthonormal and inner product with itself gives an identity matrix

$$Cov(Z) = \frac{V \Sigma U U^T \Sigma V^T}{(n-1)\Sigma}$$

$$= \frac{V \Sigma V^T}{n-1}$$

Here $\frac{\Sigma}{n-1}$ is a constan and taking the product VV^T as identity matrix we obtain the covariance matix.

$$Cov(Z) = \frac{\Sigma}{n-1} I$$

where I is identity matrix